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Limit Cycles near Vector Fields of Center Type

Proefschrift voorgelegd tot het behalen van de graad van
Doctor in de Wetenschappen, richting Wiskunde,
te verdedigen door

Magdalena CAUBERGH

Promotor : Prof. dr. Freddy Dumortier
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Magdalena Caubergh
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Preface

The subject of the thesis concerns the cyclicity and the bifurcations of limit cycles in C^∞ and analytic families $(X_\lambda)_\lambda$ of planar vector fields near vector fields of center type.

As the parameter λ varies, changes may occur in the phase portraits of the vector fields X_λ. These changes are called bifurcations and the parameter values λ^0, at which such a bifurcation occurs, are called bifurcation values; the vector field X_{λ^0} is called the bifurcation vector field. In this thesis, the bifurcation vector field X_{λ^0} is of center type, meaning that its phase portrait contains an annulus or a punctured disc of non-isolated periodic orbits; typical examples of such vector fields are Hamiltonian vector fields.

Recall that an isolated periodic orbit of X_λ is called a limit cycle. Our analysis is of a local kind, in the sense that we study bifurcations of limit cycles in the neighbourhood of a so-called limit periodic set Γ. Attention is focussed on the maximal possible number of limit cycles of X_λ, that can arise in the neighbourhood of Γ, after small perturbations of the parameter λ near λ^0; this number is referred to as the cyclicity of X_λ at (Γ, λ^0). The limit periodic sets Γ, that are considered in this thesis, are (regular) periodic orbits, non-degenerate elliptic points and 2-saddle cycles.

The study of bifurcations of limit cycles and their cyclicity is motivated by Hilbert's sixteenth problem, that asks for a bound H_n for the maximum number of limit cycles (and their relative positions) in polynomial vector fields in the plane of degree n, only depending on the degree n. Although Hilbert's sixteenth problem is of a global character, it is known that a solution to all local problems induces the existence of a finite H_n (see [R98])

Traditionally, the study of limit cycles of planar vector fields $(X_\lambda)_\lambda$ near limit periodic sets, as the ones we consider, is replaced by the study of isolated fixed points of associated 1-dimensional Poincaré-maps $(P_\lambda)_\lambda$, or equivalently, by the study of isolated zeroes or so-called displacement maps $(\delta_\lambda)_\lambda$, defined by $\delta_\lambda = P_\lambda - Id$. In such a way, configurations of isolated zeroes of δ_λ correspond to configurations of limit cycles of X_λ.

In the study of stable bifurcation diagrams near a non-degenerate elliptic singularity, people often use techniques such as normal forms or Lyapunov quantities.

To study the cyclicity near a periodic orbit or a non-degenerate elliptic singularity, when the bifurcation vector field X_{λ^0} is of center type, there is the well-known technique of computing Abelian integrals (the so-called Melnikov functions) in 1-parameter families, and the technique of the Bautin ideal in multi-parameter families. For instance, the first order Melnikov function is the coefficient in the linear approximation of the displacement map δ_λ, with respect to λ. The technique of the Bautin ideal is based on a special division of the displacement map; for 1-parameter families this technique reduces to the technique of computing Melnikov functions. In the literature, there exist algorithms to compute Melnikov functions, while the Bautin ideal is a very powerful theoretical technique, that often, in practice, is too difficult to be computed.

In this thesis, we focus on three problems, that briefly can be described as follows. The first problem deals with stable bifurcation diagrams of limit cycles near centers, where attention is focused on uniform results as well in phase plane as well as in parameter space.

The second problem is the investigation of how 1-parameter techniques, such as the computation of Melnikov functions, can be used in multi-parameter families, to compute its cyclicity near centers.

The third problem deals with families $\left(X_{(\nu,\varepsilon)}\right)$ of planar vector fields that unfold a Hamiltonian vector field for $\varepsilon = 0$, where ε is a 1-parameter; it is the investigation whether results on linear approximations I_ν of the displacement map $\delta_{(\nu,\varepsilon)}$, with respect to ε (such as the first order Melnikov function), can be transferred to valuable results on the bifurcation diagram of limit cycles and the cyclicity. Let us now describe these problems in more detail.

Related to the first problem, a well-known example of a stable bifurcation pattern is the Andronov-Hopf bifurcation in the neighbourhood of a non-degenerate elliptic singularity (i.e. with pure imaginary eigenvalues), the so-called Hopf singularity. By the implicit function theorem, it follows that under small perturbations of the vector field, the singularity persists and no new singularities are created. However, it is possible that the stability type of the singularity changes when subjected to perturbations, and then this change is usually accompanied with either the appearance or disappearance of a small limit cycle encircling the singularity. This important well-known bifurcation phenomenon is called the Andronov-Hopf bifurcation.

Generalisations of the Andronov-Hopf bifurcation, giving rise to multiple limit cycles, are called generalised Hopf bifurcations or Hopf-Takens bifurcations. A precise study of generic generalised Hopf bifurcations is done in [T], by way of normal forms, when no centers occur.

Perturbations from centers naturally show up in many problems and one constantly has to consider Hopf-Takens bifurcations that perturb from a center. In this thesis, we link the different techniques that are used in the study of a Hopf-singularity, surrounded by non-isolated periodic orbits: normal forms, Lyapunov quantities and Melnikov functions.

In the study of bifurcation diagrams of a family $(X_{(\nu,\varepsilon)})_{(\nu,\varepsilon)}$, there appears besides a 1-parameter ε, inducing centers for $\varepsilon = 0$, also an external parameter ν, that controls bifurcations from these centers. If the centers are exclusively situated at $\varepsilon = 0$, then we speak of a 'regular hypersurface of centers'. In case of a regular hypersurface of centers, we prove a result for each of these techniques, indicating precisely which verifications have to be made in order to guarantee the presence of a generic Hopf-Takens bifurcation, on a uniform domain, both in phase plane as in parameter space (i.e. a domain that does not shrink with the bifurcation value $\varepsilon \downarrow 0$).

Finally, we consider one more example in which centers are generated by a 2-parameter $\varepsilon = (\varepsilon_1, \varepsilon_2)$. In this example, besides the Hopf bifurcation also another type of bifurcation shows up. A limit cycle disappears through the boundary of the domain; this bifurcation is called a boundary bifurcation.

The second problem in this thesis deals with analytic families of planar vector fields $(X_\lambda)_\lambda$, investigating methods to detect the cyclicity in the multi-parameter family $(X_\lambda)_\lambda$ at a non-isolated closed orbit Γ, by means of 1-parameter subfamilies. In [R00], using the desingularisation theory of Hironaka, Roussarie constructed a polynomial curve $\lambda(\varepsilon)$ in parameter space, such that the first non-identical zero Melnikov function of the induced 1-parameter subfamily $\left(X_{\lambda(\varepsilon)}\right)_\varepsilon$, can be used to bound the cyclicity of the multi-parameter family. This curve $\lambda(\varepsilon)$ is called a curve of maximal index (mic). In the spirit of this result, we prove, using the theory of analytic geometry, that the multi-parameter problem can be reduced to a 1-parameter one, in the sense that there exist analytic curves in parameter space along which the maximal cyclicity can be attained. In that case one speaks about a maximal cyclicity curve (mcc) if only the number is considered and of a maximal multiplicity curve (mmc) if the multiplicity is also taken into consideration. In view of obtaining efficient algorithms for detecting the cyclicity, we investigate whether such mcc, mmc and mic can be algebraic or even linear depending on certain general properties of the families or of their associated Bautin ideal. In any case by well chosen examples we show that prudence is appropriate.

In most examples encountered in the literature, nearby vector fields, with maximal cyclicity (respectively multiplicity) are structurally stable and hence occur in open subanalytic sets of the parameter space. In case the stratum of maximal cyclicity has a non-empty interior adhering at λ^0, we show that there always exists an algebraic mcc (respectively mmc) ζ, in case the analytic family of planar vector fields has a stratum of maximal cyclicity (respectively multiplicity) with non-empty interior at λ^0. In particular, in that case there exists a 'cone of mcc's (respectively mmc's) surrounding ζ'.

For certain specific examples, we also discuss related questions such as the existence of minimal detectibility and conic degree of maximal cyclicity (respectively multiplicity).

The third problem deals with C^∞ families $\left(X_{(\nu,\varepsilon)}\right)$ of planar vector fields, that unfold a Hamiltonian vector field X_H for $\varepsilon = 0$, where ε is a 1-dimensional parameter. It asks how results on linear approximations I_ν of the displacement map $\delta_{(\nu,\varepsilon)}$, with respect to ε, can be transferred to valuable results on the bifurcation diagram of limit cycles and the cyclicity in $\left(X_{(\nu,\varepsilon)}\right)$. If $\left(\upsilon_{(\nu,\varepsilon)}\right)$ is the C^∞ family of dual 1-forms associated to the family $\left(X_{(\nu,\varepsilon)}\right)$, then it is well-known that I_ν can be computed by integration of the first order approximation of $v_{(\nu,\varepsilon)}$ with respect to ε along the level curves $\Gamma_x \subset \{H = x\}$ of the Hamiltonian H : if

$$v_{(\nu,\varepsilon)} = \mathrm{d}H + \varepsilon\bar{v}_\nu + o(\varepsilon), \varepsilon \to 0,$$

then

$$I_\nu(x) = -\int_{\Gamma_x} \bar{v}_\nu,$$

where Γ_x is oriented by the vector field X_H. Therefore, we refer to I_ν as the related Abelian integral of the family $\left(X_{(\nu,\varepsilon)}\right)$.

In case Γ is a periodic orbit or a non-degenerate elliptic singularity, then it is well-known that results on configurations of isolated zeroes of the related Abelian integral I_ν can be transferred to results on configurations of limit cycles of the family in a trivial way, at least if the Abelian integral represents an elementary catastrophe.

In dealing with a k-saddle cycle Γ (i.e. a hyperbolic polycycle with k saddle-type singular points), the transfer of the results on the related Abelian integral I_ν is no longer obvious. The difficulties are due to the fact that the displacement map is not C^∞ at the saddle points, unlike the case when Γ is a periodic orbit or a non-degenerate singular point.

In dealing with a 1-saddle cycle or a so-called saddle loop, it is known from [Mar], that under certain genericity conditions on the Abelian integral I_ν, the configuration of limit cycles of $X_{(\nu,\varepsilon)}$, for ε close to 0, is completely analoguous to the configuration of zeroes of I_ν.

In general, unlike the case of the regular periodic orbit or the saddle loop, the bifurcation diagram of limit cycles near a k-saddle cycle is no longer trivial in the ε-direction. The bifurcation diagram of a 2-saddle cycle is studied in [DRR], and more generally, the generic k-parameter unfoldings of k-saddle cycles are studied in [Mo]. Using these results, it is proven in [DR], that the Abelian integral is a very bad approximation of the displacement map as soon as the unfolding breaks more than one connection: almost all the limit cycles cannot be traced by the Abelian integral.

It is even not obvious whether it is possible to transfer results on the Abelian integral to obtain valuable results on the cyclicity along the 2-saddle cycle. Even in case the unfolding keeps one connection of the 2-saddle cycle unbroken, the transfer does not work out in a trivial way, unlike one could expect by the known results on the saddle loop.

In [DR], it is proven that there exist generic unfoldings of 2-saddle cycles leaving one connection unbroken, for which the cyclicity is 4, while the related Abelian integral I_ν is of codimension 3, and hence can produce at most 3 zeroes. As a consequence, in that case, one limit cycle is not covered by a zero of the related Abelian integral. Such a limit cycle is called an alien limit cycle.

However, the problem of transfer can be dealt with. From [DR], it is known that the Abelian integral I_ν provides a finite upperbound for the cyclicity, if it is of finite codimension. It is interesting to notice that the upperbound in this finite cyclicity result, is strictly bigger than the maximal possible zeroes of the related Abelian integral. Therefore, it is possible that, in general, the family creates more alien limit cycles near the considered 2-saddle cycle.

The thesis is organised as follows.

Chapter 1 recalls the techniques that are used in the study of the bifurcation of limit cycles and the cyclicity near centers.

In chapter 2, we present a complete and clear reference work on Hopf-Takens bifurcations (generic and near centers), aiming, in applications, at obtaining accurate results based on a minimal amount of verification. In chapter 3, we apply these results to the study of bifurcations of small-amplitude limit cycles in families originating from classical and generalized Liénard equations. The simplicity of the Liénard family is used to illustrate the advantages of the approach based on Bautin ideals. The Bautin ideal is generated by a set of Lyapunov quantities. Attention goes to the local division of a family of displacement maps, the presence of Hopf-Takens bifurcations, and the cyclicity.

In chapter 4, we examine the use of 1-parameter techniques in analytic multi-parameter families.

Chapter 5 deals with unfoldings of a 2-saddle cycle, leaving one connection unbroken, extending the results of [DR] and indicating new problems that show up in generalising this study. Special interest goes to the existence of alien limit cycles. The existence of alien limit cycles implies that knowlegde of the linear approximation I_ν, with respect to $\varepsilon = 0$, is not sufficient to transfer results on zeroes of I_ν in a trivial way to valid results on limit cycles, arbitrarily close to Γ, of the unfolding $X_{(\nu,\varepsilon)}$ (for $\lambda = (\nu, \varepsilon)$ near λ^0). The study in chapter 5 gives rise to the following conjecture: 'A generic unfolding of the 2-saddle cycle, leaving one connection unbroken, can produce $3k$ (respectively $3k-1$) limit cycles, while the related Abelian integral is of codimension $2k+1$ (respectively $2k$)'. This conjecture would imply the existence of at least $k-1$ alien limit cycles. Furthermore, we prove that a particular subfamily of the 2-saddle cycle, leaving one connection unbroken and in which the saddles remain linear at the bifurcation, can produce at least $k-2$ alien limit cycles, if the related Abelian integral is of codimension k.

Contents

Chapter 1

Preliminaries and technicalities

In this chapter, we present elementary definitions, techniques and notations, that will be used throughout this thesis. We start by recalling the sixteenth problem of Hilbert, that he proposed on the International Mathematical Congress in 1901. As explained in section 1.1, it is one of the motivations to study local bifurcation diagrams of limit cycles in analytic families of planar vector fields.

Traditionally, the study of limit cycles of vector fields is transferred into the study of zeroes of displacement maps. The cyclicity can be expressed in terms of displacement maps. In this way, the investigation of limit cycles that occur in a two-dimensional phase plane, is replaced by a problem on zeroes in a one-dimensional space. Definition and properties of these maps are recalled very quickly in section 1.2. To controll zeroes of maps locally, one can apply classical theorems as the Implicit Function Theorem, Rolle's theorem, the Preparation Theorem, as long as the map under consideration is of finite order (section 1.2.1).

If the vector field X_{λ^0} is of center type, i.e. the associated displacement map δ_{λ^0} is identically zero; as a consequence, δ_{λ^0} is of infinite order. Therefore, in this case, the above mentioned classical theorems cannot be applied in a straightforward way. In a certain sense, some degeneracies first have to be removed. In case the center type is caused by a 1-dimensional parameter, then the first non-zero Melnikov function gives information about the maximal number of limit cycles (section 1.2.2). In case a multi-dimensional parameter is responsable for the center type, then there is a powerful theoretical technique namely the Bautin ideal (section 1.2.3). This ideal serves to characterize the parameter values for which the corresponding vector field is of center type. Moreover, an upperbound for the cyclicity can be expressed in terms of this ideal. We also recall the most important properties of the Bautin ideal, that will be used later on.

Lyapunov quantities contain information about the stability of the focus and the possible presence of Hopf-Takens bifurcations, as we will see in chapter 2. Traditionally, Hopf-Takens bifurcations are characterised by normal forms. Both techniques are of algebraic nature; in section 1.2.4, we recall the definition of Lyapunov quantities and state an important relation between normal forms and Lyapunov quantities. Moreover, we clarify the connection with the Bautin ideal and the coefficients of the displacement map. We also recall a technical proposition to compute Lyapunov quantities in Liénard systems, that will be used in chapter 3. Finally, the duality between the saddle and focus is recalled.

Next, we consider the special situation of a multi-parameter family (X_λ), where $\lambda = (\nu, \varepsilon)$ and only for $\varepsilon = 0$, the vector field is of center type. We will refer to this situation by 'regular hypersurface of centers' and an exact definition is given in section 1.3. After removing the degeneracy in ε, we obtain reduced displacement maps and reduced Lyapunov quantities. These last quantities again can be used to investigate the presence of Hopf-Takens bifurcations near centers.

Finally, in chapter 5 we deal with the cyclicity problem near a graphic, more precisely a 2-saddle cycle. The displacement map at a graphic is not differentiable, but can continuously be extended there. Standard techniques as the division-derivation algorithm for differentiable functions cannot be applied. A good frame for studying such unfoldings is created in [DR] by the introduction of 'simple asymptotic scale deformation'. In section 1.4, we recall the definition of 'simple asymptotic scale deformation' and related notions, that are illustrated by several examples that will be used in chapter 5. Moreover, we show that an Abelian integral along a polycycle has an asymptotic expansion in the logarithmic scale. In sections 1.5 and 1.6, we describe its use in the study of the bifurcation pattern of limit cycles in case of the saddle loop and the 2-saddle cycle respectively, by recalling results from [Mar] and [DR] respectively.

1.1 Limit cycles

For basic definitions and theorems on differential theory, such as regular surface (or manifold), vector fields, existence and uniqueness of solutions, singular points, periodic orbits, α- and ω-limits,..., we refer the reader to e.g., [HS], [JrM], [Perko], and [W]. Recall that a vector field $X = X_1\frac{\partial}{\partial x} + X_2\frac{\partial}{\partial y}$ on $\mathbb{R}^2$, can be represented as a system of autonomous first order differential equations

$$\begin{cases} \dot{x} &= X_1(x,y) \\ \dot{y} &= X_2(x,y) \end{cases}$$

or by the so-called *dual 1-form* υ, defined as

$$\upsilon = X_1 \mathrm{d}y - X_2 \mathrm{d}x$$

Let us now give a precise definition of a limit cycle.

Definition 1 *Let X be a vector field on a regular surface S. Then we say that γ is a limit cycle if γ is an isolated periodic orbit of X, that is not a singular point. (The notion isolated means that a sufficiently small neighbourhood of γ contains no other periodic orbits than γ.)*

A famous example of vector field, presenting a limit cycle in its phase portrait, is the one corresponding to the Van der Pol equation

$$y'' - ay' + (y')^3 - y = 0$$

in the (x, y)-plane with $x = y'$:

$$\begin{cases} \dot{x} &= x + y - x^3 \\ \dot{y} &= -x \end{cases} .$$

Its phase portrait is shown in figure 1.1.

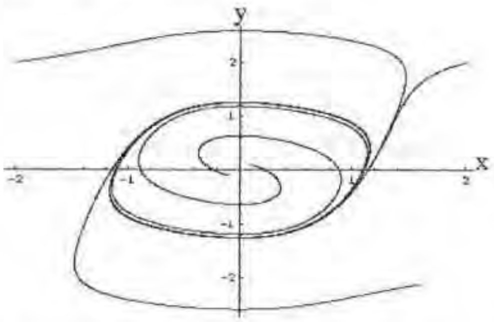

Figure 1.1: Limit cycle appearing in the phase portrait of the Van der Pol vector field

Remark 2 *Often, S will correspond to an open subset of $\mathbb{R}^2$. In this case, we speak of a one-sided isolated periodic orbit, if the property of being isolated only holds for points x on one side of γ. This definition makes sense by the Jordan Curve Theorem [D69]. In this respect, the notion 'isolated' in definition 1 is also referred to as 'two-sided isolated'. However, periodic orbits of an analytic vector field X are either two-sided isolated or not isolated. Indeed, we can associate an analytic function $\delta : I \to \mathbb{R}$ (defined on an open set $I \subset \mathbb{R}$) such that zeroes of δ correspond to periodic orbits of X (see also section 1.2.1, where δ is called a displacement map). By the principle of isolated zeroes for analytic functions [D69], an analytic function $\delta : I \subset \mathbb{R} \to \mathbb{R}$ defined on an open set I, can only have isolated zeroes, unless δ is identical 0. In the first case the periodic orbits of X are isolated, while in the last case, they are not isolated (even worse), and the vector field X is then called a vector field of center type, see section 1.2.1.*

The second part of Hilbert's sixteenth problem deals with polynomial vector fields in the plane. It essentially asks for the maximal number H_n of limit cycles, and their relative positions, in a polynomial planar vector field, in function of the degree n :

$$\sum_{i+j=0}^{n} a_{ij}x^i y^j \frac{\partial}{\partial x} + \sum_{i+j=0}^{n} b_{ij}x^i y^j \frac{\partial}{\partial y}, \text{ where } x, y, a_{ij}, b_{ij} \in \mathbb{R}, \forall 0 \leq i+j \leq n \quad (1.1)$$

The finiteness part of Hilbert's sixteenth problem exists in proving that H_n is finite, $\forall n \in \mathbb{N}$. Of course, $H_1 = 0$, since periodic orbits of linear vector fields are never isolated. It remains unsolved even for quadratic polynomial vector fields. There were several attempts to solve it, but all of them failed. So far, one only knows for sure that $H_2 \geq 4$ and $H_3 \geq 11$. Yet the problem has a source of inspiration for significant progress in the geometric theory of planar vector fields, as well as bifurcation theory, normal forms, foliations and some topics in algebraic geometry.

Let us now briefly explain how this global problem can be 'localized'. Therefore, we first need to recall some definitions.

Let (S, d) be a metric space. Then, the set of all non-empty compact subsets of S is denoted by $\mathcal{H}(S)$ (compact, for the topology induced by the metric on S). The *Hausdorff space* is defined by the metric space $\mathcal{H}(S)$, equipped with the *Hausdorff distance* d_H, defined by

$$d_H : \mathcal{H}(S) \times \mathcal{H}(S) \to \mathbb{R} : (A, B) \mapsto d_H(A, B),$$

with $d_H(A, B) = \max\{\sup\{d(a, B) : a \in A\}, \sup\{d(b, A) : b \in B\}\}$ [Barnsley]. Convergence in the sense of the Hausdorff distance will be denoted by the symbol $\underset{d_H}{\to}$. In this thesis S will be a compact subset of $\mathbb{R}^2$; if S is a compact metric space, then it follows from elementary topology that also $\mathcal{H}(S)$ is a compact metric space [Barnsley]. Now we can give a definition of the notion of limit periodic set and its cyclicity, i.e. the maximum number of limit cycles, that locally can bifurcate from Γ. From the definition it will be clear that the cyclicity only depends on the germ of the family $(X_\lambda)_\lambda$ at (Γ, λ^0). Recall that for a function $f : X \to \mathbb{R}$ defined on a metric space (X, d) and $y \in X$ the superior limit 'limsup' for $x \to y$, is defined as:

$$\underset{x \to y}{\text{limsup}} f(x) = \inf\{\sup\{f(x) : d(x, y) < \delta\} : \delta > 0\}.$$

Definition 3 *Let $P \subset \mathbb{R}^p$ be the parameter space with $\lambda^0 \in P$, and let $(X_\lambda)_{\lambda \in P}$ be a family of planar vector fields on a regular surface S. Consider the Hausdorff distance d_H on $\mathcal{H}(S)$.*

1. *A set $\Gamma \subset S$ is a limit periodic set of $(X_\lambda)_\lambda$ at λ^0 if and only if $\Gamma \in \mathcal{H}(S)$ and there exists a sequence $(\lambda_i)_{i \in \mathbb{N}}$ in P with $\lambda_i \to \lambda^0, i \to \infty$ and such that for every $i \in \mathbb{N}$, there exists a limit cycle γ_i of X_{λ_i} with $\gamma_i \underset{d_H}{\to} \Gamma$, $i \to \infty$.*

2. *Let Γ be a limit periodic set of $(X_\lambda)_{\lambda\in P}$ at λ^0, and let $S\subset\mathbb{R}^2$. Then, we say that Γ has finite cyclicity in $(X_\lambda)_\lambda$, if there exist $N\in\mathbb{N}$, and constants $\varepsilon,\delta>0$ such that for every parameter value λ with $\|\lambda-\lambda^0\|<\delta$, the vector field X_λ has at most N limit cycles γ with $d_H(\gamma,\Gamma)<\varepsilon$. If Γ has the finite cyclicity property, then we define the cyclicity of $(X_\lambda)_\lambda$ along Γ for $\lambda=\lambda^0$ by*

$$\mathrm{Cycl}\left(X_\lambda,(\Gamma,\lambda^0)\right)=\underset{\lambda\to\lambda^0,\gamma\xrightarrow[d_H]{}\Gamma}{\mathit{limsup}}\{\text{number of limit cycles }\gamma\text{ of }X_\lambda\}.$$

Notice that definition 3 includes the artificial situation that $P=\{\lambda^0\}$ and $X_\lambda\equiv X_{\lambda^0}$; then the only possible limit periodic sets are limit cycles of X_{λ^0}.

A major question is to give sufficient conditions under which the cyclicity can be finite and, in this case, to find an explicit estimation of this cyclicity.

We shortly speak of a compact family of planar vector fields $(X_\lambda)_\lambda$ if the vector fields X_λ are defined on a compact metric space, and depend on a parameter λ, that also belongs to a compact metric space. Working with a compact analytic family of planar vector fields $(X_\lambda)_\lambda$, there exists the following interesting equivalence between the global bound and local bounds [R98]: the number of limit cycles of X_λ is bounded, uniformly with respect to λ if and only if every limit periodic set Γ has finite cyclicity in $(X_\lambda)_\lambda$. Let $\left(P_{(a,b)}\right)_{(a,b)}$ denote the family of polynomial vector fields, where $P_{(a,b)}$ is defined by (1.1) and $a=(a_{ij})_{i+j=0}^n$, $b=(b_{ij})_{i+j=0}^n$. Then, the *Hilbert number* H_n is the maximum number of limit cycles in the family $\left(P_{(a,b)}\right)_{(a,b)}$. By compactification of phase space and parameter space, we can extend this family to a compact analytic family of planar vector fields $(X_\lambda)_\lambda$, in such a way that the maximum number of limit cycles in $(X_\lambda)_\lambda$ is equal to H_n. In this way, if one can solve the local cyclicity problem, then one can give an answer to the finiteness part of Hilbert's sixteenth problem [R98]. This observation motivates the study of bounding the cyclicity of limit periodic sets in analytic families of planar vector fields.

Let us also mention a non-trivial structure theorem for limit periodic sets in case there are only isolated singularities. This structure theorem is similar to the Poincaré-Bendixson theorem for α- and ω-limit sets [HS]. 'Suppose that $\lambda^0\in P\subset\mathbb{R}^p$ and $(X_\lambda)_{\lambda\in P}$ is an analytic family of vector fields on S, where S is an open subset of $\mathbb{S}^2$. Suppose further that X_{λ^0} has only isolated singular points. Then, a limit periodic set Γ of $(X_\lambda)_\lambda$ for $\lambda=\lambda^0$, is either a singular point of X_{λ^0}, or a periodic orbit of X_{λ^0}, or a graphic of X_{λ^0}, where a graphic is defined as:

Definition 4 *A graphic Γ of a vector field X is the union of singular points $p_1,\ldots,p_m$ with $p_m=p_1$, not necessarily distinct, and regular orbits $\gamma_1,\ldots,\gamma_{m-1}$, connecting these singular points, in the sense that $p_i=\alpha(\gamma_i)$ and $p_{i+1}=\omega(\gamma_i)$, $1\le i\le m-1$. (Here, we mean by regular orbit an orbit of X, that is not a singular point of X).*

Throughout the thesis, Γ might represent a singularity, a (regular) periodic orbit or a graphic.

As is common, we denote analytic by C^ω, and we always mean real analytic. Furthermore, if $C \subset \mathbb{R}^N$ ($N \in \mathbb{N}$) is a compact set, then we say that a function $f : C \to \mathbb{R}^l$ ($l \in \mathbb{N}$) is analytic on C if f can be extended analytically on an open set O in $\mathbb{R}^N$, that contains C; i.e. if there exists an analytic function $\bar{f} : O \to \mathbb{R}^l$ such that

$$\bar{f}\big|_C = f.$$

Recall that the notation $\bar{f}\big|_C$ is used for the restriction of $\bar{f}$ to C.

From the preceeding discussion, it is clear that our problems are of local nature; local, in the sense that a property has to be satisfied on arbitrarily small neighbourhoods of λ^0 and Γ. We often are not interested in the exact neighbourhood, we only need to be sure that there exists a neighbourhood with the required property. Therefore, we now summarise some notations that will be used throughout the thesis. Since the cyclicity of a limit periodic set Γ only depends on the germ of the family $(X_\lambda)_\lambda$ at (Γ, λ^0), we also speak of an unfolding of X_{λ^0} along Γ, represented by the family $(X_\lambda)_\lambda$.

We use twiddles $\tilde{}$ to denote germs of a certain C^ω function at λ^0. The notation $(\mathbb{R}^p, \lambda^0)$ will represent a neighbourhood of λ^0 in $\mathbb{R}^p$, analoguously $(\mathbb{R}, s_0)$ will represent a neighbourhood of s_0 in $\mathbb{R}$, and so on. Furthermore, $\lambda \sim \lambda^0$ (respectively $s \sim s_0$) means that λ is close to λ^0 (respectively s close to s_0), and e.g., $\forall \varepsilon \downarrow 0$ indicates that the property holds for all $\varepsilon > 0$ sufficiently close to 0.

When we speak of an analytic function $f : (\mathbb{R} \times \mathbb{R}^p, (s_0, \lambda^0)) \to \mathbb{R}$ (respectively $f : (\mathbb{R}^2 \times \mathbb{R}^p, ((x_0, y_0), \lambda^0)) \to \mathbb{R}$), it means that f is analytic on a neighbourhood $U \times W$ of (s_0, λ^0) in $\mathbb{R} \times \mathbb{R}^p$ (respectively $((x_0, y_0), \lambda^0)$ in $\mathbb{R}^2 \times \mathbb{R}^p$), that takes the form

$$\left\{(s, \lambda) \in \mathbb{R} \times \mathbb{R}^p : |s| < \rho, \left\|\lambda - \lambda^0\right\| < \rho\right\}$$

(respectively $\left\{((x, y), \lambda) \in \mathbb{R}^2 \times \mathbb{R}^p : |x - x_0| < \rho, |y - y_0| < \rho \left\|\lambda - \lambda^0\right\| < \rho\right\}$), for a certain $\rho > 0$.

Furthermore, in chapter 4, we will consider contacts between curves: we say that two curves $\zeta, \hat{\zeta} : I \subset \mathbb{R} \to \mathbb{R}^p$ have contact of order at least k at $x_0 \in I$ if

$$(\zeta - \hat{\zeta})(x) = o((x - x_0)^k), x \to x_0,$$

meaning that $\lim_{x \to x_0} \frac{(\zeta - \hat{\zeta})(x)}{(x - x_0)^k} = 0$. Related to this notion, we encounter the notion of k-jet of functions f of differentiability class C^N ($N \geq k$ or $N \in \{\infty, \omega\}$) at x_0, i.e. the Taylor polynomial P_k of f at $x = x_0$ of degree k; it is denoted by

$$j_k(f)_{x_0}(x) = P_k(x)$$

(thus having the property: $(f - P_k)(x) = o((x - x_0)^k), x \to x_0$).

Furthermore, a function $f : I \subset \mathbb{R}^p \to \mathbb{R}$ is said to be non-zero, if there exists $x \in I$ such that $f(x) \neq 0$; a germ φ of an analytic function at x^0 is said to be non-zero if every representative $f : I \subset \mathbb{R}^p \to \mathbb{R}$ (with $x^0 \in I$) for φ is non-zero.

1.2 Basic tools

Recall that throughout the thesis Γ might represent a singular point, a (regular) periodic orbit, or a graphic. In this section, we recall the notions of displacement map, vector field of center type, Melnikov functions, Bautin ideal and related notions. Besides, we state and prove some new interesting properties that will be used in this thesis. In particular, we fix notations that will be used hereafter.

1.2.1 Displacement maps

Definitions

Let $k \in (\mathbb{N} \setminus \{0\}) \cup \{\infty, \omega\}$, and let $(X_\lambda)_\lambda$ be a C^k family of planar vector fields, where the parameter λ belongs to an open neighbourhood W_0 of a fixed parameter λ^0 in $\mathbb{R}^p$.

Suppose that Γ is a periodic orbit of X_{λ^0}, and that Σ is a C^k section transverse to X_{λ^0} with $\Sigma \cap \Gamma \neq \emptyset$, i.e. the vector $X_{\lambda^0}(p)$ and an arbitrary tangent vector to Σ at p are linearly independent, $\forall p \in \Sigma$. Suppose that this section is parametrised in a regular C^k way by $s \in \mathbb{R}$ and s_0 represents the intersection of Γ with Σ.

Then, locally in a neighbourhood of Γ and λ^0, one can define a C^k family of maps $(P_\lambda)_\lambda$, induced by the flow, the so-called *Poincaré-map* or *first return map*. More precisely, there exists an open neighbourhood $W \subset W_0$ of λ^0 and an open interval I centered at s_0 in $\mathbb{R}$ such that, $\forall \lambda \in W$, the point $P_\lambda(s)$ corresponds to the first return in Σ by the orbit of X_λ that started at the point on Σ determined by s.

In this way, periodic orbits (limit cycles) of X_λ correspond to (isolated) fixed points s of the Poincaré-map P_λ. In particular, $P_{\lambda^0}(s_0) = s_0$. One also considers the C^k family of *displacement maps* $(\delta_\lambda)_\lambda$, defined by

$$\delta_\lambda(s) = P_\lambda(s) - s, s \in I$$

This value $\delta_\lambda(s)$ indicates how much the ordit 'through' s deviates from being periodic. More precisely, periodic orbits (limit cycles) of X_λ correspond to (isolated) zeroes of δ_λ. In particular, $\delta_{\lambda^0}(s_0) = 0$.

If Γ is a non-degenerate elliptic singularity, i.e. the eigenvalues of the linear part of X_{λ^0} at the singular point Γ are complex conjugate (not real), we know by the implicit function theorem, that this singularity persists for all parameter values λ sufficiently close to λ^0. Then, a polar blow up transforms the singularity into a periodic orbit, giving the possibility to considering a family of Poincaré-maps $(P_\lambda)_\lambda$ and a family of displacement maps $(\delta_\lambda)_\lambda$ as before. By the blow-up procedure, we have lost one degree of smoothness, meaning that the family $(P_\lambda)_\lambda$ as well as $(\delta_\lambda)_\lambda$ is of class C^{k-1} if the family $(X_\lambda)_\lambda$ is C^k, for $k \in \mathbb{N} \setminus \{0\}$. If the family $(X_\lambda)_\lambda$ is C^∞ (respectively C^ω), then the family $(P_\lambda)_\lambda$ and $(\delta_\lambda)_\lambda$ also are C^∞ (respectively C^ω).

Clearly, the zeroes of δ_λ are not affected by multiplication of δ_λ by a non-zero function. Therefore, we can study more generally families of functions $(\delta_\lambda)_\lambda$, having the following property: isolated zeroes of δ_λ (close to s_0) correspond to limit cycles of X_λ (close to Γ) and vice versa. Sometimes, such a family of functions will also be called a family of displacement maps for $(X_\lambda)_\lambda$.

In view of this remark, we can easily construct vector fields, given a prescribed displacement map δ_λ, as follows:

$$X_\lambda = (y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}) + \delta(s,\lambda)(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}), \tag{1.2}$$

where $s > 0, s^2 = x^2 + y^2$ and $\delta(s,\lambda) = \delta_\lambda(s)$. Notice that, in the traditional sense of the word, δ_λ is only a displacement map for X_λ, up to a non-zero factor.

Cyclicity and multiplicity

The cyclicity of the family of planar vector fields $(X_\lambda)_\lambda$ can be expressed in terms of the associated family of displacement maps $(\delta_\lambda)_\lambda$:

$$\mathrm{Cycl}\left(X_\lambda, (\Gamma, \lambda^0)\right) = \limsup_{\lambda\to\lambda^0, s\to s_0} \{\text{number of isolated zeros } s \text{ of } \delta_\lambda\}$$

Sometimes one not only counts the maximal number of possible zeroes, but one also takes multiplicity of the limit cycles into account. The multiplicity of a limit cycle is given by the multiplicity of the corresponding zero. We say that s_0 is *a multiple zero of* δ_{λ^0} *of multiplicity* k, also called a *k-uple zero of* δ_{λ^0}, if

$$\delta_{\lambda^0}^{(j)}(s_0) = 0, \forall 0 \le j \le k-1 \text{ and } \delta_{\lambda^0}^{(k)}(s_0) \neq 0, \tag{1.3}$$

If (1.3) holds, we also say that the map δ_{λ^0} *has order* k *at* s_0, and we denote it by

$$\mathrm{order}\delta_{\lambda^0}(s_0) = k$$

This number is referred to as the multiplicity:

$$\mathrm{Mult}\left(X_\lambda, (\Gamma, \lambda^0)\right) = \limsup_{\lambda\to\lambda^0, s\to s_0} \{\sum_s \mathrm{order}\delta_\lambda(s) : s \text{ zero of } \delta_\lambda\}.$$

Hence, near Γ, with Γa periodic orbit or a non-degenerate elliptic singularity, the study of the maximal possible number of limit cycles is transferred into the study of maximal number of isolated zeroes of 1-dimensional maps.

Of course,

$$\mathrm{Cycl}\left(X_\lambda, (\Gamma, \lambda^0)\right) \le \mathrm{Mult}\left(X_\lambda, (\Gamma, \lambda^0)\right).$$

In fact, the notions of cyclicity and multiplicity coincide, when there exists a parameter value λ for which the vector field X_λ has exactly $Mult(X_\lambda, (\Gamma, \lambda^0))$ limit cycles. This is what happens for generic unfoldings.

Basic theorems

We give here precise statements of some interesting classical theorems, that are the main ingredients in the study of zeroes of maps of finite order: the Implicit Function Theorem, Rolle's theorem and the Preparation Theorem. Moreover, we give two useful lemma's to investigate the submersion condition that is required by the Preparation Theorem, and that will be used later in chapter 2.

In case that s_0 is a *simple zero* of δ_{λ^0}, i.e.

$$\delta_{\lambda^0}(s_0) = 0 \text{ and } \delta'_{\lambda^0}(s_0) \neq 0$$

there is no bifurcation at all, meaning that the zero persists for all values of λ sufficiently close to λ^0. In particular, $Cycl(X_\lambda, (\Gamma, \lambda^0)) = Mult(X_\lambda, (\Gamma, \lambda^0)) = 1$. This persistence of the zero can be proven by use of the Implicit Function Theorem [D69].

Theorem 5 (Implicit Function Theorem) *Let E, F, G be three Banach spaces, let A be an open subset of $E \times F$ and let $f : A \to G$ be a function of class C^1. Let $(x_0, y_0) \in A$ such that $f(x_0, y_0) = 0$ and suppose that the partial differential $D_y f(x_0, y_0)$ is a linear homeomorphism from F in G. Then there exists an open neighbourhood U_0 of x_0 in E such that for every open connected neighbourhood U of x_0 with $U \subset U_0$, there exists a unique continuous map $u : U \to F$ such that*

$$u(x_0) = y_0$$

and $\forall x \in U$:

$$(x, u(x)) \in A \text{ and } f(x, u(x)) = 0$$

Moreover, u is of class C^1, and

$$Du(x) = -(D_y f(x, u(x)))^{-1} \circ D_x f(x, u(x))$$

Moreover, if f is of class C^k $(k \in \mathbb{N} \cup \{\infty\})$ in a neighbourhood of (x_0, y_0), then u also is of class C^k. If E, F, G are finite dimensional, and f is C^ω in A, then also u is C^ω.

Throughout the thesis, the Banach spaces E, F, G will be open subsets of a Euclidean space $\mathbb{R}^n$, where $n \in \mathbb{N} \setminus \{0\}$. The following well-known theorem from elementary calculus [D69], proves to be very useful in bounding the cyclicity. Moreover, it is the foundation of the Division-Derivation algorithm, stated below. This algorithm is frequently used in finding upperbounds for the cyclicity, e.g., [R98].

Theorem 6 (Rolle) *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a function of class C^k $(k \in \mathbb{N} \setminus \{0\})$, where I is an open interval in $\mathbb{R}$. If $f^{(n)}(s_0) \neq 0$ for $s_0 \in I$, then there exists a neighbourhood I_0 of s_0 in I_0 such that f has at most n zeroes in I_0 (multiplicity included).*

As a consequence of Rolle's theorem, in case s_0 is a *multiple zero of* δ_{λ^0} *of multiplicity* k, then δ_λ has at most k zeroes in a neighbourhood of s_0 for all parameter values λ, sufficiently close to λ^0. Hence, in this case,

$$\mathrm{Cycl}\left(X_\lambda, \left(\Gamma, \lambda^0\right)\right) \leq \mathrm{Mult}\left(X_\lambda, \left(\Gamma, \lambda^0\right)\right) \leq k.$$

Theorem 7 (Division-Derivation algorithm based on Rolle's theorem) *Let* $f : I \subset \mathbb{R} \to \mathbb{R}$ *be a function of class* C^k $(k \in \mathbb{N}\setminus\{0\})$, *where* I *is an open interval in* $\mathbb{R}$ *with* $s_0 \in I$. *Let* $a_j : I \to \mathbb{R}$ *be functions of class* C^{k-j} *with*

$$a_j(s_0) \neq 0, \forall 0 \leq j \leq n$$

Define the functions $f_j : I \to \mathbb{R}$ $(0 \leq j \leq n)$ *inductively as follows:*

$$f_0 = a_0 \cdot f \text{ and } f_j = a_j \cdot f'_{j-1}, \forall 1 \leq j \leq n$$

If $f_n(s_0) \neq 0$, *then there exists a neighbourhood* I_0 *of* s_0 *in* I *such that* f *has at most* n *zeroes in* I_0.

Notice that, by this theorem we only find an upperbound for the number of zeroes, we have no controll on the exact maximal number of zeroes, or on the relative positions of these zeroes. The following theorem is a stronger result than Rolle's theorem: it also says something about the genericity of the bifurcation diagram of the zeroes in the family $(\delta_\lambda)_\lambda$, locally at s_0 and λ^0, by linking the bifurcation diagram of the zeroes in the family $(\delta_\lambda)_\lambda$ to a family of polynomials. In this way, the family of maps is prepared to study its zeroes. For this reason, this theorem is called the Preparation Theorem [M, Malgrange66],[GG]. We will only use the C^∞ and C^ω version of this theorem.

Theorem 8 (Preparation Theorem) *Let* $U_0 \times W_0 \subset \mathbb{R} \times \mathbb{R}^p$ *be an open neighbourhood of* $(0, \lambda^0)$, *and let be given a* C^∞ *(respectively* C^ω*) function* $F : U_0 \times W_0 \to \mathbb{R}$ *such that*

$$F\left(r, \lambda^0\right) = r^l F_l(r), \quad \forall r \in U_0 \quad \text{with } F_l(0) \neq 0$$

Then, there exist an open neighbourhood $U \times W \subset U_0 \times W_0$ *of* $(0, \lambda^0)$ *and* C^∞ *(respectively* C^ω*) functions*

$$\phi : W \to \mathbb{R}^l \text{ and } h : U \times W \to \mathbb{R}$$

such that

$$F(r, \lambda) = Q(h(r, \lambda), \phi(\lambda)) \cdot V^{(l)}(h(r, \lambda), \phi(\lambda)) \tag{1.4}$$

where $Q(0,0) \cdot F_l(0) > 0$ *and*

$$V^{(l)}(R, (a_0, \ldots, a_{l-1})) = a_0 + a_1 R + \ldots + a_{l-1} R^{l-1} + R^l$$

such that

1. *$\phi(\lambda^0) = 0$ and $h(0,\lambda) = 0, \forall\lambda \in W$,*

2. *the map $h(\cdot,\lambda)$ is an orientation preserving diffeomorphism, $\forall\lambda \in W$*

3. *In particular, if the map*

$$\psi : W \subset \mathbb{R}^p \to \mathbb{R}^l : \lambda \mapsto \left(F(0,\lambda), \frac{\partial F}{\partial r}(0,\lambda), \ldots, \frac{\partial^{l-1} F}{\partial r^{l-1}}(0,\lambda)\right) \tag{1.5}$$

   defines a submersion at λ^0, then ϕ is also a submersion at λ^0.

The principal use of the Preparation Theorem in the study of zeroes consists in the fact that it allows us 'discarding the tails', i.e. it allows us to carry over various results on normal forms to the genuine expression.

If the map defined in (1.5) is a submersion, then, by (1.4), there exists an open submanifold inside the 'local' bifurcation diagram of zeroes of F, containing λ^0, that is diffeomorphic to the bifurcation diagram of $V^{(l)}$ (by 'local', we mean that zeroes of $F(\cdot,\lambda)$ belong to a neighbourhood U of 0, for parameter values λ in a neighbourhood W of λ^0). As we already know by the Implicit Function Theorem, we only have bifurcation for $l \geq 2$. Bifurcation diagrams for the polynomials $V^{(l)}$ are well-known, at least for $2 \leq l \leq 5$: these bifurcation diagrams correspond to the first 4 of the seven elementary catastrophes: the *fold* for $l = 2$, the *cusp* for $l = 3$, the *swallow-tail* for $l = 4$ and the *butterfly* for $l = 5$ ([Bro],[CH],[GG]).

The third statement in the Preparation Theorem follows from lemma 9 (chapter 1) below, whose proof relies on a straightforward calculation. We take $n = l - 1$ and

$$C_{jj}(\lambda) = j! \cdot \left(\frac{\partial h}{\partial r}(0,\lambda)\right)^j \cdot Q(0,\lambda), \quad \forall 0 \leq j \leq l-1$$

Lemma 9 *Suppose $\psi_j, \phi_j, j = 0, 1, \ldots, n$ are C^∞ (respectively C^ω) real-valued functions defined on a neighbourhood of $\lambda^0 \in \mathbb{R}^p$ such that*

$$\psi_j = \sum_{i=0}^{j} C_{ji}\phi_i,\ j = 0, 1, \ldots, n$$

for certain C^∞ (respectively C^ω) functions C_{ji} with $C_{jj}(\lambda^0) > 0$. Suppose that $\phi_n(\lambda^0) \neq 0$. Define the maps $\phi = \left(\frac{\phi_0}{\phi_n}, \frac{\phi_1}{\phi_n}, \ldots, \frac{\phi_{n-1}}{\phi_n}\right) : (\mathbb{R}^p, \lambda^0) \to \mathbb{R}^n$ and $\psi = (\psi_0, \psi_1, \ldots, \psi_{n-1}) : (\mathbb{R}^p, \lambda^0) \to \mathbb{R}^n$.

1. *$\phi_j(\lambda^0) = 0, \forall 0 \leq j \leq n-1, \phi_n(\lambda^0) < 0$ (respectively > 0) if and only if $\psi_j(\lambda^0) = 0, \forall 0 \leq j \leq n-1, \psi_n(\lambda^0) < 0$ (respectively > 0);*

2. *Under the conditions described in 1., ϕ is a submersion at λ^0 if and only if ψ is a submersion at λ^0.*

3. *If $\lambda = (\nu, \varepsilon), \varepsilon \geq 0$ and there exist C^∞ (respectively C^ω) functions $\bar{\phi}_j, \bar{\psi}_j, j = 0, 1, \ldots, n-1$ such that $\forall 0 \leq j \leq n-1$:*

$$\phi_j = \varepsilon^k \cdot \bar{\phi}_j \text{ and } \psi_j = \varepsilon^k \cdot \bar{\psi}_j,$$

then the equivalences in 1. and 2. also hold if we replace ϕ_j (respectively $\bar{\psi}_j$) by $\bar{\phi}_j(\cdot, 0)$ (respectively $\bar{\psi}_j(\cdot, 0)$) and λ^0 by ν^0.

On many occasions, one can succeed in checking that at some point λ^0 we get

$$\psi_0(\lambda^0) = \ldots = \psi_{l-1}(\lambda^0) = 0 \text{ and } \psi_l(\lambda^0) \neq 0,$$

where $\psi_l(\lambda^0) = \frac{\partial^l F}{\partial r^l}(0, \lambda^0)$. But to controll the fact that the map

$$\lambda \mapsto (\psi_0(\lambda), \ldots, \psi_{l-1}(\lambda)) \tag{1.6}$$

is a submersion at λ^0, might be quite a challenge. Sometimes one only succeeds in calculating ψ_i, with $0 < i \leq l$ at those values λ where $\psi_0(\lambda) = \ldots = \psi_{i-1}(\lambda) = 0$. This is of course not a draw-back with respect to the third statement in the Preparation Theorem. Indeed if we want the map (1.6) to be a submersion at λ^0, then certainly every component $\lambda \mapsto \psi_i(\lambda)$ has to be a submersion too. As such we need the requirement that $\lambda \mapsto \psi_0(\lambda)$ is a submersion at λ^0. This implies that for λ near λ^0 the set $Z_0 = \psi_0^{-1}(0)$ is a submanifold and we can now restrict the rest of the calculation to Z_0. It is easy to see that $\lambda \mapsto (\psi_0(\lambda), \psi_1(\lambda))$ is a submersion at λ^0 if and only if ψ_0 is a submersion at λ^0 and the restriction of ψ_1 to Z_0, $\psi_1|_{Z_0}$, is a submersion at λ^0. Of course the last map ψ_1 only needs to be known on Z_0, meaning that $\psi_1(\lambda)$ only has to be calculated for $\lambda \in Z_0$. In practice this means that we will use $\psi_0(\lambda) = 0$, near λ^0, as an equation explicitly giving some parameter λ_i as a function of the remaining parameters $(\lambda_1, \ldots, \lambda_{i-1}, \lambda_{i+1}, \ldots, \lambda_p)$, so that ψ_1 will be expressed in terms of those remaining parameters. This procedure can now be continued until reaching ψ_{l-1}, by the following lemma:

Lemma 10 *Suppose that $g_1, \ldots, g_n, n \geq 2$ are real-valued C^∞ (respectively C^ω) functions defined on a neighbourhood of $\lambda^0 \in \mathbb{R}^p$ and $g_i(\lambda^0) = 0$, $\forall i = 1, \ldots, n$. Define $Z = \bigcap_{i=1}^{n-1} g_i^{-1}(0)$. Then, the map*

$$\lambda \mapsto (g_1(\lambda), \ldots, g_n(\lambda))$$

is a submersion at λ^0, if and only if the map

$$\lambda \mapsto (g_1(\lambda), \ldots, g_{n-1}(\lambda))$$

is a submersion at λ^0 and the restriction of g_n to the submanifold Z is a submersion at λ^0.

Standard generic unfoldings

Here, we recall the definition and an equivalent characterisation of a 'standard generic unfolding' or 'catastrophy model', as encountered in chapter 2 (section 2.2.1). The displacement map $D_{\neq}^{(l)}(r,a)$, associated to the model $X_{\pm}^{(l)}$ for a standard generic Hopf-Takens bifurcation of codimension l, is given by the standard generic unfolding $(P_a^{l,\pm}(r^2))$ (section 2.2.2).

Definition 11 *Let $(f_\mu), (g_\nu)$ be C^∞ (respectively C^ω) unfoldings of functions f_μ and g_ν of the variable $y \in \mathbb{R}$, localised at $y = 0$, with parameter $\mu \in \mathbb{R}^l$ localised at $\mu = \mu^0$ and parameter $\nu \in \mathbb{R}^l$ localised at $\nu = \nu^0$ respectively. Then we say that*

1. *these unfoldings (f_μ) and (g_ν) are contact-equivalent if and only if there exist a C^∞ (respectively C^ω) germ of diffeomorphism $\nu = \psi(\mu)$:*

$$\psi : \mathbb{R}^l \to \mathbb{R}^l \text{ with } \psi(\mu^0) = \nu^0$$

and a C^∞ (respectively C^ω) germ of family of diffeomorphisms in y

$$H : (\mathbb{R} \times \mathbb{R}^l, (0,\mu^0)) \to (\mathbb{R}, 0) : (y,\mu) \to H(y,\mu) = H_\mu(y)$$

such that for all μ, the diffeomorphism H_μ sends the set of zeroes of f_μ on the set of zeroes of $g_{\psi(\mu)}$; more precisely,

$$f_\mu(y) = 0 \iff g_{\psi(\mu)}(H_\mu(y)) = 0$$

2. *the pair (ψ, H_μ) is a contact-equivalence between the unfoldings (f_μ) and (g_ν).*

Definition 12 *Let (f_μ) be a C^∞ (respectively C^ω) unfolding of functions f_μ of the variable $y \in \mathbb{R}$, localised at $y = 0$, with parameter $\mu = (\mu_0, \mu_1, \ldots, \mu_{l-1}) \in \mathbb{R}^l$ localised at $\mu = \mu^0$. We say that the unfolding (f_μ) is generic if and only if*

1. *$f^{(i)}(0,\mu^0) = 0, \forall 0 \le i \le l-1$ and $f^{(l)}(0,\mu^0) \neq 0$, where we use the notation*

$$f_\mu^{(i)} \equiv f^{(i)}(\cdot,\mu) \equiv \frac{\partial^i}{\partial y^i} f(\cdot,\mu)$$

2. *the determinant of the matrix $d = \left[\frac{\partial}{\partial \mu_j} f^{(i)}(0,\mu^0)\right]_{i,j=0,1,\ldots,l-1}$ is non-zero.*

In the following proposition, we give an equivalent characterisation of a generic unfolding (definition 12), where the derivatives with respect to the variable y are replaced by functions obtained by an algorithm of division-derivation. We see that the generic conditions expressed in terms of these functions are the same.

Proposition 13 *Let (f_μ) be a C^∞ (respectively C^ω) l-parameter unfolding of functions f_μ of the variable $y \in \mathbb{R}$, localised at $y = 0$, with parameter $\mu = (\mu_0, \mu_1, \ldots, \mu_{l-1}) \in \mathbb{R}^l$ localised at $\mu = \mu^0$. Let $w^0, w^1, \ldots, w^l$ be C^∞ (respectively C^ω) functions in (y, μ), defined in a neighbourhood of $(0, \mu^0)$ such that*

$$w^i\left(0, \mu^0\right) \neq 0, \forall 0 \leq i \leq l \tag{1.7}$$

Let us define the functions $F^i, 0 \leq i \leq l$ by recurrence as follows:

$$\left\{ \begin{array}{lcl} F^0 & = & w^0 f \\ F^i & = & w^i \frac{\partial}{\partial y} F^{i-1}, \quad \forall 1 \leq i \leq l \end{array} \right. \tag{1.8}$$

Then, the unfolding (f_μ) is generic if and only if

1. *$F^i\left(0, \mu^0\right) = 0, \forall 0 \leq i \leq l-1$ and $F^l\left(0, \mu^0\right) \neq 0$;*
2. *the determinant of the matrix $D = \left[\frac{\partial}{\partial \mu_j} F^i\left(0, \mu^0\right)\right]_{i,j=0,1,\ldots,l-1}$ is non-zero.*

Proof. From (1.8), we can prove by induction on $1 \leq i \leq l$, that there exist C^∞ (respectively C^ω) functions $v_j^i, 0 \leq j \leq i-1$ such that

$$F^i = w^0 w^1 \ldots w^i f^{(i)} + \sum_{j=0}^{i-1} v_j^i f^{(j)} \tag{1.9}$$

From (1.9) and by assumption (1.7), it follows inductively on $0 \leq k \leq l$ that

$$F^i\left(0, \mu^0\right) = 0, \forall 0 \leq i \leq k \iff f^{(i)}\left(0, \mu^0\right) = 0, \forall 0 \leq i \leq k$$

and

$$F^l\left(0, \mu^0\right) = w^0\left(0, \mu^0\right) \cdot \ldots \cdot w^l\left(0, \mu^0\right) f^{(l)}\left(0, \mu^0\right).$$

As a consequence, the first condition in definition 12 and the first one in proposition 13 are already shown to be equivalent. Next, if we suppose that

$$f^{(k)}\left(0, \mu^0\right) = 0, \forall 0 \leq k \leq l-1,$$

then we derive from (1.9) that $\forall 0 \leq i, j \leq l-1$

$$\frac{\partial}{\partial \mu_j} F^i\left(0, \mu^0\right) = w^0\left(0, \mu^0\right) \cdot \ldots \cdot w^i\left(0, \mu^0\right) \frac{\partial}{\partial \mu_j} f^{(i)}\left(0, \mu^0\right) + \sum_{j=0}^{i-1} v_j^i \frac{\partial}{\partial \mu_j} f^{(j)}\left(0, \mu^0\right)$$

As a consequence,

$$\begin{aligned} D &= \det\left[\frac{\partial}{\partial \mu_j} F^i\left(0, \mu^0\right)\right]_{i,j=0,1,\ldots,l-1} \\ &= \det\left[w^0\left(0, \mu^0\right) \cdot \ldots \cdot w^i\left(0, \mu^0\right) \frac{\partial}{\partial \mu_j} f^{(i)}\left(0, \mu^0\right)\right]_{i,j=0,1,\ldots,l-1} \\ &= \left(w^0\left(0, \mu^0\right)\right)^l \cdot \left(w^1\left(0, \mu^0\right)\right)^{l-1} \cdot \ldots \cdot \left(w^{l-1}\left(0, \mu^0\right)\right)^1 \cdot d \end{aligned}$$

where d is defined in definition 12. By assumption (1.7), it then follows that also the second generic condition in definition 12 is equivalent to the second condition in proposition 13. ■

A well-known example of generic unfolding is the (redundant) *swallow-tail catastrophe* defined by the polynomial unfolding

$$P_\nu^{4,\pm}(y) = P^{4,\pm}(y,\nu) = \nu_0 + \nu_1 y + \nu_2 y^2 + \nu_3 y^3 \pm y^4 \tag{1.10}$$

for $\nu = (\nu_0, \nu_1, \nu_2, \nu_3)$. Remark that the *standard swallow-tail catastrophe* $Q_\mu \equiv P_{(\mu,0)}^{4,\pm}$ with a 3-dimensional parameter μ is given by taking $\nu_3 \equiv 0$. In fact, the unfolding $\left(P_\nu^{4,\pm}\right)$ exhibits a curve Γ of standard swallow-tail catastrophes which are contact-equivalent, and the unfolding $\left(P_\nu^{4,\pm}\right)$ changes in a trivial way along Γ. We here introduce the redundant swallow-tail catastrophe with a 4-dimensional parameter because we will work with a 4-dimensional one in our application.

The swallow-tail catastrophe is an example of a generic unfolding with a 4-dimensional parameter. Conversely, one has the following characterisation:

Proposition 14 *Any generic unfolding with a 4-dimensional parameter is contact-equivalent to the swallow-tail catastrophe $\left(P_\nu^{4,\pm}\right)$ defined in* (1.10).

The proof of proposition 14, can be found in [DR]. Here, we prove a generalisation of this proposition for an l-dimensional parameter, $\forall l \in \mathbb{N}$.

Proposition 15 *Any generic unfolding (f_μ) with a l-dimensional parameter μ is contact-equivalent to one of the two models $\left(P_\nu^{l,\pm}\right)_\nu$:*

$$P_\nu^{l,\pm}(y) = P^{l,\pm}(y,\nu) = \nu_0 + \nu_1 y + \ldots + \nu_{l-1} y^{l-1} \pm y^l$$

where $\pm$ depends on the sign of $f^{(l)}(0,\mu^0)$.

Proof. Let us expand the function f_μ at order l in the variable y :

$$f_\mu(y) = a_0(\mu) + a_1(\mu) y + \ldots + a_{l-1}(\mu) y^{l-1} + a_l(\mu) y^l (1 + \psi(y,\mu))$$

where the functions $a_i, 1 \le i \le l$ are C^∞ (respectively C^ω) in μ, and the function ψ is C^∞ (respectively C^ω) in (y,μ). Since the unfolding (f_μ) is generic, we have that

$$\begin{cases} a_i(\mu^0) &=& \dfrac{f^{(i)}(0,\mu^0)}{i!} = 0, \quad \forall 0 \le i \le l-1 \\ a_l(\mu^0) &\neq& 0 \\ \psi(0,\cdot) &\equiv& 0 \end{cases}$$

and that the map a, defined as

$$a : (\mathbb{R}^l, \mu^0) \to (\mathbb{R}^l, \overline{0}) : \mu \mapsto (a_0(\mu), a_1(\mu), \ldots, a_{l-1}(\mu)),$$

is a local diffeomorphism at μ^0. Denote the inverse of a by $\phi = a^{-1}$. Then

$$f(y, \phi(\nu)) = \nu_0 + \nu_1 y + \ldots + \nu_{l-1} y^{l-1} + M(\nu) y^l (1 + \psi(y, \phi(\nu)))$$

where $M = a_l \circ \phi$. Since $\phi(0) = \mu^0$, $M(0) \neq 0$ and since

$$f(y, \phi(0)) = M(0) y^l (1 + O(y)), y \to 0,$$

the Preparation Theorem allows us to write

$$f(y, \phi(\nu)) = U(y, \nu) \cdot P^{l,\pm}(y, h(\nu)), \tag{1.11}$$

where U is C^∞ (respectively C^ω) in (y, ν) with $U(0,0) > 0$, and h is C^∞ (respectively C^ω) in ν such that $h(0) = 0$. The sign in $P^{l,\pm}$ is given by the sign of $M(0)$. By comparison of the coefficients corresponding to the same powers of y in previous equation, inductively on the condition that all the coefficients corresponding to lower order terms in y vanish, the map h is shown to be a diffeomorphism. From (1.11), we have that

$$f(y, \mu) = U(y, a(\mu)) \cdot P^{l,\pm}(y, h \circ a(\mu))$$

and hence, the pair $(h \circ a, \mathrm{Id}|_{\mathbb{R}})$ is a contact-equivalence between the unfoldings (f_μ) and $P_\nu^{l,\pm}$. ∎

This proposition justifies the following definition:

Definition 16 *We define the standard generic unfolding* $(P_\nu^{l,\pm})$ *or catastrophy model with a l-dimensional parameter* $\nu = (\nu_0, \ldots, \nu_{l-1}) \in \mathbb{R}^l$ *by*

$$P_\nu^{l,\pm}(y) = P^{l,\pm}(y, \nu) = \nu_0 + \nu_1 y + \ldots + \nu_{l-1} y^{l-1} \pm y^l.$$

Vector fields of center type

We say that the vector field X_{λ^0} is of *center type*, if the associated displacement map δ_{λ^0} identically vanishes. Geometrically, this means that the phase portrait of X_{λ^0} consists of a continuous band or disc of non-isolated periodic orbits, in a neighbourhood of Γ. If Γ is a non-degenerate elliptic singularity surrounded by non-isolated periodic orbits, then we say that Γ is a *center*. A typical example of a vector field of center type is given by a preditor-pray system, e.g.,

$$\dot{x} = ax - bxy, \dot{y} = -cy + dxy,$$

where x (respectively y) represents the number of prays (respectively preditors), and the parameters a (respectively d) the birth-rate of the prays (respectively preditors), and b (respectively c) represents the death-rate of the prays (respectively preditors). In figure 1.2, the phase portrait of such a system is presented for the parameter values $a = 1, b = 2, c = 3, d = 4$.

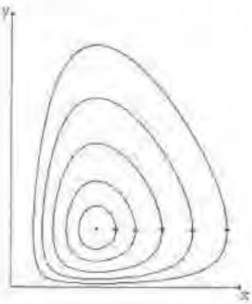

Figure 1.2: Vector field of center type

As we have seen before, to apply the Implicit Function Theorem, Rolle's theorem or the Preparation Theorem, it is necessary for the order of δ_{λ^0} at s_0 to be finite. If this is not the case, i.e.

$$\text{order}\delta_{\lambda^0}(s_0) = \infty \iff \delta_{\lambda^0}^{(j)}(s_0) = 0, \quad \forall j \in \mathbb{N}, \tag{1.12}$$

then we need other techniques, e.g., a technique in which calculation of Melnikov functions (section 1.2.2), the Bautin ideal (section 1.2.3) or analytic geometry (section 4.2) is involved. For these techniques, it is necessary that the family $(\delta_\lambda)_\lambda$ is analytic. When dealing with vector fields of center type, the families of vector fields are supposed to be analytic, unless stated otherwise. Notice however, that in view of Hilbert's 16th problem, this restriction is not stringent, as we explained in section 1.1.

