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Yetter-Drinfel'd modules for group-cograded multiplier Hopf algebras

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Abstract

We give a representation-theoretic and a categorical interpretation of the Drinfel'd double into the framework of group-cograded multiplier Hopf algebras. The Drinfel'd double as constructed by Zunino for a finite-type Hopf group-coalgebra is an example of this construction in the sense that the components of the group-cograded multiplier Hopf algebras are unital and finite-dimensional algebras and the admissible action is related with the adjoint action of the group on itself.

Key words: Drinfel'd double, Hopf group-coalgebras, group-cograded multiplier Hopf algebras, Yetter-Drinfel'd modules.
Mathematics Subject Classification: 16W30, 17B37.

1 Introduction

Let G be any group. The prototype of a G-cograded multiplier Hopf algebra is given by the multiplier Hopf algebra $K(G)$ of complex valued functions with finite support in G. Recall that the product in $K(G)$ is pointwise. The algebra $K(G)$ has no unit, except where G is finite. The multiplier algebra $M(K(G))$ of $K(G)$ is the largest algebra with unit in which $K(G)$ sits as a dense two-sided ideal. Clearly $M(K(G))$ is given by the algebra of all complex functions on G. The comultiplication Δ on $K(G)$ is given by the formula $(\Delta(f))(p,q) = f(pq)$ for all $f \in K(G)$ and $p,q \in G$. We have $\Delta(f) \in M(K(G) \otimes K(G))$. If G is finite, the multiplier algebra $M(K(G) \otimes K(G))$ equals $K(G) \otimes K(G)$.
In this paper we work with more general G-cograded multiplier Hopf algebras in the sense of [A-De-VD, Definition 1.1]. Essentially, a multiplier Hopf algebra B is G-cograded if there is a central, non-degenerate embedding $I : K(G) \to M(B)$. We require that I respects the comultiplication in the sense that $\Delta(I(f)) = (I \otimes I)(\Delta(f))$

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for all $f \in K(G)$. On the left hand side, we have extended the homomorphism Δ from B to the multiplier algebra $M(B)$, in the sense of [VD1-A5]. Similary, on the right hand side, we have extended the homomorphism $I \otimes I$ from $K(G) \otimes K(G)$ to $M(K(G) \otimes K(G))$. A G-cograded multiplier Hopf algebra is denoted as $B = \bigoplus_{p \in G} B_p$ where B_p are algebras with a non-degenerate product. For all $p, q \in G$ we have $\Delta(B_{pq})(1 \otimes B_q) = B_p \otimes B_q$. Observe that the multiplier algebra $M(B) = \prod_{p \in G} M(B_p)$.
It is shown in [A-De-VD] that a Hopf group-coalgebra as introduced by Turaev in [T], is a special case of a group-cograded multiplier Hopf algebra. Therefore, a lot of results for Hopf group-coalgebras follow from the more general results of multiplier Hopf algebras. E.g. the Drinfel'd double as constructed in [Z1] is an example of the Drinfel'd double construction D^π in [De-VD3, Theorem 3.8]. In the paper [De-VD3], we consider *any* group-cograded multiplier Hopf algebra B with an *admissible* action of the group. If we take the components of B as unital finite-dimensional algebras and we require the admissible action to be a "crossing", we recover the construction as given in [Z1].
For convenience of the reader, we recall the construction of the Drinfel'd double D^π. We start with a G-cograded multiplier Hopf algebra B. So B has the form $B = \bigoplus_{p \in G} B_p$. Assume that there is a group homomorphism $\pi : G \to \mathrm{Aut}(B)$, where $\mathrm{Aut}(B)$ denotes the group of algebra automorphisms on B.
We call π an *admissible* action of G on B if also the following requirements hold

(1) $\Delta(\pi_p(b)) = (\pi_p \otimes \pi_p)(\Delta(b))$ for all $b \in B$

(2) $\pi_p(B_q) = B_{\rho_p(q)}$ where ρ is an action of the group G on itself

(3) $\pi_{\rho_p(q)} = \pi_{pqp^{-1}}$
This means that the map π takes care of ρ not being the adjoint action. If ρ is the adjoint action itself, π is called a *crossing*.

Take B and π as above. We consider a new regular multiplier Hopf algebra on B by deforming the comultiplication while the algebra structure on B is kept. The deformation of the comultiplication of B depends on the action π, in the following way

$$\widetilde{\Delta}(b)(1 \otimes b') = (\pi_{q^{-1}} \otimes \iota)(\Delta(b)(1 \otimes b'))$$
$$(1 \otimes b')\widetilde{\Delta}(b) = (\pi_{q^{-1}} \otimes \iota)((1 \otimes b')\Delta(b))$$

for all $b \in B$ and $b' \in B_q$.

