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Yetter-Drinfel'd modules for group-cograded multiplier Hopf algebras

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Abstract

We give a representation-theoretic and a categorical interpretation of the Drinfel'd double into the framework of group-cograded multiplier Hopf algebras. The Drinfel'd double as constructed by Zunino for a finite-type Hopf group-coalgebra is an example of this construction in the sense that the components of the group-cograded multiplier Hopf algebras are unital and finite-dimensional algebras and the admissible action is related with the adjoint action of the group on itself.

Key words: Drinfel'd double, Hopf group-coalgebras, group-cograded multiplier Hopf algebras, Yetter-Drinfel'd modules. Mathematics Subject Classification: 16W30, 17B37.

1 Introduction

Let G be any group. The prototype of a G-cograded multiplier Hopf algebra is given by the multiplier Hopf algebra K(G) of complex valued functions with finite support in G. Recall that the product in K(G) is pointwise. The algebra K(G) has no unit, except where G is finite. The multiplier algebra M(K(G)) of K(G) is the largest algebra with unit in which K(G) sits as a dense two-sided ideal. Clearly M(K(G)) is given by the algebra of all complex functions on G. The comultiplication Δ on K(G)is given by the formula $(\Delta(f))(p,q) = f(pq)$ for all $f \in K(G)$ and $p,q \in G$. We have $\Delta(f) \in M(K(G) \otimes K(G))$. If G is finite, the multiplier algebra $M(K(G) \otimes K(G))$ equals $K(G) \otimes K(G)$.

In this paper we work with more general G-cograded multiplier Hopf algebras in the sense of [A-De-VD, Definition 1.1]. Essentially, a multiplier Hopf algebra B is G-cograded if there is a central, non-degenerate embedding $I: K(G) \to M(B)$. We require that I respects the comultiplication in the sense that $\Delta(I(f)) = (I \otimes I)(\Delta(f))$

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for all $f \in K(G)$. On the left hand side, we have extended the homomorphism Δ from B to the multiplier algebra M(B), in the sense of [VD1-A5]. Similary, on the right hand side, we have extended the homomorphism $I \otimes I$ from $K(G) \otimes K(G)$ to $M(K(G) \otimes K(G))$. A G-cograded multiplier Hopf algebra is denoted as $B = \bigoplus_{p \in G} B_p$ where B_p are algebras with a non-degenerate product. For all $p, q \in G$ we have $\Delta(B_{pq})(1 \otimes B_q) = B_p \otimes B_q$. Observe that the multiplier algebra $M(B) = \prod_{p \in G} M(B_p)$. It is shown in [A-De-VD] that a Hopf group-coalgebra as introduced by Turaev in [T], is a special case of a group-cograded multiplier Hopf algebra. Therefore, a lot of results for Hopf group-coalgebras follow from the more general results of multiplier Hopf algebras. E.g. the Drinfel'd double as constructed in [Z1] is an example of the Drinfel'd double construction D^{π} in [De-VD3, Theorem 3.8]. In the paper [De-VD3], we consider any group-cograded multiplier Hopf algebra B with an admissible action of the group. If we take the components of B as unital finite-dimensional algebras and we require the admissible action to be a "crossing", we recover the construction as given in [Z1].

For convenience of the reader, we recall the construction of the Drinfel'd double D^{π} . We start with a *G*-cograded multiplier Hopf algebra *B*. So *B* has the form $B = \bigoplus_{p \in G} B_p$. Assume that there is a group homomorphism $\pi : G \to \operatorname{Aut}(B)$, where $\operatorname{Aut}(B)$ denotes the group of algebra automorphisms on *B*.

We call π an *admissible* action of G on B if also the following requirements hold

- (1) $\Delta(\pi_p(b)) = (\pi_p \otimes \pi_p)(\Delta(b))$ for all $b \in B$
- (2) $\pi_p(B_q) = B_{\rho_p(q)}$ where ρ is an action of the group G on itself
- (3) $\pi_{\rho_p(q)} = \pi_{pqp^{-1}}$

This means that the map π takes care of ρ not being the adjoint action. If ρ is the adjoint action itself, π is called a *crossing*.

Take B and π as above. We consider a new regular multiplier Hopf algebra on B by deforming the comultiplication while the algebra structure on B is kept. The deformation of the comultiplication of B depends on the action π , in the following way

$$\widetilde{\Delta}(b)(1 \otimes b') = (\pi_{q^{-1}} \otimes \iota)(\Delta(b)(1 \otimes b'))$$
$$(1 \otimes b')\widetilde{\Delta}(b) = (\pi_{q^{-1}} \otimes \iota)((1 \otimes b')\Delta(b))$$

for all $b \in B$ and $b' \in B_q$.