The reason for this restriction is that condition (1.12) in general does not imply that the map δ_{λ^0} identically vanishes. Consider for instance the map defined by

$$\delta_{\lambda^0}(s) = \exp\left(-\frac{1}{s^2}\right), \forall s \neq 0 \text{ and } \delta_{\lambda^0}(0) = 0$$

Then δ_{λ^0} satisfies condition (1.12) for $s_0 = 0$, but $\neg(\delta_{\lambda^0} \equiv 0)$.

If the map δ_{λ^0} is analytic, then it is well-known that (1.12) is equivalent to the fact that the map δ_{λ^0} identically vanishes in a neighbourhood of s_0, and hence to the fact that X_{λ^0} is of center type.

1.2.2 Melnikov functions

Suppose that $(X_\varepsilon)_\varepsilon$ is a C^ω (respectively C^∞) 1-parameter family of planar vector fields, that unfolds a vector field of center type X_0. To stress that we are working with a one-dimensional parameter, we will denote the parameter by ε instead of λ. The displacement map δ_ε can locally be expanded (perhaps only formally) in terms of ε :

$$\delta(s,\varepsilon)=\sum_{i=1}^{\infty}M_i(s)\,\varepsilon^i,\, s\sim s_0,\varepsilon\sim 0 \tag{1.13}$$

where the coefficients M_i are C^ω (respectively C^∞) functions in $s, i\in\mathbb{N}^*$.

Definition 17 *The function M_i is called the i-th Melnikov function ($i\in\mathbb{N}\setminus\{0\}$).*

Suppose that M_k is the first non-zero Melnikov function, then there exist $\delta_0,\varepsilon_0>0$ such that

$$\forall(s,\varepsilon)\in\,]s_0-\delta_0,s_0+\delta_0[\,\times\,]-\varepsilon_0,\varepsilon_0[\,:\delta(s,\varepsilon)=\varepsilon^k\bar{\delta}(s,\varepsilon)$$

with

$$\bar{\delta}(s,\varepsilon)=M_k(s)+O(\varepsilon),\varepsilon\to 0.$$

The isolated zeroes of $\delta(\cdot,\varepsilon)$ correspond to those of $\bar{\delta}(\cdot,\varepsilon)$. Hence, the number of isolated zeroes of $\delta(\cdot,\varepsilon)$ close to s_0, for ε close to 0, but $\varepsilon\neq 0$, is bounded by the order of M_k at s_0 (by Rolle's theorem and a continuity argument). As a consequence, we have

$$\mathrm{Cycl}\left(X_\varepsilon,(\Gamma,0)\right)\leq \mathrm{order}M_k(s_0).$$

Moreover, by performing the blow-up in case that Γ is a non-degenerate elliptic singularity, there show up some symmetry properties; for instance, the phase portrait for $r<0$, is obtained from the phase portrait for $r>0$ after rotation over π radials. As a consequence, after blow up we find an odd number of periodic orbits (if this number is finite), say $2k+1$. One periodic orbit corresponding to the singularity Γ, and the other periodic orbits are counted twice; in fact, after blowing down, this vector field only has k periodic orbits. As a consequence, if Γ is a non-degenerate elliptic singularity, then

$$\mathrm{Cycl}\left(X_\varepsilon,(\Gamma,0)\right)\leq\frac{\mathrm{order}M_k(s_0)-1}{2}.$$

In the literature [F96],[P], one can find algorithms to compute the first non-vanishing Melnikov functions for analytic families of planar vector fields unfolding a Hamiltonian vector field of center type. This is no restriction, since every analytic vector field of center type is a Hamiltonian one, up to a non-zero analytic factor (proposition 19).

Definition 18 *Let $H:U\to\mathbb{R}$ be a C^k function ($k\in\mathbb{N}\cup\{\infty,\omega\}$), defined on an open set U of $\mathbb{R}^2$. Then, we say that the C^{k-1} vector field X, defined by*

$$X(x,y)=\frac{\partial H}{\partial y}(x,y)\frac{\partial}{\partial x}-\frac{\partial H}{\partial x}(x,y)\frac{\partial}{\partial y} \tag{1.14}$$

is a Hamiltonian vector field; the function H is called the Hamiltonian of (1.14).

As a consequence of (1.14), the Hamiltonian is a first integral of X, meaning that

$$XH = 0$$

Thus orbits of the vector field X lie on level curves of the Hamiltonian H. From (1.14), it is clear that for every constant C the function $H_1 \equiv H + C$ is a Hamiltonian of X. If Γ is a periodic orbit of X, we can choose the Hamiltonian such that $\Gamma \subset H^{-1}(0)$. The following proposition explains why we can restrict the study of centers to families $(X_\varepsilon)_\varepsilon$, that unfold a Hamiltonian vector field X_0. We restrict to $k = \infty$ or ω.

Proposition 19 *([R98],[Cau98]) Let X be an analytic (respectively C^∞) vector field of center type, and let Γ be a regular periodic orbit of X, then there exist a non-zero analytic (respectively C^∞) function K, defined in a neighbourhood of Γ, such that KX is a Hamiltonian vector field.*

To fix notations, we briefly recall these algorithms: the algorithm of J.P. Françoise for linear 1-parameter perturbations [F96], and the algorithm of Poggiale for more general 1-parameter perturbations [P]. By proposition 19, we can suppose that X_0 is a Hamiltonian vector field with Hamiltonian H, such that the periodic orbit Γ lies on the level curve $H^{-1}(0)$. Let us write

$$X_\varepsilon = X_0 + \varepsilon X^{(1)} + \varepsilon^2 X^{(2)} + \ldots + \varepsilon^k X^{(k)} + o\left(\varepsilon^k\right), \varepsilon \to 0$$

for certain analytic (respectively C^∞) vector fields $X^{(i)}\,(1 \le i \le k)$, defined on an open neighbourhood of Γ. We can also consider the dual 1-forms v_ε, and the expansion

$$v_\varepsilon = \mathrm{d}H + \varepsilon v_1 + \varepsilon^2 v_2 + \ldots + \varepsilon^k v_k + o\left(\varepsilon^k\right), \varepsilon \to 0$$

We can take a transverse section Σ and parametrise it by the value of the Hamiltonian h, meaning that the point (x, y) on Σ is parametrised by $h = H(x, y)$ in $]-h_0, h_0[$. We suppose that H has h as a regular value. Let $P :]-h_0, h_0[\to \mathbb{R}$ and $\delta = P - Id$ denote the corresponding Poincaré-map and displacement map respectively. In this way, if $h \in]-h_0, h_0[$, then $P(h, \varepsilon)$ is the value of the Hamiltonian of the point of first return in Σ, defined by the flow of X_ε and starting in $\Sigma \cap H^{-1}(h)$.

By the Fundamental Theorem of Calculus, one can write

$$\begin{aligned}\delta(h, \varepsilon) &= P(h, \varepsilon) - h \\ &= \int_{\Gamma_h(\varepsilon)} \mathrm{d}H,\end{aligned}$$

where $\Gamma_h(\varepsilon)$ is the part of the solution curve of X_ε between $H^{-1}(h) \cap \Sigma$ and $H^{-1}(P(h, \varepsilon)) \cap \Sigma$. The integration over $\Gamma_h(\varepsilon)$ uses the orientation induced by the vector field X_ε. By elementary calculations, one finds that

$$\delta(h, \varepsilon) = \varepsilon \int_{\Gamma_h} \left(-X_1^{(1)}\mathrm{d}y + X_2^{(1)}\mathrm{d}x\right) + O\left(\varepsilon^2\right), \varepsilon \to 0 \tag{1.15}$$

where Γ_h is the periodic orbit of X_0 contained in the level curve $H^{-1}(h)$. From equation (1.15), it follows that the first order Melnikov function M_1 is given by

$$\begin{aligned} M_1(h) &= -\int_{\Gamma_h} \left(X_1^{(1)} \mathrm{d}y - X_2^{(1)} \mathrm{d}x \right) \\ &= -\int_{\Gamma_h} \upsilon_1 \end{aligned}$$

Moreover, if Γ_h is the boundary of a simply connected region $S(h)$, then it follows by Green's theorem that

$$M_1(h) = \sigma \iint\limits_{S(h)} \operatorname{div}\left(X_1^{(1)}, X_2^{(1)} \right) \mathrm{d}x\mathrm{d}y$$

where $\sigma = -1$ (respectively $+1$), if the Hamiltonian vector field X_0 induces an orientation such that $S(h)$ is positively oriented (respectively negatively oriented).

Theorem 20 *[P] Suppose now that*

$$M_i \equiv 0, \forall 1 \leq i \leq k-1,$$

then

$$M_k(h) = \int_{\Gamma_h} \left(\sum_{i=1}^{k-1} g_i \upsilon_{k-i} - \upsilon_k \right),$$

where the analytic (respectively C^∞) functions g_i $(1 \leq i \leq k-1)$ are inductively defined on an open neighbourhood of Γ by

$$\upsilon_i - g_i \mathrm{d}H = \sum_{j=1}^{i-1} g_j \upsilon_{k-1-j} - \mathrm{d}R_i$$

for certain analytic (respectively C^∞) functions $R_i, 1 \leq i \leq k-1$.

We know that the displacement map δ_ε and the first non-zero Melnikov function M_k are C^ω (respectively C^∞) as functions in the variable s. It is well-known that the first non-identically zero Melnikov function can be written as a C^ω (respectively C^∞) function in the variable h (where $h = r^2$), even at $h = 0$; we recall this fact in theorem 62, including its proof, in section 1.3.2. However, the displacement map itself cannot always be written as a C^ω (respectively C^∞) function at $h = 0$, as the following example illustrates. Consider the family of polynomial vector fields given by

$$(y + \varepsilon x(1+x)) \frac{\partial}{\partial x} + (-x + \varepsilon y(1+x)) \frac{\partial}{\partial y} \tag{1.16}$$

The Hamiltonian H is given by

$$H(x,y) = \frac{1}{2}\left(x^2+y^2\right)$$

The displacement map, written in polar coordinates (r,θ) and associated to the transverse section $\{\theta=0\}$, can be written as $\delta(r,\varepsilon) = r\cdot\bar{\delta}(r,\varepsilon)$ for a C^∞ function $\bar{\delta}$ with

$$\bar{\delta}(r,\varepsilon) = \alpha_1(\varepsilon) + \alpha_2(\varepsilon)r + O\left(r^2\right), r \to 0$$

and

$$\bar{\delta}(r,\varepsilon) = \varepsilon\bar{M}_1(r) + O\left(\varepsilon^2\right), \varepsilon \to 0,$$

where

$$\alpha_1(\varepsilon) = e^{2\pi\varepsilon} - 1,$$

$$\alpha_2(\varepsilon) = \frac{\varepsilon}{\varepsilon^2+1}e^{2\pi\varepsilon}\alpha_1(\varepsilon),$$

and

$$\bar{M}_1(r) = 2\pi$$

Obviously the first non-zero (reduced) Melnikov function $\bar{M}_1$ is C^ω at h. But it is not possible to write the function $\bar{\delta}$ as a C^ω (or C^∞) map in $h = \frac{r^2}{2}$, because $\alpha_2(\varepsilon) \neq 0$ for $\varepsilon \neq 0$.

1.2.3 Bautin Ideal

In this section, we quickly recall the definition of Bautin ideal and its basic properties. We also recall the notions of minimal set of generators, factor functions and index. The displacement map can locally be divided in the minimal set of generators, in this way factor functions are introduced. The order of these functions can be used in bounding the maximal number of zeroes, and are used to define an upperbound for the cyclicity, the so-called *index*, of the family, that is independent of the choice of minimal system. Next, we consider the special cases of a principal Bautin ideal and a regular Bautin ideal. We illustrate these notions in a 1-parameter family by linking this theory to the one involving Melnikov functions.

The proofs of these facts can be found in [R98] (or the undergraduate thesis [Cau98], that is based on [R98]); otherwise they are provided here.

Throughout this section, the family of planar vector fields $(X_\lambda)_\lambda$, unfolding a vector field X_{λ^0} of center type, is supposed to be analytic. We suppose that Γ is a periodic orbit or a non-degenerate elliptic singularity, surrounded by non-isolated periodic orbits. Suppose that $(\delta_\lambda)_\lambda$ is a family of displacement maps for $(X_\lambda)_\lambda$. Then, $\delta_{\lambda^0} \equiv 0$; in this case an important tool to study the bifurcation set is the Bautin ideal [R98],[F95]. Not only does it directly serve to define the set of parameter values near λ^0, at which we have centers, it can also be used in calculating an upperbound for the cyclicity.

Definition and basic properties

Let us denote the local ring of analytic function germs at λ^0 by $\mathcal{O}_{\lambda^0}$, and its unique maximal ideal by $\mathcal{M}_{\lambda^0}$. The ideal generated by analytic function germs $\varphi_1, \ldots, \varphi_l$ is denoted by

$$(\varphi_1, \ldots, \varphi_l).$$

We use twiddles $\tilde{}$ to denote the germ of a certain analytic function at λ^0. We will not always use the twiddles to make a distinction between the germ of an analytic function at λ^0 and its representative. When we deal with ideals, we will always work in the local ring O_{λ^0} of analytic function germs at λ^0, without mentioning it each time. Furthermore, if $f, g \in O_{\lambda^0}$, and I is an ideal in O_{λ^0}, then we say that f is equal to g modulo I if and only if $f - g \in I$; briefly, this relation is denoted as:

$$f = g \bmod I.$$

The expansion of δ_λ in a Taylor series at $s = s_0$, defines a sequence of analytic functions $\alpha_0, \alpha_1, \ldots, \alpha_n, \ldots$ in a neighbourhood of λ^0 :

$$\delta_\lambda (s) = \sum_{i=0}^{\infty} \alpha_i (\lambda) (s - s_0)^i, s \sim s_0$$

Definition 21 *The ideal $\mathcal{I}$, generated by the germs of the analytic functions α_i at λ^0, is called the Bautin ideal:*

$$\mathcal{I} = (\widetilde{\alpha}_j : j \in \mathbb{N}) \equiv \mathcal{I}^{s_0}$$

The point s_0 is called the base point of $\mathcal{I}^{s_0}$. Since the local ring $\mathcal{O}_{\lambda^0}$ of analytic function germs at λ^0 is Noetherian [M], this ideal is finitely generated, i.e. there exists a number $M(s_0) \in \mathbb{N}$ such that

$$\mathcal{I} = (\widetilde{\alpha}_j : 0 \leq j \leq M(s_0)). \tag{1.17}$$

The notion *Bautin ideal* finds its origin in the study of centers in families of quadratic vector fields by N.N. Bautin, who reduced the cyclicity problem to a certain assertion concerning all Taylor coefficients of the Poincaré map. He computed the first seven Taylor coefficients explicitly and established some divisibility properties for all Taylor coefficients. As result, he found that the cyclicity from a focus or center in a family of quadratic vector fields is at most 3 ([Bautin],[Y]).

Definition 22 *We denote the zero-set of the ideal $\mathcal{I}$ by $Z(\mathcal{I})$. If $\widetilde{\varphi}_1, \ldots, \widetilde{\varphi}_l$ is a set of generators for $\mathcal{I}$, then $Z(\mathcal{I})$ is defined as the germ at λ^0 of the set*

$$\{\lambda \in W : \varphi_i (\lambda) = 0, \forall 1 \leq i \leq l\},$$

where W is a neighbourhood of λ^0 in $\mathbb{R}^p$, such that the maps $\varphi_1, \ldots, \varphi_l$ are defined on W. Notice that $Z(\mathcal{I})$ consists of all parameter values λ near λ^0 for which the vector field X_λ is of center type.

Definition 21 is meaningful by the following facts, whose proofs can be found in [R98] or [Cau98]:

1. the ideal does not depend on the chosen base point s_0.
2. the ideal is independent of the chosen analytic regular parametrisation of transverse section Σ
3. the ideal is independent of the chosen transverse section Σ
4. the ideal does not change after multiplication of the family $(X_\lambda)_\lambda$ by a non-zero analytic function (i.e. the Bautin ideal of $(X_\lambda)_\lambda$ coincides with the one of $(Y_\lambda)_\lambda$, defined by $Y_\lambda = f_\lambda \cdot X_\lambda$ where $f_\lambda(x,y) \neq 0$ for (x,y) close to Γ)
5. the ideal does not change by an analytic coordinate transformation of the phase variables (x,y)

The proof of fact 1. can be found in [R98] or [Cau98]. Fact 4. is easy to explain, since multiplication of the vector field by a non-zero function does not affect the phase portrait. Only the parametrisation of the solution curves is changed; hence, only the time necessary to return back in Σ is changed. As a consequence, the Poincaré-map, the displacement map and the Bautin ideal do not change.

Fact 5. follows from fact 2: we can take a parametrisation of Σ, corresponding to the coordinate change of the phase variables, such that the Poincaré-map is not affected.

Facts 2 and 3 are consequences of fact 1. and the following more general result (whose proof can be found in [Cau98]):

Proposition 23 *Let $(\varphi_\lambda)_\lambda$ be a C^ω family of diffeomorphisms, and let $(P_\lambda)_\lambda$ be a C^ω family of maps. Define the family $(Q_\lambda)_\lambda$ by*

$$Q_\lambda(r) = \varphi_\lambda^{-1} \circ P_\lambda \circ \varphi_\lambda(r)$$

and put $\delta_\lambda = P_\lambda - Id$ and $\delta_\lambda^1 = Q_\lambda - Id$. If

$$\delta_\lambda(s) = \sum_{i=0}^{\infty} \alpha_i(\lambda) s^i \text{ and } \delta_\lambda^1(r) = \sum_{i=0}^{\infty} \beta_i(\lambda) r^i,$$

then

$$(\widetilde{\alpha}_i : i \in \mathbb{N}) = \left(\widetilde{\beta}_i : i \in \mathbb{N}\right).$$

For later use, we also mention some more precise propositions:

Proposition 24 *Let $(\varphi_\lambda)_\lambda$ be a C^ω family of diffeomorphisms such that $\varphi_\lambda(0) = 0$, and a C^ω family of maps $(P_\lambda)_\lambda$ with $P_\lambda(0) = 0$. Define the family $(Q_\lambda)_\lambda$ by*

$$Q_\lambda(s) = \varphi_\lambda^{-1} \circ P_\lambda \circ \varphi_\lambda(r)$$

and put $\delta_\lambda = P_\lambda - Id$ *and* $\delta^1_\lambda = Q_\lambda - Id$. *Then there exist* C^ω *functions* f^i_j, $1 \leq j \leq i, \forall i \geq 1$ *such that*

$$\frac{\partial^i}{\partial r^i}\,\delta^1_\lambda(r)\big|_{r=0} = \sum_{j=1}^{i} f^i_j(\lambda)\cdot\frac{\partial^j}{\partial s^j}\,\delta_\lambda(s)\big|_{s=0}. \tag{1.18}$$

where $f^i_i(\lambda) = \left(\frac{\partial}{\partial r}\varphi_\lambda(0)\right)^{i-1}, \forall i \geq 1.$

The Bautin ideal even does not change when we multiply the displacement map itself by a non-zero function; more precisely, we have the following property:

Proposition 25 *Let* $(\delta_\lambda)_\lambda$ *be a* C^ω *family of maps and let* $(f_\lambda)_\lambda$ *a* C^ω *family of maps with* $f_\lambda(s_0) \neq 0$. *If* $(\delta^1_\lambda)_\lambda$ *is the family of maps obtained by*

$$\delta^1_\lambda(s) = f_\lambda(s)\,\delta_\lambda(s)$$

then there exist C^ω *functions* f^i_j, $1 \leq j \leq i-1, \forall i \geq 1$ *such that* $\forall i \in \mathbb{N}$:

$$\frac{\partial^i}{\partial s^i}\,\delta^1_\lambda(s)\big|_{s=s_0} = f_\lambda(s_0)\,\frac{\partial^i}{\partial s^i}\,\delta_\lambda(s)\big|_{s=s_0} + \sum_{j=1}^{i-1} f^i_j(\lambda)\,\frac{\partial^j}{\partial s^j}\,\delta_\lambda(s)\big|_{s=s_0}.$$

Remark 26 *Propositions 24 and 25 also hold when* C^ω *is replaced by* C^∞; *in fact, identities* $(1.18), 1 \leq i \leq k$, *even hold for* C^k *functions with* $k \in \mathbb{N}\setminus\{0\}$ *(the functions* f^i_j *are only* $C^{k-i+j}, \forall 1 \leq j \leq i, \forall 1 \leq i \leq k$*)*

Another interesting property is the following, its proof can be found in [R98].

Proposition 27 *If* Γ *is a non-degenerate elliptic singularity, represented by* s_0, *then the Bautin ideal is generated by the 'odd' coefficients; more precisely, suppose we have the following local expansion at* s_0 :

$$\delta(s,\lambda) = \sum_{i=1}^{\infty} \alpha_i(\lambda)(s-s_0)^i, s \sim s_0, \lambda \sim \lambda^0,$$

then $\forall p \in \mathbb{N}$:

$$\widetilde{\alpha}_{2p} \in (\widetilde{\alpha}_1, \widetilde{\alpha}_2, \ldots, \widetilde{\alpha}_{2p-1})$$

Proposition 27 can easily be checked in case the family of vector fields is expressed in normal form: after a coordinate change and multiplication by a strictly positive function, the family X_λ is expressed in polar coordinates (r,θ), up to finite order as

$$\left\{\begin{array}{lcl} \dot{r} & = & r\sum_{j=0}^{N}\beta_j(\lambda)\,r^{2j} + O\left(r^{2N+2}\right), r \to 0 \\ \dot{\theta} & = & 1 \end{array}\right.$$

The coefficients of even index in the displacement map, corresponding to the transverse section $\{\theta = 0\}$, all are zero; hence, the result of proposition 27 trivially holds in this case. From facts 2,3,4,5 and proposition 24, it now follows that proposition 27 holds in general (i.e. if Γ is a center).

Local division of the displacement map

Given a set of generators for the Bautin ideal, we can locally divide the displacement map in these generators:

Proposition 28 *Let $\varphi_1, \ldots, \varphi_l$ be a set of analytic functions that are defined on a neighbourhood W of λ^0, such that their germs generate the Bautin ideal $\mathcal{I}$. Then, there exists $R > 0$, such that $\forall s_0 \in \Sigma$, there exists a neighbourhood W_{s_0} of λ^0 with $W_{s_0} \subset W$ and there exist analytic functions*

$$h_j^{s_0} : (\Sigma \cap [s_0 - R, s_0 + R]) \times W_{s_0} \to \mathbb{R} \ \ (1 \leq j \leq l)$$

with $\forall (s, \lambda) \in (\Sigma \cap [s_0 - R, s_0 + R]) \times W_{s_0}$:

$$\delta(s, \lambda) = \sum_{i=1}^{l} \varphi_i(\lambda) \, h_i^{s_0}(s, \lambda). \tag{1.19}$$

Definition 29 *Let $\{\varphi_1, \ldots, \varphi_l\}$ be a set of generators for the Bautin ideal $\mathcal{I}$. Then we say that this set is a* minimal set of generators *for $\mathcal{I}$ if the equivalence classes modulo $\mathcal{MI}$*

$$\{\varphi_1 mod\mathcal{MI}, \ldots, \varphi_l mod\mathcal{MI}\}$$

form a basis for the real vector space $\mathcal{I}/\mathcal{MI}$. In this case, we define the dimension *of the Bautin ideal as:*

$$\dim \mathcal{I} = l.$$

The existence of such a minimal set of generators is ensured by *Nakayama's lemma* (which holds in a local ring, see [M]): if $\mathcal{I}'$ is another ideal in $\mathcal{O}_{\lambda^0}$, then

$$\mathcal{I} = \mathcal{I}' + \mathcal{MI} \Longrightarrow \mathcal{I} = \mathcal{I}'. \tag{1.20}$$

In particular, by Nakayama's lemma, one can prove that one can always select a minimal set of generators from a given set of generators for $\mathcal{I}$. The division (1.19) in a minimal set of generators has the following properties:

Proposition 30 *Let $\varphi_1, \ldots, \varphi_l$ be a set of analytic functions that are defined on a neighbourhood W of λ^0, such that their germs form a minimal set of generators for the Bautin ideal $\mathcal{I}$. Consider the expansion (1.19) in proposition 28. Then,*

1. *the analytic functions H_i, defined by*

$$H_i \equiv h_i^{s_0}(\cdot, \lambda^0), \forall 1 \leq i \leq l, \tag{1.21}$$

are independent of s_0 and, hence are globally defined.

2. *the analytic functions $H_1, \ldots, H_l$ are $\mathbb{R}$-independent, in the sense that $\forall s_1 \in \Sigma$, the associated function germs $\widetilde{H}_1, \ldots, \widetilde{H}_l$ at s_1 are $\mathbb{R}$-independent.*

From this proposition the following definition is meaningful:

Definition 31 *Let $\varphi_1, \ldots, \varphi_l$ be a minimal set of generators for $\mathcal{I}$. Then, the functions $H_i, 1 \leq i \leq l$ defined by (1.21) are called a set of factor functions corresponding to $\varphi_1, \ldots, \varphi_l$.*

Proposition 32 *There exists a minimal set of generators for $\mathcal{I}$ such that the corresponding factor functions have a strictly increasing order at s_0 :*

$$\text{order}H_1(s_0) < \text{order}H_2(s_0) < \ldots < \text{order}H_l(s_0).$$

Definition 33 *A minimal set of generators such that the corresponding factor functions have a strictly increasing order at s_0 is called a* minimal set of generators adapted at s_0.

Cyclicity, multiplicity and index.

In this section, we briefly recall the notion of relative index, that is defined in [R98], and there proven to be an upperbound for the cyclicity (and even for the multiplicity) of the family. In fact, when computing Melnikov functions in 1-parameter families, the upperbound found is exactly this index. In [R00], it is proven that there exists a 1-parameter subfamily in which the index of this family is equal to the one of the whole family. Later, in chapter 3, we come back on this notion, and in particular, attention is focused on the use of 1-parameter techniques in multi-parameter families.

Let us start by defining the index at s_0, in case s_0 represents a periodic orbit.

Definition 34 *Let $\varphi_1, \ldots, \varphi_l$ be a minimal set of generators for $\mathcal{I}$, and let $H_1, \ldots, H_l$ be a corresponding set of factor functions. Then, the* relative index of the Bautin ideal *at s_0 is defined by*

$$\text{Index}\left(X_\lambda, \left(\Gamma, \lambda^0\right)\right) = \inf\{n \in \mathbb{N} : \{j_n\left(H_i\right)_{s_0}\}_{i=1}^{l} \textit{ is } \mathbb{R}\textit{-independent}\}.$$

By proposition 30and the analyticity, the index is finite. The index only depends on δ_λ (or X_λ) and s_0. In [R98], this number is denoted by $s_\delta(s_0)$. We use the longer notation to indicate whether we work with the p-parameter family $(X_\lambda)_\lambda$ or with a 1-subfamily $\left(X_{\zeta(\varepsilon)}\right)_\varepsilon$ with $\zeta(0) = \lambda^0$; for this 1-parameter family $\left(X_{\zeta(\varepsilon)}\right)_\varepsilon$, the relative index of Γ is denoted by $\text{Index}\left(X_{\zeta(\varepsilon)}, (\Gamma, 0)\right)$. The following equivalent characterisations can easily be checked:

Proposition 35 *If $\varphi_1, \ldots, \varphi_l$ is a minimal set of generators adapted at s_0, and if $H_1, \ldots, H_l$ is a corresponding set of factor functions with*

$$\text{order}H_1(s_0) < \text{order}H_2(s_0) < \ldots < \text{order}H_l(s_0),$$

then

$$\text{Index}\left(X_\lambda, \left(\Gamma, \lambda^0\right)\right) = \text{order}H_l(s_0)$$

Proposition 36 *Consider the local expansion of δ_λ in a Taylor series at $s = s_0$:*

$$\delta_\lambda(s) = \sum_{i=0}^{\infty} \alpha_i(\lambda)(s - s_0)^i, \qquad (s,\lambda) \sim (s_0, \lambda^0)$$

If $M(s_0)$ is the smallest integer with $\mathcal{I} = (\widetilde{\alpha}_i : 0 \leq i \leq M(s_0))$, then

$$\mathrm{Index}(X_\lambda, (\Gamma, \lambda^0)) = M(s_0)$$

Let s_0 now correspond to a center. From proposition 27, it can be checked that in this case, the order of the factor functions at s_0 is odd. Therefore, the following definition is meaningful:

Definition 37 *Let $\varphi_1, \ldots, \varphi_l$ be a minimal set of generators for $\mathcal{I}$, and let $H_1, \ldots, H_l$ be a corresponding set of factor functions. Then, the* relative index of the Bautin ideal *at s_0 is defined by*

$$\mathrm{Index}(X_\lambda, (\Gamma, \lambda^0)) = \inf\left\{n \in \mathbb{N} : \{j_{2n+1}(H_i)\}_{i=1}^{l} \text{ is } \mathbb{R}\text{-independent}\right\}$$

As for the periodic orbit, the index is finite, and only depends on δ_λ (or X_λ) and s_0. Moreover, we have the following equivalent characterisations:

Proposition 38 *If $\varphi_1, \ldots, \varphi_l$ is a minimal set of generators adapted at s_0, and if $H_1, \ldots, H_l$ is a corresponding set of factor functions with*

$$\mathrm{order}H_1(s_0) < \mathrm{order}H_2(s_0) < \ldots < \mathrm{order}H_l(s_0),$$

then

$$\mathrm{Index}(X_\lambda, (\Gamma, \lambda^0)) = \frac{\mathrm{order}H_l(s_0) - 1}{2}$$

Proposition 39 *Consider the local expansion of δ_λ in a Taylor series at $s = s_0$:*

$$\delta_\lambda(s) = \sum_{i=0}^{\infty} \alpha_i(\lambda)(s - s_0)^i, \qquad (s,\lambda) \sim (s_0, \lambda^0)$$

If $M(s_0)$ be the smallest integer with $\mathcal{I} = (\widetilde{\alpha}_i : 0 \leq i \leq 2M(s_0) + 1)$, then

$$\mathrm{Index}(X_\lambda, (\Gamma, \lambda^0)) = M(s_0)$$

By use of the *division-derivation algorithm*, it is proven in [R98] that the multiplicity of X_λ along Γ for $\lambda = \lambda^0$ is bounded by the relative index at s_0, in case s_0 corresponds to a regular periodic orbit.

Theorem 40 *Let $(X_\lambda)_\lambda$ be an analytic family of planar vector fields, unfolding a vector field of center type X_{λ^0} and suppose that Γ is a non-isolated periodic orbit of X_{λ^0} (a regular periodic orbit or a center). Then,*

$$\mathrm{Cycl}(X_\lambda, (\Gamma, \lambda^0)) \leq \mathrm{Mult}(X_\lambda, (\Gamma, \lambda^0)) \leq \mathrm{Index}(X_\lambda, (\Gamma, \lambda^0)). \tag{1.22}$$

To end this section, let us illustrate in an example that the numbers in (1.22) possibly all are distinct. For instance, consider the 1-parameter family (X_ε) of type (1.2) with

$$\delta(s,\varepsilon) = \varepsilon h(s,\varepsilon)$$

and

$$h(s,\varepsilon) = (s-(s_0+\varepsilon))^2\left((s-s_0)^2 - \varepsilon^2\right)$$

where $s_0 > 0$ and $\Gamma = \left\{(x,y) : x^2+y^2 = s_0^2\right\}$. The Bautin ideal is generated by $\varphi(\varepsilon) = \varepsilon$ and the corresponding factor function H is given by

$$H(s) = (s-s_0)^4$$

Clearly,

$$\mathrm{Cycl}\left(X_\varepsilon,(\Gamma,0)\right) = 1 < \mathrm{Mult}\left(X_\varepsilon,(\Gamma,0)\right) = 2 < \mathrm{Index}\left(X_\varepsilon,(\Gamma,0)\right) = 4$$

Principal Bautin ideal

Definition 41 *The Bautin ideal is called principal, if it is a principal ideal, meaning that it can be generated by only one function.*

If the Bautin ideal is principal, then there exist analytic functions

$$\varphi : \left(\mathbb{R}^p,\lambda^0\right) \to (\mathbb{R},0) \text{ with } \varphi\left(\lambda^0\right) = 0$$

and

$$h : \left(\mathbb{R}\times\mathbb{R}^p,\left(s_0,\lambda^0\right)\right) \to \mathbb{R} \text{ with } h\left(\cdot,\lambda^0\right) \not\equiv 0$$

such that the displacement map can locally be written as

$$\delta(s,\lambda) = \varphi(\lambda)\, h(s,\lambda),\, s\to s_0, \lambda\to\lambda^0$$

Then, the set $Z(\mathcal{I}) = \{\lambda : \varphi(\lambda) = 0\}$ describes the parameter values λ, for which X_λ is of center type. Outside $Z(\mathcal{I})$, the bifurcation diagram of the limit cycles of the family $(X_\lambda)_\lambda$ is given by the one of the isolated zeroes of the family of maps $(h(\cdot,\lambda))_\lambda$ for $\lambda \notin Z(\mathcal{I})$ (i.e. $\varphi(\lambda) \neq 0$). Again, the bifurcation diagram of this last one can be studied by application of the Preparation Theorem. In particular, if Γ is a periodic orbit and if $Index\left(X_\lambda,\left(\Gamma,\lambda^0\right)\right) = l$ and if the map

$$H : \left(\mathbb{R}^p,\lambda^0\right) \to \mathbb{R}^l : \lambda \mapsto \left(h(s_0,\lambda), \frac{\partial h}{\partial s}(s_0,\lambda), \ldots, \frac{\partial^{l-1}h}{\partial s^{l-1}}(s_0,\lambda)\right)$$

is a submersion at λ^0, then

$$\mathrm{Cycl}\left(X_\lambda,\left(\Gamma,\lambda^0\right)\right) = \mathrm{Mult}\left(X_\lambda,\left(\Gamma,\lambda^0\right)\right) = \mathrm{Index}\left(X_\lambda,\left(\Gamma,\lambda^0\right)\right)$$

(if $l = 0$ (respectively $l = 1$), then the condition on the map H is omitted (always satisfied)). In case that Γ is a center, we have an analoguous result, but the map H is formed by the derivatives of odd order in s.

As an example, let us suppose that the family $(X_\lambda)_\lambda$ depends on a 1-dimensional parameter λ. As usual we write ε instead of λ to stress that the parameter is 1-dimensional.

Since the map δ is analytic, there are only two possibilities: either all Melnikov functions vanish identically or there exists a positive integer $k \in \mathbb{N}_1$ such that M_k is the first non identically zero Melnikov function. In the first case, all vector fields X_ε are of center type, and hence there are no limit cycles. The other case is the interesting one. The following proposition indicates the relation between the first non-zero Melnikov function and the Bautin ideal.

Proposition 42 *If M_k is the first non-identical zero Melnikov function, i.e.*

$$M_i \equiv 0, \forall 1 \leq i < k \text{ and } M_k \not\equiv 0 \tag{1.23}$$

Then,

$$\delta(s,\varepsilon) = \varepsilon^k (M_k(s) + O(\varepsilon)), \varepsilon \to 0$$

In particular,

1. *the Bautin ideal is principal; more precisely, the Bautin ideal is generated by the germ of the function $\varepsilon \mapsto \varepsilon^k$ at $\varepsilon = 0$;*

2. *the corresponding factor function is given by M_k; as a consequence, if Γ is a periodic orbit, then*

$$\mathrm{Index}(X_\varepsilon, (\Gamma, 0)) = \mathrm{order} M_k(s_0)$$

and if Γ is a non-degenerate elliptic singularity, then

$$\mathrm{Index}(X_\varepsilon, (\Gamma, 0)) = \frac{\mathrm{order} M_k(s_0) - 1}{2}.$$

By Nakayama's lemma, we also have the reverse of the first statement: if the Bautin ideal is generated by the germ of the analytic function $\varepsilon \mapsto \varepsilon^k$ at $\varepsilon = 0$, then M_k is the first non-zero Melnikov function.

Moreover, it follows from this proposition and analoguous properties concerning the Bautin ideal, that the index k of the first non-zero Melnikov function and its order are independent of the chosen transverse section and its parametrisation; they either do not change by multiplication of the family by a non-zero analytic function, or by chosing new coordinates in phase space.

Regular Bautin ideal

Definition 43 *The Bautin ideal is said to be regular, if it has a regular set of generators; more precisely, if the germs of the analytic functions $\varphi_1, \ldots, \varphi_l$ at λ^0 form a minimal set of generators for $\mathcal{I}$, then $\mathcal{I}$ is regular if and only if the map*

$$\varphi : (\mathbb{R}^p, \lambda^0) \to \mathbb{R}^l : \lambda \mapsto (\varphi_1(\lambda), \ldots, \varphi_l(\lambda)) \tag{1.24}$$

is a submersion at λ^0.

It is an easy calculation to check that this condition does not depend on the chosen minimal set of generators, and hence this definition is meaningful. Geometrically, the fact that the Bautin ideal is regular implies that the zero-set of the ideal forms a regular surface (or we will shortly say that 'the centers occur on a regular surface'). However, the converse is not true in general. For instance, the ideal $\mathcal{I}$ generated by the germ of the map $\lambda = (\lambda_1, \ldots, \lambda_p) \mapsto \lambda_1^2$ at $\lambda^0 = 0$ clearly is not regular, although its zero-set is a regular surface.

An interesting property of a regular Bautin ideal is that we can choose analytic coordinates in parameter space such that the generators of the minimal set of generators correspond to projections and such that the minimal system is adapted at s_0 :

Proposition 44 *Let $(X_\lambda)_\lambda$ be an analytic family of planar vector fields, unfolding a vector field of center type X_{λ^0} and suppose that Γ is a non-isolated regular periodic orbit of X_{λ^0}. If the Bautin ideal is regular, and if $\varphi_1, \ldots, \varphi_l$ is a minimal set of generators for $\mathcal{I}$, then there exists an analytic diffeomorphism $\varsigma : (\mathbb{R}^p, \lambda^0) \to (\mathbb{R}^p, \nu^0)$ such that*

$$\varphi_i \circ \varsigma(\nu_1, \ldots, \nu_p) = \nu_i, \forall 1 \le i \le l$$

and the germs of the analytic functions $\varphi_1 \circ \varsigma, \ldots, \varphi_l \circ \varsigma$ form a minimal set of generators adapted at s_0 for the Bautin ideal $\widehat{\mathcal{I}}$, where $\widehat{\mathcal{I}}$ is an ideal in $\mathcal{O}_{\nu^0}$, defined by

$$f \in \mathcal{I} \iff \widehat{f} \equiv f \circ \varsigma \in \widehat{\mathcal{I}}$$

In case the Bautin ideal is regular, the cyclicity can also be bounded from below (its proof is based on Chebyshev systems [Mar] and can be found in [R98]):

Theorem 45 *Let $(X_\lambda)_\lambda$ be an analytic family of planar vector fields, unfolding a vector field of center type X_{λ^0} and suppose that Γ is a non-isolated regular periodic orbit of X_{λ^0}. If the Bautin ideal is regular, then*

$$\dim \mathcal{I} - 1 \le \operatorname{Cycl}(X_\lambda, (\Gamma, \lambda^0)) \tag{1.25}$$

We conclude by giving two algorithms to check whether a Bautin ideal is regular or not. The first proposition states that if the map defined in (1.24) is a submersion at λ^0 for a set of generators, then this set of generators is a minimal one, and then, of course, the considered Bautin ideal is regular. The second proposition allows one to decide whether a Bautin ideal is regular or not by computation of Melnikov functions in linear 1-parameter subfamilies.

Proposition 46 *Let $\{\widetilde{\varphi}_1, \ldots, \widetilde{\varphi}_l\}$ be a set of generators for $\mathcal{I}$, such that the map φ defined by*

$$\varphi : (\mathbb{R}^p, 0) \to \mathbb{R}^l : \lambda \mapsto (\varphi_1(\lambda), \ldots, \varphi_l(\lambda))$$

is a submersion at λ^0, then the set $\{\widetilde{\varphi}_1, \ldots, \widetilde{\varphi}_l\}$ is a minimal set of generators for $\mathcal{I}$.

Proof. Suppose that

$$\alpha_1 \widetilde{\varphi}_1 + \ldots + \alpha_l \widetilde{\varphi}_l = 0 \bmod \mathcal{MI} \tag{1.26}$$

for certain $\alpha_i, 1 \leq i \leq l$. It suffices to prove that $\alpha_i = 0, \forall 1 \leq i \leq l$. From (1.26), it follows that there exist analytic functions f_i with $f_i(\lambda^0) = 0, 1 \leq i \leq l$ such that

$$(\alpha_1 - f_1)\varphi_1 + \ldots + (\alpha_l - f_l)\varphi_l = 0.$$

Taking the differential at λ^0, we obtain that

$$\alpha_1 D\varphi_1(\lambda^0) + \ldots + \alpha_l D\varphi_l(\lambda^0) = 0.$$

This equality implies that $\alpha_i = 0, \forall 1 \leq i \leq l$, since φ is a submersion at λ^0. ∎

1.2.4 Lyapunov quantities

In this section, we consider p-parameter families of planar vector fields of the form:

$$X_\lambda(x, y) = (d(\lambda)x - y + f(x, y, \lambda))\frac{\partial}{\partial x} + (x + d(\lambda)y + g(x, y, \lambda))\frac{\partial}{\partial y}. \tag{1.27}$$

in the neighbourhood of a non-degenerate elliptic singularity, that is situated in the origin, with

$$f(x, y, \lambda) = O\left(\|(x, y)\|^2\right) \text{ and } g(x, y, \lambda) = O\left(\|(x, y)\|^2\right), \text{ for } (x, y) \to (0, 0).$$

Except when dealing with the Bautin Ideal or Melnikov functions everything in this section will be C^∞; in particular the functions d, f and g in (1.27) are supposed to be C^∞ functions, unless stated otherwise.

Lyapunov quantities or focal values possess information concerning the stability of the focus. Since Lyapunov quantities can be algebraically computed (section 1.2.4, lemma 47), they form a key step for solving the stability problem of a planar system which is a perturbation of a linear center or focus at the origin. Traditionally, the presence of Hopf-Takens bifurcations are detected by computation of normal forms. In section 1.2.4, we give a technical proposition that relates Lyapunov quantities to normal forms. This relation will be used in chapter 2 to give necessary conditions on Lyapunov quantities to ensure the presence of a Hopf-Takens bifurcation.

If one deals with C^ω families of vector fields of type (1.27), unfolding a vector field of center type X_{λ^0}, the ideal generated by the germs of Lyapunov quantities at λ^0 (the so-called *Lyapunov ideal*), coincides with the Bautin ideal (section 1.2.4). As a consequence, Lyapunov quantities can also be used in the description of the set of parameter values for which centers occur.

In section 1.2.4, we give an example of calculation of Lyapunov quantities in Liénard systems, that will be used in chapter 3.

To end this section, we recall the notion of saddle quantities, and explain briefly the existing duality between saddle and focus.

Algebraic lemma

Lyapunov quantities are defined by an algebraic lemma, that is proven in [S]. The statement there is restricted to the case of an individual polynomial vector field with linear part $x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}$. We give here a small generalisation for families of vector fields of type (1.27) (hence we don't exclude here vector fields for which the linear part has a non-zero trace):

Lemma 47 *Suppose a C^∞ family of vector fields is given in the form* (1.27). *Then there exists a formal power series F_λ,*

$$F_\lambda(x,y) = \frac{1}{2}\left(x^2+y^2\right) + \sum_{j=3}^{\infty} F_j(x,y,\lambda),$$

where F_j is a homogenous polynomial of degree j in x and y,

$$F_j(x,y,\lambda) = \sum_{i=0}^{j} f_{ij}(\lambda)\, x^i y^{j-i},$$

and there exist coefficients $V_i(\lambda)$ such that:

$$X_\lambda F_\lambda(x,y) = \sum_{i=0}^{\infty} V_i(\lambda)\left(x^2+y^2\right)^{i+1}. \tag{1.28}$$

Moreover if F_λ and V_i $(i \in \mathbb{N})$ are solutions satisfying (1.28), *then the functions f_{ij} and V_i are C^∞ in λ.*

Remark 48 *The "moreover" part in lemma 47 can be generalised in the following sense: if the family given in* (1.27) *is of class C^γ $(\gamma \in \mathbb{N}\cup\{\infty,\omega\})$ in λ, then also the functions f_{ij} and V_i are C^γ.*

Definition 49 *Such coefficients $\{V_i(\lambda) : i \in \mathbb{N}\}$, as defined by* (1.28) *in lemma 47, are called* Lyapunov quantities *(or* Lyapunov coefficients *or* focal values*) of the vector field X_λ at the focus.*

From the proof of lemma 47, it follows that a set of Lyapunov quantities is not unique. However, in a certain sense, they are unique, as we will see in corollary 53 below.

In a straightforward way, one can generalise the definition of Lyapunov quantities for families of vector fields $(Y_\lambda)_\lambda$ of type

$$\begin{aligned} Y_\lambda(x,y) &= (d(\lambda)x - c(\lambda)y)\frac{\partial}{\partial x} + (c(\lambda)x + d(\lambda)y)\frac{\partial}{\partial y} \\ &\quad + f(x,y,\lambda)\frac{\partial}{\partial x} + g(x,y,\lambda)\frac{\partial}{\partial y} \end{aligned} \tag{1.29}$$

where $c : (\mathbb{R}^p, \lambda^0) \to \mathbb{R}$ is a C^∞ (respectively C^ω) nowhere-zero function. Indeed, let $\{V_i : i \in \mathbb{N}\}$ and (F_λ) be given by lemma 47 for the family of vector fields $(X_\lambda)_\lambda$, where $X_\lambda = (c(\lambda))^{-1} Y_\lambda$. Clearly, the set $\{W_i(\lambda), i \in \mathbb{N}\}$, defined by $W_i(\lambda) = c(\lambda) V_i(\lambda)$, satisfies

$$Y_\lambda F_\lambda = \sum_{i=0}^{\infty} W_i(\lambda)\left(x^2 + y^2\right)^{i+1}.$$

Therefore, such a set of coefficients is called a set of Lyapunov quantities for the family $(Y_\lambda)_\lambda$.

Relation with normal forms

The following technical proposition that relates Lyapunov quantities to normal forms is a new result, that we derived for the study of Hopf-Takens bifurcations (chapter 2). Recall that we use the notation $(g_1, \ldots, g_N)$ for the ideal generated by the set of (the germs of) the functions $g_1, \ldots, g_N$ $(N \in \mathbb{N})$.

Proposition 50 *Let $(X_\lambda)_\lambda$ be a C^∞ (respectively C^ω) family of vector fields like in (1.27), let $N \in \mathbb{N}$, φ_λ be a C^∞ (respectively C^ω) near-identity diffeomorphism, i.e.*

$$\varphi_\lambda(x,y) = (x,y) + O\left(\|(x,y)\|^2\right), \qquad (x,y) \to (0,0)$$

and let h_λ be a C^∞ (respectively C^ω) positive function such that:

$$\begin{aligned} X_\lambda^N \doteq h_\lambda \cdot (\varphi_\lambda)^* X_\lambda &= \left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right) \\ &\quad + \left(\sum_{j=0}^{N} d_j(\lambda)\left(x^2+y^2\right)^j + H(x,y,\lambda)\right)\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right) \end{aligned} \tag{1.30}$$

for certain C^∞ (respectively C^ω) functions d_j and H with

$$\begin{cases} H(x,y,\lambda) = O\left(\|(x,y)\|^{2N+1}\right), & (x,y) \to (0,0), \\ d_0 \equiv d. \end{cases}$$

Suppose we are given a set of Lyapunov quantities $\{V_i : i \in \mathbb{N}\}$ for $(X_\lambda)_\lambda$, then there exist C^∞ (respectively C^ω) functions R_{ij} ,$0 \leq i \leq N, 0 \leq j \leq i-1$ defined on a neighbourhood of λ^0 such that

$$d_i(\lambda) = V_i(\lambda) + \sum_{j=0}^{i-1} R_{ij}(\lambda) V_j(\lambda), \qquad i = 0, \ldots, N. \tag{1.31}$$

Remark 51 *This proposition remains true if we change C^∞ by C^r, with r sufficiently large.*

Proof. Suppose that F_λ is a formal power series such that equation (1.28) is satisfied. Denote the formal power series on the right-hand side of equation (1.28) by $G_\lambda(x,y)$. Then we can write:

$$X_\lambda^N\left(F_\lambda \circ \varphi_\lambda^{-1}\right) = h_\lambda \cdot \left(G_\lambda \circ \varphi_\lambda^{-1}\right). \tag{1.32}$$

Since φ_λ is a near-identity diffeomorphism and $h_\lambda(0,0) = 1$, the right-hand side of equation (1.32) -written in polar coordinates (r,θ) in the (u,v)-plane- is of the following form:

$$h_\lambda(u,v) \cdot \left(G_\lambda \circ \varphi_\lambda^{-1}\right)(u,v) = w_2(\lambda) r^2 + \sum_{j=3}^{\infty} w_j(\lambda,\theta) r^j$$

with

$$\begin{cases} w_2(\lambda) = V_0(\lambda) \\ w_{2j}(\lambda,\theta) = V_{j-1}(\lambda) \bmod (V_0, V_1, \ldots, V_{j-2}), & \forall j \geq 2, \\ w_{2j+1}(\cdot,\theta) \in (V_0, V_1, \ldots, V_{j-2}), & \forall j \geq 2. \end{cases} \tag{1.33}$$

We also have

$$F_\lambda \circ \varphi_\lambda^{-1}(u,v) = \frac{r^2}{2} + \sum_{m \geq 3} H_m(\lambda,\theta) \frac{r^m}{m}$$

for certain C^∞ (respectively C^ω) functions $H_m(\lambda,\theta)$ which are 2π-periodic in θ. After writing equation (1.32) in polar coordinates, we can identify the coefficients of $r^{2(i+1)}$ (with $0 \leq i \leq N$) in both sides of this equation. It is clear that the following relations can be deduced:

$$w_2(\lambda) = d_0(\lambda), \tag{1.34}$$

$$\begin{aligned} w_{2(i+1)}(\lambda,\theta) = {} & \frac{1}{2(i+1)} \frac{\partial}{\partial\theta} H_{2(i+1)}(\lambda,\theta) + d_0(\lambda) H_{2(i+1)}(\lambda,\theta) + d_1(\lambda) H_{2i}(\lambda,\theta) \\ & + d_2(\lambda) H_{2(i-1)}(\lambda,\theta) + \ldots + d_{i-1}(\lambda) H_4(\lambda,\theta) + d_i(\lambda). \end{aligned} \tag{1.35}$$

From (1.33) and (1.34), it follows that

$$d_0(\lambda) = V_0(\lambda).$$

This is the required expression (1.31) for $i = 0$; now we prove expression (1.31) for $1 \leq i \leq N$.

After rewriting equation (1.35) and keeping in mind properties (1.33), we obtain the following linear non-homegeneous differential equation for $H_{2(i+1)}(\lambda, \cdot)$:

$$\frac{\partial}{\partial\theta} H_{2(i+1)}(\lambda,\theta) = -2(i+1)\, d_0(\lambda)\, H_{2(i+1)}(\lambda,\theta) + 2(i+1)\, \bar{w}_{2(i+1)}(\lambda,\theta) \quad (1.36)$$

with

$$\begin{aligned}\bar{w}_{2(i+1)}(\lambda,\theta) = V_i(\lambda) - d_i(\lambda) + R_{i,0}(\lambda,\theta) V_0(\lambda) + R_{i,1}(\lambda,\theta) V_1(\lambda) \\ + \ldots + R_{i,i-1}(\lambda,\theta) V_{i-1}(\lambda) \end{aligned} \quad (1.37)$$

for certain C^∞ (respectively C^ω) functions $R_{i,j}, j = 0, \ldots, i-1$.

The solution of (1.36) is known as

$$\begin{aligned} & H_{2(i+1)}(\lambda,\theta) \\ &= e^{-2(i+1)d_0(\lambda)\theta} \left(H_{2(i+1)}(\lambda,0) + 2(i+1) \int_0^\theta \bar{w}_{2(i+1)}(\lambda,t)\, e^{2(i+1)d_0(\lambda)t} dt \right) \\ &= e^{-2(i+1)d_0(\lambda)\theta} H_{2(i+1)}(\lambda,0) + (V_i(\lambda) - d_i(\lambda)) \left(\frac{1 - e^{-2(i+1)d_0(\lambda)\theta}}{d_0(\lambda)} \right) \\ &+ e^{-2(i+1)d_0(\lambda)\theta} \sum_{j=0}^{i-1} V_j(\lambda)\, r_{ij}(\lambda,\theta) \end{aligned} \quad (1.38)$$

for certain C^∞ (respectively C^ω) functions r_{ij}. Because $H_{2(i+1)}(\lambda, \cdot)$ is 2π-periodic in θ, the equation (1.38), evaluated in $\theta = 2\pi$, is equivalent to

$$\begin{aligned} \left(1 - e^{-t_i d_0(\lambda)}\right) H_{2(i+1)}(\lambda,0) = (V_i(\lambda) - d_i(\lambda)) \left(\frac{1 - e^{-t_i d_0(\lambda)}}{d_0(\lambda)} \right) \\ + e^{-t_i d_0(\lambda)} \sum_{j=0}^{i-1} V_j(\lambda)\, r_{ij}(\lambda, 2\pi), \end{aligned} \quad (1.39)$$

where $t_i = 4(i+1)\pi$. After dividing both sides of (1.39) by the analytic function $\dfrac{1 - e^{-t_i d_0(\lambda)}}{d_0(\lambda)}$, this equation can be brought into

$$\begin{aligned} d_i(\lambda) &= V_i(\lambda) \\ &+ V_0(\lambda) \left[-H_{2(i+1)}(\lambda,0) + \left(\frac{d_0(\lambda)}{1 - e^{-t_i d_0(\lambda)}} \right) e^{-t_i d_0(\lambda)} r_{i0}(\lambda, 2\pi) \right] \\ &+ \sum_{j=1}^{i-1} V_j(\lambda) \left(\frac{d_0(\lambda)}{e^{-t_i d_0(\lambda)} - 1} \right) e^{-t_i d_0(\lambda)} r_{ij}(\lambda, 2\pi), \end{aligned}$$

which is the required expression for d_i. ■

Remark 52 *Proposition 50 can be generalised for a C^∞ (respectively C^ω) family $(X_\lambda)_\lambda$ of type* (1.29). *The main difference is that now, $h_\lambda(0,0) = (c(\lambda))^{-1} \neq 1$ in general. By the remark at the end of section 1.2.4, we can state the analogue of proposition 50 for this kind of families, where relation* (1.31) *is replaced by*

$$d_i(\lambda) = (c(\lambda))^{-1} V_i(\lambda) + \sum_{j=0}^{i-1} R_{ij}(\lambda) V_j(\lambda), \qquad i = 0, \ldots, N.$$

Clearly, proposition 50 also implies uniqueness of Lyapunov quantities in the following sense:

Corollary 53 *If both $\{W_i : i \in \mathbb{N}\}$ and $\{V_i : i \in \mathbb{N}\}$ are a set of Lyapunov quantities for a given family of vector fields of type* (1.27), *then they are related by:*

$$\begin{cases} \tilde{V}_0 = & \tilde{W}_0, \\ \tilde{V}_i = & \tilde{W}_i \operatorname{mod}(\tilde{W}_0, \ldots, \tilde{W}_{i-1}), \quad \forall i \in \mathbb{N}_1. \end{cases}$$

(recall that the twiddles in the notation denote germs).

As a consequence of corollary 53, the first non-zero Lyapunov quantity V_l is uniquely determined, but V_{l+1} and the successive ones are not.

Bautin ideal and displacement map

The notations introduced in section 1.2.3, regarding the displacement map and its coefficients, will also be used here. We call the ideal generated by the germs of the Lyapunov quantities V_i in λ^0 the *Lyapunov Ideal* and denote it by $\mathcal{L} = (\tilde{V}_i : i \in \mathbb{N})$. By corollary 53 this definition is meaningful.

Theorem 54 *Consider a C^ω family of vector fields $(X_\lambda)_\lambda$ of type* (1.27). *Then,*

1. *the Lyapunov quantities and the coefficients $\alpha_{2j+1}, j \in \mathbb{N}$ in* (27) *are related by*

$$\begin{cases} \alpha_1(\lambda) = V_0(\lambda)\left(\dfrac{e^{2\pi V_0(\lambda)} - 1}{V_0(\lambda)}\right) \text{ and } \forall j \geq 1, \\ \alpha_{2j+1}(\lambda) = V_j(\lambda)\, e^{2\pi V_0(\lambda)}\left(\dfrac{e^{4j\pi V_0(\lambda)} - 1}{2jV_0(\lambda)}\right) \operatorname{mod}(V_0, \ldots, V_{j-1}); \end{cases}$$

2. *in case the vector field X_{λ^0} is of center type, then the Bautin ideal and the Lyapunov ideal coincide, i.e., $\mathcal{I} = \mathcal{L}$; moreover,*

3. *If for $N \in \mathbb{N}$, the family $(X_\lambda^N)_\lambda$ is a C^ω normal form, like in* (1.30), *then we can write in terms of ideals of germs of C^ω functions:*

$$(\tilde{V}_0, \tilde{V}_1, \ldots, \tilde{V}_N) = (\tilde{d}_0, \tilde{d}_1, \ldots, \tilde{d}_N) = (\tilde{\alpha}_1, \tilde{\alpha}_3, \ldots, \tilde{\alpha}_{2N+1}).$$

Proof. As the ring of analytic function germs is Noetherian, it suffices by proposition 27 to prove that $\forall N \in \mathbb{N}_1$:

$$(\tilde{\alpha}_1, \tilde{\alpha}_3, \ldots, \tilde{\alpha}_{2N+1}) = (\tilde{V}_0, \tilde{V}_1, \ldots, \tilde{V}_N).$$

Now fix $N \in \mathbb{N}$ and choose a normal form X_λ^N for X_λ of type (1.30); written in polar coordinates (r, θ) the vector field X_λ^N is given by:

$$\frac{\partial}{\partial \theta} + \left(d_0(\lambda) + d_1(\lambda) r^2 + \ldots + d_N(\lambda) r^{2N} + h(r, \theta, \lambda)\right) r \frac{\partial}{\partial r} \tag{1.40}$$

for a certain analytic function h with the property that

$$h(r, \theta, \lambda) = O\left(r^{2N+1}\right), \qquad r \to 0.$$

Denote the displacement map of X_λ^N associated to the transverse section $\{\theta = 0\}$ by δ_λ^N . An expansion of this function in terms of s is given by

$$\delta_\lambda^N(s) = \sum_{i=1}^{\infty} \beta_i(\lambda) s^i, s \to 0, \tag{1.41}$$

and from proposition 24 we know that

$$(\tilde{\beta}_1, \tilde{\beta}_3, \ldots, \tilde{\beta}_{2N+1}) = (\tilde{\alpha}_1, \tilde{\alpha}_2, \ldots, \tilde{\alpha}_{2N+1}).$$

By proposition 50 it now suffices to prove that

$$\left(\tilde{\beta}_1, \tilde{\beta}_3, \ldots, \tilde{\beta}_{2N+1}\right) = \left(\tilde{d}_0, \tilde{d}_1, \ldots, \tilde{d}_N\right). \tag{1.42}$$

Consider the scalar differential equation, corresponding to (1.40) :

$$\frac{dR}{d\theta} = \sum_{j=0}^{N} d_j(\lambda) R^{2j+1} + R \cdot h(R, \theta, \lambda), \qquad R \to 0. \tag{1.43}$$

Denote the solution curve of (1.43) with initial value r at $\theta = 0$ by $R(\theta, r, \lambda)$. As solution of an analytic differential equation, R itself is analytic and we can give an expansion of R in terms of r:

$$R(\theta, r, \lambda) = \sum_{i=0}^{\infty} a_i(\lambda, \theta) r^i, \qquad r \to 0. \tag{1.44}$$

The displacement map δ_λ^N is now given by

$$\delta_\lambda^N(s) = R(2\pi, s, \lambda) - s.$$

By substitution of (1.44) in (1.43) we find the following expressions for the coefficients $\beta_i, i \geq 1$:

$$\begin{cases} \beta_1(\lambda) = a_1(\lambda, 2\pi) - 1 = e^{2\pi d_0(\lambda)} - 1 = d_0(\lambda)\left(\dfrac{e^{2\pi d_0(\lambda)} - 1}{d_0(\lambda)}\right), & \\ \beta_{2j+1}(\lambda) = a_{2j+1}(\lambda, 2\pi) = d_j(\lambda)\, e^{2\pi d_0(\lambda)}\left(\dfrac{e^{4j\pi d_0(\lambda)} - 1}{2jd_0(\lambda)}\right) & \\ \qquad\qquad + \displaystyle\sum_{k=0}^{j-1} d_k(\lambda)\, H_k^j(\lambda), & j = 1, \ldots, N \\ \beta_{2j}(\lambda) = a_{2j}(\lambda, 2\pi) = 0, & j = 1, \ldots, N \end{cases}$$

for certain analytic functions H_k^j. Therefore the ideals in (1.42) coincide. The other statements now follow from proposition 50. ■

Remark 55 *For $N \in \mathbb{N}$, the coefficients $\beta_i, 0 \leq i \leq 2N+1$, that appear in expansion (1.41) of the displacement map δ_λ^N, associated to the normal form X_λ^N, are independent of N.*

The following corollary follows immediately from theorem 54.

Corollary 56 *The analytic vector field X_λ is of center type if and only if all Lyapunov quantities vanish in λ (i.e. $\forall i \in \mathbb{N} : V_i(\lambda) = 0$).*

Remark 57 *The "moreover" statements in theorem 54 also hold when C^ω is replaced by C^∞ and the coefficients $\alpha_{2j+1}(\lambda)$, $j \in \mathbb{N}$, are read as:*

$$\frac{1}{(2j+1)!}\frac{\partial^{2j+1}}{\partial s^{2j+1}}\,\delta(r, \nu)|_{r=0}\,.$$

The proof is completely analoguous to the C^ω case.

Liénard systems

In this section, we recall a result due to C. Christopher and N. Lloyd ([CL]), concerning the computation of Lyapunov quantities for families originating from families of Liénard equations. This result will be used in chapter 3. For sake of completeness, we also provide a proof of the result. In appendix A, we give another proof, according to the algorithm used in the proof of lemma 47 in [S].

Proposition 58 *Consider the C^γ planar vector field $X = P\frac{\partial}{\partial x} + Q\frac{\partial}{\partial y}$ where*

$$\begin{cases} P(x,y) &= y + \sum_{i=1}^N a_{2i}x^{2i} + \sum_{i=k}^N a_{2i+1}x^{2i+1} + O\left(\|(x,y)\|^{2N+2}\right) \\ Q(x,y) &= -x + O\left(\|(x,y)\|^{2N+2}\right), \quad (x,y) \to (0,0) \end{cases}$$

with $k \in \mathbb{N}, k \ll N$ and $\gamma \in \mathbb{N}, \gamma \geq 2N+2$ or $\gamma \in \{\infty, \omega\}$. If $\{V_i : i \in \mathbb{N}\}$ is a set of Lyapunov quantities for X at the focus at the origin, then

$$\begin{cases} V_l \equiv 0, \forall 0 \leq l < k \\ V_k = c_k \cdot a_{2k+1} \end{cases}$$

for a certain $c_k \in \mathbb{Q}^+ \setminus \{0\}$.