Further we assume that $\langle A, B\rangle$ is a pairing of two regular multiplier Hopf algebras, in the sense of [Dr-VD]. As before, B is G-cograded and π is an admissible action of G on B. We consider a *twisted tensor product* algebra on the tensor product $A \otimes B$. This means that the trivial flip map is replaced by a more general twist map $R : B \otimes A \to A \otimes B$. This map R satisfies the appropriate compatibility conditions with respect to the multiplications on A and B. The twist map R depends on the pairing $\langle A, B\rangle$, as well as on the action π. For an explicit expression of the formula $R(b \otimes a)$, we refer to [De-VD3, Definition 3.4]. The algebra defined in this way is denoted as $A \bowtie B$. Finally, this algebra has the structure of a regular multiplier Hopf algebra if we consider the comultiplication $\overline{\Delta}$ on $A \bowtie B$ where $\overline{\Delta}(a \bowtie b) = \Delta^{cop}(a)\widetilde{\Delta}(b)$ in $M((A \bowtie B) \otimes (A \bowtie B))$. We observe that for $a \in A$, we use the opposite comultiplication of A. For $b \in B$, we use the deformation $\widetilde{\Delta}(b)$ as defined above.
In this paper, we characterize the modules of D^π. This is done from the point of view that D^π is a non-trivial twisted tensor product on the space $A \otimes B$, see above. We require a natural condition on the pairing $\langle A, B\rangle$ which generalizes the dual bases for finite-dimensional Hopf algebras. Then the characterization of the left modules over D^π can be rephrased purely in terms of the multiplier Hopf algebra B, without any reference to the multiplier Hopf algebra A. The compatibility conditions for these *π-Yetter-Drinfel'd modules* over B are given in Theorem 2.1. When we require the admissible action to be a *crossing* (this means that π is related with the adjoint action of the group G on itself) and we assume that the components are unital and finite-dimensional, the characterization in Theorem 2.1 can be put in the setting of [Z2, Section 8]. We notice that in this special situation our Drinfel'd double construction is isomorphic with the so-called mirror construction, given in [Z2, Section 9]. If π is a crossing of the group G on an arbitrary G-cograded multiplier Hopf algebra B, we have that the Drinfel'd double D^π is again G-cograded and there is a natural crossing of G on D^π, see [De-VD3, Proposition 3.13]. Furthermore, we have that D^π is π-quasitriangular, see [De-VD-W, Theorem 3.12]. The categorical interpretation of this quasitriangularity is translated to the π-Yetter-Drinfel'd modules over B, see Theorem 3.1. Our braiding is in the sense of the centre-construction of a category as given in [K, Sections XIII 4-5].

All algebras are considered over the field $\mathbb{C}$. We do not assume that an algebra A has a unit. But we require that the multiplication, considered as a bilinear map is non-degenerate. The multiplier algebra, denoted as $M(A)$, is the largest algebra with a unit in which A is contained as a dense two-sided ideal. The identity in any (multiplier) algebra is denoted by 1. The identity map is denoted as ι.

For a regular multiplier Hopf algebra A (i.e. with a bijective antipode) we denote the comultiplication by Δ. Observe that $\Delta : A \to M(A \otimes A)$. However, by the defining conditions on Δ, we have for all $a, b \in A$ that $\Delta(a)(1 \otimes b)$, $\Delta(a)(b \otimes 1)$, $(1 \otimes b)\Delta(a)$ and $(b \otimes 1)\Delta(a)$ are elements in $A \otimes A$. It can be motivated, see e.g. [Dr-VD-Z, Section 2] that these elements are denoted by *Sweedler notation*, e.g. $\Delta(a)(1 \otimes b) = \sum a_{(1)} \otimes a_{(2)}b$. In an expression, denoted by Sweedler notation, one always has to make sure that at most one factor $a_{(k)}$ is not multiplied ("covered") by an element in A.

When we consider a *module* V over an algebra A, we always mean a *left* module which is *unital.* A (left) A-module V is unital if $A \triangleright V = V$. By the regularity conditions on A, this implies that for all $x \in V$, we have an element $e \in A$ such that $x = e \triangleright x$. For details, we refer to [Dr-VD-Z, Section 3]. Therefore, the comultiplication on A can be used to make the category of (left) A-modules into a tensor category with unit.