Further we assume that $\langle A, B \rangle$ is a pairing of two regular multiplier Hopf algebras, in the sense of [Dr-VD]. As before, B is G-cograded and π is an admissible action of G on B. We consider a *twisted tensor product* algebra on the tensor product $A \otimes B$. This means that the trivial flip map is replaced by a more general twist map $R : B \otimes A \to A \otimes B$. This map R satisfies the appropriate compatibility conditions with respect to the multiplications on A and B. The twist map R depends on the pairing $\langle A, B \rangle$, as well as on the action π . For an explicit expression of the formula $R(b \otimes a)$, we refer to [De-VD3, Definition 3.4]. The algebra defined in this way is denoted as $A \bowtie B$. Finally, this algebra has the structure of a regular multiplier Hopf algebra if we consider the comultiplication $\overline{\Delta}$ on $A \bowtie B$ where $\overline{\Delta}(a \bowtie b) = \Delta^{cop}(a)\widetilde{\Delta}(b)$ in $M((A \bowtie B) \otimes (A \bowtie B))$. We observe that for $a \in A$, we use the opposite comultiplication of A. For $b \in B$, we use the deformation $\widetilde{\Delta}(b)$ as defined above.

In this paper, we characterize the modules of D^{π} . This is done from the point of view that D^{π} is a non-trivial twisted tensor product on the space $A \otimes B$, see above. We require a natural condition on the pairing $\langle A, B \rangle$ which generalizes the dual bases for finite-dimensional Hopf algebras. Then the characterization of the left modules over D^{π} can be rephrased purely in terms of the multiplier Hopf algebra B, without any reference to the multiplier Hopf algebra A. The compatibility conditions for these π -Yetter-Drinfel'd modules over B are given in Theorem 2.1. When we require the admissible action to be a crossing (this means that π is related with the adjoint action of the group G on itself) and we assume that the components are unital and finite-dimensional, the characterization in Theorem 2.1 can be put in the setting of [Z2, Section 8]. We notice that in this special situation our Drinfel'd double construction is isomorphic with the so-called mirror construction, given in [Z2, Section 9]. If π is a crossing of the group G on an arbitrary G-cograded multiplier Hopf algebra B, we have that the Drinfel'd double D^{π} is again G-cograded and there is a natural crossing of G on D^{π} , see [De-VD3, Proposition 3.13]. Furthermore, we have that D^{π} is π -quasitriangular, see [De-VD-W, Theorem 3.12]. The categorical interpretation of this quasitriangularity is translated to the π -Yetter-Drinfel'd modules over B, see Theorem 3.1. Our braiding is in the sense of the centre-construction of a category as given in [K, Sections XIII 4-5].

All algebras are considered over the field \mathbb{C} . We do not assume that an algebra A has a unit. But we require that the multiplication, considered as a bilinear map is non-degenerate. The multiplier algebra, denoted as M(A), is the largest algebra

with a unit in which A is contained as a dense two-sided ideal. The identity in any (multiplier) algebra is denoted by 1. The identity map is denoted as ι .

For a regular multiplier Hopf algebra A (i.e. with a bijective antipode) we denote the comultiplication by Δ . Observe that $\Delta : A \to M(A \otimes A)$. However, by the defining conditions on Δ , we have for all $a, b \in A$ that $\Delta(a)(1 \otimes b), \Delta(a)(b \otimes 1), (1 \otimes b)\Delta(a)$ and $(b \otimes 1)\Delta(a)$ are elements in $A \otimes A$. It can be motivated, see e.g. [Dr-VD-Z, Section 2] that these elements are denoted by *Sweedler notation*, e.g. $\Delta(a)(1 \otimes b) = \sum a_{(1)} \otimes a_{(2)}b$. In an expression, denoted by Sweedler notation, one always has to make sure that at most one factor $a_{(k)}$ is not multiplied ("covered") by an element in A.

When we consider a module V over an algebra A, we always mean a left module which is unital. A (left) A-module V is unital if $A \triangleright V = V$. By the regularity conditions on A, this implies that for all $x \in V$, we have an element $e \in A$ such that $x = e \triangleright x$. For details, we refer to [Dr-VD-Z, Section 3]. Therefore, the comultiplication on A can be used to make the category of (left) A-modules into a tensor category with unit.