Proof. The system of differential equations, corresponding to the vector field $X^R = \left(y + \sum_{i=1}^N a_{2i}x^{2i}\right)\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}$,

$$\begin{cases} \dot{x} &=& y + \sum_{i=1}^N a_{2i}x^{2i} \\ \dot{y} &=& -x \end{cases}$$

is time-reversible under the transformation $(t, x, y) \mapsto (-t, -x, y)$. As a consequence, there exists an analytic function $F(x, y) + O\left(\|(x,y)\|^3\right), \|(x,y)\| \to 0$ such that

$$X^R\left(\frac{x^2+y^2}{2} + F(x,y)\right) = 0.$$

To compute V_k, we need to find a homogeneous polynomial P of degree $2k+2$ of the form:

$$P(x,y) = \sum_{j=0}^{k} \beta_j x^{2(k-j)+1} y^{2j+1}$$

such that for $\|(x,y)\| \to 0$:

$$X\left(\frac{x^2+y^2}{2} + F(x,y) + P(x,y)\right) = V_k\left(x^2+y^2\right)^{k+1} + O\left(\left(x^2+y^2\right)^{k+2}\right).$$

Therefore,

$$\left(y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}\right)\sum_{j=0}^{k} \beta_j x^{2(k-j)+1} y^{2j+1} + a_{2k+1}x^{2k+2} = V_k\left(x^2+y^2\right)^{k+1}.$$

This yields in the following linear system in $(\beta_0, \beta_1, \ldots, \beta_k)$:

$$\begin{cases} a_{2k+1} - \beta_0 &=& V_k \\ (2(k-j)+1))\beta_j - (2j+3)\beta_{j+1} &=& V_k C_{j+1}^{k+1}, \forall j = 0, \ldots, k-1 \\ \beta_k &=& V_k \end{cases}$$

where C_{j+1}^{k+1} are the binomial coefficients:

$$C_{j+1}^{k+1} = \binom{k+1}{j+1} = \frac{(k+1)!}{(j+1)!\,(k-j)!}.$$

By backwards substitution, one finds the solution:

$$\left\{ \beta_j \;=\; d_j V_k, \forall j = 0, 1, \ldots, k \right. ,$$

where $d_k = 1, d_{k-j} = \frac{1}{2(k-j)+1}\left(C_{j+1}^{k+1} + (2j+3)\, d_{k-j+1}\right), \forall j = 1, \ldots, k$. As a consequence,

$$a_{2k+1} = \beta_0 + V_k = (d_0 + 1)\, V_k,$$

and the required result follows with $c = \frac{1}{d_0+1} \in \mathbb{Q}^+ \setminus \{0\}$. ■

Saddle quantities

There exists a strong duality between saddle and focus, and as a consequence between the generalised Hopf bifurcation and the homoclinic loop bifurcation. This duality is recalled below; furthermore, we give a precise definition of the notion of *saddle quantities* or *dual Lyapunov quantities*, that reappear in chapter 5.

It is well-known that the stability of a homoclinic loop through a saddle point in the plane is determined in first approximation by the trace of the linearisation of the vector field at the saddle point. In case of a non-zero trace, a homoclinic loop bifurcation leads to the birth (or death) of a unique limit cycle when the two separatrices of the saddle point cross each other. However, when the trace vanishes, we can have several limit cycles arising in a homoclinic loop bifurcation. As in the generalised Hopf bifurcation, this phenomenon is traditionally studied by calculation of normal forms. Another interesting tool in this study are so-called saddle quantities. Saddle quantities of a system are given by the coefficients appearing in an asymptotic expansion of the Dulac map in the neighbourhood of the saddle point. These saddle quantities are the analogues of the focal values (or Lyapunov quantities) for the focus; therefore they are also called *dual Lyapunov quantities* [JR].

Consider first the linear systems: the linear center or focus determined by

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} a & b \\ -b & a \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix}, \tag{1.45}$$

with $b \neq 0$ and the linear saddle determined by

$$\begin{bmatrix} \dot{X} \\ \dot{Y} \end{bmatrix} = \begin{bmatrix} A & B \\ B & A \end{bmatrix} \cdot \begin{bmatrix} X \\ Y \end{bmatrix} \tag{1.46}$$

with $|A| < |B|$. The matrix in (1.45) has eigenvalues $a + bi$ and $a - bi$, which are complex and conjugated. Hence, the origin is a linear center or focus for (1.45). The general solution of (1.45) with $(x(0), y(0)) = (x_0, y_0)$ can be written as

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \mathrm{e}^{at} \cdot \begin{bmatrix} \cos bt & \sin bt \\ -\sin bt & \cos bt \end{bmatrix} \cdot \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}. \tag{1.47}$$

In this way, the solution is the composition of a *rotation* and a *dilation*.

The matrix in (1.46) has eigenvalues $(A-B)$ and $(A+B)$, which are of opposite sign and non-zero. Hence, the origin is a linear saddle for (1.46). The general solution of (1.46) with $(X(0), Y(0)) = (X_0, Y_0)$ can be written as

$$\begin{bmatrix} X(t) \\ Y(t) \end{bmatrix} = \mathrm{e}^{At} \cdot \begin{bmatrix} \cosh Bt & \sinh Bt \\ \sinh Bt & \cosh Bt \end{bmatrix} \cdot \begin{bmatrix} X_0 \\ Y_0 \end{bmatrix}. \tag{1.48}$$

In analogy to (1.47), solution (1.48) can be seen as a *hyperbolic rotation* followed by a *dilation*.

Every linear saddle

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = M \begin{bmatrix} x \\ y \end{bmatrix}, M \in \mathbb{R}^{2\times 2}, \det M = \lambda_1\lambda_2 < 0 \text{ and } \mathrm{tr} M = \lambda_1 + \lambda_2.$$

can be brought into the form (1.46) after performing two linear coordinate transformations as follows: there exists a linear transformation U such that

$$UMU^{-1} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix},$$

and hence for $(v, w) = U(x, y)$, we obtain the system

$$\begin{bmatrix} \dot{u} \\ \dot{v} \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}.$$

Next, apply the linear transformation

$$\begin{bmatrix} X \\ Y \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix},$$

then we obtain the system (1.46) with

$$A = \frac{\lambda_1 + \lambda_2}{2} \text{ and } B = \frac{\lambda_1 - \lambda_2}{2}.$$

By applying the transformation

$$\begin{cases} x &= X \\ y &= \mathrm{i}Y \quad \text{where } \mathrm{i}^2 = -1 \\ t &= -\mathrm{i}T \end{cases}$$

a linear saddle can formally be transformed into a linear focus. More precisely, the system

$$\begin{cases} \dot{X} &= AX + BY + P(X,Y) \\ \dot{Y} &= BX + AY + Q(X,Y) \end{cases} \text{with} \begin{cases} P(X,Y) = O\left(\|(X,Y)\|^2\right) \\ Q(X,Y) = O\left(\|(X,Y)\|^2\right) \end{cases} \tag{1.49}$$

is transformed into

$$\begin{cases} \dot{x} &= ax + by + p(x,y) \\ \dot{y} &= -bx + ay + q(x,y) \end{cases}, \tag{1.50}$$

where

$$\begin{cases} a &= \mathrm{i}A \\ b &= B \\ p(x,y) &= \mathrm{i}P(x,-\mathrm{i}y) \\ q(x,y) &= -Q(x,-\mathrm{i}y) \end{cases}$$

Remark that the coefficients of p and q are complex-valued.

By lemma 47, we know that corresponding to the focus (1.50), there exists a formal power series

$$F(x,y) = \frac{1}{2}\left(x^2+y^2\right) + \sum_{k=3}^{\infty} F_k(x,y),$$

where F_k is a homogenous polynomial in (x,y) of degree k ($\forall k \geq 3$) such that

$$\dot{F}(x,y) = \frac{\partial F}{\partial x}(x,y)\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial F}{\partial y}(x,y)\frac{\mathrm{d}y}{\mathrm{d}t} = \sum_{k=0}^{\infty} V_k\left(x^2+y^2\right)^{k+1}.$$

The coefficients V_k are Lyapunov quantities (or focal values) of system (1.50). By the formal relation between the saddle and focus described above, there exists a formal power series

$$F^*(X,Y) = \frac{1}{2}\left(X^2-Y^2\right) + \sum_{k=3}^{\infty} F_k^*(X,Y)$$

such that

$$\begin{aligned} \dot{F}^*(X,Y) &= \frac{\partial F^*}{\partial X}(X,Y)\frac{\mathrm{d}X}{\mathrm{d}T} + \frac{\partial F^*}{\partial Y}(X,Y)\frac{\mathrm{d}Y}{\mathrm{d}T} \\ &= \frac{\partial F}{\partial x}(X,iY)\,\mathrm{i}\frac{\mathrm{d}x}{\mathrm{d}t} + \mathrm{i}\frac{\partial F}{\partial y}(X,iY)\frac{\mathrm{d}y}{\mathrm{d}t} = \mathrm{i}\dot{F}(X,\mathrm{i}Y) \\ &= \sum_{k=0}^{\infty} \mathrm{i}V_k\left(X^2-Y^2\right)^{k+1} \\ &= \sum_{k=0}^{\infty} V_k^*\left(X^2-Y^2\right)^{k+1}. \end{aligned}$$

Such coefficients V_k^* are called *dual Lyapunov quantities* or *saddle quantities* (because of their relation with the coefficients in the Dulac map). Again, the function F^* with the property:

$$\begin{cases} F^*(X,Y) &= \frac{1}{2}\left(X^2-Y^2\right) + O\left(\|(X,Y)\|^3\right), (X,Y) \to (0,0) \\ \dot{F}^*(X,Y) &= V_0^*\left(X^2-Y^2\right) + V_1^*\left(X^2-Y^2\right)^2 + \ldots + V_k^*\left(X^2-Y^2\right)^{k+1} + \ldots \end{cases} \tag{1.51}$$

is not unique, and different maps F^* give different dual Lyapunov quantities $\{V_k^* : k \in \mathbb{N}\}$. However, it can be shown that if $\{U_k^* : k \in \mathbb{N}\}$ is a set of dual Lyapunov quantities obtained with another function G^* (satisfying property (1.51), then

$$U_k^* = V_k^* + R\left(U_0^*, \ldots, U_{k-1}^*\right),$$

where R is a multi-variate polynomial in its arguments. In particular,

1. $U_i^* = 0, \forall 0 \le i \le k-1, U_k^* \neq 0 \iff V_i^* = 0, \forall 0 \le i \le k-1, V_k^* \neq 0.$

2. Suppose that the quantities depend on a parameter $\lambda \in \mathbb{R}^p$; suppose that $V_i^*\left(\lambda^0\right) = 0, \forall 0 \le i \le k-1, V_k^*\left(\lambda^0\right) \neq 0$. Define then the maps V^* and U^* as follows:

$$\left\{ \begin{array}{l} V^* : \mathbb{R}^p \to \mathbb{R}^k : \lambda \mapsto \left(V_0^*(\lambda), \ldots, V_{k-1}^*(\lambda)\right) \\ U^* : \mathbb{R}^p \to \mathbb{R}^k : \lambda \mapsto \left(U_0^*(\lambda), \ldots, U_{k-1}^*(\lambda)\right) \end{array} \right. . \text{ Then}$$

V^* is a submersion at λ^0 if and only if U^* is a submersion at λ^0.

Another interesting fact to notice is that methods to calculate Lyapunov quantities in the neighbourhood of a weak focus (1.50), can be used to compute dual Lyapunov quantities in the neighbourhood of the weak saddle (1.49): if V_k are Lyapunov quantities of (1.50), then we have derived that dual Lyapunov quantities are given by the following relation:

$$V_k^* = \mathrm{i} V_k, \forall k \in \mathbb{N} .$$

In this way, the following proposition is the dual result of the one due to C. Christopher and N. Lloyd [CL]:

Proposition 59 *Consider the C^γ planar vector field $D = P\frac{\partial}{\partial X} + Q\frac{\partial}{\partial Y}$ where*

$$\left\{ \begin{array}{lcl} P(X,Y) & = & Y + \sum_{i=1}^N A_{2i} X^{2i} + \sum_{i=k}^N A_{2i+1} X^{2i+1} + O\left(\|(X,Y)\|^{2N+2}\right) \\ Q(X,Y) & = & X + O\left(\|(X,Y)\|^{2N+2}\right), \|(X,Y)\| \to 0 \end{array} \right.$$

with $k \in \mathbb{N}, k \ll N$ and $\gamma \in \mathbb{N}, \gamma \ge 2N+2$ or $\gamma \in \{\infty, \omega\}$. If $\{V_i^* : i \in \mathbb{N}\}$ is a set of dual Lyapunov quantities for D at the saddle at the origin, then

$$\left\{ \begin{array}{lcl} V_l^* & = & 0, \forall 0 \le l < k \\ V_k^* & = & c_k \cdot A_{2k+1} \end{array} \right.$$

for a certain $c_k \in \mathbb{Q}^+ \setminus \{0\}$.

1.3 Regular hypersurface of centers

This section contains new material, that mainly will be used in chapter 2. We introduce the notion of regular hypersurface of centers; moreover, we state and prove some interesting properties.

In this section, we consider C^∞ (respectively C^ω) families of planar vector fields, unfolding a vector field X_{λ^0} of center type. In particular, we consider situations in which one has extra parameters besides the one that serves to generate the centers: we speak of *regular hypersurfaces of centers*. In such families the bifurcation diagram of limit cycles can be studied by way of *reduced displacement maps*. We start by giving a precise definition of these notions in section 1.3.1.

Since only a 1-dimensional parameter is responsable for centers in the family, it is natural to use Melnikov functions in the study of the bifurcation diagrams of limit cycles. In section 1.3.2, we introduce these Melnikov functions, depending on external parameters besides the phase variable, and, in the C^ω case, we derive the relation between these Melnikov functions and the Bautin ideal and index at the center e. Moreover, as X_{λ^0} is of center type, we can associate a Hamiltonian H to X_{λ^0} such that $e \in H^{-1}(0)$; as announced before, we here provide a proof of the fact that the first non-zero Melnikov function is C^∞ (respectively C^ω) at $h = 0$, where h denotes the value of the Hamiltonian.

Finally, in section 1.3.3, we introduce *reduced Lyapunov quantities* and show how they can be used to compute the index, and hence an upperbound for the cyclicity.

1.3.1 Reduced displacement map

Definition 60 *Suppose that $(X_\lambda)_\lambda$ is a C^∞ (respectively C^ω) family of planar vector fields, unfolding a vector field of center type X_{λ^0}. Suppose that $(\delta_\lambda)_\lambda$ is an associated C^∞ (respectively C^ω) family of displacement maps such that the parameter $\lambda = (\nu, \varepsilon) \in \mathbb{R}^p$ is close to $\lambda^0 = (\nu^0, 0)$ and such that the map $\delta_\lambda = \delta(\cdot, \lambda)$ can be divided by ε^k; more precisely:*

$$\delta(s, \nu, \varepsilon) = \varepsilon^k \bar{\delta}(s, \nu, \varepsilon) \tag{1.52}$$

for a certain C^∞ (respectively C^ω) function $\bar{\delta}$, which is not divisible by ε. Then we say that the centers in the family $(X_\lambda)_\lambda$ occur on a regular hypersurface. *The C^∞ (respectively C^ω) family of maps $(\bar{\delta}_\lambda)_\lambda$ defined by*

$$\bar{\delta}_\lambda = \bar{\delta}(\cdot, \lambda)$$

in (1.52) is called a family of reduced displacement maps.

We will only consider families in which no centers occur for $\varepsilon > 0$. Working with an analytic family, it would mean that we suppose the Bautin ideal to be generated by the germ of the map $((\nu, \varepsilon) \mapsto \varepsilon^k)$ at $(\nu^0, 0)$.

The following proposition shows that definition 60 does make sense: it is independent of the chosen family of displacement maps $(\delta_\lambda)_\lambda$. Moreover, the existence of the positive integer k that appears in (1.52) is invariant.

Proposition 61 *Suppose that a C^∞ (respectively C^ω) family of displacement maps $\delta(s,\nu,\varepsilon)$, associated to a C^∞ (respectively C^ω) family of planar vector fields $\left(X_{(\nu,\varepsilon)}\right)_{(\nu,\varepsilon)}$, satisfies the property*

$$\delta(s,\nu,\varepsilon)=\varepsilon^k\bar{\delta}(s,\nu,\varepsilon), \tag{1.53}$$

for a certain C^∞ (respectively C^ω) family of reduced displacement maps $\bar{\delta}(s,\nu,\varepsilon)$. Let δ^1 denote the family of displacement maps, associated to the C^∞ (respectively C^ω) family of planar vector fields $\left(Y_{(\Lambda,\varepsilon_1)}\right)_{(\Lambda,\varepsilon_1)}$, obtained from $\left(X_{(\nu,\varepsilon)}\right)_{(\nu,\varepsilon)}$ after a C^∞ (respectively C^ω) coordinate change ψ of the form:

$$(r,\Lambda,\varepsilon_1)=(\varphi(s,\nu,\varepsilon),\kappa(\nu,\varepsilon)) \tag{1.54}$$

for (s,ν,ε) on an open neighbourhood $W\times U\times E$ of $\left(0,\nu^0,0\right)$ in $\mathbb{R}\times\mathbb{R}^{p-1}\times\mathbb{R}$, with $\varphi(0,\nu,\varepsilon)\equiv 0$ and $\kappa(\nu,\varepsilon)=(\kappa_1(\nu,\varepsilon),\ldots,\kappa_{p-1}(\nu,\varepsilon),f(\varepsilon))$ for some C^∞ (respectively C^ω) functions $f:E\to\mathbb{R}$ with $f(0)=0$ and

$$\kappa_i:U\times E\to\mathbb{R}, i=1,\ldots,p-1.$$

Then there exist a C^∞ (respectively C^ω) function $\bar{\delta}^1(r,\Lambda,\varepsilon_1)$ such that

$$\delta^1(r,\Lambda,\varepsilon_1)=\varepsilon_1^k\bar{\delta}^1(r,\Lambda,\varepsilon_1).$$

Proof. Because the associated Poincaré-maps are conjugated, δ^1 is given by

$$\delta^1_{\kappa(\nu,\varepsilon)}\left(\varphi_{(\nu,\varepsilon)}(s)\right)+\varphi_{(\nu,\varepsilon)}(s)=\varphi_{(\nu,\varepsilon)}\left(\delta_{(\nu,\varepsilon)}(s)+s\right), \tag{1.55}$$

where we use the notations:

$$\begin{cases}\delta_{(\nu,\varepsilon)}=\delta(\cdot,\nu,\varepsilon),\\ \delta^1_{(\Lambda,\varepsilon_1)}=\delta^1(\cdot,\Lambda,\varepsilon_1),\\ \varphi_{(\nu,\varepsilon)}=\varphi(\cdot,\nu,\varepsilon).\end{cases}$$

We will prove, by induction on $j=0,1,\ldots,k-1$, that $\forall\nu\in U,\forall s\in W$:

$$\delta^1_{\kappa(\nu,0)}\left(\varphi_{(\nu,0)}(s)\right)=\varepsilon_1^{j+1}\delta^{j+2}(r,\kappa(\nu,0)) \tag{1.56}$$

for certain C^∞ functions δ^{j+2}. Because the maps $\nu\mapsto\kappa(\nu,0)$, $s\mapsto\varphi_{(\nu,0)}(s)$ are diffeomorphisms on U respectively W, this will imply the required result.

Substituting equation (1.53) in equation (1.55) gives the following equation $\forall\nu\in U$ and $\forall s\in W$:

$$\delta^1_{\kappa(\nu,0)}\left(\varphi_{(\nu,0)}(s)\right)\equiv 0.$$

Hence property (1.56) already holds for $j=0$.

Assuming that property (1.56) holds for all j with $j \leq k-2$, we will now prove that the property (1.56) also holds for $j+1$. From (1.53) it follows that

$$\frac{\partial^{j+1}}{\partial \varepsilon^{j+1}} \delta(s, \nu, \varepsilon)\bigg|_{\varepsilon=0} = 0. \tag{1.57}$$

If we write

$$G(s, \nu, \varepsilon) = \delta^1_{\kappa(\nu,\varepsilon)}\left(\varphi_{(\nu,\varepsilon)}(s)\right) + \varphi_{(\nu,\varepsilon)}(s),$$

then it follows from (1.55) that

$$\begin{aligned} \frac{\partial}{\partial \varepsilon}\delta(s, \nu, \varepsilon) &= \frac{\partial}{\partial \varepsilon}\left(\varphi^{-1}_{(\nu,\varepsilon)}\left(G(s, \nu, \varepsilon)\right)\right) \\ &= \frac{\partial}{\partial \varepsilon}\left(\varphi^{-1}_{(\nu,\varepsilon)}\right)\left(G(s, \nu, \varepsilon)\right) + \left(\varphi^{-1}_{(\nu,\varepsilon)}\right)'\left(G(s, \nu, \varepsilon)\right) \cdot \frac{\partial}{\partial \varepsilon} G(s, \nu, \varepsilon). \end{aligned} \tag{1.58}$$

From elementary calculus we can derive the following property, for given C^∞ functions ψ, g_1 and g_2 :

$$\frac{\partial^j}{\partial \varepsilon^j}\left[\psi\left(\varepsilon^{j+1} g_1(s, \nu, \varepsilon) + g_2(s, \nu, \varepsilon)\right)\right]\bigg|_{\varepsilon=0} = \frac{\partial^j}{\partial \varepsilon^j}\left[\psi\left(g_2(s, \nu, \varepsilon)\right)\right]\bigg|_{\varepsilon=0}.$$

By this property, the second line in equality (1.58) and the induction hypothesis, equation (1.57) becomes:

$$\begin{aligned} 0 = {} & \frac{\partial^j}{\partial \varepsilon^j}\left[\frac{\partial}{\partial \varepsilon}\left(\varphi^{-1}_{(\nu,\varepsilon)}\right)\left(\varphi_{(\nu,\varepsilon)}(s)\right) + \left(\varphi^{-1}_{(\nu,\varepsilon)}\right)'\left(\varphi_{(\nu,\varepsilon)}(s)\right) \cdot \frac{\partial}{\partial \varepsilon}\varphi(s, \nu, \varepsilon)\right]\bigg|_{\varepsilon=0} \\ & + \frac{\partial^j}{\partial \varepsilon^j}\left[\left(\varphi^{-1}_{(\nu,\varepsilon)}\right)'\left(\varphi_{(\nu,\varepsilon)}(s)\right) \cdot \frac{\partial}{\partial \varepsilon}\left(\delta^1_{\kappa(\nu,\varepsilon)}\left(\varphi_{(\nu,\varepsilon)}(s)\right)\right)\right]\bigg|_{\varepsilon=0}. \end{aligned} \tag{1.59}$$

By the chain rule, we find that the first term in the right-hand side of equation (1.59) is identically zero. The second term in (1.59) consists of a sum of which each term contains a factor of the form

$$\frac{\partial^{e+\sum_{q=1}^{t} n_q+\rho}}{\partial \varepsilon_1^e \partial \Lambda_{i_1}^{n_1} \ldots \partial \Lambda_{i_t}^{n_t} \partial r^\rho}\left(\delta^1_{\kappa(\nu,\varepsilon)}\left(\varphi_{(\nu,\varepsilon)}(s)\right)\right)\bigg|_{\varepsilon=0} \tag{1.60}$$

with $e + \sum_{q=1}^{t} n_q + \rho \leq j+1$, $1 \leq i_q \leq p-1$. By the induction hypothesis, the factors in (1.60) vanish for $e \leq j$. Therefore the right-hand side of equation (1.59) is reduced to only one term:

$$\left(\varphi^{-1}_{(\nu,0)}\right)'\left(\varphi_{(\nu,0)}(s)\right) \cdot \frac{\partial^{j+1}}{\partial \varepsilon_1^{j+1}}\left(\delta^1_{\kappa(\nu,\varepsilon)}\right)\bigg|_{\varepsilon=0}\left(\varphi_{(\nu,0)}(s)\right) \cdot \left(f'(0)\right)^{j+1}. \tag{1.61}$$

Since the maps $\varphi_{(0,\nu)}$ (for fixed ν) and f are diffeomorphisms (see (1.54)), the first and third factor in (1.61) are different from zero. As a consequence, (1.59) is equivalent to

$$\frac{\partial^{j+1}}{\partial \varepsilon_1^{j+1}} \left(\delta^1_{\kappa(\nu,\varepsilon)} \right) \bigg|_{\varepsilon=0} \left(\varphi_{(\nu,0)} (s) \right) \equiv 0, \qquad s \in W, \nu \in U$$

which concludes the induction step. ■

1.3.2 Melnikov functions

Let $\left(X_{(\nu,\varepsilon)} \right)_{\nu\sim\nu^0, \varepsilon\sim 0}$ be a C^ω (respectively C^∞) p-parameter family of planar vector fields, unfolding a vector field of center type $X_{(\nu^0,0)}$, and let $\left(\delta_{(\nu,\varepsilon)} \right)$ be an associated family of displacement maps. Then we can expand (perhaps only formally) this function locally $\left(\nu \sim \nu^0, s \sim s_0 \right)$ in terms of ε :

$$\delta (s, \nu, \varepsilon) = \delta_{(\nu,\varepsilon)} (s) = \sum_{k=0}^{\infty} M_k (s, \nu) \varepsilon^k, \qquad \varepsilon \to 0$$

for certain C^ω (respectively C^∞) functions M_k. Notice that, for a fixed parameter value ν^0, M_k corresponds to the k-th Melnikov function of the 1-parameter family $(X_{(\nu^0,\varepsilon)})_{\varepsilon\sim 0}$.

By Nakayama's lemma, we have the following equivalence: '$M_j \equiv 0, \forall j < k$ and $M_k^0 \equiv M_k \left(\cdot, \nu^0 \right)$ is the first non-zero Melnikov function of $X_{(\nu^0,\varepsilon)}$' if and only if 'the Bautin Ideal is generated by the germ of $\left((\nu, \varepsilon) \mapsto \varepsilon^k \right)$ at $(\nu, \varepsilon) = \left(\nu^0, 0 \right)$'. Moreover, in this case,

$$\operatorname{Index} \left(X_{(\nu,\varepsilon)}, \left(\Gamma, \left(\nu^0, 0 \right) \right) \right) = \begin{cases} \operatorname{order} M_k^0 (s_0) & \text{if } \Gamma \text{ is a periodic orbit} \\ \dfrac{\operatorname{order} M_k^0 (s_0) - 1}{2} & \text{if } \Gamma \text{ is a center} \end{cases}$$

However, let us remark that the mere computation of the first non-zero Melnikov function of the subfamily $\left(X_{(\nu^0,\varepsilon)} \right)_{\varepsilon\sim 0}$ cannot give such a result involving the Bautin Ideal of $\left(X_{(\nu,\varepsilon)} \right)_{(\nu,\varepsilon)\sim(\nu^0,0)}$. For instance, consider the 2-parameter family $\left(X_{(\nu,\varepsilon)} \right)_{(\nu,\varepsilon)\sim(0,0)}$ expressed by:

$$\left(x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x} \right) + \left(\varepsilon^2 + \varepsilon\nu \left(x^2 + y^2 \right) \right) \left(x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} \right).$$

In the notation used above, the functions M_1 and M_2 are given by: $M_1 (r, \nu) = 2\pi\nu r^3$ and $M_2 (r, \nu) = r \left(2\pi + 6\pi^2 \nu^2 r^4 \right)$. Although the first non-zero Melnikov function of $\left(X_{(0,\varepsilon)} \right)_{\varepsilon\sim 0}$ is $M_2 (\cdot, 0)$, it is obvious that the Bautin Ideal corresponding to $\left(X_{(\nu,\varepsilon)} \right)_{(\nu,\varepsilon)}$ is generated by $\varepsilon^2, \varepsilon\nu$ and not only by ε^2.

Let us now focus attention on the dependence of the first non-zero Melnikov function on the hamiltonian value h at the center e. As the vector fields $X_{(\nu,0)}$ are of center type for $(\nu \sim \nu^0)$, there exist strictly positive functions f_ν such that the vector fields $f_\nu \cdot X_{(\nu,0)}$ are Hamiltonian vector fields with some Hamiltonian H_ν (proposition 19). Hence,

$$f_\nu \cdot X_{(\nu,0)} = \frac{\partial H_\nu}{\partial y}\frac{\partial}{\partial x} - \frac{\partial H_\nu}{\partial x}\frac{\partial}{\partial y}. \tag{1.62}$$

We can suppose that $H_\nu(0,0) = 0$. The functions $H(x,y,\nu) = H_\nu(x,y)$ and $f(x,y,\nu) = f_\nu(x,y)$ are C^ω (respectively C^∞) if the family $X_{(\nu,0)}$ is C^ω (respectively C^∞). As a consequence of (1.62), the function H_ν is a first integral of $f_\nu \cdot X_{(\nu,0)}$, meaning that

$$\left(f_\nu \cdot X_{(\nu,0)}\right) H_\nu = 0. \tag{1.63}$$

Thus orbits of the vector field $f_\nu \cdot X_{(\nu,0)}$ (as well as of $X_{(\nu,0)}$) lie on the level curves of the Hamiltonian H_ν. With respect to closed orbits, we can replace the study of the family $\left(X_{(\nu,\varepsilon)}\right)_{(\nu,\varepsilon)}$ by the study of the family $\left(Y_{(\nu,\varepsilon)}\right)_{(\nu,\varepsilon)}$ defined by

$$Y_{(\nu,\varepsilon)} \equiv f_\nu \cdot X_{(\nu,\varepsilon)}.$$

Because of (1.62) and the form of $X_{(\nu,0)}$ given in (1.27), the Hessian of H_ν at $(0,0)$ has a strictly positive determinant and trace. By the Morse lemma there are coordinates $(u,v) = \varphi_\nu(x,y)$ with $\varphi_\nu(0,0) = (0,0)$ such that

$$G_\nu(u,v) = (\varphi_\nu)_* H_\nu(u,v) = H_\nu\left(\varphi_\nu^{-1}(u,v)\right) = \frac{u^2+v^2}{2}. \tag{1.64}$$

The function $\varphi(x,y,\nu) = \varphi_\nu(x,y)$ is C^ω (respectively C^∞) if the family $\left(X_{(\nu,0)}\right)_\nu$ is C^ω (respectively C^∞).

Because of (1.63) the function G_ν is a first integral of $(\varphi_\nu)_* Y_{(\nu,0)}$ and so the periodic orbits around the origin are exactly round.

We take a displacement map $\delta_{(\nu,\varepsilon)}$ of $(\varphi_\nu)^* Y_{(\nu,\varepsilon)}$, associated to the transverse section $\{\theta = 0\}$, defined in terms of r, with $r^2 = u^2 + v^2$. Clearly, this function is C^ω (respectively C^∞) in $\sqrt{h}$, where h denotes the value of the Hamiltonian. Although the displacement map is not necessarily C^ω (respectively C^∞) in h (cfr. example 1.16 above), the first non-zero Melnikov function is C^ω (respectively C^∞) in h. This fact is commonly used, but it is not easy to find a proof of it in the literature. Since it can be proven in a short way and for the sake of completeness, we provide such a proof.

From (1.52) and proposition 61 the first non-zero Melnikov function is M_k, i.e. $M_j(r,\nu) \equiv 0$, $\forall j = 1,\dots,k-1$ and $M_k(r,\nu) \neq 0$. Because $\delta_{(\nu,\varepsilon)}$ is C^ω (respectively C^∞) in r and $\delta_{(\nu,\varepsilon)}(0) = 0$, there are C^ω (respectively C^∞) functions $\bar{M}_k$ and g such that

$$\delta(r,\nu,\varepsilon) = r\varepsilon^k\left(\bar{M}_k(r,\nu) + g(r,\nu,\varepsilon)\right)$$

with $g(r,\nu,\varepsilon) = O(\varepsilon)$, $\varepsilon \to 0$. We refer to $\bar{M}_k$ as the *reduced k-th Melnikov function*, because of its relation with the k-th Melnikov function:

$$M_k(r,\nu) = r \cdot \bar{M}_k(r,\nu)$$

Theorem 62 *The reduced k-th Melnikov function $\bar{M}_k$ is C^ω (respectively C^∞) at $h, h \geq 0$; i.e. the map $\check{M}_k$ defined by*

$$\check{M}_k(h,\nu) = \bar{M}_k(\sqrt{2h},\nu), h \geq 0$$

is C^ω (respectively C^∞) at $h = 0, h \geq 0$. (This function $\check{M}_k$ is called the reduced Melnikov function in terms of the Hamiltonian, by relation (1.65) *below.)*

Proof. Denote the Poincaré map in terms of r by $P_{(\nu,\varepsilon)} = P(\cdot,\nu,\varepsilon)$; then the Poincaré map in terms of h is given by the map

$$\begin{aligned} Q(h,\nu,\varepsilon) &= \frac{1}{2}\left(P(\sqrt{2h},\nu,\varepsilon)\right)^2 \\ &= h\left(2\varepsilon^k \bar{M}_k(\sqrt{2h},\nu) + \bar{g}(\sqrt{2h},\nu,\varepsilon) + 1\right) \end{aligned}$$

for a certain C^ω (respectively C^∞) function $\bar{g}$ with

$$\bar{g}(r,\nu,\varepsilon) = O\left(\varepsilon^{k+1}\right), \qquad \varepsilon \to 0.$$

The displacement map δ^1 and the first non-zero Melnikov function M_k^1, in terms of h, are given by

$$\begin{aligned} \delta^1(h,\nu,\varepsilon) &= \varepsilon^k M_k^1(h,\nu) + h\bar{g}(\sqrt{2h},\nu,\varepsilon) = \varepsilon^k M_k^1(h,\nu) + O\left(\varepsilon^{k+1}\right), \varepsilon \to 0 \\ M_k^1(h,\nu) &= 2h\check{M}_k(h,\nu). \end{aligned} \tag{1.65}$$

Since $M_k^1(0,\nu) = 0$, it suffices by (1.65) to prove that M_k^1 is C^ω (respectively C^∞) at $h, h \geq 0$. Because $\bar{M}_k$ is C^ω (respectively C^∞) in $\sqrt{h}$, we already find that the function M_k^1 is C^1 at $h = 0, h \geq 0$.

Write the dual 1-form of $(\varphi_\nu)^* Y_{(\nu,\varepsilon)}$ as an expansion in terms of ε :

$$\upsilon_{(\nu,\varepsilon)} = d\left[(\varphi_\nu)_* H_\nu\right] + \varepsilon\upsilon_1 + \ldots + \varepsilon^k \upsilon_k + o\left(\varepsilon^k\right), \qquad \varepsilon \to 0,$$

for certain C^ω (respectively C^∞) 1-forms $\upsilon_i(x,y,\nu), 1 \leq i \leq k$. Then, we have for the first non-zero Melnikov function M_k^1 (in terms of h) the following expression obtained by J.C. Poggiale (Theorem 20)

$$M_k^1(h,\nu) = -\int_{G_\nu^{-1}(h)} \left(\upsilon_k - \sum_{i=1}^{k-1} g_i \upsilon_{k-i}\right) \tag{1.66}$$

for certain C^ω (respectively C^∞) functions g_i. Remark that

$$G_\nu^{-1}(h) = \left\{\frac{1}{2}\left(u^2 + v^2\right) = h\right\}.$$

The following lemma will finish the proof. ■

Lemma 63 *The Abelian integral*

$$\int_{\left\{\frac{1}{2}(u^2+v^2)=h\right\}} \upsilon$$

where υ is an C^ω (respectively C^∞) 1-form, is C^ω (respectively C^∞) in h at $h=0$.

Proof. We will first give a proof in the C^ω case, and then indicate which changes have to be made in the C^∞ case. Since, υ is analytic, there exist $\rho > 0$ such that the following series converge for all (u, v, ν) with $|u|, |v| < 4\rho$ and $\left\|\nu - \nu^0\right\| < 2\rho$:

$$\begin{cases} \upsilon(u,v,\nu) \equiv \left(\sum_{i,j} a_{ij}(\nu)\, u^i v^j\right) du + \left(\sum_{i,j} b_{ij}(\nu)\, u^i v^j\right) dv, \\ |\upsilon|(u,v,\nu) \equiv \left(\sum_{i,j} \left|a_{ij}(\nu)\, u^i v^j\right|\right) du + \left(\sum_{i,j} \left|b_{ij}(\nu)\, u^i v^j\right|\right) dv. \end{cases}$$

As a consequence, $\forall (h,\nu)$ with $|h|, \left\|\nu - \nu^0\right\| \leq \rho$, the integral $\int_{\{u^2+v^2=2h\}} |\upsilon| < \infty$. Then, by the dominated convergence theorem [Br], one can change sum and integral and write: $\forall (h,\nu)$ with $|h|, \left\|\nu - \nu^0\right\| \leq \rho$:

$$\int_{\left\{\frac{1}{2}(u^2+v^2)=h\right\}} \upsilon = \sum_{i,j} a_{ij}(\nu) \int_{\left\{\frac{1}{2}(u^2+v^2)=h\right\}} u^i v^j du + \sum_{i,j} b_{ij}(\nu) \int_{\left\{\frac{1}{2}(u^2+v^2)=h\right\}} u^i v^j dv. \tag{1.67}$$

Now, we use the theorem of Stokes to calculate the integrals in the right-hand side of (1.67) :

$$\begin{aligned} \int u^i v^j du &= -j \iint_{\left\{\frac{1}{2}(u^2+v^2)\leq h\right\}} u^i v^{j-1} du dv \\ &= -j \int_0^{\sqrt{2h}} \int_0^{2\pi} r^{i+j} \cos^i\theta \sin^{j-1}\theta dr d\theta \\ &= -j \frac{(\sqrt{2h})^{i+j+1}}{(i+j+1)} \int_0^{2\pi} \cos^i\theta \sin^{j-1}\theta d\theta. \end{aligned} \tag{1.68}$$

Analoguously, we find that

$$\int_{\left\{\frac{1}{2}(u^2+v^2)=h\right\}} u^i v^j dv = \frac{i(\sqrt{2h})^{i+j+1}}{(i+j+1)} \int_0^{2\pi} \cos^{i-1}\theta \sin^j\theta d\theta. \tag{1.69}$$

Integrals of the form

$$\int_0^{2\pi} \cos^m\theta \sin^l\theta d\theta$$

vanish if m or l is odd; so we can restrict to the cases

$$\int_{\left\{\frac{1}{2}(u^2+v^2)=h\right\}} u^i v^j du$$

respectively

$$\int_{\left\{\frac{1}{2}(u^2+v^2)=h\right\}} u^i v^j dv,$$

where i is even and j is odd, respectively i odd and j even; as can be seen in (1.68) and (1.69) these integrals are a multiple of an even power of $\sqrt{2h}$, or a multiple of a power of h. Consequently,

$$\int_{\left\{\frac{1}{2}(u^2+v^2)=h\right\}} \upsilon$$

is C^ω in $h=0$. This ends the proof of the lemma in the C^ω case.

In the C^∞ case one can write the 1-form $\upsilon(u,v,\nu)$ as a finite power series up to a term of order $2N+1$ in the origin, i.e.

$$\begin{aligned}\upsilon(u,v,\nu) = &\left(\sum_{0\le i+j\le 2N} a_{ij}(\nu)\,u^i v^j + g_1(u,v,\nu)\right) du \\ &+ \left(\sum_{0\le i+j\le 2N} b_{ij}(\nu)\,u^i v^j + g_2(u,v,\nu)\right) dv,\end{aligned}$$

where a_{ij}, b_{ij}, g_1, g_2 are C^∞ functions with:

$$\begin{cases} g_1(u,v,\nu) = O(\|(u,v)\|^{2N+1}), & \|(u,v)\| \to 0, \\ g_2(u,v,\nu) = O(\|(u,v)\|^{2N+1}), & \|(u,v)\| \to 0. \end{cases} \tag{1.70}$$

So the integral of υ along the circle with radius $\sqrt{2h}$ can be written as the sum of a (finite) linear combination of integrals of the form (1.68), which is a (finite) linear combination of powers of h, and an integral along the circle of a 1-form of order $2N+1$. It turns out that this integral is a function $I(\sqrt{h})$ of order $2N+1$ in $\sqrt{h}$; this implies that $I\circ\sqrt{\cdot}$ is a differentiable function of class C^N in h, for $h\ge 0$. Consequently, the integral $\int_{\{u^2+v^2=2h\}}\upsilon$ is of class C^N in h, for $h\ge 0$. N being arbitrary, we get that $\int_{\{u^2+v^2=2h\}}\upsilon$ is C^∞ in h for $h\ge 0$. ■

1.3.3 Reduced Lyapunov quantities

Suppose that $(X_\lambda)_\lambda$ is a C^∞ (respectively C^ω) family of planar vector fields of type (1.27), unfolding a vector field of center type X_{λ^0}, for which the centers occur on a regular hypersurface. Let Γ denote a non-degenerate elliptic singularity in the origin and let $\{V_i : i\in\mathbb{N}\}$ be a set of Lyapunov quantities for the family $(X_\lambda)_\lambda$, as defined in lemma 47.

As in definition 60, we write $\lambda = (\nu,\varepsilon)\in\mathbb{R}^{p-1}\times\mathbb{R}$ and $\lambda^0 = (\nu^0, 0)$. By assumption (1.52), proposition 61, and theorem 54, the Lyapunov quantities V_i too are divisible by ε^k. This means that there exist C^∞ (respectively C^ω) functions $\bar{V}_i, i\in\mathbb{N}$

with : $V_i = \varepsilon^k \bar{V}_i$. These functions $\bar{V}_i$ are called the *reduced Lyapunov quantities* (or *reduced Lyapunov coefficients* or *reduced focal values*), and they in turn can be used to determine the stability of the focus.

Suppose now that the family $(X_\lambda)_\lambda$ is C^ω; then, the Bautin Ideal is generated by the germ of the C^ω function $(\nu, \varepsilon) \mapsto \varepsilon^k$ at λ^0. The following proposition provides an expression for the relative index in terms of the reduced Lyapunov quantities:

Proposition 64 *Let $\left(X_{(\nu,\varepsilon)}\right)$ be a C^ω family of planar vector fields, where centers occur on a regular hypersurface. Let $\left\{\bar{V}_i : i \in \mathbb{N}\right\}$ be a set of reduced Lyapunov quantities. Then,*

$$\operatorname{Index}\left(X_{(\nu,\varepsilon)}, \left(\Gamma, \left(\nu^0, 0\right)\right)\right) = \inf\left\{m \in \mathbb{N} : \bar{V}_m\left(\nu^0, 0\right) \neq 0\right\}.$$

Proof. As a consequence of Nakayama's lemma (1.20), every set of generators $\{\tilde{\alpha}_1, \ldots, \tilde{\alpha}_M\}$ for the Bautin Ideal contains a minimal set of generators. Suppose j is the smallest positive integer for which $\tilde{\alpha}_j$ generates the Bautin Ideal. From proposition 27, it then follows that j is odd, say $j = 2m+1$. Then we get: $\forall i < m$

$$\left\{\begin{array}{l} \alpha_{2i+1}(\nu, \varepsilon) = \varepsilon^k H_i(\nu, \varepsilon), \\ \alpha_{2m+1}(\nu, \varepsilon) = \varepsilon^k H_m(\nu, \varepsilon), \end{array}\right. \tag{1.71}$$

for C^ω functions H_i with $H_i\left(\nu^0, 0\right) = 0, \forall i < m$ and $H_m\left(\nu^0, 0\right) \neq 0$. From theorem 54 there exists, $\forall i$, a C^ω function f_i with $f_i\left(\nu^0, 0\right) \neq 0$ such that

$$\widetilde{V}_i = \widetilde{f}_i \cdot \widetilde{\alpha}_{2i+1} \bmod \left(\widetilde{\alpha}_1, \ldots, \widetilde{\alpha}_{2i-1}\right)$$

and thus, from (1.71) : $\forall i < m : \bar{V}_i\left(\nu^0, 0\right) = 0$ and $\bar{V}_m\left(\nu^0, 0\right) \neq 0$.

Now we'll show that $Index\left(X_{(\nu,\varepsilon)}, \left(\Gamma, \left(\nu^0, 0\right)\right)\right) = m$. As a property of the Bautin Ideal, there exists a C^ω function h such that $\delta(r, \nu, \varepsilon) = \alpha_{2m+1}(\nu, \varepsilon)\, h(r, \nu, \varepsilon)$. Then the factor function associated to $\widetilde{\alpha}_{2m+1}$ reads $H(r) = h\left(r, \nu^0, 0\right)$ and because of (1.71) it is of the following form:

$$H(r) = r^{2m+1} + o\left(r^{2m+1}\right), \qquad r \to 0.$$

This ends the proof, because:

$$\operatorname{Index}\left(X_{(\nu,\varepsilon)}, \left(\Gamma, \left(\nu^0, 0\right)\right)\right) = \frac{\operatorname{order} H(0) - 1}{2} = m.$$

■

1.4 Simple asymptotic scale deformations

This section is provided for chapter 5, where we study unfoldings X_λ of a 2-saddle cycle leaving 1 connection unbroken. To create a good frame for studying these unfoldings the notions of *'simple asymptotic scale of functions'* and *'simple asymptotic scale deformation'* are introduced (in [DR]). The first notion is appropriate for the study of Abelian integrals I_ν (e.g., the first order Melnikov function) giving rise to a precise definition of codimension for I_ν; the second is appropriate for the study of the unfoldings X_λ, leading to an appropriate definition of codimension of X_λ.

This section is organised as follows. In section 1.4.1, we recall the definition of (simple) asymptotic scale deformation and related notions and properties. In section 1.4.2, we consider examples of simple asymptotic scale deformations, such as the Taylor scale, the (restricted) logarithmic scale and the enlarged logarithmic scale. For each example, we also investigate the meaning of the remainder property. Next, in section 1.4.3, we recall the fact that any Abelian integral along a polycycle admits an asymptotic expansion in the logarithmic scale; moreover, we provide a proof of this result.

In section 1.4.4, we recall the definition of (simple) asymptotic scale deformation, and related notions and properties. Next, in section 1.4.5, we recall the definition and interesting properties of 'compensator' that unfolds the function log. This compensator induces examples of simple asymptotic scale deformations of the (enlarged) logarithmic scale, as we see in section 1.4.6. To study unfoldings of 2-saddle cycles, leaving one connection unbroken, the algebra O is introduced in [DR] to lighten the writing of the building functions the there encountered simple asymptotic scale deformation.

We end this section by recalling the results in the study of the saddle loop from [Mar], that can be derived from results on the related Abelian integral.

1.4.1 Asymptotic scale of functions

In this section we recall precise definitions of the notion 'simple asymptotic scale of functions' and the related notions of 'remainder property', 'asymptotic expansion', 'codimension'. In particular, we recall the proposition that states that finite codimension implies that the maximum number of small zeroes is bounded by the codimension.

We work with the Euler differential operator ∇, since this differential operator has nice calculation rules when repeatedly derivating powers of x, $x\log x, \ldots$, as presented in propositions 80,81,113 and 114.

Definition 65 *The Euler differential operator ∇ is defined by*

$$\nabla = x\frac{\partial}{\partial x}.$$

Asymptotic scale of functions

Definition 66 *A (Chebychev) asymptotic scale of functions is a sequence of functions $\mathcal{F} = \{f_i : i \in \mathbb{N}\}$ such that there exists $h_0 > 0$ with*

1. $f_0 \equiv 1$ *and* $\forall i \in \mathbb{N}$:

2. f_i *is* C^∞ *on* $]0, h_0[\,, f_i$ *is* C^0 *at* 0

3. $f_i(x) \neq 0, \forall x \in \,]0, h_0[$

4. $\frac{f_{i+1}}{f_i}(x) = o(1), x \downarrow 0$

5. $\nabla\left(\frac{f_{i+1}}{f_i}\right)$ *has a constant sign near* 0 *in* $]0, h_0[$.

Definition 67 *A sequence of functions* $\mathcal{F} = \{f_i : i \in \mathbb{N}\}$ *is called a simple (Chebychev) asymptotic scale of functions* $\mathcal{F} = \{f_i : i \in \mathbb{N}\}$ *if each sequence of functions* $\mathcal{F}_j = \left\{f_i^j : i \in \mathbb{N}\right\}, j \in \mathbb{N}$ *inductively defined below, is an asymptotic scale of functions. The sequence* $\mathcal{F}_0$ *is defined by*

$$\mathcal{F}_0 \equiv \mathcal{F}$$

and for $j \in \mathbb{N}$, *if the sequence* $\mathcal{F}_j = \left\{f_i^j : i \in \mathbb{N}\right\}$ *is defined on some interval* $[0, h_j[$, *then the sequence* $\mathcal{F}_{j+1} = \left\{f_i^{j+1} : i \in \mathbb{N}\right\}$ *is defined by:*

$$f_i^{j+1} = \frac{\nabla f_{i+1}^j}{\nabla f_1^j}, \forall i \in \mathbb{N} \tag{1.72}$$

on some interval $[0, h_{j+1}[$ *(This supposes that* $\nabla f_1^j(x) \neq 0$ *for* $x \in [0, h_{j+1}[$*).*

Definition 68 *Let* $\mathcal{F}$ *be a simple asymptotic scale of functions and let* Φ *be a function of class* C^∞, *then we can perform a division-derivation process on* Φ *with respect to* $\mathcal{F}$ *to obtain a sequence of functions* $\Phi^j, j \in \mathbb{N}$ *as follows: put* $\Phi^0 = \Phi$ *and inductively*

$$\Phi^{j+1} = \frac{\nabla \Phi^j}{\nabla f_1^j}, \qquad \forall j \in \mathbb{N}$$

We say that the sequence $\Phi^j, j \in N$ *is defined by the division-derivation process on* Φ *with respect to* $\mathcal{F}$.

In section we give examples of simple asymptotic scales of functions. Let us here already that the Taylor scale $\mathcal{T}$ and the logarithmic scale $\mathcal{L}$ are simple asymptotic scales of functions.

Remark 69 *A simple asymptotic scale is an asymptotic scale which generates infinitely many new asymptotic scales by the inductive application of the algorithm of derivation-division as in definition 68 (derivation by* ∇ *followed by the division by the first function of the sequence).*

Remainder property and expansion in an asymptotic scale

Here, we recall the definition and some elementary properties of remainder property of order N with respect to a given simple asymptotic scale of functions. Next, we recall the definition of asymptotic expansion of a function f in a given asymptotic scale $\mathcal{F}$ of functions and the related notion 'codimension of the function with respect to $\mathcal{F}$'. We end this section by proving that the codimension bounds the maximum number of small zeroes that can be created after small perturbation of the function.

Definition 70 *Let $\mathcal{F}$ be a simple asymptotic scale of functions, let Φ be a function of class C^∞ and define the sequence $\Phi^j, j \in N$ by the division-derivation process on Φ with respect to $\mathcal{F}$.*

1. *Then we say that Φ satisfies the* remainder property of order N with respect to $\mathcal{F}$ *or* (R_N)-property with respect to $\mathcal{F}$ *if and only if*

$$\frac{\Phi^j}{f^j_{N-j}} \to 0 \text{ if } x \to 0, \qquad \forall j = 0, 1, \ldots, N \tag{1.73}$$

2. *in case that Φ depends on a parameter λ, then the limits in* (1.73) *have to be uniform in λ.*

Remark 71 *The remainder property depends on the chosen scale $\mathcal{F}$. Consider for instance the function $\Phi(x) = x$. Then, Φ satisfies the (R_1)-property with respect to $\mathcal{L}$, although Φ merely satisfies the (R_0)-property with respect to $\mathcal{T}$.*

Proposition 72 *Let $\mathcal{F} = \{f_i : i \in \mathbb{N}\}$ be a simple asymptotic scale of functions, let Φ, Φ_1, Φ_2 be functions of class C^∞ that satisfy the remainder property of order N with respect to $\mathcal{F}$. Then,*

1. *$\Phi_1 + \Phi_2$ satisfy the remainder property of order N with respect to $\mathcal{F}$.*

2. *Φ satisfies the remainder property of order k with respect to $\mathcal{F}, \forall k \leq N$.*

3. *Φ^k satisfies the remainder property of order $N - k$ with respect to $\mathcal{F}^k$, $\forall k \leq N$.*

4. *f_k satisfies the remainder property of order N with respect to $\mathcal{F}, \forall k \in \mathbb{N}, k \geq N + 1$.*

Proof. To prove the first statement, we denote $\Psi = \Phi_1 + \Phi_2$ and we remark that the sequence $\Psi^j, j \in \mathbb{N}$, defined by the division-derivation process on Ψ by $\mathcal{F}$, is given by

$$\Psi^j = \Phi_1^j + \Phi_2^j, \forall j \in \mathbb{N}$$

To prove the second statement, we remark that

$$\frac{\Phi^j}{f^j_{k-j}} = \frac{\Phi^j}{f^j_{N-j}} \cdot \frac{f^j_{N-j}}{f^j_{k-j}}$$

and

$$\frac{f^j_{N-j}}{f^j_{k-j}} = \frac{f^j_{N-j}}{f^j_{N-1-j}} \cdot \frac{f^j_{N-1-j}}{f^j_{N-2-j}} \cdot \ldots \cdot \frac{f^j_{k+1-j}}{f^j_{k-j}}$$

In the right-hand side of this equation, each of the fractions tends to 0 when $x \to 0$, since $\mathcal{F}^j$ is an asymptotic scale of functions. For the third statement, we denote $\Psi = \Phi^k$ and we remark that the sequence $\Psi^j, j \in \mathbb{N}$, defined by the division-derivation process on Ψ by $\mathcal{F}$, is given by

$$\Psi^j = \Phi^{k+j}, \forall j \in \mathbb{N}$$

To prove the fourth statement, we denote $\Phi = f_k$, and remark that $\forall 0 \leq j \leq k$:

$$\Phi^j = f^j_{k-j}$$

and as a consequence, $\forall 0 \leq j \leq N$:

$$\frac{\Phi^j}{f^j_{N-j}} = \frac{f^j_{k-j}}{f^j_{N-j}}$$

The fraction on the right-hand side of this equation tends to 0 as $x \to 0$, since is $\mathcal{F}$ is a simple asymptotic scale of functions and $k \geq N+1$. ■

Definition 73 *Let $\mathcal{F} = \{f_i : i \in \mathbb{N}\}$ be a simple asymptotic scale of functions and let f be a function of class C^∞. We say that*

1. *f has an expansion in $\mathcal{F}$ of order N if there exist $\alpha_i, 0 \leq i \leq N$ (constants or functions depending smoothly on λ) and a function Φ_{N+1} satisfying the remainder property of order N, such that*

$$f(x) = \sum_{i=0}^{N} \alpha_i f_i(x) + \Phi_{N+1}(x)$$

2. *f is asymptotic to the series*

$$\hat{f}(x) = \sum_{i=0}^{\infty} \alpha_i f_i(x), x \to 0$$

with respect to $\mathcal{F}$ if for all $N \in \mathbb{N}$, the function Φ_{N+1}, defined by

$$\Phi_{N+1}(x) = f(x) - \sum_{i=0}^{N} \alpha_i f_i(x),$$

satisfies the remainder property of order N with respect to $\mathcal{F}$.

Taylor's theorem states that every C^∞ function can be expanded asymptotically in powers of x, the Taylor scale $\mathcal{T}$. We will see in section 1.4.3 that any Abelian integral can be expanded in the logarithmic scale $\mathcal{L}$. If the scale $\mathcal{F}$ in definition 73 is the enlarged logarithmic scale $\mathcal{L}^e$ (respectively the logarithmic scale $\mathcal{L}$), then the series $\hat{f}$ is called a Dulac series (respectively a Dulac series, linear in $\log x$).

As a consequence of the remainder property, we have the following proposition:

Proposition 74 *Let $\mathcal{F} = \{f_i : i \in \mathbb{N}\}$ be a simple asymptotic scale of functions and let f be a function of class C^∞. Suppose that f is asymptotic to the series*

$$\hat{f}(x) = \sum_{i=0}^{\infty} \alpha_i f_i(x), x \to 0$$

with respect to $\mathcal{F}$. Then, for all $N \in \mathbb{N}$ with $\alpha_{N+1} \neq 0$:

$$f(x) - \sum_{i=0}^{N} \alpha_i f_i(x) = \alpha_{N+1} f_{N+1}(1 + o(1)), x \to 0$$

Definition 75 *Let $\mathcal{F} = \{f_i : i \in \mathbb{N}\}$ be a simple asymptotic scale of functions and let f be a function of class C^∞. Suppose that f is asymptotic to the series*

$$\hat{f}(x) = \sum_{i=0}^{\infty} \alpha_i f_i(x), x \to 0$$

with respect to $\mathcal{F}$. Then we say that

1. *f has a finite codimension with respect to $\mathcal{F}$ if not all the coefficients α_i are zero.*

2. *f has codimension l at $x = 0$ in the scale $\mathcal{F}$ if*

$$\alpha_0 = \alpha_1 = \ldots = \alpha_{l-1} = 0 \text{ and } \alpha_l \neq 0$$

and we write $codim_{\mathcal{F}} f = l$

3. *If f depends on a parameter λ, in a neighbourhood of $\lambda = \bar{0}$, then we define the codimension of f at $x = 0, \lambda = \bar{0}$ in the scale $\mathcal{F}$ by*

$$\operatorname{codim}_{\mathcal{F}} f = \operatorname{codim}_{\mathcal{F}} f_\lambda = \operatorname{codim}_{\mathcal{F}} f_{\bar{0}},$$

where $f_\lambda = f(\cdot, \lambda)$.

Remark 76 *The notion of codimension depends on the asymptotic scale that one considers. It may also happen that f_0 expands in some scale, but that f_0 depends on a larger scale; then we shall define the codimension in this larger scale. For instance, let f_λ be given by*

$$f_\lambda(x) = \lambda_1 + \lambda_2 x L + x$$

hence $codim_{\mathcal{L}} f = 2$, while $codim_{\mathcal{T}} f_0 = 1$.

Proposition 77 *Let $\mathcal{F} = \{f_i : i \in \mathbb{N}\}$ be a simple asymptotic scale of functions and let f be a function of class C^∞. Suppose that $codim_{\mathcal{F}} f = l < \infty$, then $\forall 0 \leq j \leq l$:*

$$\mathrm{codim}_{\mathcal{F}^j} f^j = l - j$$

Proof. This follows immediately from the following observation: if f is asymptotic to the series

$$\hat{f}(x) = \sum_{i=0}^{\infty} \alpha_i f_i(x), x \to 0$$

then f^j is asymptotic to the series

$$\hat{f^j}(x) = \sum_{i=0}^{\infty} \alpha_i^j f_i^j(x), x \to 0$$

where $\alpha_i^j = \alpha_{i+j}$. ■

Definition 78 *Let f_λ be an unfolding of f_{λ_0}. Then the cyclicity of of an isolated root x_0 of f_{λ_0} at $\lambda = \lambda_0$ is defined as the number*

$$\mathrm{Cycl}(f_\lambda, (x_0, \lambda_0)) = \underset{\lambda \to \lambda_0, x \to x_0}{limsup} \{\textit{number of isolated zeroes } x \textit{ of } f_\lambda\}$$

The cyclicity of an unfolding f_λ that expands in a simple asymptotic scale is related to its codimension as follows:

Proposition 79 *Let $\mathcal{F} = \{f_i : i \in \mathbb{N}\}$ be a simple asymptotic scale of functions and let f be a function of class C^∞ with finite codimension. Then,*

$$\mathrm{Cycl}(f_\lambda, 0) \leq codim_{\mathcal{F}} f_\lambda$$

Proof. Suppose that $\mathrm{codim}_{\mathcal{F}} f_\lambda = l$. Then, we can write

$$f = \sum_{i=0}^{l} \alpha_i f_i + R_{l+1}$$

where $\alpha_0(\bar{0}) = \ldots = \alpha_{l-1}(\bar{0}) = 0$ and $\alpha_l(\bar{0}) \neq 0$ and the function R_{l+1} satisfies the remainder property of order l with respect to the scale $\mathcal{F}$. Let the sequences of C^∞ functions f^j and R_{l+1}^j $(0 \leq j \leq l)$ be defined by the division-derivation process on f respectively R_{l+1} with respect to $\mathcal{F}$. Then, we have the following relation,$\forall 0 \leq j \leq l$:

$$f^j = \sum_{i=j}^{l} \alpha_i f_{i-j}^j + R_{l+1}^j$$

Since $f_0^j \equiv 1$, we obtain in the division-derivation with l derivations:

$$\begin{aligned} f^l &= \alpha_l + R_{l+1}^l \\ &= \alpha_l + o(1), x \downarrow 0 \end{aligned}$$

by the remainder property. Hence, after l derivations (alternated by divisions by a non-zero function), we obtain a function f^l, that does not vanish at a sufficiently small neighbourhood of $x = 0, x > 0$ and $\lambda = 0$. Rolle's theorem then implies that f_λ has at most l zeroes in a neighbourhood of $0 \in]0, h_0]$ and for λ sufficiently close to $\bar{0}$. Consequently,

$$\mathrm{Cycl}(f_\lambda, 0) \leq l = \mathrm{codim}_{\mathcal{F}} f_\lambda$$

■

1.4.2 Examples of simple asymptotic scales of functions

In this section, we first present some interesting calculation rules of the Euler differential operator ∇. Next, we consider some examples of simple asymptotic scales of functions: the Taylor scale, the traditional and restricted logarithmic scale, $\mathcal{L}$ and $\mathcal{L}^*$, and the enlarged logarithmic scale $\mathcal{L}^e$.

Proposition 80 *Let $m, p, q \in \mathbb{Z}, i_1, i_2, j \in \mathbb{N}$. Denote $L = \log x$. Then,*

1. $\nabla x^m = mx^m$
2. $\nabla L^m = mL^{m-1}$; *in particular,* $\nabla L = 1$
3. $\nabla (x^p L^q) = x^p L^q (p + qL^{-1})$
4. *There exist $i \in \mathbb{N}, A_k \in \mathbb{R}$ $(2 \leq k \leq i)$ such that*

$$\nabla \left(\left(\sum_{k=0}^{i_1} a_k L^{-k} \right) \left(\sum_{l=0}^{i_2} b_l L^{-l} \right)^{-j} \right) = \left(\sum_{k=2}^{i} A_k L^{-k} \right) \left(\sum_{l=0}^{i_2} b_l L^{-l} \right)^{-j-1}$$

This property implies that the set of rational functions in L^{-1} is invariant under the differential operator ∇.

For later use, we give some calculation rules for functions of a special type.

Proposition 81 *Let us denote by g, g_1, g_2 any rational function in L^{-1}, with denominator $1 + o(1), x \downarrow 0$, i.e. any function of the following type:*

$$\left(\sum_{k=0}^{i_1} a_k L^{-k} \right) \left(\sum_{l=0}^{i_2} b_l L^{-l} \right)^{-j}, b_0 \neq 0 \tag{1.74}$$

where $j \in \mathbb{N}, j \geq 1, a_k, b_l \in \mathbb{R}$ $(\forall 0 \leq k \leq i_1, \forall 0 \leq l \leq i_2)$. Then

1. $g_1 \cdot g_2 = g$
2. $g_1 + L^{-s} g_2 = g$ $(\forall s \in \mathbb{N}, s \geq 1)$
3. $\nabla \left(x^l L^k g \right) = x^l L^k g_1$ $(\forall l, k \in \mathbb{Z}, l \neq 0)$
4. $\nabla \left(L^k g \right) = L^{k-1} g_1$ $(k \in \mathbb{Z}, k \neq 0)$
5. *Moreover, if for the functions g (respectively g_1 and g_2) in the left-hand side in the equation also the coefficient a_0 is non-zero, then the same holds for the function g_1 (respectively g) in the right-hand side.*

Taylor scale $\mathcal{T}$

The sequence $\mathcal{T} = \{x^i : i \in \mathbb{N}\}$ is called the *Taylor scale*. If we define $\mathcal{T}_0 = \mathcal{T}$ and if for $j \geq 1$, we set $\mathcal{T}_j = \left\{ f_i^j : i \in \mathbb{N} \right\}$, and define $\mathcal{T}_{j+1}$ inductively by (1.72), then we find by induction on j explicit expressions for the functions f_i^j $(i, j \in \mathbb{N}, j \geq 1)$:

$$f_i^j(x) = \frac{(i+1)(i+2)\cdot \ldots \cdot (i+j)}{j!} x^i \tag{1.75}$$

From this formula, it is seen that $\mathcal{T}$ is a simple asymptotic scale of functions.