Basic references
The material needed for reading this paper is given in the following basic references. For (regular) multiplier Hopf algebras, we refer to [VD1] and [VD-Z1]. The group-cograded multiplier Hopf algebras are introduced in [A-De-VD] and studied in [De-VD-W]. They generalize the Hopf group-coalgebras, as introduced by Turaev in [T]. The Drinfel'd double construction into the framework of multiplier Hopf algebras is associated to a pairing, see [Dr-VD] and [De-VD1]. To have the analogous properties as for the Drinfel'd double of a finite-dimensional Hopf algebra, we assume that the pairing $\langle A, B\rangle$ of two multiplier Hopf algebras has a canonical multiplier $W \in M(A \otimes B)$. Essentially, the multiplier W takes the role of the dual bases in the finite-dimensional case. For details, we refer to [De-VD2, Section 4]. The Drinfel'd double construction for group-cograded multiplier Hopf algebras is done in [De-VD3].

2 π-Yetter-Drinfel'd modules

Let $\langle A, B\rangle$ be a pair of multiplier Hopf algebras. Let G denote a group and assume that B is G-cograded. As an algebra, we write $B = \bigoplus_{p\in G} B_p$. Let π be an admissible action of G on B, in the sense of [De-VD3, Definition 2.6]. So for all $p \in G$, we have an automorphism π_p on B which respects the multiplication and the comultiplication of B. Furthermore for all $p, q \in G$, we have $\pi_p(B_q) = B_{\rho_p(q)}$ where ρ is an automorphism of G on itself. We require that $\pi_{\rho_p(q)} = \pi_{pqp^{-1}}$ for all $p, q \in G$. In the framework of Hopf group-coalgebras, one sets $\rho_p(q) = pqp^{-1}$ for all $p, q \in G$, see [T]. Let D^π denote the Drinfel'd double as constructed in [De-VD3, Theorem 3.8]. As an algebra, D^π is a twisted tensor product on the linear space $A \otimes B$. Therefore, a left D^π-module is nothing but a linear space V with a left B-module structure, denoted as $B.V$, as well as a left A-module structure, denoted as $A \triangleright V$. For all $a \in A$, $b \in B_p$ and $x \in V$, the following compatibility equation yields

$$b \cdot (a \triangleright x) = \sum a_i \triangleright (b_i \cdot x)$$

where $\sum a_i \otimes b_i = T(b \otimes a) = \sum(\pi_{p^{-1}}(b_{(1)}) \blacktriangleright a \blacktriangleleft S^{-1}(b_{(3)})) \otimes b_{(2)}$.
The actions $\blacktriangleright$ and $\blacktriangleleft$ are the regular actions of B on A, associated to the pairing $\langle A, B\rangle$, see [Dr-VD].

We rephrase the above compatibility condition in terms of the multiplier Hopf algebra B, without any reference to the paired multiplier Hopf algebra A. We need the notion of a right-B-comodule. In Hopf algebra theory, it is possible to define the structure of a comodule on a vector space. In the setting of multiplier Hopf algebras however, more structure is needed. In [VD-Z2], the setting is that of an algebra V and a regular multiplier Hopf algebra B. Then, a right coaction of B on V is an injective linear map $\Gamma : V \to M(V \otimes B)$ satisfying

(i) $\Gamma(V)(1 \otimes B) \subseteq V \otimes B$ and $(1 \otimes B)\Gamma(V) \subseteq V \otimes B$

(ii) $(\Gamma \otimes \iota)\Gamma = (\iota \otimes \Delta)\Gamma$

The algebra structure of V is needed to be able to consider the multiplier algebra $M(V \otimes B)$. It would be too restrictive to assume that the coaction Γ has range in the tensor product itself.
Observe that Condition (i) is used to give a meaning to the left hand side of the equation in Condition (ii). The link between left A-modules and right B-comodules is given by the so-called canonical multiplier W in $M(A \otimes B)$, in the sense of [De-VD2,

Section 4] and [De2, Section 2]. A multiplier W in $M(A \otimes B)$ is called canonical for the pairing $\langle A, B\rangle$ if W is invertible in $M(A \otimes B)$ and if $\langle W, a \otimes b\rangle = \langle a, b\rangle$ for all $a \in A$ and $b \in B$. Let B be a finite-dimensional Hopf algebra and consider $A = B'$ where B' denotes the dual Hopf algebra of B. If $\{f_i\} \subset B'$ and $\{e_i\} \subset B$ are dual bases, then $W = \sum f_i \otimes e_i$ is the canonical element in $B' \otimes B$ for the natural pairing $\langle B', B\rangle$.

2.1 Theorem Consider the notations and the assumptions as above. We have that V is a (left) D^π-module if and only if V is a left B-module for the action $B.V$ and V is a right B-comodule for the right coaction $\Gamma : V \to M(V \otimes B)$ such that the left action and the right coaction of B on V satisfy the compatibility relation

$$\sum (d_{(1)} \cdot \otimes c d_{(2)})\Gamma(v) = \sum (1 \otimes c)\Gamma(d_{(2)} \cdot v)(1 \otimes \pi_{p^{-1}}(d_{(1)}))$$

for all $v \in V$, $c \in B_q$ and $d \in B_{pq}$ (where $p, q \in G$).