Basic references

The material needed for reading this paper is given in the following basic references. For (regular) multiplier Hopf algebras, we refer to [VD1] and [VD-Z1]. The group-cograded multiplier Hopf algebras are introduced in [A-De-VD] and studied in [De-VD-W]. They generalize the Hopf group-coalgebras, as introduced by Turaev in [T]. The Drinfel'd double construction into the framework of multiplier Hopf algebras is associated to a pairing, see [Dr-VD] and [De-VD1]. To have the analogous properties as for the Drinfel'd double of a finite-dimensional Hopf algebra, we assume that the pairing $\langle A, B \rangle$ of two multiplier Hopf algebras has a canonical multiplier $W \in M(A \otimes B)$. Essentially, the multiplier W takes the role of the dual bases in the finite-dimensional case. For details, we refer to [De-VD2, Section 4]. The Drinfel'd double construction for group-cograded multiplier Hopf algebras is done in [De-VD3].

2 π -Yetter-Drinfel'd modules

Let $\langle A, B \rangle$ be a pair of multiplier Hopf algebras. Let G denote a group and assume that B is G-cograded. As an algebra, we write $B = \bigoplus_{p \in G} B_p$. Let π be an admissible action of G on B, in the sense of [De-VD3, Definition 2.6]. So for all $p \in G$, we have an automorphism π_p on B which respects the multiplication and the comultiplication of B. Furthermore for all $p, q \in G$, we have $\pi_p(B_q) = B_{\rho_p(q)}$ where ρ is an automorphism of G on itself. We require that $\pi_{\rho_p(q)} = \pi_{pqp^{-1}}$ for all $p, q \in G$. In the framework of Hopf group-coalgebras, one sets $\rho_p(q) = pqp^{-1}$ for all $p, q \in G$, see [T]. Let D^{π} denote the Drinfel'd double as constructed in [De-VD3, Theorem 3.8]. As an algebra, D^{π} is a twisted tensor product on the linear space $A \otimes B$. Therefore, a left D^{π} -module is nothing but a linear space V with a left B-module structure, denoted as B.V, as well as a left A-module structure, denoted as $A \triangleright V$. For all $a \in A, b \in B_p$ and $x \in V$, the following compatibility equation yields

$$b \cdot (a \triangleright x) = \sum a_i \triangleright (b_i \cdot x)$$

where $\sum a_i \otimes b_i = T(b \otimes a) = \sum (\pi_{p^{-1}}(b_{(1)}) \triangleright a \blacktriangleleft S^{-1}(b_{(3)})) \otimes b_{(2)}$. The actions \triangleright and \blacktriangleleft are the regular actions of B on A, associated to the pairing $\langle A, B \rangle$, see [Dr-VD].

We rephrase the above compatibility condition in terms of the multiplier Hopf algebra B, without any reference to the paired multiplier Hopf algebra A. We need the notion of a right-B-comodule. In Hopf algebra theory, it is possible to define the structure of a comodule on a vector space. In the setting of multiplier Hopf algebras however, more structure is needed. In [VD-Z2], the setting is that of an algebra Vand a regular multiplier Hopf algebra B. Then, a right coaction of B on V is an injective linear map $\Gamma: V \to M(V \otimes B)$ satisfying

- (i) $\Gamma(V)(1 \otimes B) \subseteq V \otimes B$ and $(1 \otimes B)\Gamma(V) \subseteq V \otimes B$
- (ii) $(\Gamma \otimes \iota)\Gamma = (\iota \otimes \Delta)\Gamma$

The algebra structure of V is needed to be able to consider the multiplier algebra $M(V \otimes B)$. It would be too restrictive to assume that the coaction Γ has range in the tensor product itself.

Observe that Condition (i) is used to give a meaning to the left hand side of the equation in Condition (ii). The link between left A-modules and right B-comodules is given by the so-called canonical multiplier W in $M(A \otimes B)$, in the sense of [De-VD2, Section 4] and [De2, Section 2]. A multiplier W in $M(A \otimes B)$ is called canonical for the pairing $\langle A, B \rangle$ if W is invertible in $M(A \otimes B)$ and if $\langle W, a \otimes b \rangle = \langle a, b \rangle$ for all $a \in A$ and $b \in B$. Let B be a finite-dimensional Hopf algebra and consider A = B'where B' denotes the dual Hopf algebra of B. If $\{f_i\} \subset B'$ and $\{e_i\} \subset B$ are dual bases, then $W = \sum f_i \otimes e_i$ is the canonical element in $B' \otimes B$ for the natural pairing $\langle B', B \rangle$.

2.1 Theorem Consider the notations and the assumptions as above. We have that V is a (left) D^{π} -module if and only if V is a left B-module for the action B.V and V is a right B-comodule for the right coaction $\Gamma: V \to M(V \otimes B)$ such that the left action and the right coaction of B on V satisfy the compatibility relation

$$\sum (d_{(1)} \cdot \otimes cd_{(2)}) \Gamma(v) = \sum (1 \otimes c) \Gamma(d_{(2)} \cdot v) (1 \otimes \pi_{p^{-1}}(d_{(1)}))$$

for all $v \in V$, $c \in B_q$ and $d \in B_{pq}$ (where $p, q \in G$).