The remainder property with respect to the Taylor scale coincides with the notion of remainder that is used in considering Taylor polynomials:

Proposition 82 *Suppose that the function $\Phi : \mathbb{R} \to \mathbb{R}$ is C^∞, and let $\mathcal{T}$ denote the Taylor scale. Then, Φ satisfies the remainder property of order N if and only if $\Phi(x) = o\left(x^N\right), x \to 0$.*

Proof. As we calculated before in (1.75), we have

$$f_1^j = (j+1)\, x$$

As a consequence, the sequence $\Phi^j, 0 \leq j \leq N$, defined by the division-derivation process on Φ with respect to $\mathcal{T}$ is given by

$$\Phi^j = \frac{1}{j!} \cdot \frac{\partial^j}{\partial x^j} \Phi, \forall 0 \leq j \leq N.$$

By (1.75), we have that

$$\frac{\Phi^j}{f_{N-j}^j} = \frac{1}{(N-j+1)(N-j+2)\cdot \ldots \cdot N} \cdot \frac{\frac{\partial^j}{\partial x^j}\Phi}{x^{N-j}}, \forall 0 \leq j \leq N$$

As a consequence of Taylor's theorem the fact $\Phi(x) = o\left(x^N\right), x \to 0$ holds if and only if Φ can be written locally as

$$\Phi(x) = x^{N+1} g(x), x \to 0$$

for a certain C^∞ function g; as a consequence, there exist C^∞ functions g^j such that

$$\frac{\partial^j}{\partial x^j}\Phi(x) = x^{N+1-j} g^j(x), x \to 0.$$

These remarks end the proof. ■

Logarithmic scale $\mathcal{L}$

The sequence $\mathcal{L} = \{1, xL, x, x^2L, x^2, \ldots, x^iL, x^i, \ldots\}$ is called the *logarithmic scale*, where we use the shortened notation

$$L = \log x.$$

To show that the logarithmic scale is a simple asymptotic scale, we now compute by induction on $j \in \mathbb{N}$ explicit expressions for the functions $f_i^j, i \in \mathbb{N}$ in the sequence $\mathcal{L}_j$. From these sequences, it is easy to verify the conditions of definition 66 for an asymptotic scale of functions. By definition

$$\mathcal{L}_0 = \mathcal{L}$$

For $j = 1$, we find

$$\begin{aligned}\mathcal{L}_1 = \Big\{&1, L^{-1}\left(1+L^{-1}\right)^{-1}, x\left(1+L^{-1}\right)^{-1}\left(2+L^{-1}\right), 2xL^{-1}\left(1+L^{-1}\right)^{-1},\\ &\ldots, x^i\left(1+L^{-1}\right)^{-1}\left(i+1+L^{-1}\right), (i+1)x^iL^{-1}\left(1+L^{-1}\right)^{-1}, \ldots; i\in\mathbb{N}\Big\}\end{aligned}$$

For $j = 2$, we find

$$\begin{aligned}\mathcal{L}_2 = \{&1, -xL^2\left(2+2L^{-1}+2L^{-2}\right), -2xL, \ldots,\\ &-(i-1)x^{i-1}L^2\left(i+iL^{-1}+2L^{-2}\right), -ix^iL\left(i+(i-1)L^{-1}\right), \ldots; i \geq 2\}\end{aligned}$$

We go on by induction on j, and prove that $\forall j \in \mathbb{N}_1$:

$$\mathcal{L}_{2j-1} = \left\{1, L^{-1}g_1^{2j-1}, xg_2^{2j-1}, \ldots, x^{i-1}L^{-1}g_{2i-1}^{2j-1}, x^ig_{2i}^{2j-1}, \ldots; i \geq 1\right\} \quad (1.76)$$

and

$$\mathcal{L}_{2j} = \left\{1, xL^2g_1^{2j}, xLg_2^{2j}, \ldots, x^iL^2g_{2i-1}^{2j}, x^iLg_{2i}^{2j}, \ldots; i \geq 1\right\}, \quad (1.77)$$

where the functions g_i^k are rational in L^{-1}; moreover, g_i^k can be written as $T_i^k\left(L^{-1}\right)/N^k\left(L^{-1}\right)$ for certain polynomials T_i^k, N^k with the property that $T_i^k(0)\,N^k(0) \neq 0$. Let us write this more explicitly, g_i^k can be written as

$$\left(\sum_{s=0}^{m(i)} c_s^{k,i}L^{-s}\right)\left(\sum_{s=0}^{m(i)} a_s^jL^{-s}\right)^{-1} \text{ with } a_0^jc_0^{k,i} \neq 0 \quad (1.78)$$

Suppose that we have computed $\mathcal{L}_{2j-1}$ and $\mathcal{L}_{2j}$ and find (1.76) and (1.77) respectively. Starting from $\mathcal{L}_{2j}$, we now compute $\mathcal{L}_{2j+1}$. Suppose that g_i^{2j} is defined by (1.78), then $\forall i \in \mathbb{N}_1$,

$$\begin{aligned}
\nabla f_{2i}^{2j} &= \nabla\left(x^i L g_{2i}^{2j}\right) = \nabla\left(x^i L\right)\cdot g_{2i}^{2j} + x^i L \cdot \nabla g_{2i}^{2j} \\
&= x^i L \left(\sum_{k=0}^{m} a_k^{2j} L^{-k}\right)^{-2} \cdot \left[\left(i + L^{-1}\right)\sum_{k=0}^{m} a_k^{2j} L^{-k} \sum_{k=0}^{m} c_k^{2j,2i} L^{-k}\right. \\
&\left. + \sum_{k=2}^{m(m+1)} A_k^{2j,2i} L^{-k}\right] \\
&= x^i L \left(\sum_{k=0}^{m} a_k^{2j} L^{-k}\right)^{-2} \sum_{k=0}^{n} \upsilon_k^{2j+1,2i-1} L^{-k}
\end{aligned}$$

where $\upsilon_0^{2j+1,2i-1} = i a_0^{2j} c_0^{2j,2i} \neq 0$. Suppose that g_{2i-1}^{2j} is defined by (1.78), then we find analoguously, $\forall i \in \mathbb{N}_1$:

$$\nabla f_{2i-1}^{2j} = \nabla\left(x^i L^2 g_{2i-1}^{2j}\right) = x^i L^2 \left(\sum_{k=0}^{m} a_k^{2j} L^{-k}\right)^{-2} \sum_{k=0}^{n} \upsilon_k^{2j+1,2i-2} L^{-k}$$

where $\upsilon_0^{2j+1,2i-2} = i a_0^{2j} c_0^{2j,2i-1} \neq 0$. Since $f_1^{2j} = x L^2 g_1^{2j}$, we obtain

$$\begin{cases}
f_{2i-2}^{2j+1} = \dfrac{\nabla f_{2i-1}^{2j}}{\nabla f_1^{2j}} = x^{j-1} g_{2i-2}^{2j+1} \\
f_{2i-1}^{2j+1} = \dfrac{\nabla f_{2i}^{2j}}{\nabla f_1^{2j}} = x^{j-1} L^{-1} g_{2i-1}^{2j+1}
\end{cases}$$

where

$$\begin{cases}
g_{2i-2}^{2j+1} = \left(\sum_{k=0}^{n} \upsilon_k^{2j+1,2i-2} L^{-k}\right)\left(\sum_{k=0}^{m} \upsilon_k^{2j+1,0} L^{-k}\right)^{-1} \\
g_{2i-1}^{2j+1} = \left(\sum_{k=0}^{n} \upsilon_k^{2j+1,2i-1} L^{-k}\right)\left(\sum_{k=0}^{m} \upsilon_k^{2j+1,0} L^{-k}\right)^{-1}
\end{cases}$$

This is the suggested form for the functions in $\mathcal{L}_{2j+1}$. From these functions, we can compute the sequence $\mathcal{L}_{2j+2}$, and find the suggested formulas, in the same way as we did previously to calculate the functions of the sequence $\mathcal{L}_{2j+1}$. This proves the induction.

Definition 83 *Let f be a C^∞ function that can be written as*

$$f = x^i L^j g$$

where $i, j \in \mathbb{Z}$ and g takes the form (1.78)*. Then we will call $x^i L^j$ the principal part of f.*

For instance, the sequence of the principal parts of the functions of $\mathcal{L}_k, k \in \mathbb{N}$ are given by

$$\mathcal{L}_0 = \left\{1, xL, x, x^2L, \ldots, x^iL, x^i, \ldots\right\}, i \in \mathbb{N}$$

and $\forall j \in \mathbb{N}_1$:

$$\mathcal{L}_{2j-1} = \left\{1, L^{-1}, x, \ldots, x^{i-1}L^{-1}, x^i, \ldots\right\}, i \geq 1$$

and

$$\mathcal{L}_{2j} = \left\{1, xL^2, xL, \ldots, x^iL^2, x^iL, \ldots\right\}, i \geq 1.$$

Next, we exploit the remainder property with respect to the logarithmic scale. A sufficient condition is that the remainder is flat of some order at $x = 0$. To arrive at this result, we first need some technical lemma's.

Lemma 84 *Suppose that the function Φ is C^∞ and let $\mathcal{L}$ denote the logarithmic scale. Let $\Phi^j, j \in \mathbb{N}$ denote the sequence of functions defined by the division-derivation process on Φ with respect to $\mathcal{L}$. Then $\Phi^0 = \Phi$ and $\forall j \geq 1$, Φ^j can be written as*

$$\Phi^j = \sum_{r=1}^{j} G_r^j \frac{\partial^r}{\partial x^r} \Phi,$$

where G_r^j are functions of the following type (depending on j odd or even):

$$\left\{ \begin{array}{l} G_r^{2i} = x^{-i+r} L g_r^{2i}, \forall 1 \leq r \leq 2i \\ G_r^{2i+1} = x^{-i+r-1} L^{-1} g_r^{2i+1}, \forall 1 \leq r \leq 2i+1 \end{array} \right. \tag{1.79}$$

where g_r^j are functions of the type

$$\left(\sum_{k=0}^{m} c_k L^{-k}\right)\left(\sum_{k=0}^{m} a_k L^{-k}\right)^{-1} \quad \textit{with } a_0 \neq 0 \textit{ (and } c_0 \neq 0 \textit{ in case } r = j\textit{)}$$

First we check the property for Φ^1 and Φ^2 :

$$\Phi^1 = \frac{\nabla \Phi^0}{\nabla f_1^0} = g_1^1 \frac{\partial}{\partial x} \Phi^0$$

with

$$G_1^1 = L^{-1}\left(1 + L^{-1}\right)^{-1} = L^{-1} g_1^1$$

and

$$\Phi^2 = \frac{\nabla \Phi^1}{\nabla f_1^1} = G_1^2 \frac{\partial}{\partial x} \Phi + G_2^2 \frac{\partial^2}{\partial x^2} \Phi$$

with

$$\left\{ \begin{array}{l} G_1^2 = -L^2\left(1 + L^{-1}\right)^{-2} x \frac{\partial}{\partial x} G_1^1 = g_1^2 \\ G_2^2 = -L^2\left(1 + L^{-1}\right)^{-2} x G_1^1 = x L g_2^2 \end{array} \right.$$

Since Φ^{j+1} is defined as $\Phi^{j+1} = \dfrac{\nabla\Phi^j}{\nabla f_1^j}$, we find the following recursion formulas for the functions G_r^{j+1}, $r, j \in \mathbb{N}, j \geq 2, 1 \leq r \leq j+1$:

$$\left\{ \begin{array}{lcl} G_1^{j+1} & = & C_j \cdot \nabla G_1^j \\ G_r^{j+1} & = & C_j \cdot \left(x G_{r-1}^j + \nabla G_r^j \right), \forall 2 \leq r \leq j \\ G_{j+1}^{j+1} & = & C_j \cdot x G_{j+1}^{j+1} \end{array} \right. \tag{1.80}$$

where

$$C_j = \left\{ \begin{array}{ll} x^{-1} L^{-2} g_1^j & \text{if } j \text{ is odd} \\ L^2 g_1^j & \text{if } j \text{ is even} \end{array} \right.$$

with g_1^j defined by (1.78). Suppose that we already found the desired expressions (1.79) for $j = 2i$, then we prove the validity of these expressions for $j = 2i+1$ and $j = 2i+2$. Using the calculation rules given in proposition 81 and the recursion formulas (1.80), we find

$$\begin{aligned} G_1^{2i+1} &= x^{-1} L^{-2} g_1^{2i} \nabla g_1^{2i} \\ &= x^{-i} L^{-1} g_1^{2i+1} \end{aligned}$$

and for all $2 \leq r \leq 2i$:

$$\begin{aligned} G_r^{2i+1} &= x^{-1} L^{-2} g_1^{2i} \left(x G_{r-1}^{2i} + \nabla G_r^{2i} \right) \\ &= x^{-1} L^{-2} g_1^{2i} x^{-i+r} L \left(g_{r-1}^{2i} + \bar{g}_r^{2i} \right) \\ &= x^{-i+r-1} L^{-1} g_r^{2i+1} \end{aligned}$$

and

$$\begin{aligned} G_{2i+1}^{2i+1} &= x^{-1} L^{-2} g_1^{2i} x G_{2i}^{2i} \\ &= x^{i} L^{-1} g_{2i+1}^{2i+1} \end{aligned}$$

This already proves that the formula is valid for $j = 2i+1$. Now, we check the formulas for $j = 2i+2$:

$$\begin{aligned} G_1^{2i+2} &= L^2 g_1^{2i+1} \nabla G_1^{2i+1} \\ &= x^{-i} L g_1^{2i+2} \end{aligned}$$

and for all $2 \leq r \leq 2i+1$:

$$\begin{aligned} G_r^{2i+2} &= L^2 g_1^{2i+1} \left(x G_{r-1}^{2i+1} + \nabla G_r^{2i+1} \right) \\ &= L^2 g_1^{2i+1} x^{-i+r-1} L^{-1} \left(g_{r-1}^{2i+1} + \bar{g}_r^{2i+1} \right) \\ &= x^{-(i+1)+r} L g_r^{2i+2} \end{aligned}$$

and

$$\begin{aligned} G_{2i+2}^{2i+2} &= L^2 g_1^{2i+1} x G_{2i+1}^{2i+1} \\ &= x^{i+1} L g_{2i+2}^{2i+2} \end{aligned}$$

This ends the induction step; hence the proof is finished.

Lemma 85 *Denote the logarithmic scale by* $\mathcal{L} = \{f_i^0 : i \in \mathbb{N}\}$ *and* $\mathcal{L}_j = \left\{f_i^j : i \in \mathbb{N}\right\}, \forall j \geq 1$. *Let* N *be a positive integer. The sequence of functions* f_{N-j}^j *is given by:*

1. *in case* N *is odd* $(N = 2n-1)$,

$$\begin{cases} f_{2n-1}^0 = x^n L \\ f_{(2n-1)-(2j-1)}^{2j-1} = f_{2(n-j)}^{2j-1} = x^{n-j} L^{-1} g_{2n-2j}^{2j-1}, \forall 1 \leq j \leq n \\ f_{(2n-1)-2j}^{2j} = f_{2(n-j)}^{2j-1} = x^{n-j} L^2 g_{2n-2j-1}^{2j}, \forall 1 \leq j \leq n-1 \\ f_0^{2n-1} = 1 \end{cases}$$

2. *in case* N *is even* $(N = 2n)$,

$$\begin{cases} f_{2n}^0 = x^n \\ f_{2n-(2j-1)}^{2j-1} = f_{2(n-j+1)-1}^{2j-1} = x^{n-j} L^{-1} g_{2n-2j+1}^{2j-1}, \forall 1 \leq j \leq n \\ f_{2n-2j}^{2j} = f_{2(n-j)}^{2j} = x^{n-j} L g_{2n-2j}^{2j}, \forall 1 \leq j \leq n-1 \\ f_0^{2n} = 1 \end{cases}$$

where the functions g_i^j *are of type* (1.74), $\forall i, j \in \mathbb{N}$.

As a consequence of lemmas 84 and 85, we obtain the following lemma that is helpful in checking the remainder property of order N with respect to the logarithmic scale:

Lemma 86 *Let be given the logarithmic scale* $\mathcal{L}$ *and a* C^∞ *function* Φ. *Let* $\Phi^j, j \in \mathbb{N}$ *denote the sequence of functions defined by the division-derivation process on* Φ *with respect to* $\mathcal{L}$. *Then, the fractions* $\dfrac{\Phi^j}{f_{N-j}^j}$ *are given by*

1. *in case N is odd* $(N = 2n-1)$,

$$\begin{cases} \dfrac{\Phi^0}{f_N^0} = x^{-n} L^{-1} \Phi \\ \dfrac{\Phi^{2j-1}}{f_{N-(2j-1)}^{2j-1}} = \displaystyle\sum_{i=1}^{2j-1} \bar{g}_i^{2j-1} \cdot x^{i-n} \cdot \frac{\partial^i}{\partial x^i} \Phi, \forall 1 \leq j \leq n-1 \\ \dfrac{\Phi^{2j}}{f_{N-2j}^{2j}} = \displaystyle\sum_{i=1}^{2j} \bar{g}_i^{2j} \cdot x^{i-n} L^{-1} \cdot \frac{\partial^i}{\partial x^i} \Phi, \forall 1 \leq j \leq n-1 \\ \dfrac{\Phi^N}{f_0^N} = \displaystyle\sum_{i=1}^{N} \bar{g}_i^N \cdot x^{i-n} L^{-1} \cdot \frac{\partial^i}{\partial x^i} \Phi \end{cases}$$

2. *in case N is even* $(N = 2n)$,

$$\begin{cases} \dfrac{\Phi^0}{f_N^0} = \dfrac{\Phi}{x^n} \\ \dfrac{\Phi^{2j-1}}{f_{N-(2j-1)}^{2j-1}} = \displaystyle\sum_{i=1}^{2j-1} \bar{g}_i^{2j-1} \cdot x^{i-n} \cdot \frac{\partial^i}{\partial x^i} \Phi, \forall 1 \leq j \leq n \\ \dfrac{\Phi^{2j}}{f_{N-2j}^{2j}} = \displaystyle\sum_{i=1}^{2j} \bar{g}_i^{2j} \cdot x^{i-n} \cdot \frac{\partial^i}{\partial x^i} \Phi, \forall 1 \leq j \leq n-1 \\ \dfrac{\Phi^N}{f_0^N} = \displaystyle\sum_{i=1}^{N} \bar{g}_i^N \cdot x^{i-n} L \cdot \frac{\partial^i}{\partial x^i} \Phi \end{cases}$$

where the functions $\bar{g}_j^i$ are of type (1.74), $\forall i, j \in \mathbb{N}$.

Corollary 87 *Let be given the logarithmic scale $\mathcal{L}$ and let Φ be a C^∞ function. Let be $N = 2n-1$ or $N = 2n$. If Φ is flat of the order $n+1$ at $x = 0$, then Φ satisfies the remainder property of order N with respect to $\mathcal{L}$. Moreover, it is sufficient that $\exists 0 < \delta < 1$ with*

$$\Phi(x) = O\left(x^{n+1-\delta}\right), x \downarrow 0.$$

Proof. From Lemma 86, it follows that a sufficient condition for Φ to satisfy the remainder property of order N with respect to $\mathcal{L}$, is given by

$$\frac{\partial^i}{\partial x^i} \Phi(x) = O\left(x^{n-i+1-\delta}\right), x \downarrow 0, \qquad \forall 0 \leq i \leq N \tag{1.81}$$

for a certain $0 < \delta < 1$. If

$$\Phi(x) = O\left(x^{n+1-\delta}\right), x \downarrow 0$$

then the asymptotics for the derivatives of Φ in (1.81) are satisfied. ∎

Restricted logarithmic scale $\mathcal{L}^*$

In chapter 5 we encounter an asymptic scale of functions that is almost the logarithmic scale:

$$\mathcal{L}^* = \left\{1, x, x^2L, x^2, \ldots, x^iL, x^i, \ldots\right\}, \text{ where } L = \log x \tag{1.82}$$

the function xL is absent to be the logarithmic scale $\mathcal{L}$. We will call this sequence $\mathcal{L}^*$ the *restricted logarithmic scale.* Analoguously as for the logarithmic scale, we find subsequently by the derivation-division process the sequences

$$\mathcal{L}_0^* = \mathcal{L}^*$$

For $j = 1$, we find

$$\mathcal{L}_1^* = \left\{1, xL\left(2 + L^{-1}\right), 2x, x^2L\left(3 + L^{-1}\right), 3x^2, \ldots \right.$$
$$\left. \ldots, x^iL\left(i + 1 + L^{-1}\right), (i+1)\, x^i, \ldots; i \in \mathbb{N}\right\}$$

For $j \geq 1$, we find by induction on j the structure of the sequences $\mathcal{L}_{2j}^*$ and $\mathcal{L}_{2j+1}^*$

$$\mathcal{L}_{2j}^* = \left\{1, L^{-1}g_1^{2j}, xg_2^{2j}, \ldots, x^{i-1}L^{-1}g_{2i-1}^{2j}, x^ig_{2i}^{2j}, \ldots; i \geq 1\right\}$$

We go on by induction on j, and prove that $\forall j \in \mathbb{N}_1$:and

$$\mathcal{L}_{2j+1}^* = \left\{1, xL^2g_1^{2j+1}, xLg_2^{2j+1}, \ldots, x^iL^2g_{2i-1}^{2j+1}, x^iLg_{2i}^{2j+1}, \ldots; i \geq 1\right\}$$

where the functions g_i^k are rational functions in L^{-1}, as presented in (1.78).

Analoguously as for the logarithmic scale, a sufficient condition to satisfy the remainder property with respect to $\mathcal{L}^*$ can be expressed by a certain degree of flatness at $x = 0$:

Corollary 88 *Let be given the restricted logarithmic scale $\mathcal{L}^*$ and let Φ be a C^∞ function. Let be $N = 2n - 2$ or $N = 2n - 1$. If Φ is flat of the order $n + 1$ at $x = 0$, then Φ satisfies the remainder property of order N with respect to $\mathcal{L}^*$. Moreover, it is sufficient that $\exists 0 < \delta < 1$ with*

$$\Phi(x) = O\left(x^{n+1-\delta}\right), x \downarrow 0.$$

Enlarged logarithmic scale $\mathcal{L}^e$

An example of the enlarged logarithmic scale is given by the sequence $\mathcal{L}^{Se}$:

$$\mathcal{L}^{Se} = \left\{1, xL, x, x^2L^2, x^2L, x^2, \ldots, x^iL^i, x^iL^{i-1}, \ldots, x^iL, x^i, \ldots\right\}, \text{ where } L = \log x$$

and is called the *Standard enlarged logarithmic scale.* Notice that this sequence is ordered by the following lexicographic order on $\mathbb{N} \times \mathbb{N}$: $(i', j') \succ (i, j)$ if and only if:

$i' > i$ or $i' = i$ and $j' < j$. This order corresponds to the order of flatness at 0 of the sequence $f_{ij} = x^i \log^j x, 0 \leq j \leq i, i \in \mathbb{N}$.

The Standard enlarged logarithmic scale $\mathcal{L}^{Se}$ is a simple asymptotic scale. Indeed, we can calculate the principal parts of the sequences $\mathcal{L}_k^{Se}, k \in \mathbb{N}$, analoguously as we did for the logarithmic scale $\mathcal{L}$, and we find by induction that $\forall j \in \mathbb{N}$, the following pattern is repeated: let $J(j)$ be defined by $f_{J(j)} = x^j L^j$ (from an easy calculation one finds that $J(j) = \frac{1}{2} j(j-1)$; notice also that $J(0) = 0$ and $J(j+1) = J(j) + j + 1, \forall j \in \mathbb{N}$), then $\forall j \in \mathbb{N}, \forall 0 \leq s \leq j-1$: the principal parts of the functions in the sequence $\mathcal{L}_{J(j)+s}^{Se}$ are given by

$$\left\{1, L^{-1}, L^{-2}, \ldots, L^{-j+s}; x^i L^{i+2s-r} \middle| 0 \leq r \leq i+j, i \geq 1\right\}$$

In proposition 131, we encounter another example of the *enlarged logarithmic scale* $\mathcal{L}^e$:

$$\begin{aligned}\mathcal{L}^e &= \left\{1, L^{-1}, L^{-2}, xL, x, xL^{-1}, xL^{-2}, x^2L^2, \ldots\right\} \\ &= \left\{x^i L^{i-j} : 0 \leq j \leq i+2, i \in \mathbb{N}\right\}\end{aligned}$$

In an analoguous way as we did for the standard enlarged logarithmic scale $\mathcal{L}^{Se}$, one can show that $\mathcal{L}^e$ is a simple asymptotic scale. Indeed, by induction on $j \in \mathbb{N}$, we find that the following pattern is repeated: let $J(j)$ be defined by $f_{J(j)} = x^j L^j$ (from an easy calculation one finds that

$$J(j) = 1 + 3(j-1) + \frac{1}{2} j(j-1) = \frac{1}{2}\left(j^2 + 5j - 4\right); \tag{1.83}$$

notice also that $J(0) = 0$ and $J(j+1) = J(j) + j + 3$), then $\forall j \in \mathbb{N}, \forall 0 \leq s \leq j+2$: the principal parts of the functions in the sequence $\mathcal{L}_{J(j)+s}^e$ are given by

$$\left\{1, L^{-1}, L^{-2}, \ldots, L^{-(j+2)+s}; x^i L^{i+2s-r} \middle| 0 \leq r \leq i+j+2, i \geq 1\right\}.$$

Analoguously as for the logarithmic scale, one can deduce the following proposition concerning the remainder property for the enlarged logarithmic scale.

Proposition 89 *Let $\mathcal{L}^e$ be the enlarged logarithmic scale and let Φ be a C^∞ function. Let $N \in \mathbb{N}$. There exists a positive integer $M(N)$ such that if Φ is flat of the order $M(N)$ at $x = 0$, then Φ satisfies the remainder property of order N with respect to $\mathcal{L}^e$.*

1.4.3 Expansion of the Abelian integral near a hyperbolic saddle

In this section, we prove that any Abelian integral expands in the asymptotic scale $\mathcal{L}$ (proposition 92 below). First we give two lemmas. Lemma 90 is a clever application of the theorem of Taylor for functions defined in one variable on functions in 2 variables to obtain an expansion of the 1-form, that after integration gives the desired expansion of the Abelian integral in functions of the scale $\mathcal{L}$. Lemma 91 will be useful in checking the remainder property.

Lemma 90 *Let U be an open neighbourhood of $(0,0)$ in $\mathbb{R}^2$ and let $f : U \to \mathbb{R}$ be a function of class C^∞. Then $\forall n \in \mathbb{N}$, f can be written as*

$$\begin{aligned} f(X,Y) = & \sum_{\substack{0 \le i \le n+1 \\ 0 \le j \le n-1}} a_{ij} X^j Y^i + X^n \sum_{i=0}^{n} h_i(X) Y^i \\ & + Y^{n+1} \sum_{j=0}^{n-1} g_j(Y) X^j + X^n Y^{n+1} G(X,Y), \end{aligned} \tag{1.84}$$

for some C^∞ functions $h_i, g_j : U \subset \mathbb{R} \to \mathbb{R}, G : V \subset \mathbb{R}^2 \to \mathbb{R}$ (where U is an open neighbourhood of 0 in $\mathbb{R}$ and V is an open neighbourhood of $(0,0)$ in $\mathbb{R}^2$) and some constants $a_{ij} \in \mathbb{R}$ $(0 \le i \le n, 0 \le j \le n-1)$.

Proof. First we expand f in powers of Y along the $0X$-axis $(Y = 0)$ of order n, to obtain C^∞ functions F_i and G_{n+1}

$$f(X,Y) = \sum_{i=0}^{n} F_i(X) Y^i + Y^n G_{n+1}(X,Y), (X,Y) \to (0,0) \tag{1.85}$$

Then we expand the function G_{n+1} in powers of X along the $0Y$-axis $(X = 0)$ of order $n-1$, and we expand $\forall 0 \le i \le n+1$ the function F_i in powers of X at $X = 0$ of order $n-1$:

$$F_i(X) = \sum_{j=0}^{n-1} a_{ij} X^j + X^n h_i(X), X \to 0 \tag{1.86}$$

for some C^∞ functions h_i and constants $a_{ij} \in \mathbb{R}$, and

$$G_{n+1}(X,Y) = \sum_{j=0}^{n-1} g_j(Y) X^j + X^n G(X,Y), (X,Y) \to (0,0) \tag{1.87}$$

for some C^∞ functions g_j and G. Substitute (1.86) and (1.87) in (1.85), we obtain the required expansion (1.84). ■

Lemma 91 *Suppose that $G : U \subset \mathbb{R}^2 \to \mathbb{R}$ is a C^∞ function where U is an open neighbourhood of $(0,0)$ in $\mathbb{R}^2$. Define the map Φ by*

$$\Phi(x) = x^m \int_x^1 G\left(X, xX^{-1}\right) \mathrm{d}X$$

Then, $\forall 0 \leq k \leq m$, there exist C^∞ functions F_i^k, G_i^k such that

$$\begin{aligned}
\frac{\partial^k}{\partial x^k}\Phi(x) &= x^{m-k}\int_x^1 G_k^k\left(X, xX^{-1}\right)\mathrm{d}X \\
&+\sum_{j=0}^{k-1} x^{m-(k-1)+j}\left(F_{k-j-1}^k(x)+\int_x^1 G_{k-j-1}^k\left(X, xX^{-1}\right)X^{-j-1}\mathrm{d}X\right) \\
&= x^{m-k}\left(\int_x^1 G_k^k\left(X, xX^{-1}\right)\mathrm{d}X + o(1)\right), x\to 0 \\
&= O\left(x^{m-k}\right), x\to 0
\end{aligned}$$

Proof. (by induction on $0 \leq k \leq m$) For $k=0$, it is trivial. Suppose that $0 \leq k \leq m-1$, and that $\frac{\partial^k}{\partial x^k}\Phi(x)$ takes the required form. Then,

$$\begin{aligned}
\frac{\partial^{k+1}}{\partial x^{k+1}}\Phi(x) &= x^{m-k-1}\int_x^1 G_{k+1}^{k+1}\left(X, xX^{-1}\right)\mathrm{d}X \\
&+\sum_{j=0}^{k} x^{m-k+j}\left(F_{k-j}^{k+1}(x)+\int_x^1 G_{k-j}^{k+1}\left(X, xX^{-1}\right)X^{-j-1}\mathrm{d}X\right)
\end{aligned}$$

where

$$\left\{\begin{array}{lcl}
G_{k+1}^{k+1}(X,Y) &=& (m-k)\,G_k^k(X,Y) \\
F_k^{k+1}(x) &=& \sum_{j=0}^{k} G_{k-j}^k(x,1)+(m-k+1)\,F_{k-1}^k(x) \\
F_{k-j}^{k+1}(x) &=& (m-(k-1)+j)\,F_{k-j-1}^k(x)+\frac{\mathrm{d}}{\mathrm{d}x}F_{k-j}^k(x), \forall 1\leq j\leq k-1 \\
F_0^{k+1}(x) &=& \frac{\mathrm{d}}{\mathrm{d}x}F_0^k(x) \\
G_k^{k+1}(X,Y) &=& \frac{\partial}{\partial y}G_k^k(X,Y)+(m-k+1)\,G_{k-1}^k(X,Y) \\
G_{k-j}^{k+1}(X,Y) &=& (m-(k-1)+j)\,G_{k-j-1}^k(X,Y)+\frac{\partial}{\partial y}G_{k-j}(X,Y), \forall 1\leq j\leq k-1 \\
G_0^{k+1}(X,Y) &=& \frac{\partial}{\partial y}G_0^k(X,Y)
\end{array}\right.$$

To prove the part about the asymptotics, we remark that $\forall 0 \leq x \leq X \leq 1, 0 \leq x \leq xX^{-1} \leq 1$ and there exists $M>0$ such that

$$\sup\left\{G_j^k(X,Y) : 0\leq X, Y\leq 1, 0\leq j\leq k, 0\leq k\leq m\right\}\leq M$$

Hence,

$$\left|\int_x^1 G_{k-j-1}^k\left(X, xX^{-1}\right) X^{-j-1} \mathrm{d}X\right| \leq M\left|\int_x^1 X^{-j-1} \mathrm{d}X\right|$$
$$=\begin{cases} \frac{M}{j}\left(x^{-j}-1\right), & \text{if } j \neq 0 \\ -ML, & \text{if } j=0 \end{cases}$$

and the result follows. ∎

Proposition 92 *Let γ_x denote the curve $\{(X,Y) : XY = x, x \leq X \leq 1\}$ and let $f, g : U \to \mathbb{R}$ be C^∞ functions, defined on a neighbourhood U of $[0,1]^2$ in $\mathbb{R}^2$. Let G be the Abelian integral defined by*

$$G(x)=\int_{\gamma_x} f(X, Y) \mathrm{d}X+g(X, Y) \mathrm{d}Y \tag{1.88}$$

Any Abelian integral of type (1.88) *is asymptotic to a series*

$$\sum_{i=0}^{\infty} \alpha_i f_i(x) .$$

with respect to the logarithmic scale $\mathcal{L}$.

Proof. We have to prove that, $\forall N \in \mathbb{N}$, the function Φ_{N+1} defined by

$$\Phi_{N+1}(x)=G(x)-\sum_{i=0}^{N} \alpha_i f_i(x)$$

is a smooth function of class C^∞ and satisfies the remainder property of order N. By proposition 72 (1), it suffices to prove this lemma in the following two cases: $f \equiv 0$ and $g \equiv 0$. We only prove the proposition in case $g \equiv 0$, the other case is proven in an analoguous way. By lemma 90, and proposition 72 (1), it suffices to prove this lemma in case f takes one of the following forms:$\forall n \in \mathbb{N}$:

$$\begin{cases} X^j Y^i, & 0 \leq i \leq n, 0 \leq j \leq n-1 \\ X^n Y^i x_i(X), & 0 \leq i \leq n \\ X^j Y^n g_j(X), & 0 \leq j \leq n-1 \\ (XY)^n Y G(X, Y) & \end{cases}$$

where h_i, g_j, G are C^∞ functions. If f is a function of the first three forms, then the proposition is proven by direct calculation and corolary 87. Indeed, $\forall 0 \leq i \leq n, 0 \leq j \leq n-1$, we have that

$$\int_{\gamma_x} X^j Y^i \mathrm{d}X=x^i \int_x^1 X^{j-i} \mathrm{d}X$$
$$=\begin{cases} \frac{1}{j-i+1}\left(x^i-x^{j+1}\right), & \text{if } j \neq i-1 \\ -x^i L, & \text{if } j=i-1 \end{cases}$$

$\forall 0 \leq i \leq n$, we have that

$$\int_{\gamma_x} X^n Y^i h_i(X)\,\mathrm{d}X = x^i \int_x^1 X^{n-i} h_i(X)\,\mathrm{d}X$$
$$= x^i H_i(x),$$

where H_i is a smooth function of class C^∞ in h. Next, for $0 \leq j \leq n-1$, we have

$$\int_{\gamma_x} X^j Y^{n+1} g_j(Y)\,\mathrm{d}X = \int_x^1 (x/Y)^j\, Y^{n+1} g_j(Y)\, Y^{-2}\mathrm{d}Y$$
$$= x^j \int_x^1 Y^{n-1-j} g_j(Y)\,\mathrm{d}Y$$
$$= x^i G_j(x),$$

where G_j is a smooth function of class C^∞ in x. If f is a function of the fourth kind, then

$$\Phi(x) \equiv \int_{\gamma_x} (XY)^n\, YG(X,Y)\,\mathrm{d}X = x^{n-1} \int_1^x xX^{-1} G\left(X, xX^{-1}\right) \mathrm{d}x$$

If we choose $N = n+2$, then by lemma 91,$\forall 0 \leq i \leq N$:

$$\frac{\partial^i}{\partial x^i}\Phi(x) = O\left(x^{N-i+1}\right), x \to 0$$

By corollary 87, the map Φ satisfies the remainder property of order N. ■

1.4.4 Deformation of asymptotic scale

Here, we recall the definition of simple asymptotic scale deformation of a simple asymptotic scale of functions and the related notions of remainder property and asymptotic expansion.

Definition 93 *Let $\mathcal{F} = \{f_i : i \in \mathbb{N}\}$ be an asymptotic scale of functions and let $\mathcal{D} = \{F_i : i \in \mathbb{N}\}$ be a sequence of functions depending on a parameter λ, defined for $(x,\lambda) \in [0,h_0[\times V_0$, where $\lambda = \left(\bar{\lambda}, \varepsilon\right) \in V_0 = \bar{V_0} \times [0,\varepsilon_0[$ such that*

$$F_i\left(x,\bar{\lambda},0\right) \equiv f_i(x), \forall \left(x,\bar{\lambda}\right) \in [0,h_0[\times \bar{V_0}.$$

Then we say that $\mathcal{D}$ is a (Chebychev) asymptotic scale deformation of $\mathcal{F}$ if there exists $h_0 > 0$ such that

1. *$F_0(\cdot,\lambda) \equiv 1, \forall\lambda \in V_0$ and $\forall i \in \mathbb{N}$:*
2. *F_i is C^0 on $[0,h_0[\times V_0$, and $\forall\lambda \in V_0$, the map $F_i(\cdot,\lambda)$ is C^∞ on $]0,h_0[$*

3. $F_i(x,\lambda) \neq 0, \forall x \in]0,h_0[$

4. $\frac{F_{i+1}}{F_i}(x,\bar{\lambda},\varepsilon) \to 0$, *when* $(x,\varepsilon) \to (0,0)$, *uniformly in* $\bar{\lambda}$

5. $\nabla\left(\frac{F_{i+1}}{F_i}\right)(x,\lambda)$ *has a constant sign for* x *near* 0 *in* $]0,h_0[$.

Definition 94 *Let* $\mathcal{F} = \{f_i : i \in \mathbb{N}\}$ *be a simple asymptotic scale of functions and let* $\mathcal{D} = \{F_i : i \in \mathbb{N}\}$ *be a sequence of functions depending on a parameter* λ, *defined for* $(x,\lambda) \in [0,h_0[\times V_0$, *where* $\lambda = (\bar{\lambda},\varepsilon) \in V_0 = \bar{V}_0 \times [0,\varepsilon_0[$ *such that*

$$F_i(x,\bar{\lambda},0) \equiv f_i(x), \forall (x,\bar{\lambda}) \in [0,h_0[\times \bar{V}_0.$$

Then we say that $\mathcal{D}$ *is a* simple (Chebychev) asymptotic scale deformation *of* $\mathcal{F}$ *if each sequence of functions* $\mathcal{D}_j = \left\{F_i^j : i \in \mathbb{N}\right\}, j \in \mathbb{N}$ *inductively defined below, is an asymptotic scale deformation. The sequence* $\mathcal{D}_0$ *is defined by*

$$\mathcal{D}_0 \equiv \mathcal{D}$$

and for $j \in \mathbb{N}$, *if the sequence* $\mathcal{D}_j = \left\{F_i^j : i \in \mathbb{N}\right\}$ *is defined on* $[0,h_j[\times V_j$, $V_j = \bar{V}_j \times [0,\varepsilon_j[$, *then the sequence* $\mathcal{D}_{j+1} = \left\{F_i^{j+1} : i \in \mathbb{N}\right\}$ *is defined by:*

$$F_i^{j+1} = \frac{\nabla F_{i+1}^j}{\nabla F_1^j}, \forall i \in \mathbb{N}$$

on $[0,h_{j+1}[\times V_{j+1}$, $V_{j+1} = \bar{V}_{j+1} \times [0,\varepsilon_{j+1}[$ *with* $0 < h_{j+1} < h_j, 0 < \varepsilon_{j+1} < \varepsilon_j, \bar{V}_{j+1} \subset \bar{V}_j$. *(This supposes that* $\nabla F_1^j(x,\lambda) \neq 0$ *for* $(x,\lambda) \in [0,h_{j+1}[\times V_{j+1}$*).*

Definition 95 *Let* $\mathcal{D}$ *be an asymptotic scale deformation of the simple asymptotic scale* $\mathcal{F}$, *and let* Φ *be a function of class* C^∞, *then we can perform a division-derivation process on* Φ *with respect to* $\mathcal{D}$ *to obtain a sequence of functions* $\Phi^j, j \in \mathbb{N}$ *as follows: put* $\Phi^0 = \Phi$ *and inductively*

$$\Phi^{j+1} = \frac{\nabla \Phi^j}{\nabla F_1^j}, \qquad \forall j \in \mathbb{N}$$

We say that the sequence $\Phi^j, j \in N$ *is* defined by the division-derivation process on Φ with respect to $\mathcal{D}$.

Remark 96 *A simple asymptotic scale deformation is an asymptotic scale deformation which generates infinitely many new asymptotic scale deformations by the inductive application of the algorithm of derivation-division as given in definition 95 (derivation by* ∇ *followed by the division by the first function of the sequence).*

If $\mathcal{D}$ is a simple asymptotic scale deformation, the property $(R)_N$ simply expresses that the remainder has a similar asymptotic behavior as the function F_{N+1}; more precisely,

Definition 97 *Let $\mathcal{D}$ be a simple asymptotic scale deformation of the simple asymptotic scale $\mathcal{F}$, let Φ be a function of class C^∞ and define the sequence $\Phi^j, j \in N$ by the division-derivation process on Φ with respect to $\mathcal{D}$. Then we say that Φ satisfies the remainder property of order N with respect to $\mathcal{D}$ or (R_N)-property with respect to $\mathcal{D}$ if and only if*

$$\frac{\Phi^j}{F^j_{N-j}}\left(x,\bar{\lambda},\varepsilon\right) \to 0 \text{ if } (x,\varepsilon) \to (0,0), \text{ uniformly in } \bar{\lambda}, \qquad \forall j = 0,1,\ldots,N$$

Definition 98 *Let $\mathcal{D} = \{F_i : i \in \mathbb{N}\}$ be a simple asymptotic scale deformation of the asymptotic scale $\mathcal{F} = \{f_i : i \in \mathbb{N}\}$ and let F be a function of class C^∞. We say that*

1. *F has an expansion in $\mathcal{D}$ of order N if there exist $\alpha_i, 0 \leq i \leq N$ (constants or functions depending smoothly on λ) and a function Φ_{N+1} satisfying the remainder property of order N, such that*

$$F(x) = \sum_{i=0}^{N} \alpha_i F_i(x) + \Phi_{N+1}(x)$$

2. *F is asymptotic to the series*

$$\hat{F}(x) = \sum_{i=0}^{\infty} \alpha_i F_i(x), x \to 0$$

with respect to $\mathcal{D}$ if for all $N \in \mathbb{N}$, the function Φ_{N+1}, defined by

$$\Phi_{N+1}(x) = F(x) - \sum_{i=0}^{N} \alpha_i F_i(x),$$

satisfies the remainder property of order N with respect to $\mathcal{D}$.

Remark 99 *In section 1.5, we will see that the reduced displacement map near the saddle point in a saddle loop has an expansion in the simple asymptotic scale deformation $\mathcal{W}$ appearing in theorem 119 below; this property was obtained in [R86], by considering an expansion (1.108) at order $2N+1$. This allows us to write the remainder Φ_{N+1} in (1.108) as the sum of the terms in the principal part of (1.108) from order $N+1$ to order $2N+1$ plus the remainder Ψ_{2N+1} of (1.108), which is a function of class C^{2N+1} and flat at $x=0$ at the order $2N+1$. Clearly, the remainder Φ_{N+1} has the property $(R)_N$.*

Definition 100 *Let $\mathcal{D}=\{F_i : i \in \mathbb{N}\}$ be a simple asymptotic scale deformation of the asymptotic scale $\mathcal{F}=\{f_i : i \in \mathbb{N}\}$ and let F be a function of class C^∞. Suppose that F is asymptotic to the series*

$$\hat{F}\left(x,\bar{\lambda},\varepsilon\right)=\sum_{i=0}^{\infty}\alpha_i F_i\left(x,\bar{\lambda},\varepsilon\right),x\rightarrow 0$$

with respect to $\mathcal{D}$. Suppose that f is defined as

$$f(x,\bar{\lambda})\equiv F(x,\bar{\lambda},0)$$

Then we say that

1. *F has a finite codimension at $x=0,\bar{\lambda}=0,\varepsilon=0$ with respect to $\mathcal{D}$ if and only if f has a finite codimension with respect to $\mathcal{F}$ at $x=0,\bar{\lambda}=0$.*
2. *The codimension of F at $x=0,\bar{\lambda}=0,\varepsilon=0$ in the scale $\mathcal{D}$ is defined as the codimension of $f(x,\bar{\lambda})\equiv F(x,\bar{\lambda},0)$ at $x=0,\bar{\lambda}=0$:*

$$codim_{\mathcal{D}}(F)=codim_{\mathcal{F}}(f)$$

It is easy to extend the proofs given in propositions 72,74,77 and 79 for $f(x,\bar{\lambda})$ to the function $F(x,\bar{\lambda},\varepsilon)$ itself. A proof of proposition 79 is given in [R86]when F expands in the sequence $\mathcal{W}$, but the same proof works for any simple asymptotic scale deformation. So, we have the following results:

Proposition 101 *Let $\mathcal{D}=\{F_i : i \in \mathbb{N}\}$ be a simple asymptotic scale deformation of functions, let Φ,Φ_1,Φ_2 be functions of class C^∞ that satisfy the remainder property of order N with respect to $\mathcal{D}$. Then,*

1. *$\Phi_1+\Phi_2$ satisfy the remainder property of order N with respect to $\mathcal{D}$.*
2. *Φ satisfies the remainder property of order k with respect to $\mathcal{D},\forall k\leq N$.*
3. *Φ^k satisfies the remainder property of order $N-k$ with respect to $\mathcal{D}^k$, $\forall k\leq N$.*
4. *F_k satisfies the remainder property of order N with respect to $\mathcal{D},\forall k\in\mathbb{N},k\geq N+1$.*

Proposition 102 *Let $\mathcal{D}=\{F_i : i \in \mathbb{N}\}$ be a simple asymptotic scale of functions and let F be a function of class C^∞.*

1. *If F is asymptotic to the series*

$$\hat{F}\left(x\right)=\sum_{i=0}^{\infty}\alpha_i F_i\left(x\right),x\rightarrow 0$$

with respect to $\mathcal{D}$, then, for all $N \in \mathbb{N}$ with $\alpha_{N+1} \neq 0$:

$$F(x) - \sum_{i=0}^{N} \alpha_i F_i(x) = \alpha_{N+1} F_{N+1}(1 + o(1)), x \to 0$$

2. If $codim_{\mathcal{D}} F = l < \infty$, then $\forall 0 \leq j \leq l$:

$$\mathrm{codim}_{\mathcal{D}^j} F^j = l - j$$

Theorem 103 *Let $\mathcal{D} = \{F_i : i \in \mathbb{N}\}$ be a simple asymptotic scale deformation of the asymptotic scale $\mathcal{F} = \{f_i : i \in \mathbb{N}\}$ and let F be a function of class C^∞. Suppose that F has a finite codimension at $x = 0, \bar{\lambda} = 0, \varepsilon = 0$ in the scale $\mathcal{D}$. Then*

$$\mathrm{Cycl}(F, (0,0)) \leq codim(f_0) \tag{1.89}$$

1.4.5 The different compensators

The compensator ω show up in the study of the cyclicity or the bifurcation diagram of limit cycles near saddle points. It is defined as

$$\omega_\alpha : \mathbb{R}^+ \to \mathbb{R} : x \mapsto \omega_\alpha(x) = \omega(x, \alpha) = \begin{cases} \dfrac{x^\alpha - 1}{\alpha} & \text{if } \alpha \neq 0 \\ \log x & \text{if } \alpha = 0 \end{cases} . \tag{1.90}$$

Let us first recall the definition of ratio of hyperbolicity at a saddle point.

Definition 104 *Let s be a saddle point of X_λ. Then the ratio of hyperbolicity $r(\lambda)$ of the saddle point s for X_λ is defined as the ratio of the absolute value of the negative eigenvalue divided by the positive eigenvalue at s.*

Although the displacement map near a homoclinic loop is not differentiable at the saddle point s, there exists an expansion in a formal series in terms of x^i and $x^{i+1}\log x, i \in \mathbb{N}$ for $\alpha = 0$, if $1 + \alpha$ is the ratio of hyperbolicity at the saddle point s. For $\alpha \neq 0$, these monomials origin from x^i and $x^{i+1}\omega_\alpha, i \in \mathbb{N}$ respectively (see theorem 119). The introduction of compensators, which unfold the function $\log x$, is a way to avoid the divergence at $x = 0$.

In section 5.2, we consider unfoldings of a 2-saddle cycle. Theorem 127 displays an expansion for the reduced displacement map $\bar{\delta}$ for unfoldings of a 2-saddle cycle in the monomials $x^i (x\omega_1)^j (x\omega_2)^k$, where the compensators $\omega_1 = \omega(\cdot, \varepsilon\alpha^{(1)})$ respectively $\omega_2 = \omega(\cdot, \varepsilon\alpha^{(2)})$ correspond to the saddle points s_1 and s_2 in the 2-saddle cycle respectively. Taking $\varepsilon = 0$ in $\bar{\delta}$, every monomial in $x^i (x\omega_1)^j (x\omega_2)^k$ corresponding to a fixed value of i and $j + k$ converges towards the same function $x^{i+j+k} \log^{j+k} x$. In case the unfolding of the 2-saddle cycle leaves one conection unbroken, the new compensators ω_{2-1} and ω_{21} are introduced in [DR], to avoid this degeneracy phenomenon:

$$\omega_{2-1} = \omega_{\varepsilon(\alpha^{(2)}-\alpha^{(1)})} \tag{1.91}$$

and

$$\omega_{21} = \frac{\omega_2 - \omega_1}{\varepsilon(\alpha^{(2)} - \alpha^{(1)})}, \text{ if } \varepsilon\left(\alpha^{(2)} - \alpha^{(1)}\right) \neq 0. \tag{1.92}$$

In this section we recall some interesting properties of the compensators, that are useful in checking the conditions of simple asymptotic scale deformations. Finally, we give calculation rules of the Euler differential operator ∇ on functions having as 'principal part' $x^i\omega^j, i \in \mathbb{R}, j \in \mathbb{Z}$.

Let us now extend ω_{21} continuously to $\varepsilon\left(\alpha^{(2)} - \alpha^{(1)}\right) = 0$. Therefore, the analytic functions Φ and Ψ are introduced:

$$\Phi : \mathbb{R} \to \mathbb{R} : u \mapsto \Phi(u) = \begin{cases} \dfrac{e^u - 1}{u} & \text{if } u \neq 0 \\ 1 & \text{if } u = 0 \end{cases} \tag{1.93}$$

and

$$\Psi : \mathbb{R}^2 \to \mathbb{R} : (u,v) \mapsto \Psi(u,v) = \begin{cases} \dfrac{\Phi(u) - \Phi(v)}{u - v} & \text{if } u \neq v \\ \dfrac{d\Phi}{du}(u) & \text{if } u = v \end{cases} \tag{1.94}$$

These functions allow nice expressions for the different compensators. Firstly, we have

$$\omega_\alpha(x) = \log x \cdot \Phi(\alpha \log x) \tag{1.95}$$

Secondly, we have for $\varepsilon\left(\alpha^{(1)} - \alpha^{(2)}\right) \neq 0$,

$$\begin{aligned} \omega_{21}(x) &= \log x \cdot \frac{\Phi(\varepsilon\alpha^{(2)} \log x) - \Phi(\varepsilon\alpha^{(1)} \log x)}{\varepsilon(\alpha^{(2)} - \alpha^{(1)})} \\ &= \log^2 x \cdot \Psi(\varepsilon\alpha^{(1)} \log x, \varepsilon\alpha^{(2)} \log x) \end{aligned} \tag{1.96}$$

Expression (1.96) now can be used to extend ω_{21} continuously at $\varepsilon(\alpha^{(2)} - \alpha^{(1)}) = 0$ by the definition of Ψ in (1.94).

Notice also that these expressions of ω_1, ω_2 and ω_{21} by means of these functions Φ and respectively Ψ, reduce the investigation of the behaviour of functions in two and respectively three variables to the behaviour of functions in one and respectively two variables. From (1.96), we also see that $\omega_{21}|_{\varepsilon=0} = 1/2 \log^2 x$.

In corollary 106 below, we give some useful estimates for the compensators. First we give a lemma to estimate the functions Φ and Ψ and their derivatives; its proof is an elementary application of the mean-value-theorem (MVT) for differentiable real-valued functions in one variable. Corollary 106 then follows from formulas (1.95) and (1.96).

Lemma 105 *1.* $0 \leq \Phi(u) \leq e^{|u|}$

2. $0 \leq \Phi'(u) \leq e^{|u|}$

3. $0 \leq \Phi''(u) \leq e^{|u|}$

4. $0 \leq \Psi(u,v) = \Psi(v,u) \leq e^{|u|+|v|}$

5. $0 \leq \frac{\partial}{\partial u}\Psi(u,v) \leq e^{|u|+|v|}$ *and* $0 \leq \frac{\partial}{\partial v}\Psi(u,v) \leq e^{|u|+|v|}$

Proof.

1. For $u = 0$, the inequaltities are trivially satisfied. So, we can suppose that $u \neq 0$. By the MVT, there exists $c \in \mathbb{R}$ with $0 < |c| < |u|, cu > 0$ such that

$$\Phi(u) = e^{c}$$

Hence, $0 \leq \Phi(u) \leq e^{|c|} \leq e^{|u|}$.

2. Denote $T(u) = ue^{u} - e^{u}$, then

$$\Phi'(u) = \frac{T(u) - T(0)}{u^2}$$

By the MVT, there exists $c \in \mathbb{R}$ with $0 < |c| < |u|, cu > 0$ such that

$$\Phi'(u) = \frac{T'(c)}{u} = \frac{c}{u}e^{c}$$

As $0 < \frac{c}{u} < 1$, the proposed inequalities are satisfied.

3. Denote $S(u) = \left(u^2 - 2u + 2\right)e^{u}$, then

$$\Phi''(u) = \frac{S(u) - S(0)}{u} \cdot \frac{1}{u^2}$$

The inequalities are obtained in the same way as in 2.: by the MVT, there exists $c \in \mathbb{R}$ with $0 < |c| < |u|, cu > 0$ such that

$$\Phi''(u) = \frac{S'(c)}{u^2} = \left(\frac{c}{u}\right)^2 e^{c}$$

4. It is clear that $\Psi(u,v) = \Psi(v,u)$; if $u = v$, then, the inequalities surely hold by the second assertion of this lemma since, $\Psi(u,u) = \Phi'(u)$. Hence, we can suppose that $v < u$. By the MVT, there exist $c \in \mathbb{R}$ with $v < c < u$ such that

$$\begin{aligned} 0 &\leq \Psi(u,v) = \Phi'(c) \\ &\leq e^{|c|} \leq e^{|u|+|v|} \end{aligned}$$

the inequaltities follow from the second assertion in this lemma and from the fact that $v < c < u$ (implying that $|c| \leq |u| + |v|$).

5. Again, we can suppose that $v < u$. Then, by the MVT,

$$\begin{aligned}\frac{\partial}{\partial u}\Psi(u,v) &= \frac{(u-v)\,\Phi'(u)-(\Phi(u)-\Phi(v))}{(u-v)^2}\\ &= \frac{\Phi'(u)-\Phi'(c)}{u-v}\end{aligned}$$

where $c \in \mathbb{R}$ with $v < c < u$. By assertion 3. Φ' is increasing, hence $\frac{\partial}{\partial u}\Psi(u,v) \geq 0$. The upperbound is obtained by applying once again the MVT: there exist $\alpha \in \mathbb{R}$ with $c < \alpha < u$ such that

$$\frac{\partial}{\partial u}\Psi(u,v) \leq \frac{\Phi'(u)-\Phi'(c)}{u-c} = \Phi''(\alpha)$$

■

Corollary 106 $\forall 0 < x < 1:$

1. $|\omega_\alpha| \leq x^{-|\alpha|}|\log x|$
2. $0 \leq \frac{\partial}{\partial\alpha}\omega_\alpha \leq x^{-|\alpha|}\log^2 x$
3. $0 \leq \omega_{21} \leq x^{-\left(|\varepsilon\alpha^{(1)}|+|\varepsilon\alpha^{(2)}|\right)}\log^2 x$
4. $\left|\frac{\partial}{\partial\alpha_i}\omega_{21}\right| \leq x^{-\left(|\varepsilon\alpha^{(1)}|+|\varepsilon\alpha^{(2)}|\right)}\left|\log^3 x\right|, i = 1,2$

In checking the properties of a deformation of asymptotic scale the following proposition is useful.

Proposition 107 1. $\lim_{(x,\alpha)\to(0,0)} \omega_\alpha(x)^{-1} = 0$

2. $\lim_{(x,\alpha)\to(0,0)} x\omega_a(x) = 0$
3. $\lim_{(x,\alpha)\to(0,0)} x^r\omega_a(x) = 0, \forall r > 0$
4. $\lim_{(x,\alpha)\to(0,0)} x\omega_a^l(x) = 0, \forall l \in \mathbb{N}$

Proof. (1) We show that $\omega_\alpha(x)$ becomes arbitrarily large when x and α are chosen small enough. Of course, we can suppose that $0 < x < 1$. Notice that

$$|\omega_\alpha(x)| = |\log x|\,\Phi(\alpha\log x) \tag{1.97}$$

By the MVT, for all (α, x) there exists a $c = c(\alpha, x)$, that lies strictly between 0 and $\alpha\log x$, such that

$$\Phi(\alpha\log x) = \mathrm{e}^c \tag{1.98}$$

If $\alpha < 0$, then $\alpha \log x > 0$, hence $e^c > 1$, and therefore

$$|\omega_\alpha(x)| > |\log x| \to \infty, \text{ if } x \downarrow 0$$

If $\alpha > 0$, then $\alpha \log x < 0$, then we cannot use the same argument as above, since now $e^c < 1$. However, if $\alpha \log x \geq -1$, it follows that $e^c \geq e^{-1}$, hence

$$|\omega_\alpha(x)| \geq e^{-1} |\log x| \to \infty, \text{ if } x \downarrow 0$$

In the other case, if $\alpha \log x < -1$, one has

$$|\omega_\alpha(x)| = \frac{1 - e^{\alpha \log x}}{\alpha} > \frac{1 - e^{-1}}{\alpha} \to \infty \text{ if } \alpha \downarrow 0.$$

(2) Since $(x, \alpha) \to (0, 0)$, we can suppose that $0 < x < 1, 0 < |\alpha| < 1/2$. It suffices to show that

$$|x\omega_\alpha(x)| \leq \left| x^{1/2} \log x \right| \tag{1.99}$$

From (1.98), we have that $\omega_\alpha(x) = \log x \cdot e^c$ with $0 < |c| < -|\alpha| \log x$. Hence,

$$|\omega_\alpha(x)| \leq |\log x| \cdot e^{|c|} \leq |\log x| \cdot e^{-|\alpha| \log x} = |\log x| \cdot x^{-|\alpha|}$$

Since $1/2 < 1 - |\alpha| < 1$, we have $x^{1-|\alpha|} < x^{1/2}$, hence the required inequality (1.99).

Assertion (4) is a consequence of assertion (3), that can be proven analoguously as assertion (2). ■

Morover, from the proof of propostion 107, we can derive:

Proposition 108 *Denote*

$$K = \left\{ (\alpha, x) : (\alpha \leq 0 \text{ and } 0 < x < 1) \text{ or } \left(\alpha > 0 \text{ and } e^{-1/\alpha} < x < 1 \right) \right\}$$

Then,

1. $\lim_{x \downarrow 0} \omega_\alpha^{-1} = 0$, *uniformly in* $\alpha \in K$

2. $\forall 0 < x < 1, \forall |\alpha| \leq 1/2$:
$$|x\omega_\alpha| \leq x^{1/2} \log x$$
As a consequence, $\lim_{x \downarrow 0} x\omega_\alpha = 0$, *uniformly in* $\alpha \in [-1/2, 1/2]$

3. $\forall r > 0, \forall 0 < x < 1, \forall |\alpha| \leq R \equiv \max\{r - 1/2, r/2\}$:
$$|x^r \omega_\alpha| \leq x^R \log x$$
As a consequence, $\lim_{x \downarrow 0} x^r \omega_\alpha = 0$, *uniformly in* $\alpha \in [-R, R]$.

4. $\forall r > 0, \forall l \in \mathbb{N}_1, \forall 0 < x < 1, \forall |\alpha| \leq R \equiv \max\left\{\frac{(2r-1)}{2l}, \frac{r}{2l}\right\}$:

$$\left|x^r \omega_\alpha^l\right| \leq x^R \log^l x$$

As a consequence, $\lim_{x \downarrow 0} x^r \omega_\alpha^l = 0$, *uniformly in* $\alpha \in [-R, R]$.

5. $\forall r > 0 : \lim_{x \downarrow 0} x^r \omega_\alpha^{-1} = 0$, *uniformly over closed intervals in* α.

Corollary 109 *We have*

1. $\exists \tau > 0$ *such that*

$$x\omega_\alpha = o\left(x^\tau\right), x \downarrow 0$$

uniformly in $\alpha \in [-1/2, 1/2]$.

Proposition 110 1. $\forall r > 0, \forall l \in \mathbb{N}_1 : \exists \tau > 0$ *such that*

$$\left|x^r \omega_\alpha^l\right| \leq o\left(x^\tau\right), x \downarrow 0$$

uniformly in $\alpha \in [-R, R]$ *where* $R \equiv \max\left\{\frac{(2r-1)}{2l}, \frac{r}{2l}\right\}$.

2. $\forall r > 0, \forall l \in \mathbb{N} : \exists \tau > 0$ *such that* $x^r \omega_\alpha^{-l} = o\left(x^\tau\right), x \downarrow 0$, *uniformly over closed intervals in* α.