Proof. In [De2, Theorem 2.3], we have proven that a left A-module V is determined by a unique right B-comodule structure on V in the following way. Let $A \triangleright V$ denote a left A-module, then there is a right B-comodule $\Gamma : V \to M(V \otimes B)$ so that

$$a \triangleright v = (\iota \otimes \langle a, \cdot\rangle)\Gamma(v)$$

for all $A \in A$ and $v \in V$.
On the right hand side of the above equation, we have that $\Gamma(v)$ sits in the multiplier algebra $M(V \otimes B)$. However, by the regularity conditions on the pairing $\langle A, B\rangle$, there is an element $b \in B$ such that the right hand side should be read as $(\iota \otimes \langle a, \cdot\rangle)(\Gamma(v)(1 \otimes b))$. As we assume for the coaction Γ that $\Gamma(V)(1 \otimes B) \subseteq V \otimes B$, the expression $(\iota \otimes \langle a, \cdot\rangle)(\Gamma(v)(1 \otimes b))$ determines an element in V. The compatibility condition between the left B-module structure and the left A-module structure on a left D^π-module V can be rephrased as follows. For all $a \in A$, $b \in B_p$ and $v \in V$, we have

$$(\iota_V \otimes \langle a, \cdot\rangle)(b \cdot \otimes 1)\Gamma(v) = b \cdot ((\iota_V \otimes \langle a, \cdot\rangle)\Gamma(v)) = b \cdot (a \triangleright v) = \sum a_i \triangleright (b_i \cdot v)$$

$$= \sum (\iota_V \otimes \langle a_i, \cdot\rangle)\Gamma(b_i \cdot v) = \sum (\iota_V \otimes \langle \pi_{p^{-1}}(b_{(1)}) \blacktriangleright a \blacktriangleleft S^{-1}(b_{(3)}), \cdot\rangle)\Gamma(b_{(2)} \cdot v)$$

$$= \sum \langle a_{(1)}, S^{-1}(b_{(3)})\rangle \langle a_{(3)}, \pi_{p^{-1}}(b_{(1)})\rangle (\iota_V \otimes \langle a_{(2)}, \cdot\rangle)\Gamma(b_{(2)} \cdot v)$$

$$= (\iota_V \otimes \langle a, \cdot\rangle)\left(\sum (1 \otimes S^{-1}(b_{(3)}))\Gamma(b_{(2)} \cdot v)(1 \otimes \pi_{p^{-1}}(b_{(1)}))\right).$$

Observe that the above equations are in $V \otimes B$. All decompositions are well-covered by the use of the regularity conditions on the pairing $\langle A, B\rangle$. As the pairing is a non-degenerate linear form on $A \otimes B$, we obtain the following equation in $V \otimes B$. For any $p, q \in G$ and $b \in B_p$, $b' \in B_q$, we have for all $v \in V$

$$(b \cdot \otimes b')\Gamma(v) = \sum (1 \otimes b' S^{-1}(b_{(3)}))\Gamma(b_{(2)} \cdot v)(1 \otimes \pi_{p^{-1}}(b_{(1)})).$$

From the axioms on a regular multiplier Hopf algebra, we have $(1 \otimes B)\Delta(B) = B \otimes B$, see [VD1]. By the use of [VD1, Lemma 5.5], the equation above is equivalent to the following statement. For any $p, q \in G$ and $d \in B_{pq}$, $c \in B_q$ we have for all $v \in V$

$$\left(\sum d_{(1)} \cdot \otimes c d_{(2)}\right) \Gamma(v) = \sum (1 \otimes c)\Gamma(d_{(2)} \cdot v)(1 \otimes \pi_{p^{-1}}(d_{(1)})).$$

By the use of the G-cograding and the admissible action π, we have that in the left hand side $(1 \otimes c)\Delta(d) \in B_p \otimes B_q$. In the right hand side we have $(\pi_p(c) \otimes 1)\Delta(d) \in B_{\rho_p(q)} \otimes B_{\rho_p(q^{-1})pq}$. $\square$

2.2 Definition Let B be a G-cograded multiplier Hopf algebra and let π be an admissible action of G on B. An algebra V with a left B-module structure and a right B-comodule structure is called a π-Yetter-Drinfel'd module if the compatibility condition of Theorem 2.1 is satisfied. The set of all π-Yetter-Drinfel'd modules over B is denoted as ${}_B\pi(\mathcal{YD})^B$.