Proof. In [De2, Theorem 2.3], we have proven that a left A-module V is determined by a unique right B-comodule structure on V in the following way. Let $A \triangleright V$ denote a left A-module, then there is a right B-comodule $\Gamma : V \to M(V \otimes B)$ so that

$$a \triangleright v = (\iota \otimes \langle a, \cdot \rangle) \Gamma(v)$$

for all $A \in A$ and $v \in V$.

On the right hand side of the above equation, we have that $\Gamma(v)$ sits in the multiplier algebra $M(V \otimes B)$. However, by the regularity conditions on the pairing $\langle A, B \rangle$, there is an element $b \in B$ such that the right hand side should be read as $(\iota \otimes \langle a, \cdot \rangle)(\Gamma(v)(1 \otimes b))$. As we assume for the coaction Γ that $\Gamma(V)(1 \otimes B) \subseteq V \otimes B$, the expression $(\iota \otimes \langle a, \cdot \rangle)(\Gamma(v)(1 \otimes b))$ determines an element in V. The compatibility condition between the left B-module structure and the left A-module structure on a left D^{π} -module V can be rephrased as follows. For all $a \in A$, $b \in B_p$ and $v \in V$, we have

$$\begin{split} (\iota_V \otimes \langle a, \cdot \rangle)(b \cdot \otimes 1)\Gamma(v) &= b \cdot ((\iota_V \otimes \langle a, \cdot \rangle)\Gamma(v)) = b \cdot (a \triangleright v) = \sum a_i \triangleright (b_i \cdot v) \\ &= \sum (\iota_V \otimes \langle a_i, \cdot \rangle)\Gamma(b_i \cdot v) = \sum (\iota_V \otimes \langle \pi_{p^{-1}}(b_{(1)}) \blacktriangleright a \blacktriangleleft S^{-1}(b_{(3)}), \cdot \rangle)\Gamma(b_{(2)} \cdot v) \\ &= \sum \langle a_{(1)}, S^{-1}(b_{(3)}) \rangle \langle a_{(3)}, \pi_{p^{-1}}(b_{(1)}) \rangle (\iota_V \otimes \langle a_{(2)}, \cdot \rangle)\Gamma(b_{(2)} \cdot v) \\ &= (\iota_V \otimes \langle a, \cdot \rangle) \left(\sum (1 \otimes S^{-1}(b_{(3)}))\Gamma(b_{(2)} \cdot v) (1 \otimes \pi_{p^{-1}}(b_{(1)})) \right). \end{split}$$

Observe that the above equations are in $V \otimes B$. All decompositions are well-covered by the use of the regularity conditions on the pairing $\langle A, B \rangle$. As the pairing is a non-degenerate linear form on $A \otimes B$, we obtain the following equation in $V \otimes B$. For any $p, q \in G$ and $b \in B_p$, $b' \in B_q$, we have for all $v \in V$

$$(b \cdot \otimes b')\Gamma(v) = \sum (1 \otimes b'S^{-1}(b_{(3)}))\Gamma(b_{(2)} \cdot v)(1 \otimes \pi_{p^{-1}}(b_{(1)})).$$

From the axioms on a regular multiplier Hopf algebra, we have $(1 \otimes B)\Delta(B) = B \otimes B$, see [VD1]. By the use of [VD1, Lemma 5.5], the equation above is equivalent to the following statement. For any $p, q \in G$ and $d \in B_{pq}$, $c \in B_q$ we have for all $v \in V$

$$\left(\sum d_{(1)} \cdot \otimes cd_{(2)}\right) \Gamma(v) = \sum (1 \otimes c) \Gamma(d_{(2)} \cdot v) (1 \otimes \pi_{p^{-1}}(d_{(1)})).$$

By the use of the *G*-cograding and the admissible action π , we have that in the left hand side $(1 \otimes c)\Delta(d) \in B_p \otimes B_q$. In the right hand side we have $(\pi_p(c) \otimes 1)\Delta(d) \in B_{\rho_p(q)} \otimes B_{\rho_p(q^{-1})pq}$.

2.2 Definition Let *B* be a *G*-cograded multiplier Hopf algebra and let π be an admissible action of *G* on *B*. An algebra *V* with a left *B*-module structure and a right *B*-comodule structure is called a π -Yetter-Drinfel'd module if the compatibility condition of Theorem 2.1 is satisfied. The set of all π -Yetter-Drinfel'd modules over *B* is denoted as $_{B}\pi(\mathcal{YD})^{B}$.

In the characterization of Theorem 2.1, we have dispensed with the Drinfel'd double D^{π} . So we do not need to assume that B is paired with another multiplier Hopf algebra to define the π -Yetter-Drinfel'd modules over B.