To study the asymptotics of the composition of the Dulac map and a regular mapping (e.g., a regular transition map with non-zero breaking parameter), the following lemma is useful:

Lemma 111 *Let* $\omega(y, \alpha) = \dfrac{y^\alpha - 1}{\alpha}$. *Suppose that* $\Phi_\lambda(x)$ *is a* C^k *parameter family of diffeomorphisms with* $\Phi_\lambda(0) > 0, \forall \lambda$. *Then,*

$$\omega \circ \Phi_\lambda(x) \equiv \omega\left(\Phi_\lambda(x), \alpha(\lambda)\right) = c(\lambda)\,\omega + \ldots + \xi_k(x, \lambda)$$

where the $\ldots$ *denote terms in* x^i *and* $x^{i+1}\omega$ $(i \in \mathbb{N})$ *and the function* ξ_k *is* C^k *in* (x, λ) *and flat of the order* k *at* $x = 0$, *uniform in* λ :

$$\xi_k(x, \lambda) = O\left(x^{k+1}\right), x \to 0, \forall \lambda$$

Proof. We can write

$$\Phi_\lambda(x) = u(\lambda)\, x\left(P(x, \lambda) + \widetilde{\Phi}_k(x, \lambda)\right)$$

where $\forall\lambda : u(\lambda) > 0$, and $P, \widetilde{\Phi}_k$ are C^k functions such that $P(0,\lambda) = 1, \forall\lambda$ and $\widetilde{\Phi}_k$ is flat of the order k at $x = 0$, uniform in λ. Then,

$$
\begin{aligned}
\omega \circ \Phi_\lambda(x) &= \frac{\Phi_\lambda(x)^\alpha - 1}{\alpha} \\
&= \frac{u^\alpha x^\alpha \left(P(x,\lambda) + \widetilde{\Phi}_k(x,\lambda)\right)^\alpha - 1}{\alpha} \\
&= u^\alpha \frac{x^\alpha \left(P(x,\lambda) + \widetilde{\Phi}_k(x,\lambda)\right)^\alpha - 1}{\alpha} + \frac{u^\alpha - 1}{\alpha}
\end{aligned}
$$

Clearly, the second term ϕ defined by

$$\phi(\lambda) = \frac{u^\alpha - 1}{\alpha}$$

is C^∞ in λ. Now we rewrite the first term:

$$
\begin{aligned}
\frac{x^\alpha \left(P(x,\lambda) + \widetilde{\Phi}_k(x,\lambda)\right)^\alpha - 1}{\alpha} &= x^\alpha \frac{\left(P(x,\lambda) + \widetilde{\Phi}_k(x,\lambda)\right)^\alpha - 1}{\alpha} + \frac{x^\alpha - 1}{\alpha} \\
&= (\alpha\omega + 1) \frac{\left(P(x,\lambda) + \widetilde{\Phi}_k(x,\lambda)\right)^\alpha - 1}{\alpha} + \omega \\
&= (1 + \alpha\Psi(x,\lambda))\,\omega + \Psi(x,\lambda)
\end{aligned}
$$

where Ψ is the C^k function defined by

$$\Psi(x,\lambda) = \frac{\left(P(x,\lambda) + \widetilde{\Phi}_k(x,\lambda)\right)^\alpha - 1}{\alpha}$$

As a consequence, we have

$$\omega \circ \Phi_\lambda(x) = u^\alpha (1 + \alpha\Psi(x,\lambda))\,\omega + \phi(\lambda) + u^\alpha \Psi(x,\lambda)$$

Expanding the right-hand-side of this equation to the order k, we obtain the desired formula with $c(\lambda) = u^\alpha$. ■

As a consequence, one has the following corollary.

Corollary 112 *Let* $\omega(y,\alpha) = \dfrac{y^\alpha - 1}{\alpha}$ *and let* $R(x,\alpha)$ *be a* C^∞ *map with*

$$R(x,\alpha) = x(1 + O(x)), x \downarrow 0$$

Then,

$$\omega(R(x,\alpha),\alpha) = \Psi_1(x,\alpha)\omega(x,\alpha) + \Psi_2(x,\alpha)$$

where Ψ_1 *and* Ψ_2 *are smooth functions in* (x,α) *with*

$$\Psi_1(x,\alpha) = 1 + O(x) \text{ and } \Psi_2(x,\alpha) = O(x), x \downarrow 0.$$

Let us end by giving some calculation Calculation rules of ∇

Proposition 113 *Let $\omega = \omega_\alpha$, and let $m, p, q \in \mathbb{Z}, i_1, i_2, j \in \mathbb{N}$. Then,*

1. $\nabla\omega^m = m\omega^m \left(\alpha + \omega^{-1}\right)$; *in particular,*

$$\begin{cases} \nabla\omega = \omega\left(\alpha + \omega^{-1}\right) \\ \nabla\omega^{-1} = -\omega^{-1}\left(\alpha + \omega^{-1}\right) \end{cases}$$

2. $\nabla\left(x^p\omega^q\right) = x^p\omega^q\left(p + q\alpha + q\omega^{-1}\right)$; *in particular,*

$$\begin{cases} \nabla\left(x^p\omega\right) = x^p\omega\left(p + \alpha + \omega^{-1}\right) \\ \nabla\left(x^p\omega^{-1}\right) = x^p\omega^{-1}\left(p - \alpha - \omega^{-1}\right) \end{cases}$$

3. *For every $a_k, b_l, 0 \le k \le i_1, 0 \le l \le i_2$, there exist $i \in \mathbb{N}, A_k \in \mathbb{R}$ $(2 \le k \le i)$ such that*

$$\nabla\left(\left(\sum_{k=0}^{i_1} a_k\omega^{-k}\right)\left(\sum_{l=0}^{i_2} b_l\omega^{-l}\right)^{-j}\right) = \left(\sum_{k=2}^{i} A_k\omega^{-k}\right)\left(\sum_{l=0}^{i_2} b_l\omega^{-l}\right)^{-j-1}$$

This property implies that the set of rational functions in ω^{-1} is invariant under the differential operator ∇.

Proof. Notice that $\nabla\omega = x^\alpha = \alpha\omega + 1 = \omega\left(\alpha + \omega^{-1}\right)$. ■

For later use, we give some calculation rules for functions of a special type.

Proposition 114 *Let $\omega = \omega_\alpha$. Define the function g by*

$$\left(\sum_{k=0}^{i_1} a_k\omega^{-k}\right)\left(\sum_{l=0}^{i_2} b_l\omega^{-l}\right)^{-1}, b_0 \neq 0 \tag{1.100}$$

where $j \in \mathbb{N}, j \ge 1, a_k, b_l \in \mathbb{R}$ $(\forall 0 \le k \le i_1, \forall 0 \le l \le i_2)$. Then there exist functions g_1 and g_2 of type (1.100) such that

1. $\nabla\left(\omega^{-1}g\right) = \omega^{-1}g_1$
2. $\nabla\left(x^j\omega^{-1}g\right) = x^j\omega^{-1}g_2$
3. $\nabla\left(x^jg\right) = x^jg_2$
4. *Moreover, if the constant coefficient $a_0 \neq 0$, then also the 0-th order term in ω^{-1} in the nominator of g_2 is non-zero, and the 0-th order term in ω^{-1} in the nominator of g_1 can be written as $\alpha \cdot \bar{a}$ with $\bar{a} \neq 0$.*

1.4.6 Examples of simple asymptotic scale deformations

Here we consider two examples of simple asymptotic scale deformations: a deformation of the logarithmic scale and one of the enlarged logarithmic scale. In chapter 5, we also encounter another simple asymptotic scale deformation of the logarithmic scale.

Deformations of the logarithmic scale

Let us denote shortly $\omega = \omega_\alpha$, with $\alpha\left(\bar{\lambda}, \varepsilon\right) = \varepsilon\bar{\alpha}\left(\bar{\lambda}, \varepsilon\right)$. The sequence $\mathcal{W}$,

$$\mathcal{W} = \left\{1, x\omega, x, x^2\omega, x^2, \ldots, x^i\omega, x^i, \ldots\right\}$$

is a *simple asymptotic scale deformation of the logarithmic scale* $\mathcal{L}$. Indeed, using propositions 113 and 114, one can prove by induction that $\forall j \in \mathbb{N}_1$:

$$\mathcal{W}_{2j-1} = \left\{1, \omega^{-1}g_1^{2j-1}, xg_2^{2j-1}, \ldots, x^{i-1}\omega^{-1}g_{2i-1}^{2j-1}, x^i g_{2i}^{2j-1}, \ldots\right\}, i \geq 1$$

and

$$\mathcal{W}_{2j} = \left\{1, x\omega g_1^{2j}, xg_2^{2j}, \ldots, x^i\omega g_{2i-1}^{2j}, x^i g_{2i}^{2j}, \ldots\right\}, i \geq 1$$

where g_i^j are rational functions in ω^{-1}. More precisely, the functions g_i^j can be written as:

$$\begin{gathered}\left(\sum_{k=0}^{m(i)} c_k^{j,i}\omega^{-k}\right)\left(\sum_{k=0}^{m(i)} b_k^j\omega^{-k}\right)^{-1} \quad \text{with } c_0^{j,i} \neq 0 \\ \text{and} \left\{\begin{array}{ll} b_0^j \neq 0 & \text{if } j \text{ is odd} \\ b_0^j = \alpha\cdot\bar{b}_0^j, b_1^j \neq 0 & \text{if } j \text{ is even}\end{array}\right.\end{gathered} \tag{1.101}$$

The conditions in (1.101) and propositon 107 imply that every $\mathcal{W}_j$ is a deformation of the asymptotic scale $\mathcal{L}_j, j \in \mathbb{N}$ given in (1.76) and (1.77).

The expansion of the displacement map at the saddle point of a homoclinic loop is made in a sequence $\mathcal{W}$ of functions (1.102) below of the variable x and depending on the parameter $(\bar{\lambda}, \varepsilon)$. Let us denote $\omega = \omega_{\varepsilon\alpha}$, then one can prove analoguously as above that the sequence

$$\mathcal{W} = \{1, [x\omega + \varepsilon\eta_1], [x + \varepsilon\mu_1], \ldots, [x^n\omega + \varepsilon\eta_n], [x^n + \varepsilon\mu_n], \ldots\} \tag{1.102}$$

(where η_n is a polynomial in x^i and $x^{i+1}\omega, i \geq n$ and μ_n is a polynomial in $x^i\omega$ and $x^i, i \geq n+1$) is a simple asymptotic scale deformation of the logarithmic scale $\mathcal{L}$. This fact was proven in [R86] (without introducing this terminology).

Deformation of the enlarged logarithmic scale

The sequence $\mathcal{W}^{Se} = \{1, x\omega, x, \ldots, x^i\omega^i, x^i\omega^{i-1}, \ldots, x^i\omega, x^i, \ldots\}$ is a *simple asymptotic scale deformation of the standard enlarged logarithmic scale* $\mathcal{L}^{Se}$. In Proposition 131 below, we encounter a simple asymptotic scale deformation of the Enlarged logarithmic scale $\mathcal{L}^e$, that will be denoted by $\mathcal{W}^e$.

1.4.7 The algebra O

The algebra O of functions defined below, is introduced in [DR], to simplify the writing of the maps $\bar{\Delta}^i$ obtained from the reduced difference map $\bar{\Delta}$ by a division-derivation process, for the unfolding of a 2-saddle cycle, leaving one connection unbroken. Denote the space of germs of C^∞ functions at $\lambda = 0 \in \mathbb{R}^{p+1}$ by $\mathcal{C}^\infty(\lambda)$.

Definition 115 *Let f be a germ of a function defined on $(\mathbb{R}^+ \setminus \{0\}, 0) \times (\mathbb{R}^{p+1}, 0)$.*

1. *We say that $f \in \mathcal{F}$ if*
   (a) *there exists $\tau > 0$ such that $f(x,\lambda) = O(x^\tau), x \to 0$ and*
   (b) *if f is the quotient of (finite) linear combinations of generalized monomials*
   $$x^{l+\varepsilon\beta}\omega_1^m\omega_2^p\omega_{2-1}^q\omega_{21}^r \log^s x \text{ for some } \beta \in \mathcal{C}^\infty(\lambda),\ \ell, m, p, r, s \in \mathbb{Z},$$
   with coefficients in $\mathcal{C}^\infty(\lambda)$, and such that the denominator is of the form $1 + o(1), x \to 0$.
2. *We say that f belongs to the set O if $\forall N \in \mathbb{N}$, there exist functions $\phi_N \in \mathcal{F}, \Phi_N$ differentiable of class C^N and N-flat with respect to x at $x = 0$, i.e. $\forall\lambda$:*
   $$\frac{\partial^j}{\partial x^j}\Phi_N(0,\lambda) = 0, \forall 0 \le j \le N$$
   such that
   $$f(x,\lambda) = \phi_N(x,\lambda) + \Phi_N(x,\lambda).$$
3. *We write $f = g + O$ if $f - g \in O$.*

Proposition 116 *The sets $\mathcal{F}$ and O have the following properties:*

1. *$\mathcal{F}$ is an algebra closed for derivation by ∇ :*
   $$\mathcal{F} + \mathcal{F} \subset \mathcal{F}, \mathcal{F} \cdot \mathcal{F} \subset \mathcal{F}, \nabla\mathcal{F} \subset \mathcal{F}.$$
2. *$\mathcal{F} \subset O$ and $f \in \mathcal{F}$ if and only if there exists $\tau > 0$ such that $x^{-\tau}f \in \mathcal{F}$*
3. *O is an algebra closed for derivation by ∇ :*
   $$O + O \subset O, O \cdot O \subset O, \nabla O \subset O.$$

Proof. 1. By definition, $\mathcal{F}$ is an algebra. Therefore, since $\nabla(f+g) = \nabla f + \nabla g$ and $\nabla(f^{-1}) = f^{-2}\nabla f$, to show that $\mathcal{F}$ is closed for derivation by ∇, it suffices to check this property for the generalised monomials. Let f be a generalised monomial:

$$f(x,\lambda) = x^{l+\varepsilon\beta}\omega_1^m \omega_2^p \omega_{2-1}^q \omega_{21}^r \log^s x$$

where $\beta \in \mathcal{C}^\infty(\lambda)$, $\ell, m, p, r, s \in \mathbb{Z}$, then

$$\begin{aligned}\nabla f &= f \cdot \Big[(\ell+\varepsilon\beta) + m\left(\varepsilon\alpha^{(1)} + \omega_1^{-1}\right) + p\left(\varepsilon\alpha^{(2)} + \omega_2^{-1}\right) + \\ &\quad q\left(\varepsilon\left(\alpha^{(2)} - \alpha^{(1)}\right) + \omega_{2-1}^{-1}\right)) + r\left(\alpha^{(2)} - \alpha^{(1)}\right)\left(\varepsilon\alpha^{(2)} + \omega_1\omega_{21}^{-1}\right) + s\log^{-1} x\Big] \\ &= f \cdot \Big[l + \varepsilon\left(\beta + m\alpha^{(1)} + p\alpha^{(2)} + q\left(\alpha^{(2)} - \alpha^{(1)}\right) + r\alpha^{(2)}\left(\alpha^{(2)} - \alpha^{(1)}\right)\right) \\ &\quad + m\omega_1^{-1} + p\omega_2^{-1} + q\omega_{2-1}^{-1} + r\left(\alpha^{(2)} - \alpha^{(1)}\right)\omega_1\omega_{21}^{-1} + s\log^{-1} x\Big] \\ &\in \mathcal{F}\end{aligned} \tag{1.103}$$

2. By definition of $O : \mathcal{F} \subset O$. The remaining part of the second assertion follows again from the definition of $\mathcal{F}$ and checking the equivalence for a generalised monomial f. 3. Clearly, $O + O \subset O$. As $\mathcal{F}$ is closed for derivation by ∇, the result for O is a direct consequence of the following observation: if $f \in O$, then for every $N \in \mathbb{N}$ we can write $f = \varepsilon\theta + \phi_N + \Phi_N$ and hence

$$\nabla f = \phi_N^1 + \Phi_N^1$$

with

$$\phi_N^1 = \nabla\phi_N \in \mathcal{F} \text{ if } \phi_N \in \mathcal{F}$$

and

$$\Phi_N^1 = \nabla\Phi_N$$

where Φ_N^1 is differentiable of class C^{N-1} and flat of the order N at $x = 0$ if Φ_N is differentiable of class C^N and flat of the order N at $x = 0$. As a consequence, $\nabla O \subset O$. ∎

To avoid having to write positive coefficient functions of the parameter, the following definition is introduced.

Definition 117 *Suppose that f and g are germs at $(0,0)$ of functions of the variables $x > 0$ and λ. Then we will write $f \approx g$ if and only if there exists a continuous function $c(\lambda)$ with $c(0) > 0$ such that $f(x,\lambda) = c(\lambda)g(x,\lambda)$.*

Moreover, there are the following rules of derivation by ∇ :

Lemma 118 *Let us consider $\ell, m \in \mathbb{Z}$, $\ell \neq 0$ and $\alpha, \beta \in \mathcal{C}^\infty(\lambda)$. Let also G be a rational function such that $G(0) = 0$. One has:*

1. *If $F = x^{\ell+\varepsilon\alpha}(1+O)$, then $\nabla F = (\ell+\varepsilon\alpha)x^{\ell+\varepsilon\alpha}(1+O)$, i.e. $\nabla F \approx F(1+O)$.*

2. *If* $F = \omega^{\ell}_{\varepsilon\alpha}(1 + G(\omega^{-1}_{\varepsilon\alpha}))(1 + O)$, *then*
$$\nabla F = \ell\omega^{\ell-1}_{\varepsilon\alpha} x^{\varepsilon\alpha}(1 + \tilde{G}(\omega^{-1}_{\varepsilon\alpha}))(1 + O),$$
where $\tilde{G}$ *is rational function such that* $\tilde{G}(0) = 0$.

3. *If* $F = x^{\ell+\varepsilon\beta}\omega^{m}_{\varepsilon\alpha}(1 + G(\omega^{-1}_{\varepsilon\alpha}))(1 + O)$, *then*
$$\nabla F = (\ell + \varepsilon\beta)x^{l+\varepsilon\beta}\omega^{m}_{\varepsilon\alpha}(1 + \tilde{G}(\omega^{-1}_{\varepsilon\alpha}))(1 + O),$$
where $\tilde{G}$ *is rational function such that* $\tilde{G}(0) = 0$. *This means that*
$$\nabla F \approx F(1 + \tilde{G}(\omega^{-1}_{\varepsilon\alpha}))(1 + O).$$

Proof.

1. This fact follows from the observation that $\nabla x^{\ell+\varepsilon\beta} = (\ell + \varepsilon\beta)x^{\ell+\varepsilon\beta}$ and since $\nabla O \subset O, O \subset O$.

2. $\nabla F = \nabla[\omega^{\ell}_{\varepsilon\alpha}(1 + G\left(\omega^{-1}_{\varepsilon\alpha}\right))](1 + O) + O\omega^{\ell}_{\varepsilon\alpha}(1 + G\left(\omega^{-1}_{\varepsilon\alpha}\right))$ (as $\nabla O \subset O$). Let us consider the first term:
$$\nabla[\omega^{\ell}_{\varepsilon\alpha}(1+G(\omega^{-1}_{\varepsilon\alpha}))] = \ell\omega^{\ell-1}_{\varepsilon\alpha}x^{\varepsilon\alpha}[1+G\left(\omega^{-1}_{\varepsilon\alpha}\right) - \frac{1}{\ell}\frac{\mathrm{d}G}{\mathrm{d}u}(\omega^{-1}_{\varepsilon\alpha})\omega^{-1}_{\varepsilon\alpha}] = \ell\omega^{\ell-1}_{\varepsilon\alpha}x^{\varepsilon\alpha}[1+\tilde{G}\left(\omega^{-1}_{\varepsilon\alpha}\right)] \tag{1.104}$$
where $\tilde{G}(u) = G(u) - \frac{1}{\ell}\frac{\mathrm{d}G}{\mathrm{d}u}(u)\,u$. Let us consider now the second term:
$$O\omega^{\ell}_{\varepsilon\alpha}(1 + G\left(\omega^{-1}_{\varepsilon\alpha}\right)) = \ell\omega^{\ell-1}_{\varepsilon\alpha}x^{\varepsilon\alpha}(1 + \tilde{G}\left(\omega^{-1}_{\varepsilon\alpha}\right))O$$
as $\frac{1}{\ell}\omega_{\varepsilon\alpha}x^{-\varepsilon\alpha}\dfrac{1 + G\left(\omega^{-1}_{\varepsilon\alpha}\right)}{1 + \tilde{G}\left(\omega^{-1}_{\varepsilon\alpha}\right)}O \subset O$. Summing up the two terms, we have the desired formula.

3. We have:
$$\nabla(x^{\ell+\varepsilon\beta}\omega^{m}_{\varepsilon\alpha}) = (\ell + \varepsilon\beta)\cdot x^{\ell+\varepsilon\beta}\omega^{m}_{\varepsilon\alpha}\cdot(1 + \varepsilon\frac{\alpha m}{\ell + \varepsilon\beta} + \frac{m}{\ell + \varepsilon\beta}\omega^{-1}_{\varepsilon\alpha})$$
$$\begin{aligned}\nabla(1 + G(\omega^{-1}_{\varepsilon\alpha})) &= -\frac{\mathrm{d}G}{\mathrm{d}u}(\omega^{-1}_{\varepsilon\alpha})\omega^{-2}_{\varepsilon\alpha}x^{\varepsilon\alpha}\\ &= -\frac{\partial G}{\partial u}(\omega^{-1}_{\varepsilon\alpha})(\omega^{-2}_{\varepsilon\alpha} + \varepsilon\alpha\omega^{-1}_{\varepsilon\alpha})\end{aligned} \tag{1.105}$$
Using these two formulas in the computation of ∇F we obtain the desired formula in a similar way as in (2).

■

1.5 Saddle loop

1.5.1 Introduction

In chapter 5, we study unfoldings of a 2-saddle cycle, that leave one connection unbroken. As a particular case, we consider a subfamily, in which the saddle points remain lineair. For this particular subfamily, the study of limit cycles can be reduced (by rescaling) to the study of isolated zeroes of an unfolding Ξ. This unfolding Ξ has an asymptotic expansion that is very similar to the expansion of the reduced displacement map $\bar{\delta}$, in case of the saddle loop, filled with non-isolated periodic orbits, for $\varepsilon = 0$ (theorem 119). In this way, we can use results obtained by [Mar] and [R86] in case of the saddle loop, in the study of limit cycles for the particular subfamily of the unfolding of a 2-saddle cycle, as described above.

Recall that a saddle loop or saddle connection is a singular cycle, containing precisely one hyperbolic singular point (figure 1.3).

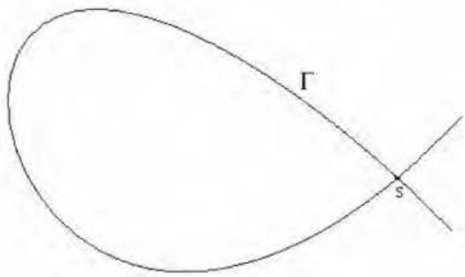

Figure 1.3: Saddle loop

The results obatined in the saddle loop case, are based on an asymptotic expansion of $\bar{\delta}$ in a simple asymptotic scale deformation (of the logarithmic scale $\mathcal{L}$), without working explicitly with this notion. It turns out that, under generic conditions, the bifurcation diagram of limit cycles of the unfolding of the saddle loop is a trivial reflection of the bifurcation diagram of zeroes of the related Abelian integral. In particular, under these generic conditions, the local bifurcation diagram of limit cycles of $X_{(\nu,\varepsilon)}$ is homeomorphic to the one of the linearisation of $X_{(\nu,\varepsilon)}$, with respect to $\varepsilon = 0$, and all limit cycles can be traced from the zeroes of the Abelian integral.

In section 1.6, we recall some results from [DR] and [DRR] for the 2-saddle cycle, that imply that the Abelian integral is a bad approximation of the displacement map, when studying limit cycles bifurcating from the 2-saddle cycle. For general generic unfoldings of the 2-saddle cycle, there are limit cycles not covered by zeroes of the Abelian integral (the so-called alien limit cycles). Even in case that the unfolding breaks only one connection, one finds in [DR] an example with one alien limit cycle. In chapter 5, we see that even the particular subfamily can produce alien limit cycles.

1.5.2 Settings

The limit periodic set Γ is supposed to be a saddle connection at a hyperbolic saddle point s of the Hamiltonian vector field X_H and we suppose that $\left(X_{(\nu,\varepsilon)}\right)_{(\nu,\varepsilon)}$ is a C^∞ unfolding of X_H. We assume that $\lambda = (\nu, \varepsilon) \in \left(\mathbb{R}^{p-1} \times \mathbb{R}, (0,0)\right)$ and that $X_{(\nu,0)} \equiv X_H$. Interest goes to limit cycles, that bifurcate from Γ at the inside of Γ.

By the implicit function theorem, we can suppose that for each parameter value $\lambda \in U$, the point s is a saddle point of X_λ. Let σ respectively τ be transverse sections to the local stable respectively unstable separatrix of s, contained in Γ. Let σ be parametrised by a C^∞ parameter $x \in [0, h_0[$. Suppose that $\Gamma \subset \{H = 0\}$ and that $\{H \geq 0\}$ corresponds to the side where the return map $P_\lambda : [0, h_0[\to [0, h_0[, \forall \lambda \in U$. Denote the Dulac map from σ in τ by D (see section 5.2.2) and the regular transition from τ in σ by R. Then, the Poincaré-map is given by $P = R \circ D$ and the associated displacement map is defined by $\delta(x, \lambda) = P(x, \lambda) - x$. This induces a C^∞ reduced displacement map $\bar{\delta}$ by

$$\delta(x, \lambda) = \varepsilon \bar{\delta}_\lambda(x) = \varepsilon \bar{\delta}(x, \nu, \varepsilon) \tag{1.106}$$

Let υ_λ be its dual 1-form unfolding, then there exists a 1-form $\bar{\upsilon}_\nu$ such that

$$\upsilon_{(\nu,\varepsilon)} = \mathrm{d}H + \varepsilon \bar{\upsilon}_\nu + o(\varepsilon), \varepsilon \to 0.$$

Using the Melnikov formula (section 1.2.2), one has:

$$\bar{\delta}(x, \nu, 0) = -\int_{\Gamma_x} \bar{\upsilon}_\nu \equiv -I_\nu(x). \tag{1.107}$$

The map I_ν is called the *related Abelian integral*. Remark that in this thesis, we use the terminology 'Abelian integral' for any integral I_ν associated to a C^∞ unfolding $\left(X_{(\nu,\varepsilon)}\right)_{(\nu,\varepsilon)}$; originally, this name was reserved for integrals of the type appearing in (1.107), where H and $\bar{\upsilon}_\nu$ are polynomials.

1.5.3 Results

We recall in theorem 119, the asymptotic expansion of δ_λ, found in [R86]. Using the theory of simple asymptotic scale deformation (in particular, theorem 103), one derives from theorem 119 a finite cyclicity result, stated in theorem 121. This result was obtained in [R86], without the introduction of the terminology of simple asymptotic scale deformation, but its proof is an example of the one to prove theorem 103.

In [Mar], using the notion of 'Chebychev property', that is owned by the logarithmic scale, one describes the bifurcation diagram of limit cycles of generic unfoldings of the saddle loop, filled with non-isolated periodic orbits for $\varepsilon = 0$. This result is recalled in theorem 125. From this result, it turns out that the Abelian integral I_ν

is a good approximation for the reduced displacement map $\bar{\delta}_\lambda$, which determines the limit cycles of the considered unfolding. Let us now recall these results.

Let $r(\lambda)$ be the ratio of hyperbolicity of the saddle point s of X_λ (see definition 104); we can write $r(\lambda) = 1 + \varepsilon\alpha(\bar{\lambda}, \varepsilon)$. Here we denote the compensator shortly by ω :

$$\omega = \omega_{\varepsilon\alpha}.$$

An asymptotic expansion for $\bar{\delta}$ at any order N in x was derived in [R86]:

Theorem 119 *There exists a sequence of germs of smooth coefficients: $\alpha_i, \beta_i, i \in \mathbb{N}$ at $\nu = 0, \varepsilon = 0$, such that, for any $N \in \mathbb{N}$:*

$$\begin{aligned}\bar{\delta}(x, \nu, \varepsilon) = \beta_0 + \alpha_1[x\omega + \varepsilon\eta_1] + \beta_1[x + \varepsilon\mu_1] + \dots \\ + \beta_N[x^N + \varepsilon\mu_N] + \alpha_{N+1}[x^{N+1}\omega + \varepsilon\eta_{N+1}] + \Phi_{N+1}(x, \bar{\lambda}, \varepsilon)\end{aligned} \tag{1.108}$$

where Φ_{N+1} is of class C^{N+1} and $(N+1)$-flat at $x = 0, \forall\lambda$, i.e;

$$\forall(\nu, \varepsilon) : \Phi_{N+1}(0, \nu, \varepsilon) = \dots = \frac{\partial^{N+1}}{\partial x^{N+1}}\Phi_{N+1}(0, \nu, \varepsilon) = 0.$$

The functions η_i, μ_i are polynomials in x and $x\omega$ with valuation strictly greater than the leading term of the bracket, and their coefficients are polynomials in the functions α_i, β_i. (The valuation is the weakest degree of the monomials in a polynomial)

Remark 120
1. *The function $\varepsilon\beta_0$ is the x-coordinate on σ of the first intersection of the unstable separatrix, and it is called the breaking parameter of the connection Γ (definition 126).*
2. *One has $\alpha_1 = (1+O(\varepsilon))\alpha$ and the other parameters α_i are related to the normal form of the unfolding X_λ at the saddle point p.*
3. *Expansion (1.108) in theorem 119, is made in a sequence $\mathcal{W}$ of functions (the brackets) of the variable x and depending on the parameter (ν, ε) :*

$$\mathcal{W} = \{1, [x\omega + \dots], [x + \dots], \dots, [x^n + \dots], [x^{n+1}\omega + \dots], \dots\} \tag{1.109}$$

The sequence $\mathcal{W}$ coincides for $\varepsilon = 0$ with the logarithmic scale $\mathcal{L}$; the expansion in the theorem coincides for $\varepsilon = 0$ with the expansion of the Abelian integral (up to its sign):

$$\bar{\delta}(x, \nu, 0) = -I(x, \nu) = \sum_{i=0}^{N}(\bar{\beta}_i(\nu)x^i + \bar{\alpha}_{i+1}(\nu)x^{i+1}\log x) + o(x^N), \tag{1.110}$$

for $x \downarrow 0$, where $\bar{\beta}_i(\nu) = \beta_i(\nu, 0)$ and $\bar{\alpha}_{i+1}(\nu) = \alpha_{i+1}(\nu, 0), 0 \le i \le N$. The fact that the sequence $\mathcal{W}$ is a simple asymptotic scale deformation of $\mathcal{L}$, was proven in [R86] (without introducing this terminology). Let us also notice that theorem 119 produces an expansion of the reduced displacement map $\bar{\delta}(x, \nu, \varepsilon)$ in this simple asymptotic scale deformation at any order, in the sense of definition 98.

Now, as a corollary of theorems 103 and 119, one has the following result, that is proven in [R86] for unfoldings of a saddle loop:

Theorem 121 *Let $(X_\lambda)_\lambda$ be a perturbation of a Hamiltonian vector field along a saddle loop Γ, where $\lambda = (\nu, \varepsilon)$ and let I_ν be the corresponding Abelian integral unfolding. Then, if $codim(I_\nu) < \infty$,*

$$\mathrm{Cycl}(X_\lambda, \Gamma) \leq codim(I_\nu)$$

For the bifurcation diagram (when $\mathcal{F} = \mathcal{L}$), we just mention a result of Mardesic [Mar], which is very hard to prove (contrarily to the regular case and the case of I_ν itself). The problem comes from the fact that the functions $F_i, i \in \mathbb{N}$ in the scale $\mathcal{W} = \{F_i, i \in \mathbb{N}\}$ depend on the parameter α_1 through ω $(\alpha_1 = \alpha)$.

It is based on the fact that the logarithmic scale $\mathcal{L}$ has the Chebychev property:

Definition 122 *Let $\mathcal{F} = \{f_i\}_{i \in \mathbb{N}}$ be a sequence of functions defined on some interval $[0, h_0]$ (for some $h_0 > 0$), such that $\forall i \in \mathbb{N} : f_i$ is smooth on $]0, h_0]$. Then we say that $\mathcal{F}$ has the Chebychev property or is a Chebychev sequence at 0 if and only if for any $n \in \mathbb{N}$ and any $a = (a_0, a_1, \ldots, a_{n-1}) \in \mathbb{R}^n$, the function*

$$f_a \equiv \sum_{i=0}^{n-1} a_i f_i + f_n$$

has at most n zeroes on $[0, h_n]$ for some $0 < h_n < h_0$.

Of course, besides $\mathcal{L}$, the Taylor scale as well has the Chebychev property. In fact, any generic unfolding, expanding in an asymptotic scale of functions with the Chebychev property, has a similar bifurcation diagram at $0 \in \mathbb{R}^+$. Let us make more precise this notion of genericity:

Definition 123 *Let $\mathcal{F} = \{f_i\}_{i \in \mathbb{N}}$ be an asymptotic scale of functions. Let f_ν be an unfolding at $0 \in \mathbb{R}^+, \nu = (\nu_0, \ldots, \nu_{q-1}) \in (\mathbb{R}^q, 0)$, expanding in $\mathcal{F}$, say*

$$f_\nu(x) = \sum_{i=0}^{q} \alpha_i(\nu) f_i(x) + o(f_q).$$

We say that f_λ is a generic unfolding with respect to $\mathcal{F}$ if

1. *$codim_{\mathcal{F}} f_\nu = q$ (i.e. $\alpha_q(0) \neq 0$)*

2. *the map*

$$(\mathbb{R}^q, 0) \rightarrow (\mathbb{R}^q, 0) : \nu \mapsto (\alpha_0(\nu), \ldots, \alpha_{q-1}(\nu))$$

is a local diffeomorphism at 0.

Remark that this definition of generic unfolding differs from the one in the regular case: if

$$\alpha_0 = \ldots = \alpha_{q-1} = 0, \alpha_q \neq 0,$$

then the codimension is q while the order is $q-1$. Now, one has the following results from [Mar]:

Theorem 124 *Let f_ν be a generic unfolding of codimension q in an asymptotic scale of functions with the Chebychev property. Then, for $h_q > 0$ small enough, the bifurcation diagram of zeroes of f_ν on $[0, h_q[$, is topologically equivalent to the bifurcation diagram of zeroes of the universal polynomial of degree q :*

$$P_\beta^\pm(x) = \beta_0 + \ldots + \beta_{q-1}x^{q-1} \pm x^q$$

(+ or − depending on the sign of $f_q(x)$ for $x > 0$ near 0). Here, topologically equivalent means that there exists a local homeomorphism β,

$$\beta : (\mathbb{R}^q, 0) \to (\mathbb{R}^q, 0) : \nu \mapsto (\beta_0(\nu), \ldots, \beta_{q-1}(\nu))$$

such that f_ν and $P^\pm_{\beta(\nu)}$ have the same zeroes on $[0, h_q[$.

Theorem 125 *Let X_λ be a perturbation of a Hamiltonian vector field, along a saddle loop, with $\lambda = (\nu, \varepsilon)$. If the corresponding Abelian integral I_ν is a generic unfolding of codimension q with q parameters (in the sense of definition 123). Then, the bifurcation set $\mathcal{D}iagram(\bar{\delta}_\lambda)$ is topologically equivalent to $\mathcal{D}iagram(I_\nu)\times[0, \varepsilon_0]$, for ε_0 small enough (i.e., the bifurcation sets are homeomorphic by a homeomorphism of the form $(\nu, \varepsilon) \to (G(\nu, \varepsilon), \varepsilon)$).*

By this result, it is natural to define the codimension of the saddle loop unfolding X_λ to be the codimension of the related Abelian integral unfolding, and to say that the unfolding X_λ is generic if and only if the Abelian integral unfolding is generic.

It is an open question if the proof of Mardesic is valuable for any $\mathcal{F}$ with the Chebychev property and any simple asymptotic scale deformation $\mathcal{D}$.

1.6 2-saddle cycle

In this section, we recall a number of results from [DR] on the relation of limit cycles bifurcating from a 2-saddle cycle and zeroes of the related Abelian integral, that is matter of subject in chapter 5.

Recall that Γ is called a k-saddle cycle or a polycycle if Γ is a compact connected curve, made by k hyperbolic saddle points, say $s_1, \ldots, s_k$ and saddle connections of X_H (the eigenvalues of the linear part of $X_{(\nu,0)}$ at the saddle points s_i $(1 \leq i \leq k)$ have a non-zero real part). In figure 1.4, we represent a 2-saddle cycle.

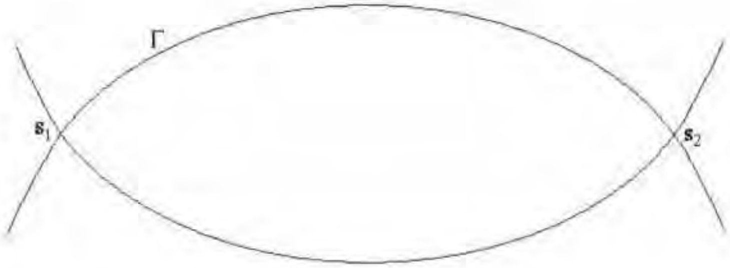

Figure 1.4: 2-saddle cycle Γ

1.6.1 Settings

Throughout section 1.6, $\left(X_{(\nu,\varepsilon)}\right)_{(\nu,\varepsilon)}$ is supposed to be a C^∞ family of planar vector fields such that $X_{(\nu,0)}$ is of center type, $\forall\nu \in W$, where W is an open and bounded set in $\mathbb{R}^p$. By proposition 19, one can suppose that the vector fields $X_{(\nu,0)}$ are Hamiltonian; more precisely, there exist C^∞ functions $H : U \times W_0 \subset \mathbb{R}^2 \times \mathbb{R}^p \to \mathbb{R}, f, g : U \times W_0 \times \left]-E, E\right[\subset \mathbb{R}^2 \times \mathbb{R}^p \to \mathbb{R}$ $(E > 0)$ such that

$$X_{(\nu,\varepsilon)} \leftrightarrow \left\{ \begin{array}{rcl} \dot{x} & = & -\frac{\partial H}{\partial y}(x,y) + \varepsilon f(x,y,\nu,\varepsilon) \\ \dot{y} & = & \frac{\partial H}{\partial x}(x,y) + \varepsilon g(x,y,\nu,\varepsilon) \end{array} \right. \tag{1.111}$$

The Hamiltonian vector field $X_{(\nu,0)}$ is denoted by X_H. In particular, the non-isolated periodic sets of X_H are supposed to have a 2-saddle cycle Γ of X_H as outer boundary of the period annulus.

Let us denote the associated family of 1-forms of $(X_{(\nu,\varepsilon)})_{(\nu,\varepsilon)}$ by $(\upsilon_{(\nu,\varepsilon)})_{(\nu,\varepsilon)}$; there exists a C^∞ family of 1-forms $(\bar{\upsilon}_\nu)_{\nu\in W}$ such that

$$\upsilon_{(\nu,\varepsilon)} = \mathrm{d}H + \varepsilon\bar{\upsilon}_\nu + o(\varepsilon), \varepsilon \to 0. \tag{1.112}$$

If Γ lies on the level curve $\{H = h_0\}$, we denote by Γ_x the periodic orbit of X_H, that belongs to the level curve $\{H = x + h_0\}$, and we consider the Abelian integral

$$I_\nu(x) = \int_{\Gamma_x} \bar{\upsilon}_\nu. \tag{1.113}$$

For $x > 0$, the Abelian integral I_ν is the first order Melnikov function (perhaps up to its sign).

1.6.2 Difference map

In the study of limit cycles bifurcating from a 2-saddle cycle, it is convenient to replace the displacement map δ by the so-called *'difference map'* Δ, which has the same properties as δ : isolated zeroes of $\Delta(\cdot,(\nu,\varepsilon))$ correspond to limit cycles of $X_{(\nu,\varepsilon)}$, and the linear part of Δ with respect to ε, corresponds to the Abelian integral I_ν.

Let us now recall the definition of the difference map Δ. Near the saddles s_1 and s_2, we can use normalizing coordinates (introduced in [R86]), denoted respectively by (x,y) and (z,w). If the ratio of hyperbolicity at s_1 of X_λ (respectively s_2 of $-X_\lambda$) is given by $1+\varepsilon\alpha^{(1)}$ (respectively $1+\varepsilon\alpha^{(2)}$), then the vector fields X_λ respectively $-X_\lambda$ in these coordinates can formally be written as

$$\begin{cases} \dot{y} &= -y \\ \dot{x} &= x(1+\varepsilon\alpha^{(1)}(\nu,\varepsilon)+\varepsilon\sum_{i=1}^{\infty} B_i^{(1)}(\nu,\varepsilon)(xy)^i) \end{cases}$$

respectively

$$\begin{cases} \dot{w} &= -w \\ \dot{z} &= z(1+\varepsilon\alpha^{(2)}(\nu,\varepsilon)+\varepsilon\sum_{i=1}^{\infty} B_i^{(2)}(\nu,\varepsilon)(zw)^i) \end{cases}$$

We consider some transverse sections C_1, C_2, C_3, C_4 corresponding to respectively $\{y=1\},\{x=1\},\{w=1\}$ and $\{z=1\}$ in the normalizing coordinates; $\{w=0\}$ (respectively $\{x=0\}$) is a point on the local stable separatrix of s_2 (respectively s_1).

The difference map Δ is the composition of the transitions D_1 and R_1 (respectively D_2 and R_2) defined by the flow of X_λ (respectively $-X_\lambda$) as follows. The transition map D_1 is the Dulac map at the saddle point s_1 from C_1 to C_2, and the transition map R_1 denotes the regular transition from C_2 to C_4. The transition map D_2 is the Dulac map at the saddle point s_2 from C_3 to C_4, and the map R_2 denotes the regular transition from C_1 to C_3. The precise definition of the Dulac map at a saddle point is recalled in chapter 5 (section 5.2.2), where we study the its coefficients in a simple asymptotic scale deformation of the enlarged logarithmic scale $\mathcal{L}^e$.

Let Δ_1 (respectively Δ_2) be the transition map from C_1 to C_4, defined by the flow of X_λ (respectively $-X_\lambda$):

$$\Delta_1 = R_1 \circ D_1 \text{ and } \Delta_2 = D_2 \circ R_2.$$

Then the difference map Δ is defined as the difference

$$\Delta = \Delta_2 - \Delta_1.$$

For $\varepsilon = 0$, the vector field is Hamiltonian, and the Hamiltonian function H is equal to xy and zw respectively in the normalizing coordinates near the saddle points s_1 and s_2. It follows that $\Delta_i - Id$ $(i=1,2)$ and Δ are divisible by ε; hence, there exist C^∞ functions $\bar{\Delta}_i$ $(i=1,2)$ and $\bar{\Delta}$ such that

$$\Delta_i(x,\nu,\varepsilon) = x + \varepsilon\bar{\Delta}_i(x,\nu,\varepsilon), \qquad \forall i=1,2$$

and

$$\Delta(x,\nu,\varepsilon) = \varepsilon\bar{\Delta}(x,\nu,\varepsilon) = \varepsilon\left(\bar{\Delta}_2(x,\nu,\varepsilon) - \bar{\Delta}_1(x,\nu,\varepsilon)\right)$$

The map $\bar{\Delta}$ is called the *reduced difference map.*

Let us now establish that the Abelian integral I_ν is the first order term in ε of the difference map Δ, as is the case for the displacement map (up to a minus sign). The difference map Δ is related to the return map $P(w,\lambda)$ on the section C_4, as follows, for $\lambda = (\nu, \varepsilon)$:

$$P(\Delta_2(x,\lambda),\lambda) - \Delta_2(x,\lambda) = \Delta_1(x,\lambda) - \Delta_2(x,\lambda) = -\Delta(x,\lambda). \tag{1.114}$$

From (1.114), we find by letting $\varepsilon \to 0$:

$$I_\nu(x) = \bar{\Delta}(x,\nu,0). \tag{1.115}$$

To describe the non-differentiability type of the difference map Δ, we have to consider two compensators ω_1 and ω_2 associated to $-X_\lambda$ and X_λ, respectively at the saddle points s_1 and s_2 :

$$\omega_1 = \omega_1(x,\nu,\varepsilon) = \frac{x^{\varepsilon\alpha^{(1)}} - 1}{\varepsilon\alpha^{(1)}} \text{ and } \omega_2 = \omega_2(z,\nu,\varepsilon) = \frac{z^{\varepsilon\alpha^{(2)}} - 1}{\varepsilon\alpha^{(2)}}.$$

The advantage of the use of the difference map Δ instead of the displacement map, is that the compensators ω_1 and ω_2 are not mixed since one just takes the difference of Δ_1 and Δ_2.

1.6.3 Problem of transfer

The result from [Mar], that is recalled in theorem 125 (section 1.5), shows that all limit cycles, bifurcating from a saddle loop, filled with non-isolated periodic orbits, can be traced by zeroes of the related Abelian integral. In [DR], using results from [DRR] and [Mo], it is shown that, in case of a polycycle, not all limit cycles are covered by zeroes of the Abelian integral, as soon as the number of saddles is at least 2 and if the unfolding breaks more than one connection. Therefore, the notions of codimension as well as genericity will not be the same for a vector field unfolding and its related Abelian integral unfolding. Here, we recall this phenomenon for unfoldings of 2-saddle cycles, breaking both connections of Γ at the bifurcation value $\varepsilon = 0$, in detail. In this case, the Abelian integral (or, linear approximation) is a bad approximation of the reduced difference map $\bar{\Delta} = \frac{1}{\varepsilon}\Delta$: the Abelian integral has only one zero while the difference map $\bar{\Delta}$ has at least 2 zeroes, which are associated to the limit cycles bifurcating from Γ. There is at least one limit cycle (a so-called 'alien limit cycle'), that can not be traced by the Abelian integral.

Under specific generic conditions given by Mourtada, the cyclicity of the Hamiltonian 2-saddle cycle is at least 2. However, these generic conditions imply that there is exactly one zero of the corresponding Abelian integral, that bifurcates from $x = 0$. To describe the generic conditions, we recall the definition of breaking parameter associated to a given connection of two saddle points. The breaking parameter measures how much the connection is broken; the breaking parameter is zero, if and only if the connection is unbroken.

Definition 126 *Let Σ be a transverse section to the separatrix Γ_0 of Γ of X_H, connecting the saddle p_1 with p_2. Then, for λ near 0, the local stable separatrix of p_2 cuts Σ_0 transversally at a point $S(\lambda)$ and the local unstable separatrix of p_1 cuts Σ_0 at $U(\lambda)$. The breaking parameter b of Γ_0 is then defined as the function $b(\lambda) = S(\lambda) - U(\lambda)$.*

Let us now denote by $b^{(1)}$ and $b^{(2)}$ the breaking parameters for X_λ at C_1 and for $-X_\lambda$ at C_4 respectively. Then the transition maps Δ_i $(i = 1, 2)$ have the following expansions:

$$\Delta_i(x, \lambda) = b^{(i)}(\lambda) + (r^{(i)}(\lambda) - 1)x\omega_i + O(x), x \downarrow 0, \tag{1.116}$$

uniformly in λ. From (1.114), it is clear that it is more easy to derive an expansion for the difference map Δ than for the displacement map δ, given expansions for Δ_1 and Δ_2.

The complete bifurcation diagram for generic 2-parameter unfoldings of 2-saddle cycles is described for instance in [DRR]. In particular, it is proven that the cyclicity of a generic 2-parameter unfolding of a 2-saddle cycle is bounded by 2.

Let us now recall the definition of genericity by Mourtada: a 2-*parameter unfolding* X_λ *is called generic* if

1. the map $\lambda \to (b^{(1)}(\lambda), b^{(2)}(\lambda))$ is a local diffeomorphism at $0 \in \mathbb{R}^2$ (in particular, one can suppose that $\lambda = b = (b^{(1)}, b^{(2)})$;

2. certain algebraic generic conditions on the values $r^{(i)}(0), i = 1, 2$ are satisfied, e.g.,

$$r^{(1)}(0) \neq 1, r^{(2)}(0) \neq 1, r^{(2)}(0)/r^{(1)}(0) \neq 1$$

Returning to the unfolding of a Hamiltonian 2-saddle cycle Γ, the generic conditions of Mourtada are translated into generic conditions for the Hamiltonian unfolding as follows. A perturbation X_λ of X_H is said to be a *generic 2-parameter unfolding* if

1. $\lambda = (\nu, \varepsilon) \in \mathbb{R}^3$ with $\nu \in (\mathbb{R}^2, 0)$ and $\varepsilon \in [0, \varepsilon_0[$;

2. the breaking parameters have the form $b^{(i)}(\nu, \varepsilon) = \varepsilon\beta^{(i)}, i = 1, 2$ and $\nu = \beta = (\beta^{(1)}, \beta^{(2)})$. The parameters $\beta^{(i)}$ $(i = 1, 2)$ are also called the *reduced breaking parameters*;

3. the hyperbolicity ratios have the form $r^{(i)}(\beta, \varepsilon) = 1 + \varepsilon\alpha^{(i)}(\beta, \varepsilon), i = 1, 2$ and certain algebraic conditions are supposed on $\alpha^{(i)}(0), i = 1, 2$, such that for $\varepsilon > 0$, the generic conditions of Mourtada on the ratios $r^{(i)}(\beta, \varepsilon), i = 1, 2$ are satisfied. E.g.,

$$\alpha^{(1)}(0) \neq 0, \alpha^{(2)}(0) \neq 0, \alpha^{(2)}(0) - \alpha^{(1)}(0) \neq 0.$$

Using these notations, it follows from (1.115) and (1.116) that the Abelian integral has the following expansion:

$$I_{\bar{\lambda}}(x) = \beta^{(2)} - \beta^{(1)} + (\alpha^{(2)}(\beta,0) - \alpha^{(1)}(\beta,0))x\log x + O(x), x \downarrow 0. \quad (1.117)$$

Now, the fact that the reduced breaking parameters $\beta^{(1)}$ and $\beta^{(2)}$ are assumed to be independent parameters and the generic condition

$$\alpha^{(1)}(0) - \alpha^{(2)}(0) \neq 0,$$

imply that the Abelian integral I_ν is a generic unfolding of codimension 1 (with respect to the logarithmic scale). By theorem 124, we know that there is exactly 1 zero of the Abelian integral that bifurcates from $x = 0$.

On the other hand, for any fixed small $\varepsilon > 0$, the unfolding $X_{(\beta,\varepsilon)}$ is generic in the sense of Mourtada, and therefore it can have a cyclicity equal to 2. In other words, for each $\varepsilon \downarrow 0$, one can find a sequence $(\beta_\varepsilon^n)_{n\in\mathbb{N}}$ converging to $0 \in \mathbb{R}^2$ such that each $X_{(\beta_\varepsilon^n,\varepsilon)}$ has 2 limit cycles which converge to Γ, when $n \to \infty$. Given a sequence $(\varepsilon_m)_m$ with $\varepsilon_m \downarrow 0$ $(m \to \infty)$, we can extract by the diagonalisation method a subsequence $(\beta_{\varepsilon_m}^{n(m)})_{m\in\mathbb{N}}$ converging to $0 \in \mathbb{R}^2$, such that the corresponding 2 limit cycles converge to Γ when $m \to \infty$. Hence, the cyclicity of the Hamiltonian unfolding $X_{(\beta,\varepsilon)}$ is at least 2.

As a conclusion, when one breaks more than one connection, the Abelian integral I_β is a very bad approximation of the displacement map $\bar{\delta}_{(\beta,\varepsilon)}$, which enters in the expression of the Poincaré-map:

$$P_{(\beta,\varepsilon)}(x) = x + \varepsilon\bar{\delta}_{(\beta,\varepsilon)}(x).$$

The Abelian integral has only one zero, while the displacement map has at least 2 zeroes that correspond to limit cycles bifurcating from Γ. Hence, not all limit cycles can be traced by the zeroes of the corresponding Abelian integral, when the unfolding breaks more than one connection of the 2-saddle cycle. There is at least one alien limit cycle.

1.6.4 Unfoldings of a 2-saddle cycle, that leave one connection unbroken

In this section, one restricts to unfoldings of the 2-saddle cycle of type (1.111), that leave one connection unbroken, because the treatment of the general case is not yet obvious, and this case looks similar to the saddle loop case. Despite this resemblance, the study is much more complicated than the saddle loop case. Moreover, in [DR], it is shown that there exist generic unfoldings of the 2-saddle cycle, for which one connection remains unbroken, of codimension 4 with 4 limit cycles, while the related Abelian integral only has 3 zeroes. Hence, there is one limit cycle that does not correspond to zeroes of the Abelian integral (a so-called alien limit cycle).

This section contains the following parts of [DR]. In case the unfolding breaks only one connection, one can obtain an asymptotic expansion for the difference map Δ in the same way as for the saddle loop. By the occurence of two different compensators, that degenerate in the same way for $\varepsilon \to 0$, this asymptotic expansion cannot be used to prove a finite cyclicity result, as in case of the saddle loop. To avoid this degeneracy phenomenon, one rearranges the expansion by introduction of new compensators. This new asymptotic expansion permits to prove a finite cyclicity result.

Asymptotic expansion of Δ in terms of ω_1 and ω_2

Theorem 127 recalls an asymptotic expansion of Δ in terms of $1, x^i\omega_1^{i-j}, x^i\omega_2^{i-j}$ $(0 \le j \le i)$ and $x^i, i \ge 1$. Unlike the Abelian integral, where we encounter only linear terms in $\log x$, we also meet unfoldings of $\log^i x, i \ge 2$.

Using results from [R86] again, one obtains asymptotic expansions for Δ_1 (or $\bar{\Delta}_1$) and Δ_2 (or $\bar{\Delta}_2$) at any order, in terms of $\omega_1(x, \bar{\lambda}, \varepsilon)$ and $\omega_2(z, \bar{\lambda}, \varepsilon)$ respectively, similar to expansion (1.108) for the map $\bar{\delta}$ in theorem 119 in section 1.5. But for $\bar{\Delta} = \bar{\Delta}_2 - \bar{\Delta}_1$, there is no reason to preserve the special grouping of terms in brackets appearing in expansion (1.108) for $\bar{\delta}$, and having the leading terms linear in the compensator. The asymptotic expansions that we can write for $\bar{\Delta}$, just have the property that their principal parts are polynomials in $x, x\omega_1, x\omega_2$ at any order N :

Theorem 127 *Let $\bar{\Delta}$ be the reduced difference function, associated to an unfolding of a 2-saddle cycle, which breaks just one connection. There exists a sequence of germs at $\nu = 0, \varepsilon = 0$ of smooth functions $\alpha_{ijk}(\nu, \varepsilon)$, $i, j, k \in \mathbb{N}$, such that for any $N \in \mathbb{N}$, one has the following expansion at order N :*

$$\bar{\Delta}(x, \nu, \varepsilon) = \sum_{i+j+k=0}^{N} \alpha_{ijk}(\nu, \varepsilon) x^i (x\omega_1)^j (x\omega_2)^k + \Psi_N(x, \nu, \varepsilon). \tag{1.118}$$

The remainder Ψ_N is of class C^N and flat at order N in $x = 0$, for all (ν, ε).

Remark 128 *In fact, each monomial in the principal part of the expansion (1.118) contains at most one of the compensators (the compensators are not mixed, since ω_1 and ω_2 come from $\bar{\Delta}_1$ and $\bar{\Delta}_2$ respectively and $\bar{\Delta} = \bar{\Delta}_2 - \bar{\Delta}_1$), but this is without importance in this work.*

The important difference between the saddle loop case which we have recalled in section 1.5 and the case of the 2-saddle cycle is that we have now two different compensators converging toward the same function $\log x$, when $\varepsilon \to 0$. Taking $\varepsilon = 0$ in $\bar{\Delta}$, every monomial $x^i (x\omega_1)^j (x\omega_2)^k$, corresponding to a given value of i and $\ell = j + k$, converges towards the same function $x^{i+l} \log^\ell x$. This degeneracy phenomenon is the obstruction in using expansion (1.118) to prove directly a result similar to theorem 103.

New asymptotic expansion of Δ with introduction of new compensators

In [DR], one finds a way to avoid this degeneracy phenomenon: one rearranges the expansion by the introduction of new compensators. We here recall this new expansion of the reduced difference map.

Performing a (parameter dependent) similarity in the coordinates (z, w), we can accomplish that

$$\frac{\partial R_2}{\partial x}(0, \nu, \varepsilon) = 1, \text{ i.e. } R_2(x, \nu, \varepsilon) = x + O\left(x^2\right), x \to 0. \tag{1.119}$$

On the other hand, we can write:

$$R_1(y, \nu, \varepsilon) = \varepsilon u_0 + (1 + \varepsilon u)\, y(1 + O(y)), y \to 0 \tag{1.120}$$

From now on, every coefficient we introduce, such as $u, u_0, \alpha^{(1)}, \alpha^{(2)}, \ldots$ are supposed to be smooth functions of the parameter (ν, ε). The Dulac maps have the following expansions:

$$D_1(x, \nu, \varepsilon) = x + \varepsilon\alpha^{(1)} x\omega_1 + O(x^2\omega_1^2), x \to 0, \tag{1.121}$$

$$D_2(z, \nu, \varepsilon) = z + \varepsilon\alpha^{(2)} z\omega_2 + O(z^2\omega_2^2), z \to 0. \tag{1.122}$$

To simplify the notations we don't write the dependence of Δ_i, R_i $(i = 1, 2)$ or Δ on (ν, ε). We start with the expansion of $D_2 \circ R_2$. From (1.122), it follows that

$$D_2 \circ R_2(x) = R_2(x) + \varepsilon\alpha^{(2)} R_2(x)\omega_2(R_2(x), \varepsilon\alpha^{(2)}) + O(R_2^2(\omega_2 \circ R_2)^2), \tag{1.123}$$

for $x \to 0$. Using lemma 112, we obtain from (1.119) and (1.123) the following expansion for Δ_2 :

$$\Delta_2(x) = D_2 \circ R_2(x) = x + \varepsilon\alpha^{(2)} x\omega_2 + O(x^2\omega_2^2), x \to 0. \tag{1.124}$$

On the other hand, from (1.120) and (1.121) we obtain directly the following expansion for Δ_1 :

$$\Delta_1(x) = R_1 \circ D_1(x) = \varepsilon u_0 + \varepsilon\alpha^{(1)}(1 + \varepsilon u)\, x\omega_1 + u_1 x + O(x^2\omega_1^2), \tag{1.125}$$

for $x \to 0$. Combining (1.124) and (1.125), we find an expansion for $\Delta = \Delta_2 - \Delta_1 = \varepsilon\bar{\Delta}$, and hence after division by ε, one finds:

$$\bar{\Delta}(x) = -u_0 + \alpha^{(2)} x\omega_2 - \alpha^{(1)}(1 + \varepsilon u)x\omega_1 - ux + O(x^2|\omega|^2), x \to 0, \tag{1.126}$$

where $|\omega| = \sup\{|\omega_1|, |\omega_2|\}$.

From theorem 127, we know that the remainder $O(x^2|\omega|^2)$ can be expressed at any order N as: $Q_N(x, x\omega_1, x\omega_2) + \Phi_N(x)$, where Q_N is a polynomial of valuation ≥ 2 in $x, x\omega_1$ and $x\omega_2$ and Φ_N is of differentiability class $\mathcal{C}^N$ and N-flat at $x = 0$. Therefore, putting $\beta = -u_0$ in (1.126), one obtains:

$$\bar{\Delta}(x) = \beta - \alpha^{(1)}(1 + \varepsilon u)x\omega_1 + \alpha^{(2)} x\omega_2 - ux + Q_N + \Phi_N. \tag{1.127}$$

In order to remove the degeneracy in the first three terms (caused by $x\omega_1$ and $x\omega_2$), the compensator ω_{2-1} is used to transform the first terms in the expansion of $\bar{\Delta}$ into the terms $1, x^{1+\varepsilon\alpha^{(1)}}\omega_{2-1}$ and $x^{1+\varepsilon\alpha^{(1)}}$. Since these last terms converge towards $1, x\omega_{2-1}$ and x respectively, when $\varepsilon \to 0$, there is no degeneracy anymore in the first three terms, and one finds:

$$\bar{\Delta}(x) = \beta + (\alpha^{(2)} - \alpha^{(1)})x^{1+\varepsilon\alpha^{(1)}}\omega_{2-1} - ux^{1+\varepsilon\alpha^{(1)}} + R(x). \tag{1.128}$$

To obtain these first three terms in (1.128), the definition of the compensators ω_1, ω_2 and ω_{2-1} and their relation are exploited as follows:

$$\begin{aligned} \varepsilon u\alpha^{(1)}x\omega_1 &= u(x^{1+\varepsilon\alpha^{(1)}} - x) \\ &= ux^{1+\varepsilon\alpha^{(1)}} - ux \end{aligned} \tag{1.129}$$

and

$$\begin{aligned} \varepsilon\alpha^{(2)}x\omega_2 - \varepsilon\alpha^{(1)}x\omega_1 &= x^{1+\varepsilon\alpha^{(2)}} - x^{1+\varepsilon\alpha^{(1)}} \\ &= x^{1+\varepsilon\alpha^{(1)}}(x^{\varepsilon(\alpha^{(2)}-\alpha^{(1)})} - 1) \\ &= x^{1+\varepsilon\alpha^{(1)}}\varepsilon(\alpha^{(2)} - \alpha^{(1)})\omega_{2-1} \end{aligned}$$

Or, after division by ε :

$$\alpha^{(2)}x\omega_2 - \alpha^{(1)}x\omega_1 = (\alpha^{(2)} - \alpha^{(1)})x^{1+\varepsilon\alpha^{(1)}}\omega_{2-1} \tag{1.130}$$

Using (1.130) and (1.129), one find

$$\begin{aligned} &\alpha^{(2)}x\omega_2 - \alpha^{(1)}(1 + u\varepsilon)x\omega_1 \\ &= (\alpha^{(2)}x\omega_2 - \alpha^{(1)}x\omega_1) - \alpha^{(1)}u\varepsilon x\omega_1 \\ &= (\alpha^{(2)} - \alpha^{(1)})x^{1+\varepsilon\alpha^{(1)}}\omega_{2-1} - ux^{1+\varepsilon\alpha^{(1)}} + ux \end{aligned} \tag{1.131}$$

Next, to remove the degeneracy in the terms of order $O\left(x^2|\omega|^2\right), x \to 0$, where $|\omega| = \max\{|\omega_1|, |\omega_2|\}$, we remark that, for each $N \in \mathbb{N}$ $(N \geq 2)$, the remainder R can be written as:

$$R(x) = Q_N(x, x\omega_1, x\omega_2) + \Phi_N(x)$$

where Q_N is a polynomial of valuation ≥ 2 in $x, x\omega_1, x\omega_2$ and Φ_N is a C^N function that is flat of the order N at $x = 0$. Hence, we are only left with the treatment of the function Q_N, where the degeneracy comes from the presence of both ω_1 and ω_2. The idea now is to introduce the compensator ω_{21} and to eliminate one of those two compensators, for instance ω_2. By the definition of ω_{21}, it is clear that

$$\omega_2 = \omega_1 + \varepsilon(\alpha^{(2)} - \alpha^{(1)})\omega_{21} \tag{1.132}$$

Substituting this relation into Q_N, we obtain

$$Q_N(x, x\omega_1, x\omega_2) = Q_N(x, x\omega_1, x\omega_1) + \varepsilon\tau x\omega_{21} F_N(x, x\omega_1, x\omega_{21}) \tag{1.133}$$

where

$$\tau = \alpha^{(2)} - \alpha^{(1)}$$

and F_N is a polynomial of valuation ≥ 2. Define

$$R_N(x_1, x_2) = Q_N(x_1, x_2, x_2);$$

then R_N is a polynomial of valuation ≥ 2. We can now rewrite the expansion given in (1.128) as follows:

$$\bar{\Delta}(x) = \beta + \tau[x^{1+\varepsilon\alpha^{(1)}}\omega_{2-1} + \varepsilon x\omega_{21}F_N] - ux^{1+\varepsilon\alpha^{(1)}} + R_N(x, x\omega_1) + \Phi_N(x) \tag{1.134}$$

In this way, the degeneracy phenomenon can be removed up to any given order N.