In the characterization of Theorem 2.1, we have dispensed with the Drinfel'd double D^π. So we do not need to assume that B is paired with another multiplier Hopf algebra to define the π-Yetter-Drinfel'd modules over B.

2.3 Remark Let B be a finite-type Hopf group-coalgebra and assume that π is a crossing, i.e. $\pi_p(B_q) = B_{pqp^{-1}}$ for all $p, q \in G$. In Theorem 2.1, the multiplier Hopf algebra A can be taken as the (usual) Hopf algebra $B^* = \bigoplus_{p \in G}(B_p)'$, where $(B_p)'$ denotes the linear dual of B_p. The formula in Theorem 2.1 is now given as in [Z2, Section 8]. When G is given by the trivial group, we recover the well-known characterization of Yetter-Drinfel'd modules for a finite-dimensional Hopf algebra. In these settings, we don't need an underlying algebra structure on the Yetter-Drinfel'd modules because the comultiplication of B is a map $\Delta : B \to B \otimes B$. Furthermore, there is always a canonical multiplier W in $M(B^* \otimes B)$. More precisely, $W = \sum_{p \in G} f_{p,i} \otimes e_{p,i}$ where for all $p \in G$, the sets $\{f_{p,i}\}$ in $(B_p)'$ and $\{e_{p,i}\}$ in B_p are dual bases.

3 The braided monoidal category ${}_B\pi(\mathcal{YD})^B$

As before, we consider a multiplier Hopf algebra B which is cograded by a group G. As an algebra we have $B = \bigoplus_{p\in G} B_p$ where B_p is a subalgebra with a non-degenerate product. Let π denote an admissible action of G on B. We consider the category ${}_B\pi(\mathcal{YD})^B$ of π-Yetter-Drinfel'd modules over B, in the sense of Definition 2.2. The morphisms in ${}_B\pi(\mathcal{YD})^B$ are linear maps which are left B-module morphisms as well as right B-comodule morphisms.

If B is paired with a multiplier Hopf algebra A, we have proven in Theorem 2.1 that the category ${}_B\pi(\mathcal{YD})^B$ is given by the left unital modules over the Drinfel'd double D^π, associated to the pair $\langle A, B\rangle$. We made use of the canonical multiplier W in $M(A \otimes B)$. The morphisms between left D^π-modules, correspond to the morphisms in ${}_B\pi(\mathcal{YD})^B$, use [De2, Theorem 2-3]. By the bialgebra structure on D^π, the modules over D^π have the structure of a monoidal tensor category. Therefore, it is expected that the category ${}_B\pi(\mathcal{YD})^B$ is also a monoidal tensor category. Let V be in ${}_{B_p}\pi(\mathcal{YD})^B$ and let V' be in ${}_{B_q}\pi(\mathcal{YD})^B$, then $V \otimes V'$ is in ${}_{B_{\rho_q(p)q}}\pi(\mathcal{YD})^B$ in the following way

$$b \cdot (v \otimes v') = \sum \pi_{q^{-1}}(b_{(1)}) \cdot v \otimes b_{(2)} \cdot v'$$

for all $b \in B_{\rho_q(p)q}$, $v \in V$ and $v' \in V'$. To determine the right B-comodule structure on the tensor product $V \otimes V'$, we translate the A-module structure on $V \otimes V'$ (A^{cop} is embedded in D^π). This translation is done by the use of the canonical multiplier in $M(A \otimes B)$. We have denoted this multiplier by the letters W and P. For $a \in A$ and $b \in B$, we write $(a \otimes 1)W(1 \otimes b)$ as $\sum aW^{(1)} \otimes W^{(2)}b$ in $A \otimes B$. Following [De2, Proposition 2.2], we have for all $b \in B$, $v \in V$ and $v' \in V'$

$$\begin{aligned}\Gamma(v \otimes v')(1 \otimes 1 \otimes b) &= \sum (W^{(1)} \triangleright (v \otimes v')) \otimes W^{(2)}b \\ &= \sum (P^{(1)} \triangleright v) \otimes (W^{(1)} \triangleright v') \otimes W^{(2)}P^{(2)}b.\end{aligned}$$

We made use of the formula $(\Delta \otimes \iota)(W) = W^{13}W^{23}$. We have obtained the following right B-comodule structure on $V \otimes V'$

$$\Gamma(v \otimes v')(1 \otimes 1 \otimes b) = \Gamma(v')_{23}\Gamma(v)_{13}(1 \otimes 1 \otimes b)$$

for all $v \in V$, $v' \in V'$ and $b \in B$. In the right hand side, we use the leg-numbering notation in the usual way.