2.3 Remark Let *B* be a finite-type Hopf group-coalgebra and assume that π is a crossing, i.e. $\pi_p(B_q) = B_{pqp^{-1}}$ for all $p, q \in G$. In Theorem 2.1, the multiplier Hopf algebra *A* can be taken as the (usual) Hopf algebra $B^* = \bigoplus_{p \in G} (B_p)'$, where $(B_p)'$ denotes the linear dual of B_p . The formula in Theorem 2.1 is now given as in [Z2, Section 8]. When *G* is given by the trivial group, we recover the well-known characterization of Yetter-Drinfel'd modules for a finite-dimensional Hopf algebra. In these settings, we don't need an underlying algebra structure on the Yetter-Drinfel'd modules because the comultiplication of *B* is a map $\Delta : B \to B \otimes B$. Furthermore, there is always a canonical multiplier *W* in $M(B^* \otimes B)$. More precisely, $W = \sum_{p \in G} f_{p,i} \otimes e_{p,i}$ where for all $p \in G$, the sets $\{f_{p,i}\}$ in $(B_p)'$ and $\{e_{p,i}\}$ in B_p are dual bases.

3 The braided monoidal category ${}_B\pi(\mathcal{YD})^B$

As before, we consider a multiplier Hopf algebra B which is cograded by a group G. As an algebra we have $B = \bigoplus_{p \in G} B_p$ where B_p is a subalgebra with a non-degenerate product. Let π denote an admissible action of G on B. We consider the category ${}_B\pi(\mathcal{YD})^B$ of π -Yetter-Drinfel'd modules over B, in the sense of Definition 2.2. The morphisms in ${}_B\pi(\mathcal{YD})^B$ are linear maps which are left B-module morphisms as well as right B-comodule morphisms.

If *B* is paired with a multiplier Hopf algebra *A*, we have proven in Theorem 2.1 that the category ${}_{B}\pi(\mathcal{YD})^{B}$ is given by the left unital modules over the Drinfel'd double D^{π} , associated to the pair $\langle A, B \rangle$. We made use of the canonical multiplier *W* in $M(A \otimes B)$. The morphisms between left D^{π} -modules, correspond to the morphisms in ${}_{B}\pi(\mathcal{YD})^{B}$, use [De2, Theorem 2-3]. By the bialgebra structure on D^{π} , the modules over D^{π} have the structure of a monoidal tensor category. Therefore, it is expected that the category ${}_{B}\pi(\mathcal{YD})^{B}$ is also a monoidal tensor category. Let *V* be in ${}_{B_{p}}\pi(\mathcal{YD})^{B}$ and let *V'* be in ${}_{B_{q}}\pi(\mathcal{YD})^{B}$, then $V \otimes V'$ is in ${}_{B_{pq(p)q}}\pi(\mathcal{YD})^{B}$ in the following way

$$b \cdot (v \otimes v') = \sum \pi_{q^{-1}}(b_{(1)}) \cdot v \otimes b_{(2)} \cdot v'$$

for all $b \in B_{\rho_q(p)q}$, $v \in V$ and $v' \in V'$. To determine the right *B*-comodule structure on the tensor product $V \otimes V'$, we translate the *A*-module structure on $V \otimes V'$ (A^{cop} is embedded in D^{π}). This translation is done by the use of the canonical multiplier in $M(A \otimes B)$. We have denoted this multiplier by the letters *W* and *P*. For $a \in A$ and $b \in B$, we write $(a \otimes 1)W(1 \otimes b)$ as $\sum aW^{(1)} \otimes W^{(2)}b$ in $A \otimes B$. Following [De2, Proposition 2.2], we have for all $b \in B$, $v \in V$ and $v' \in V'$

$$\begin{split} &\Gamma(v \otimes v')(1 \otimes 1 \otimes b) = \sum (W^{(1)} \triangleright (v \otimes v')) \otimes W^{(2)}b \\ &= \sum (P^{(1)} \triangleright v) \otimes (W^{(1)} \triangleright v') \otimes W^{(2)}P^{(2)}b. \end{split}$$

We made use of the formula $(\Delta \otimes \iota)(W) = W^{13}W^{23}$. We have obtained the following right *B*-comodule structure on $V \otimes V'$

$$\Gamma(v \otimes v')(1 \otimes 1 \otimes b) = \Gamma(v')_{23}\Gamma(v)_{13}(1 \otimes 1 \otimes b)$$

for all $v \in V$, $v' \in V'$ and $b \in B$. In the right hand side, we use the leg-numbering notation in the usual way.