On the other hand, since $R_N(x, x\omega_1)$ is a polynomial in x and $x\omega_1$ of valuation ≥ 2, and F_N is a polynomial of $x, x\omega_1$ and $x\omega_{21}$ of valuation ≥ 1, we have

$$\begin{aligned}
&x^{1+\varepsilon\alpha^{(1)}}\omega_{2-1} + \varepsilon x\omega_{21}F_N(x, x\omega_1, x\omega_{21}) \\
&= x^{1+\varepsilon\alpha^{(1)}}\omega_{2-1}\left(1 + \varepsilon x^{-\varepsilon\alpha^{(1)}}\omega_{2-1}^{-1}\omega_{21}F_N(x, x\omega_1, x\omega_{21})\right) \\
&= x^{1+\varepsilon\alpha^{(1)}}\omega_{2-1}(1 + O)
\end{aligned}$$

where the notation O is introduced for functions of the type specified in definition 115. Hence, equation (1.134) gives an expansion of $\bar{\Delta}$ in the (ordered) sequence $\mathcal{W} = \{F_{ij} : 0 \leq j \leq i, i \in \mathbb{N}\}$, defined by

$$\begin{aligned}
F_{00} = 1, F_{11} = x^{1+\varepsilon\alpha^{(1)}}\omega_{2-1}(1 + O), F_{10} = x^{1+\varepsilon\alpha^{(1)}} \text{ and} \\
F_{i,i-j} = x^i\omega_1^{i-j}, \forall 0 \leq j \leq i, \forall i \geq 2
\end{aligned} \tag{1.135}$$

where the functions in the right-hand side are evaluated at (x, ν, ε). Summarizing, the following result from [DR] is proven:

Theorem 129 *Let $X_{(\nu,\varepsilon)}$ be any unfolding of Hamiltonian perturbation type, along a 2-saddle cycle Γ. We suppose that the unfolding leaves one connection unbroken. Then, there exists a sequence of functions $\mathcal{W}$:*

$$F_{i,i-j}(x, \nu, \varepsilon), \ 0 \leq j \leq i, i \in \mathbb{N},$$

defined in (1.135), which are polynomials in $x, x\omega_1, x\omega_{2-1}, x\omega_{21}$ with coefficients smooth in the parameter (ν, ε) such that

$$F_{i,i-j}(x, \nu, 0) = f_{i,i-j}(x) = x^i \log^{i-j} x,$$

and such that for any $N \in \mathbb{N}$ one has an expansion of $\bar{\Delta}$ as follows:

$$\bar{\Delta}\left(x,\nu,\varepsilon\right)=\sum_{0\leq j\leq i\leq N}\alpha_{i,i-j}(\nu,\varepsilon)F_{i,i-j}(x,\nu,\varepsilon)+\Psi_N(x,\nu,\varepsilon) \tag{1.136}$$

The coefficients α_{ij} are smooth functions of the parameter and the remainder Ψ_N is C^N and flat of order N at $x=0$, for all (ν,ε).

Notice that, for $\varepsilon=0$, expansion (1.134) reduces to

$$I_\nu\left(x\right)=\bar{\Delta}\left(x,\nu,0\right)=\bar{\beta}+(\bar{\alpha}^{(2)}-\bar{\alpha}^{(1)})x\log x-\bar{u}x+\bar{R}_N(x,x\log x)+\bar{\Phi}_N(x),$$

where $\bar{\beta}\equiv\left.\beta\right|_{\varepsilon=0},\bar{\alpha}^{(i)}\equiv\left.\alpha^{(i)}\right|_{\varepsilon=0},i=1,2,\bar{u}\equiv\left.u\right|_{\varepsilon=0}$ and $\bar{R}_N\equiv\left.R_N\right|_{\varepsilon=0},\ \bar{\Phi}_N\equiv\left.\Phi_N\right|_{\varepsilon=0}$. This is the expected expansion of $I_{\bar{\lambda}}$ in the logarithmic scale $\mathcal{L}$, proven in proposition 92.

Simple asymptotic scale deformation

For $\varepsilon=0$, the sequence $\mathcal{W}$ coincides with the standard enlarged logarithmic scale $\mathcal{L}^{Se}$. However, the sequence $\mathcal{W}$ probably is not a simple asymptotic scale deformation. In fact, if one tries to prove it, at some point one has to control linear combinations of $x^{\varepsilon\alpha^{(1)}}$ and $x^{\varepsilon(\alpha^{(2)}-\alpha^{(1)})}$ (coming from the derivations of ω_1 and ω_{2-1} respectively). But, fortunately enough, after three applications of the algorithm of division-derivation to $\bar{\Delta}$ (with the operator $\nabla=x\frac{\partial}{\partial x}$ used instead of $\frac{\partial}{\partial x}$), one can make the compensator ω_{2-1} to disappear from the formulas. In fact, it remains hidden in terms O.

Proposition 130 *After three steps in the division-derivation algorithm (each step consists in the division by a function, smooth and positive for $x>0$, followed by the derivation by ∇, starting with the function $\bar{\Delta}$), one obtains a new function, $\bar{\Delta}^3$, with expansions at any order in the sequence $\mathcal{W}_3$ given by*

$$\mathcal{W}_3=\{x^i\omega_1^{i-j}(1+G_{ij}(\omega_1^{-1}))(1+O),\ 0\leq j\leq i,\ i\geq 2\}, \tag{1.137}$$

where the functions $G_{ij}(u)$ are rational functions of u and $G_{ij}(0)=0$.

Proof. It suffices to perform the three successive steps in the division-derivation algorithm starting from the sequence $\mathcal{W}=\{F_{ij}:0\leq j\leq i,i\in\mathbb{N}\}$ given in (1.135). Using the derivation properties of the algebra O (lemma 118), we will deduce the elements of each sequence up to the relation $\approx$.

First step One divides the sequence $\mathcal{W}_0\equiv\mathcal{W}$ by $x^{1+\varepsilon\alpha^{(1)}}$ and one then derives this sequence by ∇ to obtain the sequence $\mathcal{W}_1$:

$$\mathcal{W}_1=\{x^{-1-\varepsilon\alpha^{(1)}},x^{\varepsilon(\alpha^{(2)}-\alpha^{(1)})}(1+O),\ 0,\ x^{i-1-\varepsilon\alpha^{(1)}}\omega_1^{i-j}(1+G_{ij}^1(\omega_1^{-1}))\}$$

where G_{ij}^1 are rational functions with $G_{ij}^1(0)=0$, for $i\geq 2,\ 0\leq j\leq i$.

Second step One divides the sequence $\mathcal{W}_1$ by the second function

$$x^{\varepsilon(\alpha^{(2)}-\alpha^{(1)})}(1+O)$$

and one then derives this sequence by ∇, to obtain the sequence $\mathcal{W}_2$:

$$\mathcal{W}_2 = \{x^{-1-\varepsilon\alpha^{(2)}}(1+O),\ 0,\ 0,\ x^{i-1-\varepsilon\alpha^{(2)}}\omega_1^{i-j}(1+G_{ij}^2(\omega_1^{-1}))(1+O)\} \quad (1.138)$$

where G_{ij}^2 are rational functions with $G_{ij}^2(0)=0$, for $i \geq 2,\ 0 \leq j \leq i$.

Third step One divides the sequence $\mathcal{W}_2$ by the first function

$$x^{-1-\varepsilon\alpha^{(2)}}(1+O)$$

and one then derives this sequence by ∇, to obtain the sequence $\mathcal{W}_3$:

$$\{0,\ 0,\ 0,\ x^i\omega_1^{i-j}(1+G_{ij}(\omega_1^{-1}))(1+O),\ i \geq 2,\ 0 \leq j \leq i\} \quad (1.139)$$

where G_{ij} are rational functions with $G_{ij}(0) = 0,\ i \geq 2,\ 0 \leq j \leq i$. The 0's are maintained in the successive sequences to give more transparancy to the operations. If one now applies these successive steps to the function $\bar{\Delta}$ itself, expanded in an asymptotic series, one obtains an asymptotic series expansion of the resulting function $\bar{\Delta}^3$, in the sequence $\mathcal{W}_3$. ■

One can now deduce a simple asymptotic scale deformation from the scale $\mathcal{W}_3$ obtained in proposition 130:

Proposition 131 *Consider the sequence* $\mathcal{W}_3$:

$$\mathcal{W}_3 = \left\{x^i\omega_1^{i-j}(1+G_{ij}(\omega_1^{-1}))(1+O)\right|\ 0 \leq j \leq i,\ i \geq 2\Big\},$$

where each $G_{ij}(u)$ *is a rational function in* u *with* $G_{ij}(0)=0$. *After division by the first function in this sequence,*

$$x^2\omega_1^2(1+G_{20}(\omega_1^{-1}))(1+O),$$

one obtains a sequence $\mathcal{W}^e$ *that is a simple asymptotic scale deformation of the simple asymptotic scale* $\mathcal{L}^e = \left\{x^i \log^{i-j} x : i \in \mathbb{N}, 0 \leq j \leq i+2\right\}$:

$$\mathcal{W}^e = \{x^i\omega_1^{i-j}(1+\tilde{G}_{ij}(\omega_1^{-1}))(1+O)\Big|\ 0 \leq j \leq i+2,\ i \geq 0\},$$

where each $\tilde{G}_{ij}(u)$ *is a rational function in* u *with* $\tilde{G}_{ij}(0)=0$.

Proof. First observe that (due to lemma 107) for $\varepsilon \to 0$, the sequence $\mathcal{W}^e$ reduces to the simple asymptotic scale $\mathcal{L}^e$.

The first three conditions in definition 93 are trivially verified. The quotient of two consecutive functions in the sequence $\mathcal{W}^e$ takes the form $\omega_1^{-1}(1+G(\omega_1^{-1}))(1+O)$

or $x\omega_1^{i+3}(1 + G(\omega_1^{-1})(1 + O)$, where G is a rational function, $G(0) = 0$, and $i \in \mathbb{N}$. From Lemma 107, it follows that these ratios go to zero when $(x, \varepsilon) \to 0$, uniformly in $\bar{\lambda}$. The derivation by ∇ of these ratios gives functions equivalent (in the $\approx$ sense) to $-\omega_1^{-2}x^{\varepsilon\alpha^{(1)}}(1 + \tilde{G})(1 + O)$ and $x\omega_1^{i+3}(1 + \tilde{G})(1 + O)$ respectively with a constant sign for $x > 0$ small enough. So, also the fourth condition in definition 93 is verified.

We now proceed to prove that this asymptotic scale deformation is simple (definition 94). To simplify the notations, we will just write the principal term for each function in the different sequences. In fact, these functions are obtained by multiplying the principal term P by a factor $M = (1 + G(\omega_1^{-1}))(1 + O)$. The operation of division and derivation transforms this term into a similar one. Precisely, from lemma 118 part 3, one has that $\nabla(PM) \approx (\nabla P)\tilde{M}$, where $\tilde{M}$ is similar to M. Moreover, we indicate the functions just up to the relation $\approx$. We begin with the sequence $\mathcal{W}^e$:

$$\mathcal{W}_0^e \equiv \mathcal{W}^e \approx \{1, \omega_1^{-1}, \omega_1^{-2}; x^i\omega_1^{i-j} \Big| 0 \le j \le i+2, i \ge 1\} \tag{1.140}$$

Application of the operator ∇ gives

$$\{0, \omega_1^{-2}x^{\varepsilon\alpha^{(1)}}, \omega_1^{-3}x^{\varepsilon\alpha^{(1)}}; x^i\omega_1^{i-j} \Big| 0 \le j \le i+2, i \in \mathbb{N}_1\} \tag{1.141}$$

We now divide by the first function to obtain

$$\mathcal{W}_1^e \approx \{1, \omega_1^{-1}; x^{i-\varepsilon\alpha^{(1)}}\omega_1^{i+2-j} \Big| 0 \le j \le i+2, i \in \mathbb{N}_1\} \tag{1.142}$$

Next (after derivation by ∇ and division), we will find

$$\begin{aligned}
\mathcal{W}_2^e &\approx \{1; x^{i-2\varepsilon\alpha^{(1)}}\omega_1^{i+4-j} \Big| 0 \le j \le i+2, i \in \mathbb{N}_1\} \\
\mathcal{W}_3^e &\approx \{1, \omega_1^{-1}, \omega_1^{-2}, \omega_1^{-3}, x\omega_1, x, x\omega_1^{-1}, x\omega_1^{-2}, x\omega_1^{-3}, x^2\omega_1^2, \ldots\} \\
&\approx \{1; x^i\omega_1^{i-j} \Big| 0 \le j \le i+3, i \in \mathbb{N}\}, \text{ and so on.}
\end{aligned}$$

Let us again define for $r \in \mathbb{N}$, the index $J(r)$ as the order of the function $f_{rr} = x^r \log^r x$ in the scale $\mathcal{L}^e$ as in (1.83). Then we can prove by induction on $r \in \mathbb{N}$, that $\forall r \in \mathbb{N}, \forall 0 \le s \le r+2$, the principal parts of the functions in the sequence $\mathcal{W}_{J(r)+s}^e$ are given by

$$\left\{1, \omega_1^{-1}, \omega_1^{-2}, \ldots, \omega_1^{s-(r+2)}; x^{i-s\varepsilon\alpha^{(1)}}\omega_1^{i+2s-j} \Big| 0 \le j \le i+r+2, i \ge 1\right\}$$

Therefore, as the sequence $\mathcal{W}_0^e$, each sequence $\mathcal{W}_i^e$ $(i \in \mathbb{N})$ verifies the conditions of definition 93, once that one restricts x to some interval $[0, h_i[$. ■

Finite cyclicity

As we have already seen in (1.115), the Abelian integral is given by

$$I(x, \nu) \equiv \bar{\Delta}(x, \nu, 0)$$

In previous sections, we have recalled from [DR] that, $\bar{\Delta}$ expands in the enlarged asymptotic scale $\mathcal{L}^e$. However, the coefficients according to $x^i \log^j x, j \geq 2$ in $\bar{\Delta}$, are all divisible by ε; therefore, those terms $x^i \log^j x, j \geq 2$ disappear in the expansion of I. The consequence will be that now the Abelian integral cannot completely control the zeroes of $\bar{\Delta}$, reflecting the limit cycles of X_λ bifurcating from Γ. However, information on the Abelian integral can be used to give an upperbound for the cyclicity.

Here, we recall from [DR] the notions of codimension for X_λ and the corresponding Abelian integral in definition 132: the codimension of the Abelian integral is defined with respect to the logarithmic scale $\mathcal{L}$ while the codimension of the vector field is defined by the codimension of the Abelian integral with respect to the enlarged logarithmic scale $\mathcal{L}^e$ (in the sense of definition 75).

Next, proposition 133 expresses the relation between the codimensions of the vector field and the related Abelian integral. This definition is motivated by theorem 134, that gives an upperbound for the cyclicity in terms of the Abelian integral, by use of the non-standard division-derivation algorithm on the asymptotic expansion of $\bar{\Delta}$ (recalled in proposition 130).

In particular, finite codimension of the Abelian integral implies finite cyclicity. It is also striking to notice that the upperbound stated in theorem 134, grows more than linearly in codim (I_ν), as soon as codim $(I_\nu) \geq 3$.

In [DR], it is proven that the upperbound for the cyclicity, stated by theorem 134, is optimal in case the Abelian integral is of codimension 3. More precisely, in [DR], it is proven that there exist generic unfoldings of a 2-saddle cycle for which one connection remains unbroken with cyclicity 4, although the related Abelian integral has codimension 3, and can have at most 3 zeroes. However, from its proof it follows that knowledge of both the first and the second Melnikov function are sufficient to transfer results to the unfolding X_λ.

Definition 132 *The integral I_ν is said to be of finite codimension if $codim_{\mathcal{L}} I_\nu < \infty$. Moreover, in this case*

1. *the codimension of I_ν is defined as*

$$\text{codim} I_\nu \equiv \text{codim}_{\mathcal{L}} I_\nu$$

2. *the codimension of X_λ is defined as*

$$\text{codim} X_\lambda = \text{codim}_{\mathcal{L}^e} I_\nu$$

Proposition 133 *Suppose that the Abelian integral I_ν is of finite codimension, i.e., $codim I_\nu = q < \infty$. Then $codim X_\lambda = q_e < \infty$; moreover, the relation between these two numbers is given by:*

$$\begin{array}{lll} \textit{If } q = 2p, & \textit{then} & q_e = 2p + p(p-1)/2 \\ \textit{If } q = 2p+1, & \textit{then} & q_e = 2p+1+p(p+1)/2 \end{array} \tag{1.143}$$

Proof. Write $\mathcal{L}^e$ as a table, ordered from below to above and from the left to the right.

$$\begin{array}{cccccc}
1 & x & x^2 & x^3 & x^4 & \cdots \\
 & xL & x^2L & x^3L & x^4L & \cdots \\
 & & x^2L^2 & x^3L^2 & x^4L^2 & \cdots \\
 & & & x^3L^3 & x^4L^3 & \cdots \\
 & & & & x^4L^4 & \cdots \\
 & & & & & \ddots
\end{array}$$

where $L = \log x$. Notice, that the first two rows of this table correspond to $\mathcal{L}$. If $q = 2p$, then the principal term belongs to the first row of the table; hence

$$\begin{aligned}
q_e &= 2p + \sum_{j=2}^{p} (j-1) \\
&= 2p + \sum_{j=1}^{p-1} j = 2p + \frac{1}{2}(p-1)\,p
\end{aligned}$$

If $q = 2p+1$, then the principal term belongs to the second row; hence

$$\begin{aligned}
q_e &= \left[2(p+1) + \frac{1}{2}p(p+1)\right] - 1 \\
&= 2p + 1 + \frac{1}{2}p(p+1)
\end{aligned}$$

■

For instance, we have

$$\begin{array}{c|cccccccccccc}
q & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & \cdots \\
\hline
q_e & 1 & 2 & 4 & 5 & 8 & 9 & 13 & 14 & 19 & 20 & \cdots
\end{array} \tag{1.144}$$

Theorem 134 *Let X_λ be a C^∞ unfolding of a 2-saddle cycle, breaking just the lower connection. If codimX_λ is finite, then:*

$$\mathrm{Cycl}(X_\lambda, \Gamma) \leq codim X_\lambda \tag{1.145}$$

Chapter 2

Hopf-Takens bifurcations and centers

2.1 Introduction

The Hopf bifurcation, also called Andronov-Hopf bifurcation is a very well-known generic and structurally stable 1-parameter bifurcation. It unfolds a non-degenerate singularity of codimension 1 and it gives birth to a limit cycle. Since it can be determined by algebraic techniques (positioning the singularity and calculating the 3-jet of the unfolding) it gives rise to a powerful instrument to detect small amplitude periodic dynamics. As such, its use goes beyond the theory of ordinary differential equations.

A generalisation giving rise to more than one limit cycle and to related multiple limit cycle bifurcations has been studied in [T]. The generic p-parameter structurally stable bifurcation is called *Hopf bifurcation* or *Hopf-Takens bifurcation of codimension p*. These bifurcations are very well understood and once again can be detected using algebraic techniques. The latter does not mean that there are no more problems left, since in many concrete problems it is very hard to bring the calculation to a good end. Alternative algebraic techniques, like the use of Lyapunov quantities, have been introduced to simplify the calculations but even this method is still very hard to deal with in concrete problems.

In many studies an extra complication shows up, because the generic Hopf bifurcations show up in perturbations from systems exhibiting a center, like from a Hamiltonian system or more generally from a system having a first integral. This is often the case if one has to deal with a family of vector fields obtained by rescaling. The perturbations from an integrable system can be described by one parameter ε, in the sense that for $\varepsilon = 0$, we have an integrable system. The set of integrable systems could also have a more complicated structure, in which case it is interesting to use the notion of the Bautin Ideal in the study of the bifurcation.How can one study the bifurcation sets in uniform neighbourhoods as well in the phase plane as in the parameter space and obtain "stable bifurcation diagrams" when bifurcating from the set of integrable systems? This is the subject of this chapter.

A lot of results in this chapter are for sure well-known among specialists, if not explicitly then at least implicitly. However as far as we know they have never been written down thoroughly.

In [BoLe] and [GW], one can find the relation between the normal form of a Hopf point or Hopf singularity and associated Lyapunov quantities. However these papers do not take care about the relations on parameters, hence on the bifurcations as such. In [T] we have a well elaborate study of the generic Hopf bifurcation of any codimension, but in this paper nothing is said about the relation with Lyapunov quantities. Moreover none of these papers deal with the more degenerate bifurcations in which centers can occur; in that case people use Lyapunov quantities as well as Melnikov functions but again a clear description of the relation between these notions does not seem to be present in the literature. For a simple Hopf bifurcation of codimension 1 there exists a result using normal forms in the case of a center [CLW]. Without claiming to be complete, we state and prove a number of theorems that can be used as a firm theoretical base for the calculations that are generally made in treating concrete examples.

In this chapter, we deal with C^∞ families of vector fields if the study is (completely) analoguous to the C^ω case. However certain results are restricted to C^ω families. The main reason is that these results cannot be generalised to non-analytic families. Also in most concrete examples the families to deal with are for sure analytic if not to say polynomial.

Throughout this chapter, we only deal with a local study of families near a center. The right framework for this study relies on the notion of germ. We of course mean the notion of "germ of a family" and not "family of germs". However to make the study simpler we will sometimes forget to state the results in terms of germs and use the families of vector fields themselves.

In the last paragraph of [Chic] a few ideas developed here are already present. However in [Chic] attention only goes to the cyclicity and not to the genericity of the unfoldings, which is our main concern.

This chapter is organised as follows. In section 2.2 (respectively section 2.3), we look for sufficient and necessary conditions to ensure the presence of a generic Hopf-Takens bifurcation (respectively near centers). We start by recalling the definition of the generic Hopf-Takens bifurcation of codimension l in section 2.2.1. In sections 2.2.2 and 2.2.3, we state sufficient and necessary conditions in terms of normal forms and Lyapunov quantities respectively. In section 2.3.1, we restrict to situations where centers occur on the regular hypersurface $\{(\nu,\varepsilon)\in\mathbb{R}^{p-1}\times\mathbb{R} : \varepsilon = 0\}$, and we generalise the results in section 2.2 in a uniform way with respect to $\varepsilon > 0$. In addition, we give a uniform result with respect to $\varepsilon > 0$ in terms of the first non-zero Melnikov function. Finally in section 2.3.2, we investigate a more complicated situation, in which the Bautin ideal is not anymore principal (but this time it is regular). In the considered example, two bifurcation types occur: the usual Hopf bifurcation and a so-called boundary bifurcation (meaning that a limit cycle possibly can escape from the boundary of the chosen neighbourhood of the Hopf singularity).

2.2 Hopf-Takens bifurcations

2.2.1 Standard models

The generic Hopf-Takens bifurcation is defined by way of a weak morphism to one of the standard models, $X_{\pm}^{(l)}$, the so-called standard generic Hopf-Takens bifurcation of codimension l. First, we briefly discuss the bifurcation diagrams of $X_{\pm}^{(l)}$.

When one studies systems depending on parameters the study of bifurcations of small codimension is a key to the understanding of the different behaviours. Therefore, let us start by considering the codimension 1 case. In polar coordinates (r,θ), the family $X_{+}^{(1)}$ is written as:

$$X_{+}^{(1)} \leftrightarrow \begin{cases} \dot{r} &= r\left(r^2+a_0\right) \\ \dot{\theta} &= 1 \end{cases}, \ a_0 \in \mathbb{R}$$

Clearly, for each value of the parameter a_0, the origin is a singularity (a so-called *Hopf singularity* or *Hopf point*). As noticed before the bifurcation diagram of zeroes $r_0 > 0$ of the map $r^2 + a_0$, is a reflection of the bifurcation diagram of the limit cycles of the family $X_{+}^{(1)}$. In particular, if r_0 is a root of $r^2 + a_0 = 0$, then the circle $C\left(r_0\right)$ centered at the origin with radius r_0 is the corresponding limit cycle. Its stability is determined by the sign of $\dot{r}$.

Clearly, for $a_0 \geq 0$, there are no limit cycles. For $a_0 < 0$, there is exactly one (repelling or unstable) limit cycle: $C\left(\sqrt{-a_0}\right)$. Hence, passing from positive values to negative values a_0 through the bifurcation value $a_0 = 0$, a repelling limit cycle is born from the singularity in the origin. Simultaneously, the singularity changes stability: for $a_0 \geq 0$ (respectively $a_0 < 0$), the singularity is of repelling (respectively attracting) nature. The phenomenon of birth or death of a limit cycle from a singular point, switching its stability, is well-known as Hopf-bifurcation. The same phenomenon happens in the family $X_{-}^{(1)}$, that in polar coordinates (r,θ) is given by:

$$X_{-}^{(1)} \leftrightarrow \begin{cases} \dot{r} &= -r\left(r^2+a_0\right) \\ \dot{\theta} &= 1 \end{cases}, \ a_0 \in \mathbb{R}$$

The only difference is contained in the minus-sign in the equation of $\dot{r}$, that is responsable for the stability of the Hopf singularity and the possible limit cycle. More precisely, the Hopf bifurcation that appears in $X_{-}^{(1)}$ gives rise to an attracting (or stable) limit cycle, and simultaneously, the singularity switches from stable to unstable. Notice that the bifurcation diagram in these models can be described by a

1-dimensional parameter; therefore we speak of a codimension 1 bifurcation. We denote the *standard generic Hopf bifurcation of codimension* 1, giving rise to either a stable or an unstable limit cycle, at once by $X_{\pm}^{(1)}$. The bifurcation diagram of limit cycles of $X_{+}^{(1)}$ is presented in figure 2.1.

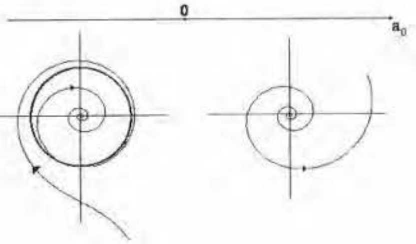

Figure 2.1: Standard generic Hopf-Takens bifurcation of codimension 1 (type $X_{+}^{(1)}$)

The simplicity of this model relies on the fact that limit cycles of the vector field are exact circles centered at the origin, with radius determined by zeroes of a polynomial of degree 2; the stability of the singularity and the possible limit cycles are found by investigating the sign of this polynomial.

In general, bifurcation diagrams of zeroes of polynomials are matter of subject in Catastophy theory, and in an analoguous way, the bifurcation diagram of limit cycles in the standard models $X_{\pm}^{(l)}$, defined by

$$X_{\pm}^{(l)} \leftrightarrow \left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right) \pm \left(r^{2l} + a_{l-1}r^{2(l-1)} + \ldots + a_1 r^2 + a_0\right)\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right), \quad (2.1)$$

where $r^2 = x^2 + y^2$ and $a_0, \ldots, a_{l-1} \in \mathbb{R}$, is represented by the one of the zeroes of the governing polynomial of degree $2l$ in r. The occuring bifurcation phenomenon in the family $X_{\pm}^{(l)}$, is referred as *'Standard generic Hopf-Takens bifurcation'* or *'Standard generic generalised Hopf bifurcation' of codimension* l, since it can be described by a l-dimensional parameter $a = (a_0, \ldots, a_{l-1})$. Clearly, we deal with a multiple limit cycle bifurcation, meaning that $\forall 0 \leq i \leq l$, there is a region in the bifurcation diagram of limit cycles of $X_{\pm}^{(l)}$ in $\mathbb{R}^l$, adhering to $a = 0$, with i limit cycles bifurcation.

The bifurcation diagram of $X_{\pm}^{(2)}$ is shown in figure 2.2. Its bifurcation diagram exhibits regions with $0, 1$ or 2 limit cycles.

Let us now consider a family of planar vector fields $(X_\lambda)_\lambda$ with singularity e, that unfolds the vector field X_{λ^0}, where $\lambda^0 \in \mathbb{R}^p$. Then, we say that the family $(X_\lambda)_\lambda$ exhibits a generic Hopf-Takens bifurcation of codimension l at (e, λ^0), if there exist an open submanifold $W \subset \mathbb{R}^p$ containing λ^0 and an open neighbourhood $U \subset \mathbb{R}^2$ of e such that the bifurcation diagram of limit cycles of (X_λ) on $U \times W$ is diffeomorphic to the one of $X_{\pm}^{(l)}$ on $\mathbb{R}^2 \times \mathbb{R}^l$, preserving the repelling and attracting nature of the singular point and possible limit cycles. More precisely,

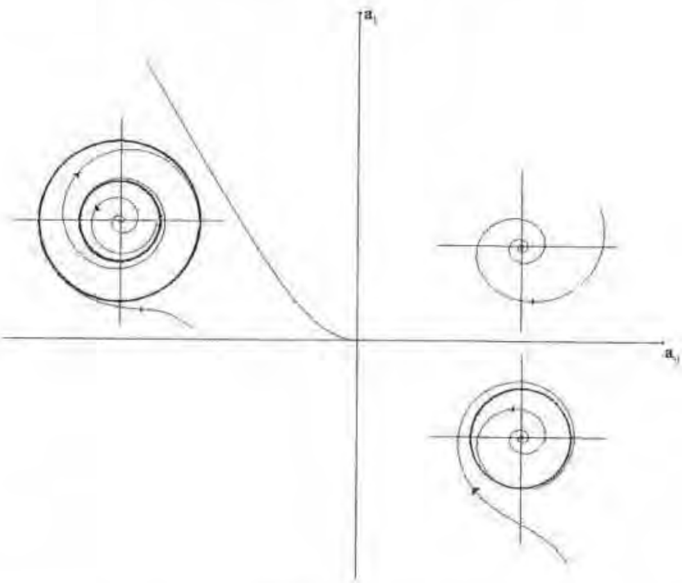

Figure 2.2: Standard generic Hopf-Takens bifurcation of codimension 2 (type $X_+^{(2)}$)

Definition 135 *Let $(X_\lambda)_\lambda$ be a family of planar vector fields with singularity e (i.e., $X_\lambda(e) = 0, \forall\lambda$), that unfolds the vector field X_{λ^0}, where $\lambda^0 \in \mathbb{R}^p$. Then, the family $(X_\lambda)_\lambda$ exhibits a generic generalised Hopf bifurcation or generic Hopf-Takens bifurcation of codimension l at (e, λ^0) if and only if there exists a weak morphism from $(X_\lambda)_\lambda$ to $X_\pm^{(l)}$, meaning that there exist a neighbourhood $U \times W$ of (e, λ^0) in $\mathbb{R}^2 \times \mathbb{R}^p$ and a C^∞ (respectively C^ω) map $\Phi : U \times W \to \mathbb{R}^2 \times \mathbb{R}^l$ such that*

1. *there exist a family of diffeomorphisms $h_\lambda : U \to h_\lambda(U), \lambda \in W$ and a submersion $\phi : W \to \phi(W)$ at λ^0 such that the map Φ can be written as*

$$\Phi((x, y), \lambda) = (h_\lambda(x, y), \phi(\lambda)), \forall((x, y), \lambda) \in U \times W,$$

   and

2. *for every $((x, y), \lambda) \in U \times W$:*

$$\mathrm{d}(h_\lambda)_{(x,y)}(X_\lambda(x, y)) = X_\pm^{(l)}(\Phi((x, y), \lambda)).$$

The conditions given by this definition are not easy to check in practice. In [T], an equivalent characterisation is given in terms of normal forms. In this chapter equivalent characterisations of the generic Hopf-Takens bifurcation of codimension l, are given in terms of Lyapunov quantities and, in case centers occur on a regular hypersurface, Melnikov functions.

2.2.2 Normal forms

In this section, we recall the main result and the ideas of the proof given in [T], to fix notations and to clarify how this result can be generalised to situations where centers occur on regular hypersurfaces (section 1.3). We first give a summary of this section. F. Takens reduced the investigation of the presence of generic Hopf-Takens bifurcations' to the one for families of 'symmetric normal forms' according to the standard generic Hopf-Takens bifurcation $X_{\pm}^{(l)}$, that is an example of such a family of symmetric normal forms. For such families he defined 'symmetric displacement maps', in such a way that the governing polynomial in (2.1) multiplied by r^2, is a symmetric displacement map for $X_{\pm}^{(l)}$, the so-called standard symmetric displacement map $D_{\pm}^{(l)}$. In this way, the family $(X_\lambda)_\lambda$ exhibits a generic Hopf-Takens bifurcation if there exists a weak morphism from its associated symmetric displacement map D and the standard symmetric displacement map $D_\sigma^{(l)}$ (where σ is $+$ or $-$). Moreover, for this purpose, he gave a generalisation of the Preparation Theorem for symmetric functions. Then, by this theorem and the relation between the coefficients in the expansion of D and the ones in D_σ^l, one can give an equivalent characterisation of the generic Hopf-Takens bifurcation of codimension l in terms of normal forms. This last step is neither worked out, nor explicitly mentioned in [T].

Firstly, the family $(X_\lambda)_\lambda$ is reduced to a *symmetric normal form* $(X_\lambda^{(N)})_\lambda$; it means that $(X_\lambda^{(N)})_\lambda$ satisfies the following *'symmetry property'*: in polar coordinates (r,θ) the vector field $X_\lambda^{(N)}$ is expressed as

$$\left\{\begin{array}{rcl} \dot{r} &=& r\left(f\left(r^2,\lambda\right)+g\left(r\cos\theta, r\sin\theta,\lambda\right)\right) \\ \dot{\theta} &=& 1 \end{array}\right.$$

where f,g are C^∞ functions with

$$\left\{\begin{array}{rcl} f\left(0,0\right) &=& 0 \\ j_\infty\left(g\left(\cdot,\cdot,\lambda\right)\right)_{(0,0)} &=& 0, \forall\lambda \end{array}\right.$$

and all closed orbits γ of $X_\lambda^{(N)}$ sufficiently close to e are exactly round (i.e., there exists a positive constant ρ such that

$$\gamma=\left\{(x,y)\in\mathbb{R}^2 : x^2+y^2=\rho\right\}).$$

This reduction is obtained by C^∞ diffeomorphisms of the form:

$$(x,y,\lambda)\mapsto\left(K_\lambda\left(x,y\right),\lambda\right)$$

and by multiplying the family by a strictly positive function F :

$$X_\lambda^{(N)}\left(K_\lambda\left(x,y\right)\right)=F\left(x,y,\lambda\right)\cdot d\left(K_\lambda\right)_{(x,y)}\left(X_\lambda\left(x,y\right)\right).$$

In this way, the singularity e of X_λ is sent to the singularity $(0,0)$ of the vector field $X_\lambda^{(N)}$ preserving its type (sink or source); in a neighbourhood of e, closed orbits of X_λ are mapped onto closed orbits of $X_\lambda^{(N)}$, also preserving their attracting or repelling nature. Therefore, by this reduction, the presence of a generic Hopf-Takens bifurcation of codimension l is preserved.

Next Takens defined a "symmetric" displacement map [T]. Denote the circle centered at the origin with radius r_0 by $C(r_0)$:

$$C(r_0) = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 = r_0^2\}$$

Definition 136 *For a family $(X_\lambda)_\lambda$ having the symmetry property we define a family of "symmetric displacement maps" by a C^∞ function $D : \mathbb{R} \times \mathbb{R}^p \to \mathbb{R}$ with the following properties:*

1. $D(r,\lambda) = D(-r,\lambda)$, $\quad \forall (r,\lambda) \in \mathbb{R} \times \mathbb{R}^p$;
2. $D(0,\lambda) = 0$, $\quad \forall \lambda \in \mathbb{R}^p$;
3. *(x,y) lies on a closed orbit of $X_\lambda \iff D(\sqrt{x^2+y^2},\lambda) = 0$;*
4. *If $D(r_0,\lambda) > 0$ (respectively $D(r_0,\lambda) < 0$) then all points (x,y,λ) with $(x,y) \in C(r_0)$ are wandering, the ω-limit of such points is contained in*
$$\{(x,y) : x^2+y^2 > r_0^2\} \text{ (respectively } \{(x,y) : x^2+y^2 < r_0^2\}\text{)}$$
and the α-limit in
$$\{(x,y) : x^2+y^2 < r_0^2\} \text{ (respectively } \{(x,y) : x^2+y^2 > r_0^2\}\text{)}.$$

Notice that the standard models $X_\pm^{(l)}$ satisfy the symmetry property (with $\lambda = a = (a_0, a_1, \ldots, a_{l-1})$) and that a symmetric displacement map for $X_\pm^{(l)}$ is given by the so-called standard symmetric displacement map $D_\pm^{(l)}$:

$$D_\pm^{(l)}(r,a) = \pm r^2 \left(r^{2l} + a_{l-1} r^{2(l-1)} + \ldots + a_1 r^2 + a_0\right)$$

where $a = (a_0, a_1, \ldots, a_{l-1}) \in \mathbb{R}^l$.

From now on, we suppose that the family $(X_\lambda)_\lambda$ itself already satisfies the symmetry property. Clearly, if $D_\lambda \equiv D(\cdot,\lambda), \lambda \in \mathbb{R}^p$ is a symmetric displacement map for X_λ, then the family $(X_\lambda)_\lambda$ exhibits a generic Hopf-Takens bifurcation of codimension l at (e,λ^0), if and only if there exist an open submanifold $W \subset \mathbb{R}^p$ containing λ^0 and an open neighbourhood $U \subset \mathbb{R}^2$ of e such that the bifurcation diagram of positive zeroes of $(D_\lambda)_\lambda$ on $U \times W$ is diffeomorphic to the one of $D_\pm^{(l)}$ on $\mathbb{R} \times \mathbb{R}^l$, preserving the repelling and attracting nature of its zeroes. More precisely,

Proposition 137 *Let $(X_\lambda)_\lambda$ be a family of planar vector fields satisfying the symmetry property with symmetric displacement maps $(D_\lambda)_\lambda$. Then, the family $(X_\lambda)_\lambda$ exhibits a generic Hopf-Takens bifurcation of codimension l if and only if there exist a neighbourhood $\bar{U} \times W \subset \mathbb{R} \times \mathbb{R}^p$ of $(0,\lambda^0)$ and a C^∞ map $\bar{\Phi} : \bar{U} \times W \to \mathbb{R} \times \mathbb{R}^l$ such that*

1. *there exist a family of* C^∞ *maps* $H_\lambda : \bar{U} \to \mathbb{R}$, *with* $H_\lambda(0) > 0, \lambda \in W$ *and a submersion* $\phi : W \to \phi(W)$ *at* λ^0, *such that the map* $\bar{\Phi}$ *can be written as*

$$\bar{\Phi}(r, \lambda) = \left(H_\lambda\left(r^2\right), \phi(\lambda)\right), \forall (r, \lambda) \in U \times W,$$

and

2. *for every* $(r, \lambda) \in \bar{U} \times W$:

$$D(r, \lambda) = D_{\pm}^{(l)}\left(\bar{\Phi}(r, \lambda)\right).$$

Proof. The map $\Phi((x, y), \lambda) \equiv \left(H_\lambda\left(x^2 + y^2\right) \cdot (x, y), \phi(\lambda)\right)$ defines a weak morphism from (X_λ) to $X_{\pm}^{(l)}$, and its action is presented in figure 2.3:

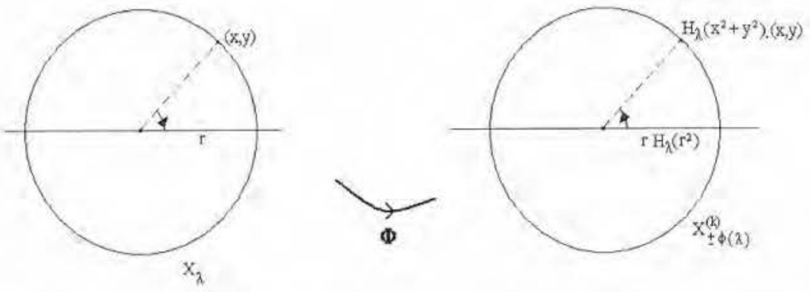

Figure 2.3: Weak morphism Φ

Proposition 137 gives an equivalent characterisation of a Hopf-Takens bifurcation in terms of the symmetric displacement map. In practice, if we already can find an explicit expression for the symmetric displacement map, it is although no evidence to find a map $\bar{\Phi}$ satisfying the prescribed conditions. However, Takens proved a generalisation of the Preparation Theorem of Mather for symmetric functions and this result will be the key in stating generic conditions for a Hopf-Takens bifurcation in terms of normal forms.

Theorem 138 (Preparation Theorem of Mather for symmetric functions) *Let* $\bar{U}_0 \times W_0 \subset \mathbb{R} \times \mathbb{R}^p$ *be an open neighbourhood of* $\left(0, \lambda^0\right)$, *and let be given a* C^∞ *(respectively* C^ω*) function* $D : \bar{U}_0 \times W_0 \to \mathbb{R}$ *such that*

$$\begin{array}{lll} i) & D\left(r, \lambda^0\right) = r^{2l+2} F_l(r), \quad \forall r \in \bar{U}_0 & \textit{with } F_l(0) \neq 0 \\ ii) & D(0, \lambda) = 0, \quad \forall \lambda \in W_0 & \\ iii) & D(r, \lambda) = D(-r, \lambda), \quad \forall (r, \lambda) \in \bar{U}_0 \times W_0 & \end{array}$$

Then, there exist an open neighbourhood $\bar{U} \times W \subset \bar{U}_0 \times W_0$ of $(0, \lambda^0)$ and C^∞ (respectively C^ω) functions

$$\phi : W \to \mathbb{R}^l \text{ and } h : \bar{U} \times W \to \mathbb{R}$$

such that

$$D(r, \lambda) = D_\sigma^{(l)}(h(r, \lambda), \phi(\lambda)) \tag{2.2}$$

where σ is the sign of $F_l(0)$ and such that

1. *$\phi(\lambda^0) = 0$ and $h(0, \lambda) = 0, \forall \lambda \in W$,*
2. *the map $h(\cdot, \lambda)$ is an orientation preserving diffeomorphism, $\forall \lambda \in W$, and*
3. *$h(-r, \lambda) = -h(r, \lambda), \quad \forall (r, \lambda) \in \bar{U} \times W$*

Remark 139 1. *Conclusions* 2. *and* 3. *of the Preparation Theorem of Mather for symmetric functions imply that there exists a C^∞ (respectively C^ω) function $H : (\mathbb{R} \times \mathbb{R}^p, (0, \lambda^0)) \to \mathbb{R}$ such that*

$$h(r, \lambda) = rH(r^2, \lambda) \text{ with } H(0, \lambda) > 0$$

2. *Suppose that the function D in the Preparation Theorem of Mather for symmetric functions has the following Taylor expansion at $r = 0$ (locally for (r, λ) near $(0, \lambda^0)$)*

$$D(r, \lambda) = r^2 \left(\sum_{i=0}^{l} f_i(\lambda) r^{2i} + o(r^{2l}) \right), r \to 0,$$

If the map

$$W \subset \mathbb{R}^p \to \mathbb{R}^l : \lambda \mapsto (f_0(\lambda), f_1(\lambda), \ldots, f_{l-1}(\lambda))$$

defines a submersion at λ^0, then one can check by comparison of coefficients corresponding to equal powers of r after expansion of (2.2) *that the map ϕ also is a submersion at λ^0.*

Combining proposition 137, the Preparation Theorem of Mather for symmetric functions and the above remarks, we can draw the following result from the study in [T].

Theorem 140 *Let $(X_\lambda)_\lambda$ be a family of planar vector fields satisfying the symmetry property with symmetric displacement maps $(D_\lambda)_\lambda$. Suppose that $\bar{U} \times W$ is an open neighbourhood of $(0, \lambda^0)$ such that $\forall (r, \lambda) \in \bar{U} \times W$:*

$$D_\lambda(r) = r^2 \left(\sum_{i=0}^{l} f_i(\lambda) r^{2i} + o(r^{2l}) \right), r \to 0.$$

Then the family $(X_\lambda)_\lambda$ exhibits a generic Hopf-Takens bifurcation of codimension l (the sign $\pm$ of its type $X_\pm^{(l)}$ is given by the sign of $f_l(\lambda^0)$) if and only if

1. $(0,0)$ *is a Hopf singularity or Hopf point of codimension* l, *i.e.*

$$f_0(\lambda^0) = \ldots = f_{l-1}(\lambda^0) = 0 \text{ and } f_l(\lambda^0) \neq 0$$

2. The map $f = (f_0, f_1, \ldots, f_{l-1})$ *is a submersion at* λ^0.

Theorem 140 can be rephrased in terms of normal forms using the technical proposition derived in chapter 1. We provide a proof of this theorem since it is not given in [T].

Theorem 141 *Let* $(X_\lambda)_\lambda$ *be a* C^∞ *(respectively* C^ω*) family of planar vector fields* $(X_\lambda)_\lambda$ *of type* (1.27). *Furthermore, let* (1.30) *be a normal form for the family with* $N \gg l$. *Then the family* $(X_\lambda)_\lambda$ *exhibits a generic Hopf-Takens bifurcation of codimension* l *(the sign* $\pm$ *of its type* $X_\pm^{(l)}$ *is given by the sign of* $d_l(\lambda^0)$*) if and only if*

1. $(0,0)$ *is a Hopf singularity of codimension* l, *i.e.*

$$d_0(\lambda^0) = \ldots = d_{l-1}(\lambda^0) = 0 \text{ and } d_l(\lambda^0) \neq 0$$

2. The map $d = (d_0, d_1, \ldots, d_{l-1})$ *is a submersion at* λ^0.

Proof. For the family $(X_\lambda^{(N)})_\lambda$ Takens constructed a symmetric displacement map as follows: in polar coordinates the family $(X_\lambda^{(N)})_\lambda$ corresponds to the following scalar differential equation:

$$\dot{R} = R \cdot \left(f\left(R^2, \lambda\right) + G\left(R\cos\theta, R\sin\theta, \lambda\right)\right) \tag{2.3}$$

with $j_\infty\left(G\left(\cdot,\cdot,\lambda\right)\right)_{(0,0)} = 0$ and $f(0,\lambda) = d_0(\lambda)$.

If $R(\theta, r, \lambda)$ is an integral curve of (2.3) with $R(0, r, \lambda) = r$, then

$$D(r,\lambda) = r\left(R(2\pi, r, \lambda) - r - R(2\pi, -r, \lambda) + r\right) \tag{2.4}$$

defines a symmetric displacement map for the family $(X_\lambda^{(N)})_\lambda$. Denote the displacement map of $X_\lambda^{(N)}$, corresponding to the transverse section $\{\theta = 0\}$ and parametrised by the radial variable r, by

$$\delta^1(r,\lambda) = \delta^1_\lambda(r).$$

Now there exists the following relation between D and δ^1:

$$D(r,\lambda) = r\left(\delta^1(r,\lambda) - \delta^1(-r,\lambda)\right). \tag{2.5}$$

As a consequence of (2.5), we have for all $j \in \mathbb{N}$:

$$\left\{\begin{array}{ll} \left.\dfrac{\partial^{2j}}{\partial r^{2j}} D(r,\lambda)\right|_{r=0} & = 2 \left.\dfrac{\partial^{2j-1}}{\partial r^{2j-1}} \delta^1(r,\lambda)\right|_{r=0} \\ \left.\dfrac{\partial^{2j+1}}{\partial r^{2j+1}} D(r,\lambda)\right|_{r=0} & = 0. \end{array}\right. \tag{2.6}$$

By equalities (2.6), proposition 50, theorem 54 (and remark 57 in the C^∞ case), and lemma 9 (1,2), the conditions of theorem 140 are equivalent to the ones stated in the current theorem. ■

2.2.3 Lyapunov quantities

Proposition 50 enables us to describe a generic Hopf-Takens bifurcation in terms of Lyapunov quantities. By lemma 9 and proposition 50, theorem 141 is equivalent to the following one:

Theorem 142 *Let $(X_\lambda)_\lambda$ be a C^∞ family of planar vector fields of the form* (1.27) *with a given set of Lyapunov quantities $V_i, 0 \le i \le l$. Then, the family $(X_\lambda)_\lambda$ exhibits a generic Hopf-Takens bifurcation of codimension l (the sign $\pm$ of its type $X_\pm^{(l)}$ is given by the sign of $V_l(\lambda^0)$) if and only if*

1. *$e = (0,0)$ is a Hopf singularity of codimension l i.e.*

$$V_0(\lambda^0) = \ldots = V_{l-1}(\lambda^0) = 0 \text{ and } V_l(\lambda^0) \neq 0,$$

2. *the map $V := (V_0, V_1, \ldots, V_{l-1})$ is a submersion at λ^0*

Although in many problems it is easier to work with Lyapunov quantities $V_0, \ldots, V_l$ than to work directly with the normal form, often it remains too difficult to calculate $V_0, \ldots, V_l$ directly. On many occasions one can succeed in checking that at some point λ^0 the origin is a Hopf singularity of codimension l. But to controll that the map V in theorem 142 is a submersion at λ^0 might be quite a challenge. Sometimes one only succeeds in calculating V_i, with $0 < i \le l$ at those values λ where $V_0(\lambda) = \ldots = V_{i-1}(\lambda) = 0$. Using lemma 10, theorem 142 can be translated into the following more practical theorem:

Theorem 143 *Let $(X_\lambda)_\lambda$ be a C^∞ family of planar vector fields as given in* (1.27)*. Let $V_0 = d$ be a submersion at λ^0 with $d(\lambda^0) = 0$ and let $Z_0 = d^{-1}(0)$. Let V_1 be the first Lyapunov quantity defined for $\lambda \in Z_0$ and suppose that $V_1|_{Z_0}$ is a submersion at λ^0; let us suppose, by induction on $i = 2, \ldots, l-2$, that it is possible to define $Z_i = Z_{i-1} \cap g_i^{-1}(0)$ and $V_{i+1} : Z_i \to \mathbb{R}$ with $V_{i+1}(\lambda^0) = 0$ such that $V_{i+1}|_{Z_i}$ is a submersion at λ^0. Suppose that the l-th Lyapunov quantity V_l has the property $V_l(\lambda^0) \neq 0$. Then the family $(X_\lambda)_\lambda$ exhibits a generic Hopf-Takens bifurcation of codimension l. Moreover the sign of $V_l(\lambda^0)$ determines the sign of its type $X_\pm^{(l)}$.*

Under the generic conditions expressed by the submersion requirement we know that the generic Hopf-Takens bifurcation of codimension l will exhibit systems having l limit cycles. In the literature -in the presence of a Hopf singularity of codimension l at a parameter value $\lambda = \lambda^0$- people often only check the last property by exhibiting Lyapunov quantities $V_0(\tilde\lambda^0), V_1(\tilde\lambda^0), \ldots, V_l(\tilde\lambda^0)$ that for $\tilde\lambda^0 \sim \lambda^0$ are of alternating sign and satisfy

$$\left|V_0(\tilde\lambda^0)\right| \ll \left|V_1(\tilde\lambda^0)\right| \ll \ldots \ll \left|V_l(\tilde\lambda^0)\right|.$$

This for sure guarantees the occurence of a system with l limit cycles for $\lambda = \tilde{\lambda}^0$, but it for sure does not imply to have a full unfolding of the considered Hopf singularity. As a trivial counterexample it suffices to consider the 1-parameter family

$$\frac{\partial}{\partial\theta} + (r^2 - \varepsilon^2)r\frac{\partial}{\partial r}.$$

Similar examples can be found for any value of l.

2.3 Hopf-Takens bifurcations near centers

2.3.1 Regular hypersurface of centers

In this section we deal with a C^∞ (respectively C^ω) family of planar vector fields $\left(X_{(\nu,\varepsilon)}\right)_{(\nu,\varepsilon)}$ of type (1.27), $(\nu,\varepsilon) \sim (\nu^0, 0) \in \mathbb{R}^{p-1} \times \mathbb{R}$, such that centers in this family occur on a regular hypersurface (as defined in section 1.3). Hence, there exists a strictly positive integer k such that if δ is a family of displacement maps for this family, then there exists a C^∞ (respectively C^ω) function $\bar{\delta}$, which is not divisible by ε, with

$$\delta(s,\nu,\varepsilon) = \varepsilon^k \bar{\delta}(s,\nu,\varepsilon). \tag{2.7}$$

We will only consider families in which no centers occur for $\varepsilon > 0$.

Of course for $\varepsilon = 0$, the origin $e = (0,0)$ is always a center and no limit cycle occurs in a neighbourhood of the origin. For fixed $\varepsilon > 0$, one can apply theorems 140, 141, 142 or 143 in order to guarantee that the subfamily $\left(X_{(\nu,\varepsilon)}\right)_{\nu\in U(\varepsilon)}$ exhibits a generic Hopf-Takens bifurcation with respect to certain neighbourhoods $U(\varepsilon)$ of ν^0 in $\mathbb{R}^{p-1}$ and $W(\varepsilon)$ of the origin e in $\mathbb{R}^2$ (depending on ε). When ε decreases to 0, these neighbourhoods $U(\varepsilon)$ (respectively $W(\varepsilon)$) might however shrink to ν^0 (respectively e). However, in many cases, it is imperative to have the result in a uniform way, i.e. on a fixed $\bar{U} \subset U(\varepsilon)$ and $\bar{W} \subset W(\varepsilon)$, for all $0 < \varepsilon < \bar{\varepsilon}$, with $\bar{\varepsilon} > 0$.

Chow, Li and Wang [CLW] have proved such a uniform result (with respect to ε) for the Hopf bifurcation of codimension 1. We state here five general uniform results. The conditions are expressed in terms of either the reduced symmetric displacement map (theorem 144), or normal forms (theorem 145), or reduced Lyapunov quantities (theorems 146 and 147), or the first non-zero (reduced) Melnikov function (theorem 148). Depending on the concrete situation, one can apply one of these theorems to investigate the presence of a generic Hopf-Takens bifurcation (uniformly with respect to $\varepsilon > 0$).

To obtain the announced results, we reduce the current situation to a non-degenerate one, where again the result of Takens applies. Then we can give sufficient conditions on the *'reduced symmetric displacement map'* in order to have a uniform result on generic Hopf-Takens bifurcations near a regular hypersurface of centers.

Firstly, we reduce the family $\left(X_{(\nu,\varepsilon)}\right)_{(\nu,\varepsilon)}$ to a family of symmetric normal forms, $(X^N_{(\nu,\varepsilon)})_{(\nu,\varepsilon)}$; let δ^1 and D be the traditional displacement map and the symmetric displacement map as in the proof of theorem 141, where $\lambda = (\nu, \varepsilon)$. Then we know that there exists a family of near-identity diffeomorphisms $\varphi^{-1}_{(\nu,\varepsilon)}$ such that

$$\varphi^{-1}_{(\nu,\varepsilon)}\left(\delta^1_{(\nu,\varepsilon)}\left(\varphi_{(\nu,\varepsilon)}(s)\right) + \varphi_{(\nu,\varepsilon)}(s)\right) = \varepsilon^k \bar{\delta}_\nu(s) + s;$$

as a consequence, by (2.7) and proposition 61, there exists a C^∞(respectively C^ω) function $\overline{\delta}^1$ such that

$$\delta^1(r, \nu, \varepsilon) = \varepsilon^k \overline{\delta}^1(r, \nu, \varepsilon). \tag{2.8}$$

From (2.3) and (2.8), we can define a C^∞ (respectively C^ω) function $\bar{D}$ by

$$\begin{aligned} D(r, \nu, \varepsilon) &= \varepsilon^k \bar{D}(r, \nu, \varepsilon) && (2.9)\\ &= \varepsilon^k \cdot \left(r\overline{\delta}^1(r, \nu, \varepsilon) - r\overline{\delta}^1(-r, \nu, \varepsilon)\right). && (2.10) \end{aligned}$$

It's clear that the function $\bar{D}$ too is a family of symmetric displacement maps. Indeed, for $\varepsilon > 0$, the bifurcation diagram of zeroes of $\bar{D}(\cdot, \nu, \varepsilon)$ in function of (ν, ε) is the same as the one corresponding to $D(\cdot, \nu, \varepsilon)$ and therefore, represents the bifurcation diagram of the limit cycles of $\left(Y_{(\nu,\varepsilon)}\right)$ in function of (ν, ε) (for $\varepsilon > 0$). By relation (2.9), the function $\bar{D}$ is called a family of *reduced symmetric displacement maps.* Moreover, as a consequence of 2.10, the following equalities hold for each $j \in \mathbb{N}$:

$$\begin{cases} \left.\dfrac{\partial^{2j}}{\partial r^{2j}}\bar{D}(r, \nu, \varepsilon)\right|_{r=0} = 2\left.\dfrac{\partial^{2j-1}}{\partial r^{2j-1}}\overline{\delta}^1(r, \nu, \varepsilon)\right|_{r=0}, \\ \left.\dfrac{\partial^{2j+1}}{\partial r^{2j+1}}\bar{D}(r, \nu, \varepsilon)\right|_{r=0} = 0. \end{cases} \tag{2.11}$$

From now on, we'll work with the reduced symmetric displacement map $\bar{D}$, as playing the role of D in the proof given in [T], and in the proof of theorem 141. As a consequence, we have the following uniform results with respect to $\varepsilon > 0$:

Theorem 144 *Suppose that the centers of a given C^∞ (respectively C^ω) family of planar vector fields $\left(X_{(\nu,\varepsilon)}\right)$ of type (1.27), occur on a regular hypersurface, as described by (2.7). Furthermore, let $\bar{D}$ be a family of reduced displacement maps for the family $\left(X_{(\nu,\varepsilon)}\right)_{(\nu,\varepsilon)}$. Then the family $\left(X_{(\nu,\varepsilon)}\right)_{(\nu,\varepsilon)}$ exhibits a generic Hopf-Takens bifurcation of codimension l, uniformly with respect to $\varepsilon > 0$, if*

1. *$\bar{D}(r, \nu^0, 0) = \alpha r^{2l+2} + o\left(r^{2l+2}\right), r \to 0$ with $\alpha \neq 0$ (in other words, if $e = (0,0)$ is a Hopf singularity of codimension l),*
2. *the map $\nu \mapsto \left(\frac{\partial^2 \bar{D}}{\partial r^2}(0, \nu, 0), \frac{\partial^4 \bar{D}}{\partial r^4}(0, \nu, 0), \ldots, \frac{\partial^{2l} \bar{D}}{\partial r^{2l}}(0, \nu, 0)\right)$ is a submersion at ν^0.*

Moreover, the sign of α determines the sign of its type $X^{(l)}_\pm$.

We can restate this theorem where the conditions are expressed in terms of normal forms:

Theorem 145 *Suppose that the centers of a given C^∞ (respectively C^ω) family of planar vector fields $\left(X_{(\nu,\varepsilon)}\right)$ of type* (1.27)*, occur on a regular hypersurface, as described by* (2.7)*. Furthermore, let* (1.30) *be a normal form for the family with $N \gg l$. Then, there exist C^∞ functions $\bar{d}_j$ such that*

$$d_j = \varepsilon^k \bar{d}_j \quad ,j \in \mathbb{N}; \tag{2.12}$$

and the family $\left(X_{(\nu,\varepsilon)}\right)_{(\nu,\varepsilon)}$ exhibits a generic Hopf-Takens bifurcation of codimension l, uniformly with respect to $\varepsilon > 0$ (the sign $\pm$ of its type $X_{\pm}^{(l)}$ is given by the sign of $\bar{d}_l\left(\nu^0, 0\right)$), if

1. $(0,0)$ *is a Hopf singularity of codimension l, i.e.*

$$\bar{d}_j\left(\nu^0, 0\right) = 0, \forall 0 \leq j \leq l-1 \text{ and } \bar{d}_l\left(\nu^0, 0\right) \neq 0$$

2. *The map $\bar{d} : \left(\mathbb{R}^{p-1}, \nu^0\right) \to \mathbb{R} : \nu \mapsto \left(\bar{d}_0\left(\nu, 0\right), \ldots, \bar{d}_{l-1}\left(\nu, 0\right)\right)$ is a submersion at ν^0.*

Proof. Theorem 54 (3.) implies the existence of functions $\bar{d}_j$, defined by (2.12) (therefore the so-called reduced coefficients in the expansion of the considered normal form). The statement on the generic Hopf-Takens bifurcations follows immediately from theorem 144 using the relations in (2.11), lemma 9 (3.) and theorem 54 (3.) (as in the proof of theorem 141 where $\bar{D}$ plays the role of D). ■

Next, we would like to give a uniform result with respect to $\varepsilon > 0$ in terms of Lyapunov quantities. Recall from section 1.3.3 that in current situation a given set of Lyapunov quantities defines a set of reduced Lyapunov quantities (after division by ε^k).

Theorem 146 *Suppose that the centers of a given C^∞ (respectively C^ω) family of planar vector fields $\left(X_{(\nu,\varepsilon)}\right)$ of type* (1.27)*, occur on a regular hypersurface, as described by* (2.7)*. Furthermore, let $\left\{\bar{V}_i : i \in \mathbb{N}\right\}$ be a set of reduced Lyapunov quantities for this family. The family $\left(X_{(\nu,\varepsilon)}\right)_{(\nu,\varepsilon)}$ exhibits a generic Hopf-Takens bifurcation of codimension l, uniformly with respect to $\varepsilon > 0$, (the sign $\pm$ of its type $X_{\pm}^{(l)}$ is given by the sign of $\bar{V}_l\left(\nu^0, 0\right)$) if*

1. $(0,0)$ *is a Hopf singularity of codimension l, i.e.*

$$\bar{V}_j\left(\nu^0, 0\right) = 0, \forall 0 \leq j \leq l-1 \text{ and } \bar{V}_l\left(\nu^0, 0\right) \neq 0$$

2. *the map $\bar{V} : \left(\mathbb{R}^{p-1}, \nu^0\right) \to \mathbb{R} : \nu \mapsto \left(\bar{V}_0\left(\nu, 0\right), \ldots, \bar{V}_{l-1}\left(\nu, 0\right)\right)$ is a submersion at ν^0.*

Theorem 147 *Suppose that the centers of a given C^{∞} (respectively C^{ω}) family of planar vector fields $\left(X_{(\nu,\varepsilon)}\right)$ of type (1.27), occur on a regular hypersurface, as described by (2.7). Furthermore, let $\left\{\bar{V}_i : i \in \mathbb{N}\right\}$ be a set of reduced Lyapunov quantities for this family. Assume that*

$$\bar{V}_0\left(\nu^0, 0\right) = 0$$

and the map $\bar{V}_0\left(\nu^0, 0\right)$ is a submersion at ν^0. Denote the submanifold $\bar{V}_0^{-1}(0)$ by Z_0, and the restriction of the first reduced Lyapunov quantity $\bar{V}_1\big|_{Z_0}$ by W_1. Define $\bar{W}_1$ by

$$W_1 = \varepsilon^k \bar{W}_1.$$

Suppose, by induction on $i = 2, \ldots, l-2$, that we can define submanifolds

$$Z_i = Z_{i-1} \cap \bar{W}_i^{-1}(0),$$

and the maps

$$W_{i+1}, \bar{W}_{i+1} : Z_i \to \mathbb{R}$$

by

$$\begin{cases} W_{i+1} := V_{i+1}|_{Z_i} \\ W_{i+1} = \varepsilon^k \bar{W}_{i+1} \end{cases}$$

with the properties $\bar{W}_{i+1}\left(\nu^0, 0\right) = 0$ and the map $\bar{W}_{i+1}(\cdot; 0)$ is a submersion at ν^0. Presuming that $W_l : Z_{l-1} \to \mathbb{R}$ is the restriction of the l-th Lyapunov quantity

$$W_l = V_l|_{Z_{l-1}},$$

and $\bar{W}_l$ defined by

$$W_l = \varepsilon^k \bar{W}_l$$

has the property

$$\bar{W}_l\left(\nu^0, 0\right) \neq 0.$$

Then, the family $\left(X_{(\nu,\varepsilon)}\right)_{(\nu,\varepsilon)}, (\nu, \varepsilon) \sim \left(\nu^0, 0\right), \varepsilon > 0$ exhibits a generic Hopf-Takens bifurcation of codimension l, uniformly with respect to $\varepsilon > 0$. Moreover the sign of $\bar{W}_l\left(\nu^0, 0\right)$ determines the sign of its type $X_{\pm}^{(l)}$.

Proof. By theorem 54 (3.) and lemma 9 (3.), the conditions in this theorem are equivalent to the conditions of theorem 145. ∎

Finally, we would like to state a uniform result (with respect to $\varepsilon > 0$) in terms of Melnikov functions. Recall from section 1.3.2, that in this situation we can suppose that the vector fields $X_{(0,\nu)}$ of center type are Hamiltonian ones, for $\nu \sim \nu^0$ with Hamiltonian H_ν such that $H_\nu(0,0) = 0$. If $M_k^1(\cdot, \nu)$ is the first non-zero Melnikov function of the family $\left(X_{(\nu,\varepsilon)}\right)_{\varepsilon}$, then

$$M_k^1(h, \nu) = 2h\bar{M}_k(h, \nu)$$

defines the C^{∞} (respectively C^{ω}) reduced first non-zero Melnikov function (cfr. (1.65) in section 1.3.2).