One can check that the compatibility condition holds for the tensor object $V \otimes V'$. Moreover, we have that ${}_B\pi(\mathcal{YD})^B$ is a monoidal category. We omit these proofs because we would be repeating the construction of D^π as bialgebra, see [De-VD3].

So far, we have "translated" the multiplier Hopf algebra structure on D^π to determine the monoidal category ${}_B\pi(\mathcal{YD})^B$. Further structures on the multiplier Hopf algebra D^π will correspond directly to properties of its category of modules and can be translated towards the category ${}_B\pi(\mathcal{YD})^B$. Further in this sequel, we assume that the admissible action of G on B is given as a crossing. This means that $\pi_p(B_q) = B_{pqp^{-1}}$ for all $p, q \in G$. However, the components of the G-cograded multiplier Hopf algebra B, denoted as B_p for all $p \in G$, are arbitrary algebras with a non-degenerate multiplication. In this setting, we have that D^π is G-cograded and there is a natural crossing of G on D^π. More precisely, in [De-VD3, Proposition 3.13], we have proven that D^π is G-cograded as follows

$$D^\pi = \bigoplus_{p \in G} (D^\pi)_p \qquad\qquad \text{with } (D^\pi)_p = A \bowtie B_{p^{-1}}.$$

For all $p \in G$, define π'_p on A via the formula $\langle \pi'_p(a), b\rangle = \langle a, \pi_{p^{-1}}(b)\rangle$ for all $a \in A$, $b \in B$. Then, the maps $\pi'_p \otimes \pi_p$ define a crossing of G on D^π. Let W in $M(A \otimes B)$ denote the canonical multiplier of the pair $\langle A, B\rangle$. Let σ be the twist map on $A \otimes B$, extended to $M(A \otimes B)$. It is proven in [De-VD-W, Theorem 3.12] that the embedding $\sigma(W)$ in $M(D^\pi \otimes D^\pi)$ is a generalised π-matrix for D^π, in the sense of [De-VD-W, Definition 3.1]. By the use of [De-VD-W, Section 3.4], this π-quasitriangularity of D^π corresponds to the following properties of the category of (left) modules over D^π. For all $p \in G$, let ${}_p\mathcal{M}$ denote the modules over the algebra $(D^\pi)_p$. Then we have that the category of left modules over D^π is given as ${}_{D^\pi}\mathcal{M} = \prod_{p \in G} {}_p\mathcal{M}$. For all $p \in G$, there is an invertible functor F_p on ${}_{D^\pi}\mathcal{M}$. If $V \in {}_q\mathcal{M}$, then $F_p(V) \in {}_{pqp^{-1}}\mathcal{M}$. As a linear space, we have that $F_p(V)$ equals V. Let the D^π-module structure on V be denoted as $D^\pi \to V$. For an element $(a \bowtie b) \in (D^\pi)_{pqp^{-1}}$, we have $(a \bowtie b) \to F_p(v) = F_p((\pi'_{p^{-1}}(a) \bowtie \pi_{p^{-1}}(b)) \to v)$. A morphism in ${}_q\mathcal{M}$ is sent to itself, now considered as a morphism in ${}_{pqp^{-1}}\mathcal{M}$.
Finally, the π-quasitriangularity of D^π gives the following π-braiding in ${}_{D^\pi}\mathcal{M}$. Let V (resp. V') be in ${}_p\mathcal{M}$ (resp. ${}_q\mathcal{M}$). Then we have

$$\begin{aligned} &t_{V,V'} : V \otimes V' \to F_p(V') \otimes V \text{ such that} \\ &t_{V,V'}(v \otimes v') = \sum F_p(W^{(1)} \triangleright v') \otimes (W^{(2)} \cdot v) \end{aligned}$$

where $A \triangleright V$ (resp. $B \cdot V$) denotes the A-module (resp. B-module) structure on V.

In Theorem 2.1, we have given a characterization for the π-Yetter-Drinfel'd modules, without the use of a pairing and a Drinfel'd double. So, associated to any G-cograded multiplier Hopf algebra B and a crossing π of G on B, we have the following π-braided monoidal tensor category ${}_B\pi(\mathcal{YD})^B$.

3.1 Theorem Let B be a G-cograded multiplier Hopf algebra and let π denote a crossing of G on B. The monoidal category ${}_B\pi(\mathcal{YD})^B$ is π-braided.