One can check that the compatibility condition holds for the tensor object $V \otimes V'$. Moreover, we have that ${}_B\pi(\mathcal{YD})^B$ is a monoidal category. We omit these proofs because we would be repeating the construction of D^{π} as bialgebra, see [De-VD3].

So far, we have "translated" the multiplier Hopf algebra structure on D^{π} to determine the monoidal category ${}_{B}\pi(\mathcal{YD})^{B}$. Further structures on the multiplier Hopf algebra D^{π} will correspond directly to properties of its category of modules and can be translated towards the category ${}_{B}\pi(\mathcal{YD})^{B}$. Further in this sequel, we assume that the admissible action of G on B is given as a crossing. This means that $\pi_{p}(B_{q}) = B_{pqp^{-1}}$ for all $p, q \in G$. However, the components of the G-cograded multiplier Hopf algebra B, denoted as B_{p} for all $p \in G$, are arbitrary algebras with a non-degenerate multiplication. In this setting, we have that D^{π} is G-cograded and there is a natural crossing of G on D^{π} . More precisely, in [De-VD3, Proposition 3.13], we have proven that D^{π} is G-cograded as follows

$$D^{\pi} = \bigoplus_{p \in G} (D^{\pi})_p \qquad \text{ with } (D^{\pi})_p = A \bowtie B_{p^{-1}}.$$

For all $p \in G$, define π'_p on A via the formula $\langle \pi'_p(a), b \rangle = \langle a, \pi_{p^{-1}}(b) \rangle$ for all $a \in A$, $b \in B$. Then, the maps $\pi'_p \otimes \pi_p$ define a crossing of G on D^{π} . Let W in $M(A \otimes B)$ denote the canonical multiplier of the pair $\langle A, B \rangle$. Let σ be the twist map on $A \otimes B$, extended to $M(A \otimes B)$. It is proven in [De-VD-W, Theorem 3.12] that the embedding $\sigma(W)$ in $M(D^{\pi} \otimes D^{\pi})$ is a generalised π -matrix for D^{π} , in the sense of [De-VD-W, Definition 3.1]. By the use of [De-VD-W, Section 3.4], this π -quasitriangularity of D^{π} corresponds to the following properties of the category of (left) modules over D^{π} . For all $p \in G$, let ${}_{p}\mathcal{M}$ denote the modules over the algebra $(D^{\pi})_p$. Then we have that the category of left modules over D^{π} is given as ${}_{D^{\pi}}\mathcal{M} = \prod_{p \in G} {}_{p}\mathcal{M}$. For all $p \in G$, there is an invertible functor F_p on ${}_{D^{\pi}}\mathcal{M}$. If $V \in {}_{q}\mathcal{M}$, then $F_p(V) \in {}_{pqp^{-1}}\mathcal{M}$. As a linear space, we have that $F_p(V)$ equals V. Let the D^{π} -module structure on V be denoted as $D^{\pi} \to V$. For an element $(a \bowtie b) \in (D^{\pi})_{pqp^{-1}}$, we have $(a \bowtie b) \to F_p(v) = F_p((\pi'_{p^{-1}}(a) \bowtie \pi_{p^{-1}}(b)) \to v)$. A morphism in ${}_{q}\mathcal{M}$ is sent to itself, now considered as a morphism in ${}_{pqp^{-1}}\mathcal{M}$.

Finally, the π -quasitriangularity of D^{π} gives the following π -braiding in $_{D^{\pi}}\mathcal{M}$. Let V (resp. V') be in $_{p}\mathcal{M}$ (resp. $_{q}\mathcal{M}$). Then we have

$$t_{V,V'}: V \otimes V' \to F_p(V') \otimes V \text{ such that}$$

$$t_{V,V'}(v \otimes v') = \sum F_p(W^{(1)} \triangleright v') \otimes (W^{(2)} \cdot v)$$

where $A \triangleright V$ (resp. $B \cdot V$) denotes the A-module (resp. B-module) structure on V.

In Theorem 2.1, we have given a characterization for the π -Yetter-Drinfel'd modules, without the use of a pairing and a Drinfel'd double. So, associated to any *G*-cograded multiplier Hopf algebra *B* and a crossing π of *G* on *B*, we have the following π -braided monoidal tensor category ${}_B\pi(\mathcal{YD})^B$.

3.1 Theorem Let *B* be a *G*-cograded multiplier Hopf algebra and let π denote a crossing of *G* on *B*. The monoidal category ${}_{B}\pi(\mathcal{YD})^{B}$ is π -braided.