Theorem 148 *Suppose that the centers of a given C^∞ (respectively C^ω) family of planar vector fields $\left(X_{(\nu,\varepsilon)}\right)$ of type (1.27), occur on a regular hypersurface, as described by (2.7). Furthermore, let $\check{M}_k$ be the reduced first non-zero Melnikov function. Then, the family $\left(X_{(\nu,\varepsilon)}\right)_{(\nu,\varepsilon)}, (\nu,\varepsilon) \sim \left(\nu^0, 0\right), \varepsilon > 0$ exhibits a generic Hopf-Takens bifurcation of codimension l, uniformly with respect to $\varepsilon > 0$, if*

1. *$(0,0)$ is a Hopf singularity of codimension l, i.e.*

$$\frac{\partial^j \check{M}_k}{\partial h^j}\left(\nu^0, 0\right) = 0, \forall 0 \leq j \leq l-1 \text{ and } \frac{\partial^l \check{M}_k}{\partial h^l}\left(\nu^0, 0\right) \neq 0;$$

2. *the map $\nu \mapsto \left(\check{M}_k(\nu,0), \frac{\partial}{\partial h}\check{M}_k(\nu,0), \ldots, \frac{\partial^{l-1}}{\partial h^{l-1}}\check{M}_k(\nu,0)\right)$ is a submersion at ν^0.*

Moreover, the sign of of its type $X_{\pm}^{(l)}$ is determined by the sign of $\dfrac{\partial^l \check{M}_k}{\partial h^l}\left(\nu^0, 0\right)$.

Proof. As we did in section 1.3.2, we can introduce coordinates like in (1.64), in which $h = \frac{r^2}{2}$, and continue working in these coordinates. Since $M_k(r,\nu) = r\check{M}_k(\frac{r^2}{2},\nu)$, we have the following relations: $\forall j \in \mathbb{N}_1$, there exist $A_j, B_j \in \mathbb{Q}_0^+$ such that

$$\begin{aligned}\left.\frac{\partial^j}{\partial h^j}\check{M}_k(h,\nu)\right|_{h=0} &= A_j \cdot \left.\frac{\partial^{2j+1}}{\partial r^{2j+1}} M_k(r,\nu,\varepsilon)\right|_{r=0}, \\ &= B_j \left.\frac{\partial^k}{\partial \varepsilon^k}\left(\frac{\partial^{2j+1}}{\partial r^{2j+1}}\delta(0,\nu,\varepsilon)\right)\right|_{\varepsilon=0}. \end{aligned} \tag{2.13}$$

Combining relations (2.13) and theorem 54 (3.), it follows from lemma 9 (3.) that the conditions stated in this theorem are equivalent to those of theorem 145. ■

2.3.2 Bautin ideal with more than one generator

When working with analytic families, the Bautin Ideal can be more complicated than merely generated by ε or ε^k $(k \in \mathbb{N})$. It does not get more complicated to study the local cyclicity, or the diversity of phase portraits; only the bifurcation diagram gets more involved. We do not want to make a complete general study but introduce one example with some interesting properties. We suppose that the Bautin Ideal is regular having two generators, let us call them $(\varepsilon_1, \varepsilon_2)$. To keep the conditions as non-degenerate as possible we take factor functions having as order respectively 1 and 3. As a model we consider

$$\delta(r,\nu,\varepsilon_1,\varepsilon_2) = \varepsilon_1 r(\nu + r^2) + \varepsilon_2 r \tag{2.14}$$

with $\nu, \varepsilon_1, \varepsilon_2 \in \mathbb{R}$ and small. Clearly the local phase portraits near the origin exhibit at most one limit cycle. Also the bifurcation diagram is easy to obtain. There are however two bifurcation types and not one as in the classical Hopf bifurcations. On one hand there is the usual Hopf bifurcation, but there also is the possibility for a limit cycle to disappear from the chosen neighbourhood of $(x,y)=(0,0)$ on which we make the study. Indeed the Hopf bifurcations are situated at

$$\varepsilon_2+\varepsilon_1\nu=0,$$

and if we fix the boundary of the chosen neighbourhood at $\{r=r_0\}$, then we see that the limit cycle can leave the neighbourhood at values

$$\varepsilon_2+r_0^2\varepsilon_1+\varepsilon_1\nu=0.$$

The Hopf bifurcations occur on a regular surface H in parameter space with $\{\varepsilon_2=0\}$ as tangent plane at $(0,0,0)$; the boundary bifurcations occur on another regular surface B with $\{\varepsilon_2=-r_0^2\varepsilon_1\}$ as tangent plane at $(0,0,0)$. The two surfaces delimit four open regions, which in an alternating way contain systems with respectively 0 and 1 limit cycle.

Let us emphasize the fact that one can hence find sequences of parameter values $(\nu^i,\varepsilon_1^i,\varepsilon_2^i)$ tending to $(0,0,0)$ for $i\to\infty$, for which the associated limit cycle does not shrink to the origin, but has $\{r=r_0\}$ as a limit (in the Hausdorff metric). This kind of phenomenon had first been encountered in [DRS] in the study of the local bifurcations near a codimension 3 nilpotent singularity of elliptic type. It is not present in the generic Hopf-Takens bifurcations neither in the degenerate Hopf-Takens bifurcations studied in section 2.3.1.

Let us now finish by showing that -as we might expect- this bifurcation diagram is stable for higher order perturbation if we do not change the Bautin Ideal. More precisely we consider situations in which the displacement map is

$$\delta(r,\nu,\varepsilon_1,\varepsilon_2)=r\left(\varepsilon_1\left(\nu+r^2+r^3f(r)\right)+\varepsilon_2\left(1+r^2g(r)\right)+A(r,\nu,\varepsilon_1,\varepsilon_2)\right) \quad (2.15)$$

with f,g,A of class C^ω and

$$\frac{\partial A}{\partial r}(0,\nu,\varepsilon_1,\varepsilon_2)\equiv 0 \text{ and } A(r,\nu,\varepsilon_1,\varepsilon_2)=O(\|(\varepsilon_1,\varepsilon_2)\|^2), \text{ for } (\varepsilon_1,\varepsilon_2)\to 0$$

The absence of r^2-terms in (2.15) is a consequence of the theory developed in section 1.2.3, and more precisely of proposition 27.

For an expression like in (2.15) the Hopf bifurcation is expressed by

$$\varepsilon_2+\nu\varepsilon_1+A(0,\nu,\varepsilon_1,\varepsilon_2)=0. \quad (2.16)$$

If one chooses the neighbourhood appropriately then the boundary bifurcation is expressed by the condition

$$\varepsilon_1\left(\nu+r_0^2+r_0^3f(r_0)\right)+\varepsilon_2\left(1+r_0^2g(r_0)\right)+A(r_0,\nu,\varepsilon_1,\varepsilon_2)=0. \quad (2.17)$$

It is clear that no other bifurcation types can occur, since away from the origin a limit cycle is necessarily unique and hyperbolic. Since the two bifurcation surfaces expressed by (2.16) and (2.17) are transverse, it is easy to prove that the bifurcation diagram of (2.15) is C^ω diffeomorphic to the one described by the model (2.14).

Now checking on the analytic family that we have a displacement map like in (2.15) can again be done either by calculating a normal form of the vector field or by calculating the Lyapunov quantities. One has to check the folowing three conditions on the Lyapunov quantities:

1. The Lyapunov ideal has to be $(\varepsilon_1, \varepsilon_2)$, i.e. it is generated by the germs at $(0,0,0)$ of the analytic functions
$$(\nu, \varepsilon_1, \varepsilon_2) \mapsto \varepsilon_1 \text{ and } (\nu, \varepsilon_1, \varepsilon_2) \mapsto \varepsilon_2$$
2. The divergence (i.e., twice the 0-th Lyapunov quantity) has to be $\varepsilon_2 + \nu\varepsilon_1 + O\left(\|(\varepsilon_1, \varepsilon_2)\|^2\right)$, up to a strictly positive function;
3. The first Lyapunov quantity, under the condition that the divergence is zero, has to be $\varepsilon_1 + O\left(\varepsilon_1^2\right) + O\left(\varepsilon_2\right)$, up to a strictly positive function.

Chapter 3

Generalized Liénard equations

3.1 Introduction

Recently many contributions have been devoted to analyse Bautin's approach to the local Hilbert's 16th problem. This involves algebraic techniques such as the use of the Bautin ideal and Lyapunov quantities and bifurcation techniques such as normal forms. The aim of this chapter is to illustrate the advantage of this approach. As particular examples of unfoldings, we will consider some families of plane vector fields associated with second-order differential equations, namely Liénard equations. There are many interests to consider Liénard equations (see for instance [F95, FP, F02]). Furthermore, the (generalized) Liénard equation provides the simplest settings to illustrate the advantages of the approach based on the Bautin ideal or equivalently, Lyapunov quantities (theorem 54).

The Bautin ideal, cyclicity, Lyapunov quantities, and generic Hopf-bifurcations for a family of Liénard equations is defined by the corresponding family of planar vector fields:

Definition 149 *Let $(E_\lambda)_\lambda$ be a C^ω (or C^∞ or $C^k, k \in \mathbb{N}$) family of autonomous second order differential equations, i.e., there exists a C^ω (or C^∞ or $C^k, k \in \mathbb{N}$) family of functions $(f_\lambda)_\lambda$ such that E_λ can be written as:*

$$x'' = f_\lambda(x, x'),$$

and let e be a center of the corresponding family of planar vector fields $(X_\lambda)_\lambda$. Then,

1. *Lyapunov quantities, the cyclicity and the Bautin ideal of $(E_\lambda)_\lambda$ at (e, λ^0), are defined by the ones for the family $(X_\lambda)_\lambda$.*

2. *we say that $(E_\lambda)_\lambda$ exhibits a generic Hopf-Takens bifurcation of codimension l if and only if the family $(X_\lambda)_\lambda$ does.*

In article ([F02]), the cyclicity was computed following the Françoise-Yomdin approach based on a recurrency relation for the coefficients of the return map and a complex analysis method (Bernstein's inequality) for the classical Liénard equations. Here, in section 3.2, we use instead techniques of R. Roussarie, based on a minimal system of generators for the Bautin ideal which provides a lower bound for the cyclicity (section 1.2.3). Finding this lower bound is closely related to the existence of Hopf-Takens bifurcations. This last approach is developed later, in section 3.3, for generalized Liénard equations. It would be certainly interesting to discuss also the Françoise-Yomdin approach for generalized Liénard equations in the future.

Here, attention not only goes to bounding the cyclicity, but also to local division of the displacement map and presence of Hopf-Takens bifurcations. To the best of our knowledge, these three aspects have not been jointly discussed previously. But perhaps the most original contribution of this chapter is section 3.3, where we discuss these topics for a family of generalized Liénard equations.

In our approach, we use results from sections 1.2.3, 1.2.4, 2.2.3 and 2.3.1, concerning Bautin ideal, Lyapunov quantities and Hopf-Takens bifurcations. In particular, we use a slightly generalised result on Hopf-Takens bifurcations (cfr. theorems 142, 143, 146 and 147), in case the first r Lyapunov quantities identically vanish (or equivalently, the traditional displacement maps δ_λ all are divisible by s^{2r+3}). Let us now gather together these results.

Let $(X_\lambda)_\lambda$ be a C^ω family of planar vector fields of type (1.27), let $(\delta_\lambda)_\lambda$ be an associated family of displacement maps (parametrised by s such that $s=0$ corresponds to $e=(0,0)$), and let $\{V_i : i \in \mathbb{N}\}$ be a set of Lyapunov quantities for the focus e. Assume that

$$V_i \equiv 0, \forall 0 \leq i \leq r \text{ and } V_{r+1} \not\equiv 0 \tag{3.1}$$

Notice that fact (3.1) is equivalent to the fact that the displacement maps δ_λ are divisible by s^{2r+3} and $\delta_\lambda^{(2r+3)} \neq 0$. There are two possible situations: the non-degenerate situation,

$$\exists l \in \mathbb{N} : \forall j < r+l : V_j\left(\lambda^0\right) = 0 \text{ and } V_{r+l}\left(\lambda^0\right) \neq 0 \tag{3.2}$$

or the degenerate situation,

$$\forall j \in \mathbb{N} : V_j\left(\lambda^0\right) = 0 \tag{3.3}$$

Since the family $(\delta_\lambda)_\lambda$ is analytic, the vector field X_{λ^0} is of center type, in case of (3.3). Let l be a positive integer such that the Bautin ideal at λ^0 is generated by the germs of the analytic functions $V_{r+j}, 1 \leq j \leq l$ at λ^0. Then, the following properties hold:

1. In both cases (3.2) and (3.3), the family $(\delta_\lambda)_\lambda$ can locally be expanded as:

$$\delta(s,\lambda) = \delta_\lambda(s) = s^{2r+3} \sum_{j=1}^{l} V_{r+j}(\lambda)\, h_j(s,\lambda), s \sim 0, \lambda \sim \lambda^0 \tag{3.4}$$

for certain C^ω functions $h_j, 1 \leq j \leq l$ with

$$h_j(s,\lambda) = c_j(\lambda)\, s^{2j-2} + o\left(s^{2j-2}\right), s \to 0$$

for C^ω functions c_j with $c_j\left(\lambda^0\right) > 0, 1 \leq j \leq l$. In particular,

$$\mathrm{Cycl}\left(X_\lambda, \left(e, \lambda^0\right)\right), \mathrm{Mult}\left(X_\lambda, \left(e, \lambda^0\right)\right) \leq l-1.$$

2. If, in case of (3.2), the map

$$\left(\mathbb{R}^p, \lambda^0\right) \to \mathbb{R}^{l-1} : \lambda \mapsto \left(V_{r+1}(\lambda), \ldots, V_{r+l-1}(\lambda)\right)$$

is a submersion at λ^0, then the family $(X_\lambda)_\lambda$ exhibits a generic Hopf-Takens bifurcation of codimension $l-1$ (the sign of its type $X_{\pm}^{(l-1)}$ is given by the sign of $V_{r+l}\left(\lambda^0\right)$). In particular,

$$\mathrm{Cycl}\left(X_\lambda, \left(e, \lambda^0\right)\right) = \mathrm{Mult}\left(X_\lambda, \left(e, \lambda^0\right)\right) = l-1.$$

3. If, in case of (3.3), the map

$$\left(\mathbb{R}^p, \lambda^0\right) \to \mathbb{R}^{l} : \lambda \mapsto \left(V_{r+1}(\lambda), \ldots, V_{r+l}(\lambda)\right)$$

is a submersion at λ^0, then

$$\mathrm{Cycl}\left(X_\lambda, \left(e, \lambda^0\right)\right) = \mathrm{Mult}\left(X_\lambda, \left(e, \lambda^0\right)\right) = l-1.$$

4. In the special case of situation (3.3), that centers occur on a regular hypersurface (section 1.3), we can reduce the current situation to (3.2) in terms of reduced Lyapunov quantities $\bar{V}_i, i \in \mathbb{N}$, to arrive at properties 1. and 2., written in terms of an associated family of reduced displacement maps $\left(\bar{\delta}_\lambda\right)_\lambda$ and $\left\{\bar{V}_i : i \in \mathbb{N}\right\}$.

Finally, remark that properties 1., 2. and 4. also hold for families of planar vector fields of class C^γ, where $\gamma \in \mathbb{N}, \gamma \geq 2(r+l)+1$ or $\gamma = \infty$, and we replace C^ω everywhere in these properties by C^γ. Although, in case of (3.3), the vector field X_{λ^0} is not necessarily of center type this time, property 1. remains valid when we replace the left-hand side of (3.4) by $j_\gamma(\delta_\lambda)(0)$ (and $j_\gamma\left(\bar{\delta}_\lambda\right)(0)$ in property 4.).

3.2 Classical Liénard equations

Throughout this section, we will study (locally) the family of Liénard equations:

$$\ddot{x} + f(x,\lambda)\,\dot{x} + x = 0, \tag{3.5}$$

where $f(x,\lambda)$ is a function of class C^γ ($\gamma \in \{\infty,\omega\}$) with $f(0,\lambda) \equiv 0$. If f is sufficiently differentiable, then we can write

$$f(x,\lambda) = \sum_{j=1}^{2N} f_j(\lambda) x^j + O\left(x^{2N+1}\right), x \to 0,$$

for $N \in \mathbb{N}$ and certain functions $f_j, j \in \mathbb{N}$ of class C^γ.

In section 3.2.1, we see that the first N Lyapunov quantities are given by $f_{2j}, 1 \leq j \leq N$, to arrive in section 3.2.2, where we give results concerning cyclicity, displacement map, and presence of a (degenerate) Hopf-Takens bifurcation, as was explained at the end of section 3.1.

3.2.1 Calculation of Lyapunov quantities

Lyapunov quantities of the Liénard equation (3.5) are defined as Lyapunov quantities of the corresponding system of first order differential equations:

$$X_\lambda \leftrightarrow \left\{ \begin{array}{lcl} \dot{x} & = & y \\ \dot{y} & = & -x - f(x,\lambda) y \end{array} \right. \tag{3.6}$$

As usual, system (3.6), is transformed into:

$$\left\{ \begin{array}{lcl} \dot{X} & = & Y + F_1(X,\lambda) \\ \dot{Y} & = & -X \end{array} \right. \tag{3.7}$$

by use of the near-identity transformation

$$(x,y) \mapsto (x, y - F_1(x,\lambda)), \text{ where } F_1(x,\lambda) = -\int_0^x f(u,\lambda)\,\mathrm{d}u. \tag{3.8}$$

From proposition 58, it follows that Lyapunov quantities of (3.7) are given by the coefficients of odd order in x of $F_1(x,\lambda)$, and hence, by the coefficients of even order in x of the function $f(x,\lambda)$. More precisely:

Corollary 150 *Let $\{V_i : i \in \mathbb{N}\}$ be a set of Lyapunov quantities of (3.5). Then $V_0 \equiv 0$, and $\forall 1 \leq i \leq N$:*

$$V_i = c_i f_{2i} \bmod (f_2, f_4, \ldots, f_{2N-2}),$$

where $c_i \in \mathbb{Q}^- \setminus \{0\}$.

3.2.2 Conclusions

By corollary 150 and the remarks at the end of section 3.1, we have the following results for a general family of classical Liénard equations

$$\ddot{x} + f(x,\lambda)\dot{x} + x = 0. \tag{3.9}$$

In the *non-degenerate situation* (3.2) :

Theorem 151 *Consider the family of Liénard equations* (3.9), *where* $f(x,\lambda)$ *is a function of class* C^γ, $\gamma \in \{\infty, \omega\}$, *with*

$$f(x,\lambda) = \sum_{j=1}^{2N} f_j(\lambda) x^j + O\left(x^{2N+1}\right), x \to 0,$$

for $N \in \mathbb{N}$ *and certain functions* $f_j, 1 \leq j \leq 2N$ *of class* C^γ. *Suppose that* $\lambda^0 \in \mathbb{R}^p$ *such that*

$$f_{2j}\left(\lambda^0\right) = 0, \forall 1 \leq j \leq N-1, \text{ and } f_{2N}\left(\lambda^0\right) \neq 0.$$

1. *Then, there are at most* $N-1$ *limit cycles in system* (3.6) *that bifurcate from the non-degenerate elliptic singularity; i.e.*

$$\operatorname{Cycl}\left(X_\lambda, \left(e, \lambda^0\right)\right) \leq N-1.$$

2. *Furthermore, if the map*

$$\lambda \mapsto \left(f_2(\lambda), f_4(\lambda), \ldots, f_{2N-2}(\lambda)\right)$$

is a submersion at λ^0, *then the family* $(X_\lambda)_\lambda$ *exhibits a* generic Hopf-Takens *bifurcation of codimension* $N-1$ *at the origin* e. *Moreover, the sign of its type* $X_\pm^{(N-1)}$ *is given by the sign of* $-f_{2N}\left(\lambda^0\right)$.

Next, in the *degenerate situation* (3.3) :

Theorem 152 *Consider the family of Liénard equations* (3.9), *where the function* f *is* C^ω *with*

$$f(x,\lambda) = \sum_{j=1}^{\infty} f_j(\lambda) x^j, x \to 0$$

for certain C^ω *functions* $f_j, j \geq 1$. *Suppose that* $\lambda^0 \in \mathbb{R}^p$ *and*

$$f_{2j}\left(\lambda^0\right) = 0, \forall j \in \mathbb{N}.$$

Then, the vector field X_{λ^0}, *defined by* (3.6), *is of center type, and there exists* $N \in \mathbb{N}$ *such that* $\forall j \in \mathbb{N} : f_{2j} \in (f_2, f_4, \ldots, f_{2N})$. *In particular,*

1. *the Bautin ideal is generated by the germs of the analytic functions* $f_2, f_4, \ldots, f_{2N}$ *at* λ^0, *and the displacement map can be written as:*

$$\delta(s,\lambda) = s^3 \sum_{j=1}^{N} f_{2j}(\lambda) h_j(s,\lambda),$$

for analytic functions h_j *with*

$$h_j(s,\lambda) = \eta_j(\lambda) s^{2j-2} + o\left(s^{2j-2}\right), s \to 0,$$

for certain $\eta_j\left(\lambda^0\right) < 0, \forall 1 \leq j \leq N$.

2. *If* $\{f_{2j} : 1 \leq j \leq N\}$ *is a set of generators, then*

$$\mathrm{Cycl}\left(X_\lambda, \left(e, \lambda^0\right)\right) \leq N - 1.$$

If, furthermore, the map $\lambda \mapsto \left(f_2(\lambda), f_4(\lambda), \ldots, f_{2N}(\lambda)\right)$ *is a submersion at* λ^0*, then*

$$\mathrm{Cycl}\left(X_\lambda, \left(e, \lambda^0\right)\right) = N - 1.$$

Theorems 151 and 152 are stated in the most general case, in the sense that we consider families that do not necessarily depend in a polynomial way on (x, y), but are sufficiently differentiable in (x, y). Theorem 153 below contains the analoguous results in case the Liénard equations are polynomials. Let the integer part of $N/2$ be denoted by $[N/2]$.

Theorem 153 *Consider a family of polynomial Liénard equations* (3.9) *with*

$$f(x, \lambda) = \sum_{j=1}^{N} \lambda_j x^j, \lambda = (\lambda_1, \ldots, \lambda_N) \in \mathbb{R}^N$$

Fix a parameter value $\lambda^0 = \left(\bar{\lambda}_1, \bar{\lambda}_2, \ldots, \bar{\lambda}_N\right)$.

1. *If there exists* $1 \leq l \leq [N/2]$ *such that* $\bar{\lambda}_{2i} = 0, \forall 1 \leq i \leq l-1$ *and* $\bar{\lambda}_{2l} \neq 0$*, then the family of Liénard equations exhibits a generic Hopf-Takens bifurcation of codimension* $l-1$*. Moreover, the sign of its type* $X_{\pm}^{(l-1)}$ *is given by the sign of* $-\bar{\lambda}_{2l}$*. Particularly,*

$$\mathrm{Cycl}\left(X_\lambda, \left(e, \lambda^0\right)\right) = l - 1$$

2. *If for all* $1 \leq l \leq [N/2]$: $\bar{\lambda}_{2l} = 0$*, then the Bautin ideal is generated by the germs of the analytic functions* $\lambda_2, \lambda_4, \ldots, \lambda_{2[N/2]}$ *at* λ^0*, and the displacement map can be written as:*

$$\delta(s, \lambda) = s^3 \sum_{j=1}^{[N/2]} \lambda_{2j} h_j(s, \lambda),$$

for analytic functions h_j *with*

$$h_j(s, \lambda) = \eta_j(\lambda) s^{2j-2} + o\left(s^{2j-2}\right), s \to 0,$$

for certain $\eta_j\left(\lambda^0\right) < 0, \forall 1 \leq j \leq [N/2]$*. Particularly,*

$$\mathrm{Cycl}\left(X_\lambda, \left(e, \lambda^0\right)\right) = [N/2] - 1.$$

The following theorem deals with the special situation of (3.3), in which centers occur on the regular hypersurface $\{(\nu, \varepsilon) \in \mathbb{R}^p : \varepsilon = 0\}$.

Theorem 154 *Consider a family of polynomial Liénard equations* (3.9) *with*

$$f(x,\nu,\varepsilon)=\varepsilon^k\sum_{j=1}^{N}\nu_j x^j,\nu=(\nu_1,\ldots,\nu_N),\lambda=(\nu,\varepsilon)$$

Fix a parameter $\nu^0=(\bar{\nu}_1,\bar{\nu}_2,\ldots,\bar{\nu}_N)$, *and put* $\lambda^0=(\nu^0,0)$. *If* $1\leq l\leq[N/2]$ *such that* $\bar{\nu}_{2i}=0,\forall 1\leq i\leq l-1$ *and* $\bar{\nu}_{2l}\neq 0$, *then the family of Liénard equations exhibits a generic Hopf-Takens bifurcation of codimension* $l-1$, *uniformly with respect to* $\varepsilon\neq 0$. *Moreover, the sign of its type* $X_{\pm}^{(l-1)}$ *is given by the sign of* $-\varepsilon^k\bar{\nu}_{2l}$. *In particular,*

$$\mathrm{Cycl}\left(X_\lambda,\left(e,\lambda^0\right)\right)=l-1.$$

3.3 Generalised Liénard equations

In this section, we consider a family of generalized Liénard equations:

$$\ddot{x}+f(x,\lambda)\dot{x}+g(x,\lambda)=0 \tag{3.10}$$

where f,g are C^γ with $\gamma\in\{\infty,\omega\}$, and

$$\begin{cases} f(0,\lambda) &= 0,\forall\lambda \\ g(x,\lambda) &= x+o(x),x\to 0\end{cases}$$

For this family of generalized Liénard equations, we will give similar results as in case of the classical family of Liénard equations: investigating the presence of a generic Hopf-Takens bifurcation and bounding the cyclicity near the focus $e=(0,0)$, by calculation of Lyapunov quantities. However, this time, more elaborate calculations are involved.

Again, we would like to use proposition 58 in order to find expressions for Lyapunov quantities in terms of f and g. For this purpose, in section 3.3.1, the generalized Liénard equation is first reduced to the classical one by a near-identity transformation similar to the so-called Cherkas transform. Next, we prove by computations that this generalized Cherkas transformation induces a 'triangular' transformation on the generators of the ideal. Finally, we find explicit expressions for a set of Lyapunov quantities (modulo the preceeding ones), in case the function f is odd with respect to x and f starts with a non-vanishing linear term in x. Finally, in section 3.3.2, we state results, analoguously as in section 3.2.2.

Let us write the (possibly only formal) series

$$\begin{cases} f(x,\lambda) &= \sum_{i=1}^{\infty}f_i(\lambda)x^i \\ g(x,\lambda) &= x+\sum_{i=2}^{\infty}g_i(\lambda)x^i\end{cases}$$

3.3.1 Calculation of Lyapunov quantities

The generalized Liénard equation can be written as the following system of first order differential equations:

$$\begin{cases} \dot{x} &= y \\ \dot{y} &= -g(x,\lambda) - f(x,\lambda)\, y \end{cases}$$

Again, we perform transformation (3.8) to obtain the system:

$$\begin{cases} \dot{x} &= Y + F_1(x,\lambda) \\ \dot{Y} &= -g(x,\lambda) \end{cases} \tag{3.11}$$

To transform the family (3.10) to a family of type (3.7), we will introduce (inspired by Cherkas'article ([Cher]) a new (locally defined) coordinate $X = xA(x,\lambda)$ such that (3.11) is equivalent to a system of type (3.7). In the coordinates (X,Y), system (3.11) is transformed into

$$\begin{cases} \dot{X} &= (Y + F_1(XB(X),\lambda))\left(A(x,\lambda) + x\frac{\partial}{\partial x}A(x,\lambda)\right) \\ \dot{Y} &= -X\left(A(x,\lambda) + x\frac{\partial}{\partial x}A(x,\lambda)\right)\dfrac{g(x,\lambda)}{X\left(A(x,\lambda) + x\frac{\partial}{\partial x}A(x,\lambda)\right)} \end{cases}$$

where $x = XB(X,\lambda)$ denotes the inverse transformation of $X = xA(x,\lambda)$. Hence A has to be defined such that

$$\frac{g(x,\lambda)}{xA(x,\lambda)\left(A(x,\lambda) + x\frac{\partial}{\partial x}A(x,\lambda)\right)} = 1$$

This is equivalent to

$$2g(x,\lambda) = \frac{\partial}{\partial x}\left((xA(x,\lambda))^2\right).$$

Therefore, A can formally be written as

$$A(x,\lambda) = \sqrt{1 + \sum_{i=1}^{\infty} q_i(\lambda)\, x^i},$$

where $\forall i \geq 1$:

$$q_i = \frac{2g_{i+1}}{i+2}. \tag{3.12}$$

After division by $A(x,\lambda) + x\frac{\partial}{\partial x}A(x,\lambda)$, we obtain the desired system with

$$F(X,\lambda) = F_1(XB(X),\lambda).$$

Now, it is clear that Lyapunov quantities for the generalized Liénard equation are given by the coefficients of odd order in X appearing in the expansion of F. Let us write the (possibly only formal) series

$$\begin{cases} A(x,\lambda) &= \sum_{i=0}^{\infty} a_i(\lambda)\, x^i \text{ with } a_0 = 1 \\ B(X,\lambda) &= \sum_{i=0}^{\infty} b_i(\lambda)\, X^i \text{ with } b_0 = 1 \\ F(X,\lambda) &= \sum_{i=2}^{\infty} d_i(\lambda)\, X^i \\ F_1(x,\lambda) &= \sum_{i=2}^{\infty} p_i(\lambda)\, x^i \end{cases}$$

where $\forall i \geq 2$:

$$p_i = -\frac{f_{i-1}}{i}.$$

We will derive successively general structure formulas for the coefficients a_i, b_i and d_i. Furthermore, we investigate for each of these coefficients what happens when we start from the case where $f(x,\lambda)$ is an odd function with respect to x with $f(x,\lambda) = 2x + o(x), x \to 0$ (i.e. $f_1 = 2$, or equivalently, $p_2 = -1$). Notice that this corresponds to the fact that $F_1(x,\lambda)$ is even with respect to x and starts in quadratic terms of x :

$$\begin{cases} F_1(x,\lambda) &= F_1(-x,\lambda) \\ F_1(x,\lambda) &= -x^2 + o(x^2), x \to 0. \end{cases} \tag{3.13}$$

Throughout the rest of this section, we do not mention any more the dependence of the coefficients on λ explicitly.

Proposition 155 *The coefficients of $A(x)$ are given by the following recurrence relations:*

$$\begin{cases} a_0 &= 1 \text{ and } \forall l \geq 1: \\ a_l &= \sum_{k=1}^{l} \bar{a}_k \sum_{\substack{i_1+\ldots+i_k=l \\ 1\leq i_s \leq l}} q_{i_1}\cdot\ldots\cdot q_{i_k} \end{cases} \tag{3.14}$$

where $\bar{a}_k \in \mathbb{Q}\setminus\{0\}$ arise from the Taylor expansion of $\sqrt{1+u}$ at $u=0$; hence they are given by

$$\bar{a}_k = (-1)^{k-1}\left(\frac{1}{2}\right)^k \frac{1\cdot 3\cdot\ldots\cdot(2k-3)}{k!}, \forall k \geq 1.$$

As a consequence, the coefficients a_l of $A(x)$ are polynomials of degree $\leq l$ in $q_1, q_2, \ldots, q_l$. Moreover, we have the following corollary:

Corollary 156 *There exists the following relation between the coefficients $a_l, l \geq 1$ and the coefficients $q_l, l \geq 1$:*

$$\begin{cases} a_1 &= \frac{1}{2}q_1, \text{ and } \forall l \geq 2: \\ a_l &= \frac{1}{2}q_l \operatorname{mod}(q_1,\ldots,q_{l-1}) \end{cases}$$

Moreover,

$$\begin{cases} a_1 & = & \frac{1}{2}q_1, \text{ and } \forall k \geq 1 : \\ a_{2k+1} & = & \frac{1}{2}q_{2k+1} \operatorname{mod}(q_1, q_3, \ldots, q_{2k-1}) \end{cases}$$

Proof. The relation between the coefficients of odd order follows from the observation that if the sum $i_1 + \ldots + i_k$ is odd, then there has to be at least one odd index i_s. ■

By identifying coefficients of equal powers in X of the equation:

$$XB(X)A(XB(X)) = X,$$

we can deduce recurrence formulas for the coefficients b_i of B :

Proposition 157 *The coefficients of $B(X)$ are given by the following recurrence formulas:*

$$\begin{cases} b_0 = 1, \text{ and } \forall l \geq 1 : \\ b_l + b_0 C_l + b_1 C_{l-1} + \ldots + b_{l-1} C_1 = 0 \end{cases} \tag{3.15}$$

where $C_l, l \geq 1$ is defined by

$$C_l = a_1 b_{l-1} + a_2 \sum_{\substack{i_1+i_2=l-2 \\ 0 \leq i_s \leq l-2}} b_{i_1} b_{i_2} + \cdots \tag{3.16}$$

$$+ a_r \sum_{\substack{i_1+\ldots+i_r=l-r \\ 0 \leq i_s \leq l-r}} b_{i_1} \cdot \ldots \cdot b_{i_r} + \ldots + a_l.$$

As a consequence, the coefficients b_l of $B(X)$ are polynomials of degree $\leq l$ in $q_1, \ldots, q_l$. We also have the following corollary:

Corollary 158 *There exists the following relation between the coefficients $a_l, l \geq 1$ and $b_l, l \geq 1$:*

$$\begin{cases} b_1 & = & -a_1, \text{ and } \forall k \geq 1 : \\ b_{2k+1} & = & -a_{2k+1} \operatorname{mod}(b_1, b_3, \ldots, b_{2k-1}) \end{cases}$$

Proof. It can be easily checked that $b_1 = -a_1$ and $C_1 = -b_1$. Using (3.15) and (3.16) we now prove by induction on k that,$\forall k \geq 1$,

$$\begin{cases} C_{2k+1} & = & a_{2k+1} \operatorname{mod}(b_1, b_3, \ldots, b_{2k-1}) \\ b_{2k+1} & = & -a_{2k+1} \operatorname{mod}(b_1, b_3, \ldots, b_{2k-1}) \end{cases} \tag{3.17}$$

Suppose that property (3.17) holds for all $1 \leq k \leq j$. From (3.16), we have

$$C_{2j+3} = a_1 b_{2j+2} + a_2 \sum_{\substack{i_1+i_2=2j+1 \\ 0 \leq i_s \leq 2j+1}} b_{i_1} b_{i_2} + \cdots$$

$$+ a_r \sum_{\substack{i_1+\ldots+i_r=2j+3-r \\ 0 \leq i_s \leq 2j+3-r}} b_{i_1} \cdot \ldots \cdot b_{i_r} + \ldots + a_{2j+3}.$$

We now show that the terms

$$a_r \sum_{\substack{i_1+\ldots+i_r=2j+3-r \\ 0\leq i_s\leq 2j+3-r}} b_{i_1}\cdot\ldots\cdot b_{i_r}$$

belong to the ideal $(b_1, b_3, \ldots, b_{2k+1})$. For $1 \leq r \leq 2j+1$ odd, these terms belong to the ideal $(a_r) \subset (b_1, b_3, \ldots, b_{2k+1})$, by corollary 156 and the induction hypothesis. For $2 \leq r \leq 2j+2$ even, the sum $i_1 + \ldots + i_r = 2j+3-r$ is odd, and hence there has to be at least one odd index i_s, and $0 \leq i_s \leq 2j+3-r \leq 2j+1$. Hence, it follows that property (3.17), concerning the coefficients C_{2k+1}, holds for all $k \geq 1$.

For the result on the coefficient b_{2j+3}, we write by use of (3.15):

$$b_{2j+3} = -C_{2j+3} - \sum_{\substack{r=1 \\ r \text{ odd}}}^{2j+1} b_j C_{2j+3-r} - \sum_{\substack{r=2 \\ r \text{ even}}}^{2j+2} b_j C_{2j+3-r}$$

It is clear that the second term in the right-hand side of this equation belongs to the ideal $(b_1, b_3, \ldots, b_{2j+1})$; by the induction hypothesis, also the third term belongs to this ideal, since $2j+3-r$ is odd and $2j+3-r \leq 2k+1$. ■

From corollaries 156 and 158, we now obtain the following relation between the coefficients b_{2k+1} and $q_{2k+1}, \forall k \geq 1$:

Corollary 159 *There exists the following relation between the coefficients b_{2k+1} and the coefficients $q_{2k+1}, k \in \mathbb{N}$:*

$$\begin{cases} b_1 & = & -\frac{1}{2}q_1 \\ b_{2k+1} & = & -\frac{1}{2}q_{2k+1} \operatorname{mod}(q_1, q_3, \ldots, q_{2k+1}) \end{cases}$$

The following proposition says that the coefficients d_l of the function F are polynomials of degree $\leq l$ in $p_2, \ldots, p_l, q_1, \ldots, q_{l-2}$; the structure is given more precisely in the following proposition:

Proposition 160 *1. The coefficients d_l of $F(X)$ are given by the formulas,*

$$d_l = \sum_{\substack{t+j=l \\ j\geq 2, t\in\mathbb{N}}} p_j \sum_{i_1+\ldots+i_j=t} b_{i_1}\cdot\ldots\cdot b_{i_j}, \quad \forall l \geq 2 \tag{3.18}$$

2. Moreover, if $p_{2j+1} = 0, \forall j \in \mathbb{N}$, then $\forall k \geq 1$:

$$d_{2k+1} = p_2 b_{2k-1} \operatorname{mod}(b_1, b_3, \ldots, b_{2k-3}).$$

Proof. The first part follows from the fact that d_l corresponds to the coefficient of X^l in

$$F(X, \lambda) = \sum_{j=2}^{\infty} p_j X^j B(X, \lambda)^j.$$

The second part follows from the following observation. If $l = 2k+1$ is odd in (3.18), then t has to be odd (since j even); as a consequence, there must be at least one odd index i_s. ∎

From proposition 160 and corollary 159, we can deduce the following corollaries.

Corollary 161 *Suppose that $p_{2j+1} = 0, \forall j \in \mathbb{N}$.*

1. *Then*
$$\begin{cases} d_3 &= -\frac{1}{2}p_2q_1, \text{ and } \forall k \geq 2: \\ d_{2k+1} &= -\frac{1}{2}p_2q_{2k-1} \operatorname{mod}(q_1, q_3, \ldots, q_{2k-3}) \end{cases}$$

2. *If, for instance, $p_2 = -1$, then*
$$\begin{cases} d_3 &= \frac{1}{2}q_1, \text{ and } \forall k \geq 2: \\ d_{2k+1} &= \frac{1}{2}q_{2k-1} \operatorname{mod}(q_1, q_3, \ldots, q_{2k-3}) \end{cases}$$

Corollary 162 *Lyapunov quantities $V_i, i \geq 1$, in the generalized Liénard system*

$$\ddot{x} + f(x,\lambda)\dot{x} + g(x,\lambda) = 0$$

with

$$\begin{cases} f(x,\lambda) &= 2x + o(x), x \to 0 \\ f(x,\lambda) &= -f(-x,\lambda) \\ g(x,\lambda) &= x + \sum_{i=2}^{\infty} g_i(\lambda) x^i \end{cases}$$

are given by:

$$\begin{cases} V_1(\lambda) &= \frac{1}{3}g_2(\lambda), \text{ and } \forall k \geq 2 \\ V_k(\lambda) &= \frac{1}{2k+1}g_{2k}(\lambda) \operatorname{mod}(g_2(\lambda), g_4(\lambda), \ldots, g_{2k-2}(\lambda)) \end{cases}$$

3.3.2 Conclusions

The conclusions deal with the following family of generalized Liénard equations:

$$\ddot{x} + f(x,\lambda)\dot{x} + g(x,\lambda) = 0 \tag{3.19}$$

where f, g are C^∞ or C^ω with

$$\begin{cases} f(x,\lambda) &= 2x + o(x), x \to 0 \\ f(x,\lambda) &= -f(-x,\lambda) \\ g(x,\lambda) &= x + \sum_{i=2}^{\infty} g_i(\lambda) x^i \end{cases}$$

Combining corollary 162 and the results of section 3.1, we can draw conclusions on the presence of Hopf-Takens bifurcations in these generalized Liénard equations, as we did in section 3.2.2 for the classical Liénard equations.

Again, we start by stating the results for general families, and next, for polynomial ones. In situation (3.2), we arrive at the following theorem:

Theorem 163 *Suppose we are given a family of generalized Liénard equations* (3.19). *Suppose* $\lambda^0 \in \mathbb{R}^p$ *such that*

$$g_{2j}\left(\lambda^0\right) = 0, \forall 1 \leq j \leq N-1, \text{ and } g_{2N}\left(\lambda^0\right) \neq 0.$$

1. *Then there are at most* N *limit cycles in system* (3.19) *that bifurcate from the non-degenerate elliptic singularity; i.e.*

$$\mathrm{Cycl}\left(X_\lambda, \left(e, \lambda^0\right)\right) \leq N-1.$$

2. *Furthermore, if the map*

$$\lambda \mapsto \left(g_2\left(\lambda\right), g_4\left(\lambda\right), \ldots, g_{2N-2}\left(\lambda\right)\right)$$

is a submersion at λ^0, *then the family* $\left(X_\lambda\right)_\lambda$ *exhibits a* generic Hopf-Takens bifurcation of codimension $N-1$. *Moreover, the sign of its type* $X_{\pm}^{(N-1)}$ *is given by the sign of* $g_{2N}\left(\lambda^0\right)$.

In situation (3.3), we obtain the following result:

Theorem 164 *Suppose we are given a family of generalized Liénard equations* (3.19); *we suppose that the functions* f *and* g *are analytic. Assume that for a given* $\lambda^0 \in \mathbb{R}^p$ *all* $g_{2j}\left(\lambda^0\right) = 0, j \in \mathbb{N}$. *Then, the vector field* X_{λ^0}, *defined by* (3.19), *is of center type, and there exists* $N \in \mathbb{N}$ *such that* $\forall j > N : g_{2j} \in \left(g_2, g_4, \ldots, g_{2N}\right)$.

1. *Then, the Bautin ideal is generated by* $g_2, g_4, \ldots, g_{2N}$, *and the displacement map can be written as:*

$$\delta\left(s, \lambda\right) = s^3 \sum_{j=1}^{N} g_{2j}\left(\lambda\right) h_j\left(s, \lambda\right),$$

for analytic functions h_j *with*

$$h_j\left(s, \lambda\right) = \eta_j\left(\lambda\right) s^{2j-2} + o\left(s^{2j-2}\right), s \to 0,$$

with $\eta_j\left(\lambda^0\right) > 0, \forall 1 \leq j \leq N$.

2. *If* $\{g_{2j} : 1 \leq j \leq N\}$ *is a set of generators, then*

$$\mathrm{Cycl}\left(X_\lambda, \left(e, \lambda^0\right)\right) \leq N-1. \tag{3.20}$$

Furthermore, the inequality in (3.20) *becomes an equality, if the map*

$$\lambda \mapsto \left(g_2, g_4, \ldots, g_{2N}\right)$$

is a submersion at λ^0.

For a polynomial family, previous theorems are reduced to the following one:

Theorem 165 *Suppose we are given a family of polynomial generalized Liénard equations* (3.19) *with*

$$\begin{cases} f(x,\lambda) &= 2x + o(x), x \to 0 \\ f(x,\lambda) &= -f(-x,\lambda) \\ g(x,\lambda) &= x + \sum_{i=2}^{N} \lambda_i x^i, \lambda = (\lambda_2, \lambda_3, \ldots, \lambda_N) \in \mathbb{R}^{N-1} \end{cases}$$

Fix $\lambda^0 = (\bar{\lambda}_2, \ldots, \bar{\lambda}_N) \in \mathbb{R}^{N-1}$.

1. *If there exists* $1 \leq l \leq [N/2]$ *such that*

$$\bar{\lambda}_{2j} = 0, \forall 1 \leq j \leq l-1, \text{ and } \bar{\lambda}_{2l} \neq 0,$$

then the family $(X_\lambda)_\lambda$ *exhibits a generic Hopf-Takens bifurcation of codimension* $l-1$. *Moreover, the sign of its type* $X_{\pm}^{(l-1)}$ *is given by the sign of* $\bar{\lambda}_{2l}$. *Particularly,*

$$\mathrm{Cycl}\left(X_\lambda, (e, \lambda^0)\right) = l-1.$$

2. *If for all* $1 \leq l \leq [N/2]$: $\bar{\lambda}_{2l} = 0$. *Then, the Bautin ideal is generated by* $\{\lambda_{2i} : 1 \leq i \leq [N/2]\}$, *and the displacement map can be written as:*

$$\delta(s,\lambda) = s^3 \sum_{j=1}^{[N/2]} \lambda_{2j} h_j(s,\lambda),$$

for C^ω *functions* h_j *with*

$$h_j(s,\lambda) = \eta_j(\lambda) s^{2j-2} + o\left(s^{2j-2}\right), s \to 0,$$

with $\eta_j > 0, \forall 1 \leq j \leq [N/2]$. *Moreover,*

$$\mathrm{Cycl}\left(X_\lambda, (e, \lambda^0)\right) = [N/2] - 1.$$

The following theorem deals with the special situation of (3.3), in which centers occur on a regular hypersurface.

Theorem 166 *Suppose we are given a family of polynomial generalized Liénard equations* (3.19) *with*

$$\begin{cases} f(x,\lambda) &= 2x + o(x), x \to 0 \\ f(x,\lambda) &= -f(-x,\lambda) \\ g(x,\lambda) &= x + \varepsilon^k \sum_{i=2}^{N} \nu_i x^i \end{cases}$$

where $\lambda = (\nu, \varepsilon), \nu = (\nu_2, \dots, \nu_N) \in \mathbb{R}^{N-1}$. *Fix* $\lambda^0 = (\bar{\nu}_2, \dots, \bar{\nu}_N, 0) \in \mathbb{R}^N$. *If there exists* $1 \le l \le [N/2]$ *such that*

$$\bar{\nu}_{2j} = 0, \forall 1 \le j \le l-1, \text{ and } \bar{\nu}_{2l} \neq 0,$$

then the family $(X_\lambda)_\lambda$ *exhibits a generic Hopf-Takens bifurcation of codimension* $l-1$, *uniformly with respect to* $\varepsilon \neq 0$. *Moreover, the sign of its type* $X_{\pm}^{(l-1)}$ *is given by the sign of* $\varepsilon^k \bar{\nu}_{2l}$. *Particularly,*

$$\operatorname{Cycl}(X_\lambda, (e, \lambda^0)) = l - 1.$$

Chapter 4

Algebraic curves of maximal cyclicity

4.1 Introduction

Throughout this chapter, we consider analytic p-parameter families of planar vector fields $(X_\lambda)_\lambda$, unfolding a vector field X_{λ^0} of center type. Furthermore, we suppose that Γ is a non-isolated regular periodic orbit of X_{λ^0}. Let $(\delta_\lambda)_\lambda$ be an analytic family of displacement maps with $\delta_{\lambda^0} \equiv 0$. Checking properties on the derivatives of this map leads to a good understanding of the bifurcation diagram of the limit cycles of X_λ close to Γ, in terms of the parameter λ near λ^0. However, in most concrete situations, finding an explicit expression or even computing derivatives of the displacement map, remains a tough problem, especially when a multi-dimensional parameter is involved. In 1-parameter families, there is the well-known technique of computing Melnikov functions of Françoise and Poggiale (section 1.2.2). This technique enables one to compute an upperbound for the multiplicity in 1-parameter unfoldings.

In treating multi-parameter bifurcation problems from centers, one can restrict to a study of 1-parameter subfamilies, sometimes even to algebraic or to linear subfamilies, depending on the kind of problem one treats and the kind of family one works with. In any case, it needs to be checked carefully that one respects the necessary conditions. In this part, we describe a number of results that can be used for that purpose.

In section 4.2, profit is taken from the theory of analytic geometry to explore the structure of the bifurcation diagrams of zeroes of analytic functions (and hence limit cycles of analytic planar vector fields). In particalur, we show that 1-parameter techniques can be used to find upperbounds for the cyclicity in multi-parameter families. Let us already mention the main result of section 4.2, which will be the center of discussion in the rest of this chapter: 'There exist analytic curves $\zeta_c, \zeta_m : I \to \mathbb{R}^p$

through λ^0 (i.e. $\zeta_c(0) = \zeta_m(0) = \lambda^0$) such that

$$\operatorname{Cycl}\left(X_\lambda, (\Gamma, \lambda^0)\right) = \operatorname{Cycl}\left(X_{\zeta_c(\varepsilon)}, (\Gamma, 0)\right) \tag{4.1}$$

$$\operatorname{Mult}\left(X_\lambda, (\Gamma, \lambda^0)\right) = \operatorname{Mult}\left(X_{\zeta_m(\varepsilon)}, (\Gamma, 0)\right) \tag{4.2}$$

Throughout this chapter we will not really make an explicit distinction between the parametrization $\gamma : I \to \mathbb{R}^n$ and its image $\gamma(I)$. Neither will we always specify I since we are essentially interested in germs. In any case "analytic curve" will mean that $\gamma : I \to \mathbb{R}^n$ is analytic, and the same holds for "algebraic curve" and "linear curve", in the sense that the components of γ are respectively polynomial and affine. The main concern is the existence of an algebraic or linear curve of maximal cyclicity (respectively multiplicity) in each of these families. We now give a precise definition of a curve of maximal cyclicity (respectively multiplicity):

Definition 167 *Consider an analytic p-parameter family of planar vector fields $(X_\lambda)_\lambda$ unfolding a vector field of center type X_{λ^0} $(\lambda \in \mathbb{R}^p)$, and let Γ be a non-isolated periodic orbit of X_{λ^0}. An analytic curve $\zeta(\varepsilon)$ having property (4.1) (respectively (4.2)) is called a curve of maximal cyclicity (respectively a curve of maximal multiplicity). Shortly, we say that ζ is an mcc (respectively an mmc).*

The problem is interesting, since, in studying bifurcations of vector fields of center type (e.g., Hamiltonian ones), the technique of Melnikov functions is generally used. Often one only computes Melnikov functions for 1-parameter subfamilies, sometimes merely induced by straight lines in parameter space. In this chapter, it becomes clear that one needs more information than only these computations to derive the right conclusions on the cyclicity of the whole family.

Besides general analytic families of planar vector fields $(X_\lambda)_\lambda$, our attention goes to the following ones: *algebraic systems*, i.e.

$$X_0 + \sum_{i_1+\ldots+i_p=1}^{i_1+\ldots+i_p=N} \lambda_1^{i_1} \ldots \lambda_p^{i_p} X_{i_1 \ldots i_p} \tag{4.3}$$

and *linear systems*, i.e.

$$X_\lambda = X_0 + \lambda_1 X_1 + \ldots + \lambda_p X_p, \tag{4.4}$$

where the analytic vector fields $X_0, X_{i_1 \ldots i_p}$ $(1 \le i_1 + \ldots + i_p \le N)$, X_i $(1 \le i \le p)$ only depend on (x, y), and the vector field X_0 is a vector field of center type; attention also goes to these kind of families with a *regular* or *principal* Bautin ideal.

Using the algorithms of Françoise or Poggiale (section 1.2.2), the index of 1-parameter subfamilies can be computed; therefore, we also investigate the existence of curves in parameter space such that the index of the induced 1-parameter family is equal to the index of the p-parameter family. In [R00], using the desingularisation theory of Hironaka, R. Roussarie proved that there exists an analytic curve ζ with

$$\operatorname{Index}\left(X_\lambda, (\Gamma, \lambda^0)\right) = \operatorname{Index}\left(X_{\zeta(\varepsilon)}, (\Gamma, 0)\right). \tag{4.5}$$

Definition 168 *Consider an analytic p-parameter family of planar vector fields $(X_\lambda)_\lambda$ unfolding a vector field of center type X_{λ^0} $(\lambda \in \mathbb{R}^p)$, and let Γ be a non-isolated periodic orbit of X_{λ^0}. An analytic curve $\zeta(\varepsilon)$ having property (4.5) is called a curve of maximal index. We shortly say that ζ is an mic.*

In this chapter, we aim to provide a serious warning that passing results from 1- to multi-parameter families is not trivial.

First, we investigate the existence of a linear, algebraic mcc, mmc and mic, in analytic, algebraic and linear families. In section 4.3, we will treat the most general case (without conditions on the Bautin ideal or stratum of maximal cyclicity respectively multiplicity). In sections 4.4 and 4.5, we continue the investigation in case the Bautin ideal is regular and principal respectively. In section 4.6, we consider the case when the family $(X_\lambda)_\lambda$ has a stratum of maximal cyclicity (respectively multiplicity) with non-empty interior at λ^0.

Let us now summarize the most important results. In general, there always exists an algebraic mic. In case of a regular or principal Bautin Ideal, there even exists a linear mic. Furthermore,

	Linear system			Algebraic system			Analytic system		
	I	II	III	I	II	III	I	II	III
Linear mic		yes	yes	no	yes	yes	no	yes	yes
Linear mmc				no	no	no	no	no	no
Linear mcc	no	no		no	no	no	no	no	no
Algebraic mmc							no	no	no
Algebraic mcc							no	no	no

In this table, I, II repectively III corresponds to the general case, in case of a regular respectively a principal Bautin ideal. The word 'yes' means 'always exists', while 'no' means 'does not always exist'. Proofs and counterexamples are provided in sections 4.3, 4.4 respectively 4.5. In case of an empty space, the question of existence still remains open. Important to observe is that even in a linear system there does not need to exist a linear mcc. Counterexamples even occur if the Bautin ideal is regular.

In section 4.6, we determine a condition that guarantees the existence of algebraic mcc's and mmc's. In section 4.6.1, two useful specifications of the curve selection lemma for *open* subanalytic sets are derived. Recall that the curve selection lemma ensures the existence of an analytic curve ζ entering the considered subanalytic set V. By use of the Łojasiewicz inequality (to ensure that certain closed subanalytic stay at some distance from each other), we here prove that if the subanalytic set V is open, there exists of a 'cone of curves surrounding ζ' entering V; by a 'cone of curves surrounding ζ', we mean the union of analytic curves $\hat{\zeta}$ that have a contact of some order n with ζ (see figure 4.1).

If the family $(X_\lambda)_\lambda$ has an mc-stratum (respectively mm-stratum) with non-empty interior at λ^0, then there exists an open subanalytic set W, such that curves entering W at λ^0 are mcc's (respectively mmc's). The precise definition of this condition can

Figure 4.1: Cone of curves surrounding ζ

be found in section 4.6.2. In section 4.6.2, the open subanalytic set W with the 'good' properties is constructed. Next, we apply the curve selection lemma for open subanalytic sets on W, to guarantee the existence of algebraic mcc's (respectively mmc's).

If the family $(X_\lambda)_\lambda$ does not satisfy the condition, then we do not yet know whether an algebraic mcc (respectively mmc) always do exist. In the algebraic example (4.28), there is an algebraic mcc (respectively mmc) present. Moreover, for the moment we don't have an example of a linear family that does not satisfy the condition.

Finally, in section 4.7, we discuss some extra remaining problems, such as the problem of the minimal degree of an algebraic mcc (respectively mmc) for certain specific families. However, we do not know whether a uniform bound exists on the degree (only depending on the degree of algebraicity in λ). Yet we still have one result for analytic families of planar vector fields with a l-dimensional regular Bautin ideal and *Index*$= l-1$: there exists an algebraic mcc (and mmc) of degree $\leq \left[\frac{l+2}{2}\right]$, and every analytic curve with the same $\left[\frac{l+2}{2}\right]$-jet enters the mc-stratum (and mm-stratum) at λ^0 (precise conditions on the Bautin ideal are formulated in theorem 214). As a consequence, for a 2-dimensional regular Bautin ideal with index 1, there exists a linear mcc (mmc).

4.2 Curves of maximal cyclicity and multiplicity

In this section, it turns out that the theory of analytic geometry is an interesting tool in the study of zeroes of analytic functions (and hence limit cycles of analytic planar vector fields). For instance the set B_n of all parameter values with n zeroes can be described by a subanalytic set. In this way, the parameter space can be partitioned in subanalytic sets with respect to the number of zeroes (multiplicity included or not) of the associated function in a fixed interval $I \subset \mathbb{R}$. A very useful lemma is the curve selection lemma (lemma 170), that ensures the existence of an analytic 1-parameter subfamily such that every map in this family has exactly n zeroes in I. Moreover, under certain conditions on $s_0 \in I$, we can achieve that these zeroes all converge to s_0. These results are extended and discussed in detail in section 4.2.1. Afterwards, in section 4.2.2, this study is translated to analytic families of planar vector fields by way of the associated family of displacement maps. As a consequence, there exists an analytic curve in parameter space such that for each parameter value on this curve the maximum number of limit cycles is attained; moreover the cyclicity (respectively multiplicity) of the 1-parameter subfamily induced by this analytic curve is equal to the one of the multi-parameter family.

4.2.1 Configuration of zeroes in analytic families of 1-dimensional functions

In this section we deal with an analytic family $(f_\lambda)_\lambda$ of real-valued functions, i.e.

$$f : I \times B \to \mathbb{R} : (s, \lambda) \mapsto f_\lambda(s), \tag{4.6}$$

where I is an open interval in $\mathbb{R}$ and the parameter space B is an open ball around λ^0 in $\mathbb{R}^p$. We assume that the function f_{λ^0} only has non-isolated zeroes, i.e. f_{λ^0} is identically zero in I_0. Attention then goes to all possible configurations of zeroes that can arise from a non-isolated zero $s_0 \in I$ of f_{λ^0} after small variations of the parameter value λ^0.

If for a certain parameter value λ the function f_λ has exactly n zeroes in I, say $\xi_1 < \ldots < \xi_n$, with respective multiplicities $m_1, \ldots, m_n$, then we will say that λ gives rise to the configuration $(n, m_1, \ldots, m_n)$ in I. For other parameter values λ the function f_λ can have no zeroes at all, or can have infinitely many zeroes in I. We will say that those parameter values give rise to the configuration 0, respectively ∞ in I.

How is the parameter space B partitioned by these configurations? What is the structure of the sets, determined by the configurations? Is there an analytic 1-parameter subfamily that has just one particular configuration? These questions will be investigated in this section. Of particular interest are those configurations near s_0 that appear for parameters arbitrarily close to λ^0.

The set B_α of all parameters that give rise to the same configuration α in I is a subanalytic set (proposition 169 below). The curve selection lemma (lemma 170 below) applied to this subset implies that every accumulation point of B_α can be attained by an analytic curve that lies in B_α. It is worth noticing that the curve selection lemma does not give any information on how to find this curve, only the existence is guaranteed by this lemma.

Hence, given a configuration $\alpha = (n, m_1, \ldots, m_n)$ that appears for (infinitely many) parameters arbitrarily close to λ^0, there exists an analytic curve ζ in parameter space such that $\zeta(0) = \lambda^0$ and such that for $\varepsilon > 0$, the function $f_{\zeta(\varepsilon)}$ has exactly n zeroes in I, say $\xi_1(\varepsilon) < \ldots < \xi_n(\varepsilon)$, with respective multiplicities $m_1, \ldots, m_n$. Although we know the existence of these points, we don't know how these points depend on ε. We don't even know if these points converge at all (for $\varepsilon \downarrow 0$); and if they do, whether they all converge to the same point or not? Convergence to the same point means in terms of bifurcations, that all these zeroes originate from one single non-isolated zero of f_{λ^0} after small variations of the parameter λ. The following example shows that this certainly doesn't have to be the case. Consider the analytic 1-parameter family $(f_\varepsilon)_\varepsilon$ defined by $f_\varepsilon(s) = \varepsilon\left(s^2 - \varepsilon^2\right)\left(s - \frac{1}{2} + \varepsilon\right)$ on the interval $I = \left]-\frac{1}{2}, \frac{1}{2}\right[$. For $\varepsilon = 0$ the function f_0 is identically zero, and we are interested in configurations of zeroes that arise after perturbation from a non-isolated zero $s_0 \in I$ of f_0, e.g., $s_0 = 0$. For $0 < \varepsilon < \frac{1}{4}$ the function f_ε has exactly three simple roots in I, namely $\varepsilon, -\varepsilon, \frac{1}{2} - \varepsilon$. The first two zeroes converge both to s_0. However the last zero tends to the boundary of I when $\varepsilon \downarrow 0$, and escapes from I for $\varepsilon = 0$. Hence only the first two zeroes result from s_0 after perturbation of f_0. These zeroes are the interesting ones, as being the zeroes that arise after perturbation of a given non-isolated zero s_0 of f_{λ^0}.

To avoid that some of the zeroes $\xi_i(\varepsilon)$ in the configuration α do not converge to a given non-isolated zero s_0 (for instance because they escape through the boundary of I), we will look at the set C_α in the product space $B \times I^n$, that contains all ordered $(p+n)$-tuples $(\lambda, \xi_1, \ldots, \xi_n)$, of which the first p coordinates refer to a parameter λ that gives rise to the configuration α, and the other coordinates are the ordered zeroes $\xi_1 < \ldots < \xi_n$ of f_λ in I with prescribed multiplicities $m_1, \ldots, m_n$. This set C_α also is subanalytic (proposition 169 below). Again due to the curve selection lemma for subanalytic sets, we know that for every configuration that appears for infinitely many parameters arbitrarily close to λ^0, and for which the zeroes in this configuration approximate s_0 infinitely close, there exists an analytic curve ζ in parameter space B through λ^0 (i.e. $\zeta(0) = \lambda^0$) such that every parameter $\zeta(\varepsilon)$ on this curve gives rise to the configuration α in I. Moreover, this time, the ordered zeroes $\xi_1(\varepsilon), \ldots, \xi_n(\varepsilon)$ of $f_{\zeta(\varepsilon)}$ in I in this configuration depend analytically on ε, and since $\xi_i(0) = s_0$ $(\forall i = 1, \ldots, n)$ they all tend to s_0. In other words, these zeroes originate from the same non-isolated zero s_0 of f_{λ^0} after perturbation. This is the content of theorem 171.

We give now precise statements of the propositions announced above, the curve selection lemma and theorem 171. For a complete description of the properties of subanalytic sets and the curve selection lemma, we refer the reader to the literature, e.g., [BM], [DS], [Loj]. We just recall the basic facts.

A set will be called subanalytic if it is (locally) a linear projection of a relatively compact semi-analytic set. A semi-analytic set is determined by a finite number of equalities and inequalities of analytic functions; i.e. it can be written as

$$\bigcup_{j=1}^{k} \bigcap_{i=1}^{l} \{x : f_{ij}(x)\, \sigma_{ij} 0\}$$

where $k, l \in \mathbb{N}$, f_{ij} are analytic functions and σ_{ij} corresponds to one of the signs $<, >$ or $=$. Proposition 169 can be checked using basic properties of subanalytic sets: a finite union or intersection of subanalytic sets is again subanalytic, a projection or a complement of a subanalytic set results again in a subanalytic set.