Proof. Let V be in ${}_B\pi(\mathcal{YD})^B$. The left action of B on V is denoted as $B \cdot V$. The right coaction of B on V is denoted as $\Gamma : V \to M(V \otimes B)$. If B is paired with another multiplier Hopf algebra A, we already have that the category of the left modules over D^π is a braided tensor category, see above. We rephrase the results on this category, but we dispense with the Drinfel'd double D^π itself. Let V be in ${}_{B_{p^{-1}}}\pi(\mathcal{YD})^B$ and V' is in ${}_{B_{q^{-1}}}\pi(\mathcal{YD})^B$. Then $V \otimes V'$ is a π-Yetter Drinfel'd module over the subalgebra $B_{q^{-1}p^{-1}}$.
For all $p \in G$, there is an invertible function F_p on ${}_B\pi(\mathcal{YD})^B$. For V in ${}_{B_{q^{-1}}}\pi(\mathcal{YD})^B$, we have $F_p(V)$ in ${}_B\pi(\mathcal{YD})^B$. As an algebra, we have that $F_p(V)$ equals V. As a left B-module, $F_p(V)$ lies over the subalgebra $B_{pq^{-1}p^{-1}}$. For $b \in B_{pq^{-1}p^{-1}}$ and $v \in F_p(V)$, we have $b \cdot F_p(v) = F_p(\pi_{p^{-1}}(b) \cdot v)$.
We now find the right B-comodule structure on $F_p(V)$. If B is paired with a multiplier Hopf algebra A, we assume that $W \in M(A \otimes B)$ is the canonical multiplier of this pair. By the uniqueness of the canonical multiplier W, we have for all $p \in G$, $(\pi_p^{'} \otimes \pi_p)(W) = W$. For $v \in V$ and $b \in B$ we have

$$\Gamma(F_p(v))(1 \otimes b) = \sum (W^{(1)} \triangleright F_p(v)) \otimes W^{(2)} b$$

$$= \sum F_p(\pi_{p^{-1}}^{'}(W^{(1)}) \triangleright v) \otimes W^{(2)} b = \sum F_p(W^{(1)} \triangleright v) \otimes \pi_p(W^{(2)}) b$$

$$= (F_p \otimes \pi_p) \left(\sum (W^{(1)} \triangleright v) \otimes W^{(2)} \pi_{p^{-1}}(b) \right) = (F_p \otimes \pi_p)(\Gamma(v)(1 \otimes \pi_{p^{-1}}(b))).$$

Finally the braiding in the category of left D^π-modules gives the following braiding on ${}_B\pi(\mathcal{YD})^B$. For V in ${}_{B_{p^{-1}}}\pi(\mathcal{YD})^B$ and V' in ${}_{B_{q^{-1}}}\pi(\mathcal{YD})^B$, we have $t_{V,V'} : V \otimes V' \to F_p(V') \otimes V$ such that for $v \in V$, $v' \in V'$

$$t_{V,V'}(v \otimes v') = \sum F_p(W^{(1)} \triangleright v') \otimes W^{(2)} \cdot v = \sum F_p(v'^{(1)}) \otimes (v'^{(2)} \cdot v).$$

In the right hand side of this formula, the tensor $v'^{(2)} \cdot v$ should be read as $v'^{(2)} b \cdot v$ where b is chosen in $B_{p^{-1}}$ such that $b \cdot v = v$, see [Dr-VD-Z]. The summation

$\sum v'^{(1)} \otimes v'^{(2)} b$ stands for the element $\Gamma(v')(1 \otimes b)$ in $V' \otimes B$. $\square$

3.2 Remark Suppose that G is given by the trivial group $G = \{e\}$. The G-cograded multiplier Hopf algebra B is a usual multiplier Hopf algebra. In the case that B is finite-dimensional (and so B is a Hopf algebra), Theorem 3.1 recovers the categorical interpretation of the usual Drinfel'd double of B which is equivalent with the centre-construction of B-mod as given in [K, Section XIII.5].

3.3 Examples

3.3.1 G-cograded multiplier Hopf algebras

We first give examples of G-cograded multiplier Hopf algebras with a crossing.

- The Hopf group-coalgebras and their crossing, as considered in [T], are examples of G-cograded multiplier Hopf algebras. This point of view is explained in [A-De-VD, Theorem 1.5]. Let $K(G)$ denote the multiplier Hopf algebra of the complex valued functions with a finite support in G. The product is pointwise and the coproduct is dual to the product in the group. We write $K(G) = \bigoplus_{p \in G} \mathbb{C}\delta_p$. In this case all the components are equal to the trivial algebra $\mathbb{C}$. The natural crossing on $K(G)$ is related with the adjoint action of G on itself.