Proof. Let V be in ${}_B\pi(\mathcal{YD})^B$. The left action of B on V is denoted as $B \cdot V$. The right coaction of B on V is denoted as $\Gamma : V \to M(V \otimes B)$. If B is paired with another multiplier Hopf algebra A, we already have that the category of the left modules over D^{π} is a braided tensor category, see above. We rephrase the results on this category, but we dispense with the Drinfel'd double D^{π} itself. Let V be in $B_{p^{-1}}\pi(\mathcal{YD})^B$ and V' is in $B_{q^{-1}}\pi(\mathcal{YD})^B$. Then $V \otimes V'$ is a π -Yetter Drinfel'd module over the subalgebra $B_{q^{-1}p^{-1}}$.

For all $p \in G$, there is an invertible function F_p on ${}_B\pi(\mathcal{YD})^B$. For V in ${}_{B_{q^{-1}}}\pi(\mathcal{YD})^B$, we have $F_p(V)$ in ${}_B\pi(\mathcal{YD})^B$. As an algebra, we have that $F_p(V)$ equals V. As a left B-module, $F_p(V)$ lies over the subalgebra $B_{pq^{-1}p^{-1}}$. For $b \in B_{pq^{-1}p^{-1}}$ and $v \in F_p(V)$, we have $b \cdot F_p(v) = F_p(\pi_{p^{-1}}(b) \cdot v)$.

We now find the right *B*-comodule structure on $F_p(V)$. If *B* is paired with a multiplier Hopf algebra *A*, we assume that $W \in M(A \otimes B)$ is the canonical multiplier of this pair. By the uniqueness of the canonical multiplier *W*, we have for all $p \in G$, $(\pi'_p \otimes \pi_p)(W) = W$. For $v \in V$ and $b \in B$ we have

$$\Gamma(F_p(v))(1 \otimes b) = \sum (W^{(1)} \triangleright F_p(v)) \otimes W^{(2)}b$$

= $\sum F_p(\pi'_{p^{-1}}(W^{(1)}) \triangleright v) \otimes W^{(2)}b = \sum F_p(W^{(1)} \triangleright v) \otimes \pi_p(W^{(2)})b$
= $(F_p \otimes \pi_p) \left(\sum (W^{(1)} \triangleright v) \otimes W^{(2)}\pi_{p^{-1}}(b) \right) = (F_p \otimes \pi_p)(\Gamma(v)(1 \otimes \pi_{p^{-1}}(b)).$

Finally the braiding in the category of left D^{π} -modules gives the following braiding on ${}_{B}\pi(\mathcal{YD})^{B}$. For V in ${}_{B_{p^{-1}}}\pi(\mathcal{YD})^{B}$ and V' in ${}_{B_{q^{-1}}}\pi(\mathcal{YD})^{B}$, we have $t_{V,V'}: V \otimes V' \to F_{p}(V') \otimes V$ such that for $v \in V, v' \in V'$

$$t_{V,V'}(v \otimes v') = \sum F_p(W^{(1)} \triangleright v') \otimes W^{(2)} \cdot v = \sum F_p(v'^{(1)}) \otimes (v'^{(2)} \cdot v).$$

In the right hand side of this formula, the tensor $v^{(2)} \cdot v$ should be read as $v^{(2)}b \cdot v$ where b is chosen in $B_{p^{-1}}$ such that $b \cdot v = v$, see [Dr-VD-Z]. The summation $\sum v'^{(1)} \otimes v'^{(2)}b$ stands for the element $\Gamma(v')(1 \otimes b)$ in $V' \otimes B$.

3.2 Remark Suppose that G is given by the trivial group $G = \{e\}$. The G-cograded multiplier Hopf algebra B is a usual multiplier Hopf algebra. In the case that B is finite-dimensional (and so B is a Hopf algebra), Theorem 3.1 recovers the categorical interpretation of the usual Drinfel'd double of B which is equivalent with the centre-construction of B-mod as given in [K, Section XIII.5].

3.3 Examples

3.3.1 G-cograded multiplier Hopf algebras

We first give examples of G-cograded multiplier Hopf algebras with a crossing.

- The Hopf group-coalgebras and their crossing, as considered in [T], are examples of G-cograded multiplier Hopf algebras. This point of view is explained in [A-De-VD, Theorem 1.5]. Let K(G) denote the multiplier Hopf algebra of the complex valued functions with a finite support in G. The product is pointwise and the coproduct is dual to the product in the group. We write K(G) = ⊕ Cδ_p. In this case all the components are equal to the trivial algebra C. The natural crossing on K(G) is related with the adjoint action of G on itself.
- Let (A, Δ) denote any multiplier Hopf algebra. Let G be a group which acts on the multiplier Hopf algebra A by means of automorphisms α_p for all p ∈ G. We assume α_e = ι, α_p(α_q(a)) = α_{pq}(a) for all p, q ∈ G and a ∈ A. Further, the automorphism α_p respects the comultiplication of A in the sense that Δ(α_p(a)) = (α_p ⊗ α_p)Δ(a) for all p ∈ G and a ∈ A. Consider the tensor product algebra B = K(G) ⊗ A with the trivial product. In [De1, Example 3.3] is given a non-trivial coproduct on K(G) ⊗ A as follows

$$\Delta(\delta_p \otimes a)((1 \otimes 1) \otimes (\delta_q \otimes a')) = \sum (\delta_{pq^{-1}} \otimes \alpha_q(a_{(1)})) \otimes (\delta_q \otimes a_{(2)}a')$$

for all $p, q \in G$ and $a, a' \in A$.