- Let (A, Δ) denote any multiplier Hopf algebra. Let G be a group which acts on the multiplier Hopf algebra A by means of automorphisms α_p for all $p \in G$. We assume $\alpha_e = \iota$, $\alpha_p(\alpha_q(a)) = \alpha_{pq}(a)$ for all $p, q \in G$ and $a \in A$. Further, the automorphism α_p respects the comultiplication of A in the sense that $\Delta(\alpha_p(a)) = (\alpha_p \otimes \alpha_p)\Delta(a)$ for all $p \in G$ and $a \in A$. Consider the tensor product algebra $B = K(G) \otimes A$ with the trivial product. In [De1, Example 3.3] is given a non-trivial coproduct on $K(G) \otimes A$ as follows

$$\Delta(\delta_p \otimes a)((1 \otimes 1) \otimes (\delta_q \otimes a')) = \sum (\delta_{pq^{-1}} \otimes \alpha_q(a_{(1)})) \otimes (\delta_q \otimes a_{(2)} a')$$

  for all $p, q \in G$ and $a, a' \in A$.
  The multiplier Hopf algebra $B = K(G) \otimes A$ is G-cograded. We have $B = \bigoplus_{p \in G} B_p$ where $B_p = \mathbb{C}\delta_p \otimes A$. Let $\{f_p \mid p \in G\}$ denote a family of automorphisms on A which respect the comultiplication of (A, Δ) and assume furthermore that $f_{pq} = f_p \circ f_q$ and $f_p \circ \alpha_q = \alpha_{pqp^{-1}} \circ f_p$ for all $p, q \in G$. Then, a crossing of G on

B is given by the automorphisms π_p on B where $\pi_p(\delta_q \otimes a) = \delta_{pqp^{-1}} \otimes f_p(a)$ for all $p, q \in G$ and $a \in A$. Observe that the family $\{\alpha_p \mid p \in G\}$ can always be taken to define a crossing on B. In this example all components are equal to the (possible infinite-dimensional) multiplier Hopf algebra A. However, the comultiplication on B in not trivially given by the comultiplication on A. We notice that (B, Δ) has integrals if (A, Δ) has integrals, see [De1, Theorem 1.16.1]. So, in these situations we can consider the pairing $\langle \widehat{B}, B\rangle$ where $\widehat{B}$ denotes the dual multiplier Hopf algebra, in the sense of [VD2]. The pairing $\langle \widehat{B}, B\rangle$ has a canonical multiplier W in $M(\widehat{B} \otimes B)$, see [De-VD2, Proposition 4.12].

3.3.2 π-Yetter-Drinfel'd modules

Let B be a G-cograded multiplier Hopf algebra and let π denote a crossing of G on B. Assume that $\langle A, B\rangle$ is a pair of multiplier Hopf algebras with a canonical multiplier W in $M(A \otimes B)$. The tensor algebra $A \otimes B$ (with trivial product) can be made into a π-Yetter-Drinfel'd module over B as follows. For all $p \in G$ and $b \in G_p$ we set

$$b \cdot (x \otimes y) = \sum (\pi_{p^{-1}}(b_{(1)}) \blacktriangleright x \blacktriangleleft S^{-1}(b_{(3)})) \otimes b_{(2)} y$$

for all $x \in A$ and $y \in B$. Observe that $\blacktriangleright$ and $\blacktriangleleft$ are the regular actions of B on A, associated to the pairing $\langle A, B\rangle$.

$$\Gamma(x \otimes y)((1 \otimes 1) \otimes b) = \sum (W^{(1)} x \otimes y) \otimes W^{(2)} b$$

for all $x \in A$ and $y, b \in B$. This π-Yetter-Drinfel'd module for B corresponds with the left regular module of the Drinfel'd double D^{π} on itself.

Let B be a finite-type Hopf group-coalgebra with a crossing π, in the sense of [T, Section 11]. Then we have $B = \bigoplus_{p \in G} B_p$ where for all $p \in G$, the algebra B_p is unital and finite-dimensional. This multiplier Hopf algebra B is paired with the (usual) Hopf algebra $A = \bigoplus_{p \in G} (B_p)'$ where $(B_p)'$ is the linear dual of B_p. The canonical multiplier W in $M(A \otimes B)$ is given by the formal summation $\sum_{p \in G} f_{p,i} \otimes e_{p,i}$ where $\{f_{p,i}\} \subset (B_p)'$ and $\{e_{p,i}\} \subset B_p$ are dual bases. Consider the tensor algebra $\bigoplus_{p,q \in G} ((B_q)' \otimes B_p)$. This algebra is a π-Yetter-Drinfel'd module for B in the following way. For all $p \in G$ and $b \in B_p$, $f \in A$ and $y \in B$ we set

$$b \cdot (f \otimes y) = \sum f(S^{-1}(b_{(3)}) \ \cdot \ \pi_{p^{-1}}(b_{(1)})) \otimes b_{(2)} y$$

$$\Gamma(f \otimes y)((1 \otimes 1) \otimes b) = \sum_i (f_{p,i} f \otimes y) \otimes e_{p,i} b.$$

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