The multiplier Hopf algebra $B = K(G) \otimes A$ is *G*-cograded. We have $B = \bigoplus_{p \in G} B_p$ where $B_p = \mathbb{C}\delta_p \otimes A$. Let $\{f_p \mid p \in G\}$ denote a family of automorphisms on *A* which respect the comultiplication of (A, Δ) and assume furthermore that $f_{pq} = f_p \circ f_q$ and $f_p \circ \alpha_q = \alpha_{pqp^{-1}} \circ f_p$ for all $p, q \in G$. Then, a crossing of *G* on

B is given by the automorphisms π_p on *B* where $\pi_p(\delta_q \otimes a) = \delta_{pqp^{-1}} \otimes f_p(a)$ for all $p, q \in G$ and $a \in A$. Observe that the family $\{\alpha_p \mid p \in G\}$ can always be taken to define a crossing on *B*. In this example all components are equal to the (possible infinite-dimensional) multiplier Hopf algebra *A*. However, the comultiplication on *B* in not trivially given by the comultiplication on *A*. We notice that (B, Δ) has integrals if (A, Δ) has integrals, see [De1, Theorem 1.16.1]. So, in these situations we can consider the pairing $\langle \hat{B}, B \rangle$ where \hat{B} denotes the dual multiplier Hopf algebra, in the sense of [VD2]. The pairing $\langle \hat{B}, B \rangle$ has a canonical multiplier *W* in $M(\hat{B} \otimes B)$, see [De-VD2, Proposition 4.12].

3.3.2 π -Yetter-Drinfel'd modules

Let *B* be a *G*-cograded multiplier Hopf algebra and let π denote a crossing of *G* on *B*. Assume that $\langle A, B \rangle$ is a pair of multiplier Hopf algebras with a canonical multiplier *W* in $M(A \otimes B)$. The tensor algebra $A \otimes B$ (with trivial product) can be made into a π -Yetter-Drinfel'd module over *B* as follows. For all $p \in G$ and $b \in G_p$ we set

$$b \cdot (x \otimes y) = \sum (\pi_{p^{-1}}(b_{(1)}) \blacktriangleright x \blacktriangleleft S^{-1}(b_{(3)})) \otimes b_{(2)}y$$

for all $x \in A$ and $y \in B$. Observe that \blacktriangleright and \blacktriangleleft are the regular actions of B on A, associated to the pairing $\langle A, B \rangle$.

$$\Gamma(x \otimes y)((1 \otimes 1) \otimes b) = \sum (W^{(1)}x \otimes y) \otimes W^{(2)}b$$

for all $x \in A$ and $y, b \in B$. This π -Yetter-Drinfel'd module for B corresponds with the left regular module of the Drinfel'd double D^{π} on itself.

Let B be a finite-type Hopf group-coalgebra with a crossing π , in the sense of [T, Section 11]. Then we have $B = \bigoplus_{p \in G} B_p$ where for all $p \in G$, the algebra B_p is unital and finite-dimensional. This multiplier Hopf algebra B is paired with the (usual) Hopf algebra $A = \bigoplus_{p \in G} (B_p)'$ where $(B_p)'$ is the linear dual of B_p . The canonical multiplier W in $M(A \otimes B)$ is given by the formal summation $\sum_{p \in G} f_{p,i} \otimes e_{p,i}$ where $\{f_{p,i}\} \subset (B_p)'$ and $\{e_{p,i}\} \subset B_p$ are dual bases. Consider the tensor algebra $\bigoplus_{p,q \in G} ((B_q)' \otimes B_p)$. This algebra is a π -Yetter-Drinfel'd module for B in the following

way. For all $p \in G$ and $b \in B_p$, $f \in A$ and $y \in B$ we set

$$b \cdot (f \otimes y) = \sum f(S^{-1}(b_{(3)}) \cdot \pi_{p^{-1}}(b_{(1)})) \otimes b_{(2)}y$$
$$\Gamma(f \otimes y)((1 \otimes 1) \otimes b) = \sum_{i} (f_{p,i}f \otimes y) \otimes e_{p,i}b.$$

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