Proposition 169 *Let α denote an existing configuration in I, then*

1. *the subset B_α is subanalytic (the set B_∞ is even analytic); moreover,*

2. if $\alpha = (n, m_1, \ldots, m_n)$, the set C_α is subanalytic, where

$$C_\alpha = \{(\lambda, \xi_1, \ldots, \xi_n) \in B \times I^n : \lambda \in B_\alpha \text{ such that } \xi_1 < \ldots < \xi_n \text{ are the zeroes of } f_\lambda \text{ with respective multiplicities } m_1, \ldots, m_n \}.$$

Proof. Let B_n denote the set of parameter values λ for which f_λ has exactly n zeroes in $I, n \in \mathbb{N}$. We next prove that B_n is subanalytic. Let D_n denote the set of parameter values λ for which f_λ has at least n zeroes in $I, n \in \mathbb{N}^*$. Then $B_0 = D_1^c$, $B_n = D_n \setminus D_{n+1}$. Define the semi-analytic set E_n by

$$E_n = \{(\lambda, \xi_1, \ldots, \xi_n) \in B \times I^n : f(\xi_i, \lambda) = 0, \xi_1 < \ldots < \xi_n\},$$

then D_n is the projection of E_n onto the parameter space $\mathbb{R}^p$. Therefore, the sets D_n and B_n are subanalytic for all $n \in \mathbb{N}$. As a consequence, also the set C_n defined as

$$C_n = \{(\lambda, \xi_1, \ldots, \xi_n) \in E_n : \lambda \in B_n\}$$

is subanalytic. We now take also the multiplicity into account. Let $B_{n,m_1,\ldots,m_n}$, $n \in \mathbb{N}^*$, denote the set of parameter values $\lambda \in B_n$, for which the multiplicities of the zeroes $\xi_1 < \ldots < \xi_n$ are given by $m_1, \ldots, m_n$. Define the subanalytic set

$$C_{n,m_1,\ldots,m_n} = \left\{(\lambda, \xi_1, \ldots, \xi_n) \in C_n : \frac{\partial^k f}{\partial s^k}(\xi_i, \lambda) = 0, \forall k < m_i, \right.$$
$$\left. \frac{\partial^{m_i} f}{\partial s^{m_i}}(\xi_i, \lambda) \neq 0, \forall i = 1, \ldots, n \right\},$$

then $B_{n,m_1,\ldots,m_n}$ is the projection of $C_{n,m_1,\ldots,m_n}$ on $\mathbb{R}^p$, and hence is subanalytic.

The set B_∞ that consists of all parameter values λ for which $f(\cdot, \lambda)$ is identically zero, even is analytic. We can write

$$f(s, \lambda) = \sum_{i=0}^{\infty} a_i(\lambda)(s - s_0)^i, s \to s_0, \lambda \to \lambda^0.$$

The analytic function $f(\cdot, \lambda)$ is identically zero if and only if $a_i(\lambda) = 0$, $\forall i \in \mathbb{N}$. Consider the ideal generated by all analytic function germs of a_i at λ^0. Since the local ring of analytic function germs is Noetherian, this ideal is finitely generated. Therefore the ideal is generated by $a_0, a_1, \ldots, a_N$, for a certain $N \in \mathbb{N}$. Hence, $\lambda \in B_\infty$ if and only if these generators all vanish at λ. ■

Lemma 170 (Curve selection lemma for subanalytic sets [BM],[DS],[Loj]) *Suppose that V is a subanalytic set in $\mathbb{R}^p$ and λ^0 is an accumulation point of V, then there exists an analytic curve $\gamma : [0,1] \to \mathbb{R}^p$ such that $\gamma(]0,1]) \subset V$ and $\gamma(0) = \lambda^0$.*

Theorem 171 *Suppose we are given an analytic p-parameter family of real-valued functions $(f_\lambda)_\lambda$ such as (4.6), with $s_0 \in I, \lambda^0 \in B$ and $f_{\lambda^0} \equiv 0$. Denote by $\mathcal{A}$ the set of all possible configurations of zeroes that appear for parameter values arbitrarily close to λ^0 in any arbitrarily small open interval $I_0 \subset I$ around s_0. Then,*

1. *for every configuration $\alpha \in \mathcal{A}$, there exists an analytic curve $\zeta : [0, E] \to B$ with $\zeta(0) = \lambda^0$ such that $\forall \varepsilon \in]0, E] : \zeta(\varepsilon)$ gives rise to the configuration α.*

2. *Moreover, if $\alpha = (n, m_1, \ldots, m_n)$, then there exist analytic curves $\xi_i:[0, E] \to I_0$ with $\xi_i(0) = s_0, \forall 1 \le i \le n$, such that $\forall \varepsilon \in]0, E] : \xi_1(\varepsilon) < \ldots < \xi_n(\varepsilon)$ are the zeroes of $f_{\zeta(\varepsilon)}$ in I_0 with respective multiplicities $m_1, \ldots, m_n$.*

4.2.2 Configuration of limit cycles in analytic families of planar vector fields

Let $(X_\lambda)_\lambda$ be an analytic family of planar vector fields, unfolding a vector field of center type X_{λ^0}, with an associated analytic family of displacement maps $(\delta_\lambda)_\lambda$. Then the following corollary 172 is the analoguous one of theorem 171 for limit cycles in analytic families of planar vector fields. As a consequence, any configuration of limit cycles of X_λ arbitrarily close to Γ, λ near λ^0, can be detected by studying all one-parameter subfamilies $\left(X_{\zeta(\varepsilon)}\right)_{\varepsilon\sim 0}$, induced by an analytic curve ζ through λ^0 (i.e. $\zeta(0) = \lambda^0$). In particular, this implies the existence of an analytic curve ζ through λ^0 such that the cyclicity $\mathrm{Cycl}\left(X_{\zeta(\varepsilon)}, (\Gamma, 0)\right)$ is attained for each parameter value on this curve, a so-called curve of maximal cyclicity.

Corollary 172 *Suppose we are given an analytic family of planar vector fields $(X_\lambda)_\lambda$, that unfolds an analytic vector field of center type X_{λ^0}, and Γ a non-isolated regular periodic orbit of X_{λ^0}. Denote by $\mathcal{A}$ the set of all configurations of limit cycles that appear in any sufficiently small neighbourhood of Γ for any small variation of the parameter value λ^0.*

1. *For every configuration $\alpha \in \mathcal{A}$, there exists an analytic curve ζ through λ^0, such that for each parameter value $\zeta(\varepsilon) \neq \lambda^0$, the vector field $X_{\zeta(\varepsilon)}$ has this configuration of limit cycles in a small neighbourhood of Γ; moreover when $\zeta(\varepsilon)$ tends to λ^0, the limit cycles of $X_{\zeta(\varepsilon)}$ in this configuration tend to Γ.*

2. *In particular, there exists a curve $\zeta : [0, 1] \to \mathbb{R}^p$ with $\zeta(0) = \lambda^0$ of maximal cyclicity (respectively multiplicity).*

An important consequence of this corollary is that one can find upperbounds for the cyclicity of the family X_λ by studying 1-parameter subfamilies $\left(X_{\zeta(\varepsilon)}\right)_\varepsilon$, induced by analytic curves ζ through λ^0.

4.3 General case

4.3.1 Linear curves

Theorem 173 *1. There are algebraic families without a linear mic, mmc or mcc.*

2. Moreover, there are algebraic families with

$$\mathrm{Cycl}\left(X_\lambda, \left(\Gamma, \lambda^0\right)\right) > \max_{\bar{\lambda}\in\mathbb{R}^p, \bar{\lambda}\neq 0} \left\{\mathrm{Index}\left(X_{\lambda^0+\bar{\lambda}\varepsilon}, (\Gamma, 0)\right)\right\}$$

and

$$\mathrm{Mult}\left(X_\lambda, \left(\Gamma, \lambda^0\right)\right) > \max_{\bar{\lambda}\in\mathbb{R}^p, \bar{\lambda}\neq 0} \left\{\mathrm{Index}\left(X_{\lambda^0+\bar{\lambda}\varepsilon}, (\Gamma, 0)\right)\right\}$$

Proof. 1. Consider the family $\left(X_{(\lambda_1,\lambda_2)}\right)$ of type (1.2) with

$$\delta\left(s, (\lambda_1, \lambda_2)\right) = \varphi_1(\lambda_1, \lambda_2) + \varphi_2(\lambda_1, \lambda_2)(s - s_0) \tag{4.7}$$

where

$$\begin{cases} \varphi_1(\lambda_1, \lambda_2) & = & \lambda_2^2 + \lambda_1^4 - 2\lambda_1^2\lambda_2 \\ \varphi_2(\lambda_1, \lambda_2) & = & \lambda_1^6 \end{cases}$$

and let be $\lambda^0 = (0, 0)$. Clearly,

$$\mathrm{Cycl}\left(X_\lambda, \left(\Gamma, \lambda^0\right)\right) \leq \mathrm{Mult}\left(X_\lambda, \left(\Gamma, \lambda^0\right)\right) \leq \mathrm{Index}\left(X_\lambda, \left(\Gamma, \lambda^0\right)\right) = 1.$$

On the other hand, the cyclicity in the 1-parameter subfamily $\left(X_{\zeta(\varepsilon)}\right)_\varepsilon$, induced by $\zeta(\varepsilon) = \left(\varepsilon, \varepsilon^2\right)$ is equal to 1. Hence,

$$\mathrm{Cycl}\left(X_\lambda, \left(\Gamma, \lambda^0\right)\right) = \mathrm{Mult}\left(X_\lambda, \left(\Gamma, \lambda^0\right)\right) = \mathrm{Index}\left(X_\lambda, \left(\Gamma, \lambda^0\right)\right) = 1 \tag{4.8}$$

However, the region of parameter values (λ_1, λ_2), for which the associated displacement map can have isolated zeroes near s_0, has a quadratic contact at λ^0. By straightforward calculation, one finds that the cyclicity (and hence the multiplicity) for any 1-parameter subfamily, induced by a straight line through λ^0, is equal to 0. Indeed, take any $\bar{\lambda} = \left(\bar{\lambda}_1, \bar{\lambda}_2\right) \neq (0, 0)$, then

$$\delta\left(s, \bar{\lambda}\varepsilon\right) = \varepsilon^2\left(\left(\bar{\lambda}_2^2 + \varepsilon^2\bar{\lambda}_1^4 - 2\varepsilon\bar{\lambda}_1^2\bar{\lambda}_2\right) + \varepsilon^4\bar{\lambda}_1^6(s - s_0)\right)$$

If $\bar{\lambda}_2 \neq 0$, then

$$\delta\left(s, \bar{\lambda}\varepsilon\right) = \varepsilon^2\left(\bar{\lambda}_2^2 + o(1)\right), \varepsilon \to 0, s \to s_0$$

If $\bar{\lambda}_2 = 0$, then for $\varepsilon \to 0, s \to s_0$:

$$\delta\left(s, \bar{\lambda}\varepsilon\right) = \varepsilon^4\bar{\lambda}_1^4\left(1 + \varepsilon^2\bar{\lambda}_1^2(s - s_0)\right) = \varepsilon^4\bar{\lambda}_1^4(1 + o(1))$$

Hence, in eather case, $\delta\left(\cdot, \bar{\lambda}\varepsilon\right)$ has no isolated zeroes for ε small enough. As a consequence, $\forall\bar{\lambda} \neq (0, 0)$,

$$\mathrm{Cycl}\left(X_{\bar{\lambda}\varepsilon}, (\Gamma, 0)\right) = \mathrm{Mult}\left(X_{\bar{\lambda}\varepsilon}, (\Gamma, 0)\right) = \mathrm{Index}\left(X_{\bar{\lambda}\varepsilon}, (\Gamma, 0)\right) = 0. \tag{4.9}$$

Comparing (4.8) and (4.9), it follows that there doesn't exist any linear mic, mmc or mcc.

2. From (4.9), it follows that

$$\sup_{0\neq\bar{\lambda}\in\mathbb{R}^2} \text{Index}\left(X_{\lambda^0+\bar{\lambda}\varepsilon}, (\Gamma, 0)\right) = 0$$

and hence, by (4.8), it follows that the cyclicity (respectively multiplicity) of the p-parameter family can not be bounded by calculating only the Index of linear 1-parameter subfamilies. ■

Theorem 174 *There are linear families without a linear mcc.*

The key in proving this theorem relies on the following proposition, that even provides examples in case of a regular Bautin ideal.

Proposition 175 *Suppose that the displacement map of a given analytic family of vector fields $(X_\lambda)_\lambda$ can locally be written as:*

$$\delta(s,\lambda) = \sum_{i=1}^{l} \varphi_i(\lambda) h_i(s,\lambda), \text{ for } (s,\lambda) \text{ close to } \left(s_0, \lambda^0\right)$$

where $3 \leq l \leq p, \lambda^0 = \left(\lambda_1^0, \ldots, \lambda_p^0\right)$ and up to reordering of the parameter variabels $\lambda_1, \ldots, \lambda_p$:

$$\left\{\begin{array}{lcl} \varphi_1(\lambda) & = & \left(\lambda_1 - \lambda_1^0\right) c_1^1(\lambda) \\ \varphi_2(\lambda) & = & \left(\lambda_2 - \lambda_2^0\right) c_2^2(\lambda) + \left(\lambda_1 - \lambda_1^0\right) c_1^2(\lambda) \\ & \vdots & \\ \varphi_l(\lambda) & = & \left(\lambda_l - \lambda_l^0\right) c_l^l(\lambda) + \ldots + \left(\lambda_1 - \lambda_1^0\right) c_1^l(\lambda) \end{array}\right. \tag{4.10}$$

where the functions c_i^j are analytic in λ, with $c_i^i(\lambda^0) \neq 0, \forall i = 1, \ldots, l$, and $\forall i = 1, \ldots, l$:

$$h_i(s,\lambda) = (s-s_0)^{n_i} + g_i(s,\lambda)$$

for some analytic functions g_i with

$$\left\{\begin{array}{lcl} g_i\left(s,\lambda^0\right) & = & 0 \\ g_i(s,\lambda) & = & o\left((s-s_0)^{n_i}\right), s \to s_0 \end{array}\right.$$

and $n_1 < \ldots < n_l$ are non-negative integers. Then, there does not exist any linear mcc.

Remark 176 *It follows from lemma 9, that, under the conditions of proposition 175, the corresponding Bautin ideal is regular.*

Proof. From (4.10) it follows that

$$\text{Cycl}\left(X_\lambda, \left(\Gamma, \lambda^0\right)\right) \geq l - 1 \geq 2$$

Now let ζ be an arbitrary linear curve in parameter space through λ^0 :

$$\zeta : \varepsilon \mapsto \lambda^0 + \varepsilon\bar{\lambda}, \qquad \bar{\lambda} \neq 0.$$

If j is the smallest integer such that $\bar{\lambda}_j \neq 0$, then we can write

$$\delta(s, \zeta(\varepsilon)) = \varepsilon(s - s_0)^{n_j}\left(\bar{\lambda}_j c_j^j(\lambda^0) + G(s, \varepsilon)\right)$$

for an analytic function G with

$$G(s, \varepsilon) = O(s - s_0), s \to s_0.$$

Therefore, for $\varepsilon \neq 0$, the displacement map $\delta(\cdot, \zeta(\varepsilon))$ has at most one isolated zero in a sufficiently small neighbourhood of s_0. As a consequence, for a linear curve ζ through λ^0 :

$$\mathrm{Cycl}\left(X_{\zeta(\varepsilon)}, (\Gamma, 0)\right) \leq 1 < \mathrm{Cycl}\left(X_\lambda, (\Gamma, 0)\right)$$

■

We now give two concrete examples of linear families, that satisfy the conditions (4.10) in proposition 175, and hence that don't have any linear mcc. Although in the rest of the chapter, we consider bifurcations from regular periodic orbits, in the first example below we consider a bifurcation from a non-degenerate center. After blowing-up the elliptic point, we get a regular periodic orbit [R98], and we obtain an example that we are looking for. We prefer not to perform the blow-up here, and we will deal with the bifurcation from a center only in this example. Notice that all the examples that we give in this chapter could correspond to the bifurcation of a non-degenerate center after blowing-down the regular periodic orbit. In the second example below, we consider the bifurcation from a regular periodic orbit.

Example 1 Consider the polynomial family of planar vector fields $X_\lambda, \lambda \in \mathbb{R}^3$ given by

$$\begin{cases} \dot{x} &= -y \\ \dot{y} &= x + \bar{Q}(x, \lambda) y \end{cases}, \tag{4.11}$$

with $\lambda = (\lambda_1, \lambda_2, \lambda_3)$ and

$$\bar{Q}(x, \lambda) = \lambda_1 x^2 + \lambda_2 x^4 + \lambda_3 x^6.$$

Using techniques of section 1.2.3, we find that the number of limit cycles that can originate from the center point $(0, 0)$ for λ close to $(0, 0, 0)$, equals 2. Applying proposition 175, we now show that there is no linear mcc. Denote the displacement map associated to the transversal section $\{x > 0, y = 0\}$, as

$$\delta(s, \lambda) = \sum_{i=2}^{\infty} \alpha_i(\lambda) s^i.$$

Then, one can easily compute generators for the Bautin ideal $\mathcal{I} = (\alpha_{i+2} : i \in \mathbb{N})$:

$$\begin{cases} \alpha_{2k}(\lambda) &= 0, \qquad \forall k \geq 1 \\ \alpha_3(\lambda) &= \frac{1}{4}\pi\lambda_1 \\ \alpha_5(\lambda) &= \frac{1}{8}\pi\lambda_2 + \frac{3}{32}\pi^2\lambda_1^2 \\ \alpha_7(\lambda) &= \frac{5}{64}\pi\lambda_3 + \lambda_1\left(\frac{1}{8}\pi^2\lambda_2 + \frac{163}{6144}\pi\lambda_1^2 + \frac{5}{128}\pi^3\lambda_1^2\right) \end{cases}$$

It is clear that the set $\{\alpha_3, \alpha_5, \alpha_7\}$ is a minimal set of generators adapted at $s = 0$. From proposition 175, it follows that there does not exist any linear mcc.

Example 2 Consider now the analytic family of planar vector fields X_λ, depending linearly on the small parameter $\lambda = (\lambda_1, \lambda_2, \lambda_3)$ given by (1.2) with

$$\delta(s, (\lambda_1, \lambda_2, \lambda_3)) = \lambda_1 + \lambda_2 s^2 + \lambda_3 s^4. \tag{4.12}$$

Denote Γ the periodic orbit of the linear center X_0 with radius $s_0 > 0$. The computations below show that the cyclicity $\mathrm{Cycl}(X_\lambda, (\Gamma, 0))$ equals 2, and that there exists a quadratic mcc and a linear mmc. However, using proposition 175, we can conclude that there does not exist a linear mcc.

Up to a non-zero analytic factor, the displacement map is given by

$$\delta(s, \lambda) = \sum_{i=0}^{4} a_i(\lambda)(s - s_0)^i,$$

where the coefficients a_i $(0 \leq i \leq 4)$ can be found by use of Taylor's theorem:

$$\begin{cases} a_0(\lambda) &= \lambda_1 + \lambda_2 s_0^2 + \lambda_3 s_0^4 \\ a_1(\lambda) &= 2\lambda_2 s_0 + 4\lambda_3 s_0^3 \\ a_2(\lambda) &= \lambda_2 + 6\lambda_3 s_0^2 \\ a_3(\lambda) &= 4\lambda_3 s_0 \\ a_4(\lambda) &= \lambda_3 \end{cases}$$

It is seen that $(\lambda_1, \lambda_2, \lambda_3) \mapsto (a_0(\lambda), a_1(\lambda), a_2(\lambda))$ defines a linear transformation in parameter space; its inverse is given by the matrix

$$\begin{bmatrix} 1 & -\dfrac{5}{8}s_0 & \dfrac{1}{4}s_0^2 \\ 0 & \dfrac{3}{4s_0} & -\dfrac{1}{2} \\ 0 & -\dfrac{1}{8s_0^3} & \dfrac{1}{4s_0^2} \end{bmatrix} \tag{4.13}$$

Clearly, the set $\{a_0, a_1, a_2\}$ is a minimal set of generators for the Bautin ideal; using (4.13), one can expand the displacement map δ in a_0, a_1, a_2 :

$$\delta(s, \lambda) = \sum_{i=0}^{2} a_i(\lambda) h_i(s),$$

with

$$\begin{cases} h_0(s) &= 1 \\ h_1(s) &= (s-s_0) - \frac{1}{2s_0^2}(s-s_0)^3 - \frac{1}{8s_0^2}(s-s_0)^4 \\ h_2(s) &= (s-s_0)^2 + \frac{1}{s_0}(s-s_0)^3 + \frac{1}{4s_0^2}(s-s_0)^4 \end{cases}$$

Conditions (4.10) of proposition 175 are satisfied, and hence, there is no linear mcc in the coordinates (a_0, a_1, a_2). Therefore, and since the transformation between the coordinates $(\lambda_1, \lambda_2, \lambda_3)$ and (a_0, a_1, a_2) is linear, there cannot exist a linear mcc in the coordinates $(\lambda_1, \lambda_2, \lambda_3)$.

However, the quadratic curve λ^c, defined by

$$\lambda^c(\varepsilon) = (\lambda_1^c(\varepsilon), \lambda_2^c(\varepsilon), \lambda_3^c(\varepsilon)) = \left(-\frac{5}{8}s_0\varepsilon^2 + \frac{1}{4}s_0^2\varepsilon, \frac{3}{4s_0}\varepsilon^2 - \frac{1}{2}\varepsilon, -\frac{1}{8s_0^3}\varepsilon^2 + \frac{1}{4s_0^2}\varepsilon\right)$$

is an mcc. Indeed, after performation of the linear transformation defined by the matrix in (4.13), this curve corresponds to the quadratic curve

$$a^c(\varepsilon) = (a_0^c(\varepsilon), a_1^c(\varepsilon), a_2^c(\varepsilon)) = (0, \varepsilon^2, \varepsilon).$$

Hence, for $s \to s_0$, the displacement map can be written as:

$$\begin{aligned} &\delta(s, \lambda^c(\varepsilon)) = \varepsilon(s-s_0) f(s,\varepsilon) \\ &\text{with } f(s,\varepsilon) = \varepsilon\left(1 + O\left((s-s_0)^2\right)\right) + \left((s-s_0) + O\left((s-s_0)^2\right)\right) \end{aligned}$$

Now, it is clear that for $\varepsilon \neq 0$, the zeroes of the function $\delta(\cdot, \lambda^c(\varepsilon))$ correspond to $s = s_0$ and the zeroes of the function $f(\cdot, \varepsilon)$. Since $f(s_0, 0) = 0$ and $\frac{\partial}{\partial \varepsilon} f(s_0, 0) = 1 \neq 0$, the Implicit Function Theorem guarantees the existence of an analytic curve $s(\varepsilon)$ such that $f(s(\varepsilon), \varepsilon) = 0, \forall \varepsilon \downarrow 0$ and $s(0) = s_0$; in particular, $f(s_0, \varepsilon) \neq 0$ if $\varepsilon \neq 0$; hence, $s(\varepsilon) \neq s_0$ if $\varepsilon \neq 0$. As a consequence, the curve λ^c is an mcc.

Notice that there exist a linear mic and mmc; for instance, we claim that the linear curve ζ defined by

$$\zeta(\varepsilon) = (\lambda_1(\varepsilon), \lambda_2(\varepsilon), \lambda_3(\varepsilon)) = \varepsilon\left(\frac{1}{4}s_0^2, -\frac{1}{2}, \frac{1}{4s_0^2}\right)$$

is an mic and mmc. Indeed, in the coordinates (a_0, a_1, a_2), this curve corresponds to

$$\varepsilon(0,0,1)$$

and hence, the displacement map can be expressed as:

$$\delta(s, \zeta(\varepsilon)) = \varepsilon(s-s_0)^2(1 + O(s-s_0)), s \to s_0.$$

In section 4.4, we will show that there always exists a linear mic, if the Bautin ideal is regular (as is clearly the case here).

To end this section we give another technical proposition, that gives some insight in the search for examples without a linear mic; it lies on the ground of the example given in the proof of theorem 173.

Proposition 177 *Suppose that $\{\varphi_1, \ldots, \varphi_l\}$ is a minimal set of generators for the Bautin ideal of the analytic family $(X_\lambda)_\lambda$ of planar vector fields such that the map*

$$\lambda \overset{\varphi}{\mapsto} (\varphi_1(\lambda), \ldots, \varphi_l(\lambda))$$

is a submersion at λ^0 and suppose furthermore that the corresponding factor functions have an increasing order at s_0 such that

$$0 = orderH_1(s_0) < \ldots < orderH_l(s_0) = l-1,$$

Furthermore, suppose that the semi-analytic set

$$V_l = \{\lambda : |\varphi_j(\lambda)| \leq |\varphi_l(\lambda)|, \forall j = 1, \ldots, l\}$$

does not contain the germ of a linear curve at λ^0. Then there doesn't exist a linear mic (neither a linear mcc, neither a linear mmc).

Proof. Since the map φ is a submersion at λ^0, it follows that the Bautin ideal is regular. As a consequence, by theorems 40 and 45, it follows that

$$\mathrm{Cycl}(X_\lambda, (\Gamma, \lambda^0)) = \mathrm{Mult}(X_\lambda, (\Gamma, \lambda^0)) = \mathrm{Index}(X_\lambda, (\Gamma, \lambda^0)) = l - 1 \qquad (4.14)$$

If we define analoguously the semi-analytic sets $V_i, i = 1, \ldots, l$:

$$V_i = \{\lambda : |\varphi_j(\lambda)| \leq |\varphi_i(\lambda)|, \forall j = 1, \ldots, l\},$$

then by the division-derivation algorithm (theorem 7), it follows that

$$\mathrm{Mult}((X_\lambda)_{\lambda \in V_i}, (\Gamma, \lambda^0)) \leq \mathrm{Index}((X_\lambda)_{\lambda \in V_i}, (\Gamma, \lambda^0)) = orderH_i(s_0) = i - 1.$$

Take a straight line R through λ^0, then by assumption there exists a neighbourhood $W_1 \subset W$ of λ^0 such that $R \cap W_1$ is contained in some V_i $(i \in \{1, \ldots, l-1\})$. Therefore,

$$\mathrm{Index}((X_\lambda)_{\lambda \in R}, (\Gamma, \lambda^0)) = i - 1 < l - 1 \qquad (4.15)$$

Comparing (4.14) and (4.15), it follows that there does not exist a linear mic; furthermore by (1.22) ,there does not exist a linear mmc, or a linear mcc. ∎

4.3.2 Algebraic curves

Theorem 178 *There does always exist an algebraic mic.*

Proof. From the result of R. Roussarie in [R00], we know that there does always exist an analytic mic ζ, i.e. ζ is an analytic curve with

$$\mathrm{Index}\left(X_\lambda, \left(\Gamma, \lambda^0\right)\right) = \mathrm{Index}\left(X_{\zeta(\varepsilon)}, (\Gamma, 0)\right) = \mathrm{order}(M^\zeta_{k(\zeta)})\,(s_0)\,,$$

where $M^\zeta_{k(\zeta)}$ is the first non-zero Melnikov function of $\left(X_{\zeta(\varepsilon)}\right)_\varepsilon$. Then we can write

$$\zeta(\varepsilon) = \underbrace{\lambda^0 + a_1\varepsilon + \ldots + a_{k(\zeta)}\varepsilon^{k(\zeta)}}_{\zeta_1(\varepsilon)} + O\left(\varepsilon^{k+1}\right), \varepsilon \to 0, a_i \in \mathbb{R}^p,$$

and we define the algebraic curve ζ_1 by the Taylor polynomial of degree $k = k(\zeta)$ at $\varepsilon = 0$. One can easily check that the first non-zero Melnikov function of $\left(X_{\zeta_1(\varepsilon)}\right)$ also is M^ζ_k. Indeed, if $\zeta(\varepsilon) = \zeta_1(\varepsilon) + O\left(\varepsilon^{k+1}\right)$, then also $\delta(\zeta(\varepsilon)) = \delta(\zeta_1(\varepsilon)) + O\left(\varepsilon^{k+1}\right), \varepsilon \to 0$. Therefore, we obtain the same index:

$$\mathrm{Index}\left(X_{\zeta_1(\varepsilon)}, (\Gamma, 0)\right) = \mathrm{order}(M^\zeta_{k(\zeta)})\,(s_0) = \mathrm{Index}\left(X_\lambda, \left(\Gamma, \lambda^0\right)\right).$$

∎

Theorem 179 *There are analytic p-parameter families of planar vector fields without an algebraic mmc or algebraic mcc.*

Proof. Consider the analytic 2-parameter family of planar vector fields $X_{(\lambda_1,\lambda_2)}$ of type (1.2) with

$$\delta(s, \lambda_1, \lambda_2) = \lambda_1 \left((s - s_0)^2 + (\lambda_2 - \sin\lambda_1)^2\right) \tag{4.16}$$

where $s_0 > 0$. We can easily draw the bifurcation diagram of limit cycles in function of (λ_1, λ_2) for this family. In the (λ_1, λ_2)-plane, there is exactly one limit cycle Γ with multiplicity 2 for parameter values (λ_1, λ_2) on the graph of $\lambda_2 = \sin\lambda_1$:

$$\Gamma = \left\{(x, y) : x^2 + y^2 = s_0^2\right\};$$

for all other parameter values, there are no limit cycles. Clearly, if we denote $\overline{0} = (0, 0)$, then

$$\mathrm{Cycl}\left(X_{(\lambda_1,\lambda_2)}, \left(\Gamma, \overline{0}\right)\right) = 1 \text{ and } \mathrm{Mult}\left(X_{(\lambda_1,\lambda_2)}, \left(\Gamma, \overline{0}\right)\right) = 2,$$

and these numbers are only attained on the analytic curve $\zeta(\varepsilon) = (\varepsilon, \sin\varepsilon), \varepsilon \in \mathbb{R}$. It is easy to show that there is no algebraic reparametrisation of this curve. Moreover, we can prove that there are no algebraic curves ζ_m or ζ_c through $\overline{0}$ such that

$$\mathrm{Mult}\left(X_{\zeta_m(\varepsilon)}, (\Gamma, 0)\right) = 2 \text{ and } \mathrm{Cycl}\left(X_{\zeta_c(\varepsilon)}, (\Gamma, 0)\right) = 1. \tag{4.17}$$

If there was such an algebraic curve ζ_m (respectively ζ_c), this would imply that ζ_m (respectively ζ_c) had infinitely many intersections with the analytic curve ζ, accumulating on $\overline{0}$. In fact, any analytic curve ζ_m (respectively ζ_c) through $\overline{0}$ satisfying (4.17), is a reparametrisation of ζ. We make these statements more precise in next lemma. ∎

Lemma 180 *Suppose that $\hat{\zeta}:I \to \mathbb{R}^2$ is an algebraic curve, where I is an open interval, $0 \in I, \hat{\zeta}(0) = \overline{0}$. Then there exists a neighbourhood V of $\overline{0}$ such that*

$$\hat{\zeta}(I) \cap \{(\varepsilon, \sin\varepsilon) : \varepsilon \in \mathbb{R}\} \cap V = \{\bar{0}\}.$$

Proof. Let the algebraic curve be given by

$$\hat{\zeta}(\varepsilon) = \left(a_1\varepsilon + \ldots + a_k\varepsilon^k, b_1\varepsilon + \ldots + b_l\varepsilon^l\right), \varepsilon \in I,$$

where $k, l \in \mathbb{N}^*, a_i, b_j \in \mathbb{R}$ $(\forall i = 1, \ldots, k, \forall j = 1, \ldots l)$. We can restrict to the case $a_k b_l \neq 0$. Suppose now that $\hat{\zeta}(I)$ has infinitely many intersection points with the graph of sin that accumulate on $\overline{0}$; then, the following analytic equation would hold for infinitely many ε close to 0,

$$b_1\varepsilon + \ldots + b_l\varepsilon^l = \sin\left(a_1\varepsilon + \ldots + a_k\varepsilon^k\right), \tag{4.18}$$

implying that the equality must hold on all of $\mathbb{R}$. This leads into a contradiction, because the left-hand side of (4.18) has at most l zeroes. ■

Remark 181 *In this example, the mc-stratum (and mm-stratum) is nowhere dense, since for each $M > 0$, the interior of the subanalytic set Z_M is given by*

$$\begin{aligned} Z_M &= \{(\lambda_1, \lambda_2) : \exists r \in]R - M, R + M[\text{ such that } f(r, \lambda_1, \lambda_2) = 0\} \\ &= \{(\lambda_1, \lambda_2) : \lambda_1 = \sin\lambda_2\} \end{aligned}$$

is empty (i.e. $\mathring{Z}_M = \emptyset$). In section 4.6, it will be proven that there exists an algebraic mmc (repectively mcc) if the family $(X_\lambda)_\lambda$ has a stratum of maximal multiplicity (respectively cyclicity) with non-empty interior at λ^0.

4.4 Regular Bautin Ideal

If the Bautin ideal is regular, then we show below, for instance, that there does always exist a linear mic. Hence, in bounding the cyclicity, we can restrict to linear 1-parameter subfamilies, i.e. induced by a straight line through λ^0. However, there remains the problem of checking whether the Bautin ideal is regular. Therefore, we provide an equivalent characterisation for the Bautin ideal to be regular, in terms of linear 1-parameter subfamilies in section 4.4.1 (in case the dimensions of the Bautin ideal and the parameter space coincide).

4.4.1 Linear curves

Theorem 182 *If the Bautin ideal is regular, then there exists a linear mic ζ^i :*

$$\zeta^i(\varepsilon) = \lambda^0 + a\varepsilon, \qquad a \in \mathbb{R}^p, \|a\| = 1.$$

In particular,

$$\operatorname{Cycl}\left(X_\lambda,(\Gamma,\lambda^0)\right) \le \operatorname{Mult}\left(X_\lambda,(\Gamma,\lambda^0)\right) \le \operatorname{Index}\left(X_{\zeta^i(\varepsilon)},(\Gamma,0)\right) = order M_1(s_0),$$

where M_1 is the first Melnikov function of $\left(X_{\zeta^i(\varepsilon)}\right)_{\varepsilon\sim 0}$.

Proof. We can take a minimal set of generators for $\mathcal{I}$, say $\{\varphi_1,\ldots,\varphi_l\}$, adapted to s_0, such that the differentials $\{D\varphi_1(\lambda^0),\ldots,D\varphi_l(\lambda^0)\}$ are linearly independent (section 1.2.3). Therefore,

$$\operatorname{Cycl}\left(X_\lambda,(\Gamma,\lambda^0)\right) \le \operatorname{Mult}\left(X_\lambda,(\Gamma,\lambda^0)\right) \le \operatorname{Index}\left(X_\lambda,(\Gamma,\lambda^0)\right) = \operatorname{order}H_l(s_0).$$

Now we consider a linear curve $\zeta(\varepsilon) = \lambda^0 + a\varepsilon, a \in \mathbb{R}^p$. A displacement map for the induced 1-parameter family $\left(X_{\zeta(\varepsilon)}\right)_{\varepsilon\sim 0}$ is given by

$$\delta(s,\zeta(\varepsilon)) = \varepsilon M_1(s) + O\left(\varepsilon^2\right), \varepsilon \to 0,$$

where

$$M_1(s) = \sum_{i=1}^{l} D(\varphi_i)_{\lambda^0}(a)\, H_i(s).$$

Since $\mathcal{I}$ is regular, there exists a constant $\bar{a} \in \mathbb{R}^p$ with $\|\bar{a}\| = 1$ such that $M_1(s) = cH_l(s)$, for some $c > 0$. Hence, the linear curve $\zeta^i(\varepsilon) = \lambda^0 + \bar{a}\varepsilon$ is an mic. ■

To check whether the Bautin ideal is regular, we have the following characterisation in terms of linear 1-parameter subfamilies.

Theorem 183 *Suppose that $(X_\lambda)_\lambda$ is a family of planar vector fields unfolding a vector field of center type X_{λ^0}.*

1. *The Bautin ideal is regular and its dimension equals the dimension of the parameter space if and only if for every linear curve in parameter space through λ^0, the first Melnikov function is not identical to zero.*

2. *If all first order Melnikov functions (induced by a linear curve through λ^0 in parameter space) are identical to 0, then the Bautin ideal is not regular.*

Proof. The second statement follows by contraposition from the proof of theorem 182. We here only prove the first statement. Take a minimal set of generators $\varphi_1,\ldots,\varphi_l$ adapted to s_0. By Taylor's theorem, we can write, $\forall j = 1,\ldots,l$:

$$\varphi_j(\lambda) = \sum_{i=1}^{p} \frac{\partial\varphi_j}{\partial\lambda_i}(\lambda^0)\left(\lambda_i - \lambda_i^0\right) + O\left(\left\|\lambda - \lambda^0\right\|^2\right), \lambda \to \lambda^0.$$

Let $(\delta_\lambda)_\lambda$ be a C^ω family of dsplacement maps for $(X_\lambda)_\lambda$. Then by proposition 28, we can write locally for $s \sim s_0, \lambda \sim \lambda^0$:

$$\begin{aligned}\delta(s,\lambda) &= \sum_{j=1}^{l} \varphi_j(\lambda) h_j(s,\lambda) \\ &= \sum_{j=1}^{l}\sum_{i=1}^{p} \frac{\partial \varphi_j}{\partial \lambda_i}(\lambda^0)\left(\lambda_i - \lambda_i^0\right) H_j(s) + O\left(\left\|\lambda - \lambda^0\right\|^2\right), \lambda \to \lambda^0,\end{aligned}$$

where $H_j \equiv h_j(\cdot,\lambda^0), \forall j = 1,\dots,l$. Take a linear curve through $\bar{0}$, $\lambda = \lambda^0 + \varepsilon\bar{\lambda}$, with $\bar{\lambda} \neq \bar{0}$, then we have

$$\delta\left(s,\lambda^0 + \varepsilon\bar{\lambda}\right) = \varepsilon \sum_{j=1}^{l}\sum_{i=1}^{p} \frac{\partial \varphi_j}{\partial \lambda_i}(\lambda^0)\bar{\lambda}_i H_j(s) + O\left(\varepsilon^2\right), \varepsilon \to 0.$$

Therefore, the first order Melnikov function is defined as:

$$M_1\left(s,\bar{\lambda}\right) = \sum_{j=1}^{l}\sum_{i=1}^{p} \frac{\partial \varphi_j}{\partial \lambda_i}(\lambda^0)\bar{\lambda}_i H_j(s).$$

$\boxed{\textit{If}\text{-part:}}$ Since the factor functions H_j have a strictly increasing order at $s = s_0$, the first order Melnikov function $M_1\left(\cdot,\bar{\lambda}\right)$ is not identical to zero if there exists an integer $j \in \{1,\dots,l\}$ such that

$$\sum_{i=1}^{p} \frac{\partial \varphi_j}{\partial \lambda_i}(\bar{0})\bar{\lambda}_i \neq 0.$$

Therefore, the fact that for every $\bar{\lambda} \neq \bar{0}$, the first order Melnikov function $M_1\left(\cdot,\bar{\lambda}\right)$ is not identical to zero, is equivalent to the fact that the map $\varphi = (\varphi_1,\dots,\varphi_l)$ is an immersion at λ^0. Since the set of generators is minimal, this fact implies that $l = p$; as a consequence, the Bautin ideal is regular.

$\boxed{\textit{Only if}\text{-part:}}$ now we take $l = p$: suppose to the contrary that $M_1\left(\cdot,\bar{\lambda}\right) \equiv 0$, for some $\bar{\lambda} \neq \bar{0}$. Since the factor functions are independent over $\mathbb{R}$, it follows that

$$\forall j = 1,\dots,l: \qquad \sum_{i=1}^{p} \frac{\partial \varphi_j}{\partial \lambda_i}(\lambda^0)\bar{\lambda}_i = 0.$$

This is impossible since the matrix of the linear system is non-singular. ■

Theorem 184 *There exist algebraic families having a regular Bautin ideal, without a linear mmc.*

Proof. Consider the 3-parameter family of planar vector fields X_λ of type (1.2) with

$$\delta(s,\lambda) = \left(\lambda_1 + \lambda_2 (s - s_0) + \lambda_3 (s - s_0)^2\right) \cdot \left((s - s_0)^2 + \left(\lambda_1 - \lambda_3^2\right)^2\right)$$

where $\lambda = (\lambda_1, \lambda_2, \lambda_3) \in \mathbb{R}^3$ and $s_0 > 0$ and let $\lambda^0 = (0,0,0)$. Then, the Bautin ideal is generated by the germs at λ^0 of the maps

$$\varphi_i : \mathbb{R}^3 \to \mathbb{R} : \lambda = (\lambda_1, \lambda_2, \lambda_3) \mapsto \lambda_i, \qquad i = 1, 2, 3$$

For $\Gamma = \left\{(x,y) : x^2 + y^2 = s_0^2\right\}$, we have

$$\mathrm{Mult}\left(X_\lambda, \left(\Gamma, \lambda^0\right)\right) \leq \mathrm{Index}\left(X_\lambda, \left(\Gamma, \lambda^0\right)\right) = 4.$$

On the other hand, the 1-parameter subfamily induced by the algebraic curve $\zeta(\varepsilon) = \left(\varepsilon^4, 2\varepsilon^3, \varepsilon^2\right)$, has multiplicity 4, since the displacement map for this 1-parameter family is given by

$$\delta(s, \zeta(\varepsilon)) = \varepsilon^2 \left(s - (s_0 - \varepsilon)\right)^2 (s - s_0)^2$$

Hence,

$$\mathrm{Mult}\left(X_\lambda, \left(\Gamma, \lambda^0\right)\right) = 4$$

We now show that for any linear curve

$$R(\varepsilon) = \bar{\lambda}\varepsilon = \left(\bar{\lambda}_1\varepsilon, \bar{\lambda}_2\varepsilon, \bar{\lambda}_3\varepsilon\right),$$

one has that

$$\mathrm{Mult}\left(X_{R(\varepsilon)}, (\Gamma, 0)\right) < 4 = \mathrm{Mult}\left(X_\lambda, \left(\Gamma, \lambda^0\right)\right).$$

Indeed, if $\bar{\lambda}_1 \neq 0$, then $\mathrm{Mult}\left(X_{R(\varepsilon)}, (\Gamma, 0)\right) = 0$. If $\bar{\lambda}_1 = 0$, then the displacement map takes the form

$$\delta(s, R(\varepsilon)) = \varepsilon (s - s_0)\left(\bar{\lambda}_2 + \bar{\lambda}_3 (s - s_0)\right) \cdot \left((s - s_0)^2 + \bar{\lambda}_3^4 \varepsilon^4\right)$$

In case $\bar{\lambda}_3 \neq 0$, then $\mathrm{Mult}\left(X_{R(\varepsilon)}, (\Gamma, 0)\right) \leq 2$. In case $\bar{\lambda}_3 = 0$ (to have isolated zeroes, it is then necessary that $\bar{\lambda}_2 \neq 0$), the displacement map can be written as:

$$\delta(s, R(\varepsilon)) = \varepsilon \bar{\lambda}_2 (s - s_0)^3;$$

hence, in this case, $\mathrm{Mult}\left(X_{R(\varepsilon)}, (\Gamma, 0)\right) = 3$. ■

Theorem 185 *There exist linear families having a regular Bautin ideal, without a linear mcc.*

Proof. As already observed, the examples (4.11) and (4.12) also apply here. ■

4.4.2 Algebraic curves

Theorem 186 *There exist analytic families having a regular Bautin ideal, without an algebraic mcc, or an algebraic mmc.*

Proof. The example given in (4.16) also applies here, since the Bautin ideal of the family is generated by λ_1, and therefore the Bautin ideal is regular. ■

4.5 Principal Bautin Ideal

4.5.1 Linear curves

Theorem 187 *If the Bautin ideal $\mathcal{I}$ is principal, then there exists a linear mic:*

$$\zeta^i(\varepsilon) = \lambda^0 + \bar{a}\varepsilon, \quad \bar{a} \in \mathbb{R}^p \tag{4.19}$$

with $\|\bar{a}\| = 1$. *In particular,*

$$\operatorname{Cycl}(X_\lambda, (\Gamma, \lambda^0)) \le \operatorname{Mult}(X_\lambda, (\Gamma, \lambda^0)) \le \operatorname{Index}(X_{\zeta^i(\varepsilon)}, (\Gamma, \varepsilon))$$

Proof. Let φ be a generator for $\mathcal{I}$. We can write, for a certain $k \ge 1$,

$$\varphi(\lambda) = P_k(\lambda - \lambda^0) + R_k(\lambda),$$

where P_k is a homogeneous polynomial in $(\lambda - \lambda^0)$ of degree k, and

$$R_k(\lambda) = O\left(\|\lambda - \lambda^0\|^{k+1}\right), \lambda \to \lambda^0.$$

Hence, the displacement map can locally be written as

$$\begin{aligned}\delta(s, \lambda) &= \varphi(\lambda) h(s, \lambda) \\ &= (P_k(\lambda - \lambda^0) + R_k(\lambda)) h(s, \lambda)\end{aligned}$$

For a linear curve ζ as in (4.19), the displacement map of $(X_{\zeta(\varepsilon)})_\varepsilon$ can be written as

$$\delta(s, \zeta(\varepsilon)) = \varepsilon^k P_k(a) H(s) + O(\varepsilon^{k+1}), \varepsilon \to 0.$$

We can choose $\bar{a} \in \mathbb{R}^p$ with $\|\bar{a}\| = 1$ and $P_k(\bar{a}) \ne 0$; let $\zeta^i(\varepsilon) = \lambda^0 + \bar{a}\varepsilon$, then the k-th Melnikov function is given by $M_k(s) = P_k(a) H(s)$, and the required result follows. ∎

Theorem 188 *There exist algebraic families with a principal Bautin ideal, without a linear mmc or a linear mcc.*

Proof. Consider the 2-parameter family of planar vector fields of type (1.2) with

$$\delta(s, \lambda) = \lambda_2((s - s_0)^2 + (\lambda_2^2 - \lambda_1)^2), \qquad \text{where } \lambda = (\lambda_1, \lambda_2) \in \mathbb{R}^2$$

Let be $\lambda^0 = (0, 0)$. Then, the Bautin ideal is generated by the germ at λ^0 of the map

$$\varphi : \mathbb{R}^2 \to \mathbb{R} : \lambda = (\lambda_1, \lambda_2) \mapsto \lambda_2$$

It is easily seen that for $\Gamma = \{(x, y) : x^2 + y^2 = s_0^2\}$,

$$\operatorname{Cycl}(X_\lambda, (\Gamma, \lambda^0)) = 1 \text{ and } \operatorname{Mult}(X_\lambda, (\Gamma, \lambda^0)) = 2,$$

while $\mathrm{Cycl}\left(X_{R(\varepsilon)},(\Gamma,0)\right)=\mathrm{Mult}\left(X_{R(\varepsilon)},(\Gamma,0)\right)=0$, for any linear curve

$$R(\varepsilon)=\bar{\lambda}\varepsilon=\left(\bar{\lambda}_1\varepsilon,\bar{\lambda}_2\varepsilon\right).$$

Hence, for this family of planar vector fields having a principal Bautin ideal, there doesn't exist a linear mmc, neither a linear mcc. This ends the proof.

Another example of an algebraic family with a principal Bautin ideal, without a linear mcc is provided by a family $(X_\lambda)_\lambda$ of type (1.2) with

$$\delta(s,\lambda)=\delta_\lambda(s)=\lambda_3\left(\lambda_1+\lambda_2(s-s_0)+\lambda_3(s-s_0)^2\right), \tag{4.20}$$

where $\lambda=(\lambda_1,\lambda_2,\lambda_3)\in\mathbb{R}^3$ and $s_0>0$. Notice that the dependence on λ in this example is only quadratic. For $\lambda^0=(0,0,0)$, the vector field is clearly of center type. Let us consider the number of limit cycles of X_λ, close to $\Gamma=\left\{(x,y):x^2+y^2=s_0^2\right\}$ and for λ close to λ^0. Limit cycles of X_λ correspond to positive isolated zeros of δ_λ. Hence, the Bautin ideal is generated by the germ at $(0,0,0)$ of the map $\lambda=(\lambda_1,\lambda_2,\lambda_3)\mapsto\lambda_3$. Using the same arguments as in the proof of proposition 175, one shows that

$$\mathrm{Cycl}\left(X_\lambda,(\Gamma,\lambda^0)\right)=2$$

while $\mathrm{Cycl}\left(X_{\bar{\lambda}\varepsilon},(\Gamma,0)\right)\le 1,\forall\bar{\lambda}\neq 0$. ■

4.5.2 Algebraic curves

Theorem 189 *There exist analytic families with a principal Bautin ideal, without an algebraic mmc or an algebraic mcc.*

Proof. Example (4.16) that is given in section 4.3.2, also applies here, since the Bautin ideal of the family is generated by λ_1, and therefore the Bautin ideal is principal. ■

4.6 Open subanalytic sets and algebraic curves

In this section, we prove the existence of an algebraic mcc ζ_c (respectively mmc ζ_m) for analytic families of planar vector fields having a stratum of maximal cyclicity (respectively multiplicity) with non-empty interior at λ^0. In section 4.6.2, one can find a precise definition of this condition.

First, in section 4.6.1, we derive two interesting specifications of the curve selection lemma for open subanalytic sets, implying the existence of an algebraic curve entering a given subanalytic set. This section leaves the theory of dynamical systems: the results and proofs of this section are purely situated in the theory of analytic geometry. Moreover, the subanalytic sets that appear here, don't need to be defined by the number of zeroes of analytic functions.

Next, we apply these results on analytic families of vector fields having a stratum of maximal cyclicity (respectively multiplicity) with non-empty interior at λ^0. Since the stratum of maximal cyclicity (respectively multiplicity) cannot be defined by a subanalytic set, the existence of an algebraic mcc (respectively mmc) cannot be proven in a straightforward way.

This problem is situated in section 4.6.2. Next, in section 4.6.2, the condition described above is defined precisely. Then, in section 4.6.2, we can construct, under this condition, an appropriate open subanalytic set W, and we can apply the specifications of the curve selection lemma to this open subanalytic set W, leading to the existence of an algebraic mcc ζ_c (respectively mmc ζ_m). Moreover, we prove the existence of a *'cone of mcc's surrounding ζ_c'* (respectively *'cone of mmc's surrounding ζ_m'*); this *'cone'* is defined as the union of all analytic curves having a certain contact with ζ_c (respectively ζ_m).

4.6.1 Algebraic curves and determining jets

In section 4.6.1, we state two interesting specifications of the curve selection lemma for subanalytic sets (theorems 190 and 191), that will be proven in three steps (sections 4.6.1, 4.6.1 and 4.6.1). To end, in section 4.6.1, we give an example to illustrate the proof of theorem 191. The framework of this section is the theory of analytic geometry.

Statement of the results

Theorem 190 *For any open subanalytic set $V \subset \mathbb{R}^p$, that accumulates at $\lambda^0 \notin V$, there always exists an algebraic curve γ,*

$$\gamma(t) = (P_1(\varepsilon), \ldots, P_p(\varepsilon)), \gamma(0) = \lambda^0$$

where $P_1, \ldots, P_p$ are polynomials in ε, such that the curve $\gamma(\varepsilon)$ lies in V for all $\varepsilon > 0$ small enough.

This theorem is a consequence of the curve selection lemma (lemma 170) and the following improvement of it:

Theorem 191 *Let V be an open subanalytic set in $\mathbb{R}^p$, that accumulates on λ^0, and let γ be an analytic curve that starts at λ^0 (i.e. $\gamma(0) = \lambda^0$). Suppose that $\gamma(\varepsilon) \in V \setminus \{\lambda^0\}$, for all $\varepsilon > 0$ small enough. Then there exists a positive integer n such that for every analytic curve $\hat{\gamma}$ with $j_n(\gamma - \hat{\gamma})_0 = \bar{0}$, we also have*

$$\hat{\gamma}(\varepsilon) \in V \setminus \{\lambda^0\}, \text{ for } \varepsilon > 0 \text{ sufficiently small.}$$

Theorem 191 will be proven in the following three steps: first when γ is a regular parametrisation of a straight line (section 4.6.1), next when γ is merely regular (section 4.6.1), finally when γ is not regular (section 4.6.1). The existence of an analytic curve entering the subanalytic set $V \setminus \{\lambda^0\}$ is ensured by the *curve selection lemma for subanalytic sets* (lemma 170).

Clearly, the openess of V, as required in this section, is invariant under translations and rotations. We may also shrink V at any stage. As a consequence and by the theory of subanalytic sets, we can, from the start on, assume that $\gamma(0) = \lambda^0 = \bar{0} \in \mathbb{R}^p$, and that V lies entirely in the first quadrant $\mathbb{I}$:

$$\mathbb{I} = \{(\lambda_1, \ldots, \lambda_p) \in \mathbb{R}^p : \lambda_i > 0, \forall i = 1, \ldots, p\}.$$

It is clear that every non-constant analytic curve, entering the first quadrant, has a so-called *"Puiseux expansion"* [BM] of the form

$$\tau \longmapsto \left(\tau^k, \varphi_2(\tau), \ldots, \varphi_p(\tau)\right), \tag{4.21}$$

for a certain $k \in \mathbb{N}$, where $\tau = h(\varepsilon)$ is an analytic regular reparametrisation with $h(0) = 0$, and the φ_j are analytic in τ, for τ close to 0 $(\forall j = 2, \ldots, p)$.

The analytic curve γ is a regular parametrisation of a straight line

The key in proving theorem 191 is proposition 194 below, that is based on the Łojasiewicz inequality. Let us therefore start by recalling the Łojasiewicz's inequality and a corollary, that essentially says that two closed subanalytic sets cannot become too close [BM],[Loj].

Theorem 192 (Lojasiewicz's inequality) *Let M be a real analytic manifold and let K be a subset of M. Let $f, g : K \to \mathbb{R}$ be subanalytic functions with compact graphs. If $f^{-1}(0) \subset g^{-1}(0)$, then there exist $C, r > 0$ such that, for all $x \in K$,*

$$|f(x)| \geq C \cdot |g(x)^r|.$$

Let us denote the closed ball in $\mathbb{R}^{p-1}$, with center $\bar{0}$ and radius 1, by

$$\bar{B}(\bar{0}, 1) = \left\{\mu \in \mathbb{R}^{p-1} : \|\mu\| \leq 1\right\}.$$

Corollary 193 *Let U be an open subset of $\mathbb{R}^n$. Then any two closed subanalytic subsets X, Y of U are regularly situated; this means that, $\forall x^0 \in X \cap Y$, there exist a neighbourhood V of x^0 and $C, r > 0$ such that for all $x \in V$,*

$$d(x, X) + d(x, Y) \geq C \cdot d(x, X \cap Y)^r$$

Proposition 194 *Let V be an open subanalytic set, $V \subset \mathbb{R}^p$ such that*

$$(]0,1] \times \{\bar{0}\}) \cup (\{1\} \times \bar{B}(\bar{0}, 1)) \subset V.$$

Then there exists an integer $n \in \mathbb{N}_1$ with $C_n \subset V$, where

$$C_n = \left\{(\lambda, \mu) \in]0,1] \times \bar{B}_1(\bar{0}) : \|\mu\| \leq \lambda^n\right\}.$$

Proof. By a compactness-argument we clearly see that for any $0 < r_1 < 1$, there must exist $n \in \mathbb{N}_1$ with the property that

$$C_n \cap \{(\lambda, \mu) \in \mathbb{R}^p : r_1 \leq \lambda \leq 1\} \subset V.$$

Hence, the problem is concentrated near the origin. As such, if we suppose that no $C_n \subset V$, we do not only find a sequence $(\lambda_n, \mu_n) \in C_n \setminus V$, but we also may assume that a certain subsequence (λ_{n_k}) tends to 0 for $k \to \infty$. Since V^c and $[0,1] \times \{\bar{0}\}$ are closed subanalytic sets, they are regularly situated (cfr. corollary 193); hence, there exist a neighbourhood W of $(0, \bar{0})$ in $\mathbb{R}^p$, and positive constants C, r such that $\forall (\lambda, \mu) \in W$:

$$d((\lambda, \mu), V^c) + d((\lambda, \mu), [0,1] \times \{\bar{0}\}) \geq Cd((\lambda, \mu), (0, \bar{0}))^r .$$

Since $(\lambda_{n_k}, \mu_{n_k}) \to (0, \bar{0})$, there is a positive integer $N \in \mathbb{N}$ such that $\forall k \geq N$: $(\lambda_{n_k}, \mu_{n_k}) \in W \cap V^c$ and $\lambda_{n_k}^{n_k - r} < C$. Therefore, $\forall k \geq N$:

$$\|\mu_{n_k}\| = d((\lambda_{n_k}, \mu_{n_k}), [0,1] \times \{\bar{0}\}) \geq C \|(\lambda_{n_k}, \mu_{n_k})\|^r \geq C\lambda_{n_k}^r > \lambda_{n_k}^{n_k}.$$

This is in contradiction with $(\lambda_{n_k}, \mu_{n_k}) \in C_{n_k}$. ■

Remark 195 *The sequence of cones $(C_n)_{n \in \mathbb{N}_1}$ is nested (i.e. $C_{n+1} \subset C_n, \forall n \in \mathbb{N}_1$.)*

We return to the proof of theorem 191 in case γ is a straight line. Since V is open, we obtain after a linear coordinate transformation that

$$(]0,1] \times \{\bar{0}\}) \cup (\{1\} \times \bar{B}(\bar{0}, 1)) \subset V,$$

and that the image of γ belongs to the positive λ-axis. This coordinate transformation does not matter when determining the positive integer n of theorem 191. This is expressed in the following proposition that we state without proof.

Proposition 196 *Let $h : \mathbb{R}^p \to \mathbb{R}^p$ be a C^ω diffeomorphism with $h(\bar{0}) = \bar{0}$. Suppose that γ and $\hat{\gamma}$ are C^ω curves with $\gamma(a) = \hat{\gamma}(a) = \bar{0}$. If $j_n(\gamma - \hat{\gamma})_a = \bar{0}$, then also $j_n(h \circ \gamma - h \circ \hat{\gamma})_a = \bar{0}$.*

Let us also recall that a regular analytic reparametrisation of a curve has no influence on the contact of the jets:

Proposition 197 *Suppose $\varphi : I \subset \mathbb{R} \to \mathbb{R}^p$ is an analytic function with*

$$j_n(\varphi)_0 = \bar{0},$$

and h is a diffeomorphism of the form

$$h(\tau) = a\tau + o(\tau), \tau \to 0,$$

with $a \neq 0$, then also

$$j_n(\varphi \circ h)_0 = \bar{0}.$$

As a consequence of proposition 194, there is a positive integer n such that $C_n \subset V$. Now, we claim that for every analytic curve $\hat{\gamma}$ with $j_n(\gamma - \hat{\gamma})_0 = 0$, the germ at 0 is contained in $V \cup \{\bar{0}\}$. From proposition 197, it is clear that this claim only needs to be proven for $\gamma : I \to \mathbb{R}^p : \varepsilon \mapsto (\varepsilon, 0, \ldots, 0)$:

Lemma 198 *For every analytic curve* $\hat{\gamma} : I \to \mathbb{R} \times \mathbb{R}^{p-1}$

$$\hat{\gamma}(\varepsilon) = (\varepsilon + g_1(\varepsilon), g_2(\varepsilon))$$

with

$$\begin{cases} g_1(\varepsilon) = O\left(\varepsilon^{n+1}\right), \varepsilon \to 0 \\ g_2(\varepsilon) = O\left(\varepsilon^{n+1}\right), \varepsilon \to 0 \end{cases},$$

there exists a $\delta > 0$ *such that* $\forall 0 < \varepsilon < \delta : \hat{\gamma}(\varepsilon) \in C_n$.

Proof. We can write

$$(\varepsilon + g_1(\varepsilon))^n = \varepsilon^n \bar{g}(\varepsilon),$$

with

$$\bar{g}(\varepsilon) = 1 + O(\varepsilon^n), \varepsilon \to 0.$$

For $\varepsilon \geq 0$ sufficiently small, we have $\bar{g}(\varepsilon) > \frac{1}{2}$ and

$$|g_2(\varepsilon)| \leq \frac{1}{2}\varepsilon^n \leq \varepsilon^n \bar{g}(\varepsilon) = (\varepsilon + g_1(\varepsilon))^n.$$

■

The analytic curve γ is regular

When the analytic curve is regular (i.e. $\gamma'(\varepsilon) \neq 0$), we can reduce the curve to a straight line by an analytic coordinate transformation. Indeed, by the Implicit Function Theorem, we can rewrite the analytic curve locally as the graph of an analytic function. After a linear coordinate transformation and an analytic reparametrisation, the curve then takes the form

$$\tau \mapsto (\tau, \gamma_2(\tau), \ldots, \gamma_p(\tau)).$$

Next, we perform the analytic coordinate transformation

$$(\lambda, \mu_2, \ldots, \mu_p) \mapsto (\lambda, \mu_2 - \gamma_2(\lambda), \ldots, \mu_p - \gamma_p(\lambda)).$$

In these coordinates, the curve γ is a regular parametrisation of a straight line. Remark that the image of a(n open) subanalytic set by an analytic diffeomorphism is again a(n open) subanalytic set. Therefore, it is clear that theorem 191 also holds in case γ is a regular analytic curve.

The analytic curve γ is not regular

Upon a linear coordinate transformation and an analytic reparametrisation, we can suppose that the curve γ is parametrised by

$$\varepsilon \mapsto (\varepsilon^{n_1}, \gamma_2(\varepsilon), \ldots, \gamma_p(\varepsilon)),$$

with $\forall i = 2, \ldots, p$:

$$\gamma_i(\varepsilon) = \varepsilon^{n_i} + o(\varepsilon^{n_i}), \varepsilon \downarrow 0,$$

and $n_i \in \mathbb{N}_1$ $(\forall i = 1, \ldots, p)$. Consider the blow-up map Φ defined by

$$\Phi(\bar{\mu}_1, \bar{\mu}_2, \ldots, \bar{\mu}_p) = (\bar{\mu}_1^{n_1}, \bar{\mu}_2, \ldots, \bar{\mu}_p).$$

Then V and γ are blown up to respectively the open subanalytic set $\Phi^{-1}(V)$ and the regular curve $\sigma(\varepsilon) = (\varepsilon, \gamma_2(\varepsilon), \ldots, \gamma_p(\varepsilon))$. Hence, there exists a positive integer n such that every analytic curve $\hat{\sigma}$, with $j_n(\sigma - \hat{\sigma})(0) = \bar{0}$, enters $\Phi^{-1}(V) \cup \{\bar{0}\}$.

Now we claim that every analytic curve $\hat{\gamma}$ with $j_{n_1+n-1}(\gamma - \hat{\gamma})_0 = \bar{0}$, enters $V \cup \{\bar{0}\}$. Indeed, if we write $\hat{\gamma}(\varepsilon) = (\hat{\gamma}_1(\varepsilon), \ldots, \hat{\gamma}_p(\varepsilon))$, then

$$\hat{\gamma}_1(\varepsilon) = \varepsilon^{n_1}(1 + g(\varepsilon))$$

with

$$g(\varepsilon) = O\left(\varepsilon^{n-1}\right), \varepsilon \to 0,$$

and $\forall i = 2, \ldots, p$:

$$\hat{\sigma}_i(\varepsilon) = \hat{\gamma}_i(\varepsilon) = \gamma_i(\varepsilon) + O\left(\varepsilon^{n_1+n-1}\right) = \sigma_i(\varepsilon) + O(\varepsilon^n), \varepsilon \to 0.$$

Since $\hat{\sigma}_1$ is defined by $\hat{\sigma}_1(\varepsilon)^{n_1} = \hat{\gamma}_1(\varepsilon)$, we have

$$\begin{aligned}\hat{\sigma}_1(\varepsilon) &= \varepsilon(1 + g(\varepsilon))^{1/n_1} \\ &= \varepsilon(1 + \bar{g}(\varepsilon)) = \varepsilon + O(\varepsilon^n),\end{aligned}$$

from which the result follows.

Example

In figure 4.2, we illustrate the proof of theorem 191 for the open subanalytic set V, defined by

$$\left\{(\lambda, \mu) \in \mathbb{R}^2 : \lambda^2 < (e^{\mu} - 1)^3 < 27\lambda^2\right\}$$

(in fact this set even is semi-analytic). The analytic curve γ, implicitly defined by

$$(e^{\mu} - 1)^3 = 8\lambda^2, \qquad \lambda, \mu \geq 0,$$

lies inside of V; an analytic parametrisation of γ is given by

$$\zeta(\varepsilon) = \left(\varepsilon^3, \log\left(1 + 2\varepsilon^2\right)\right), \varepsilon \downarrow 0.$$

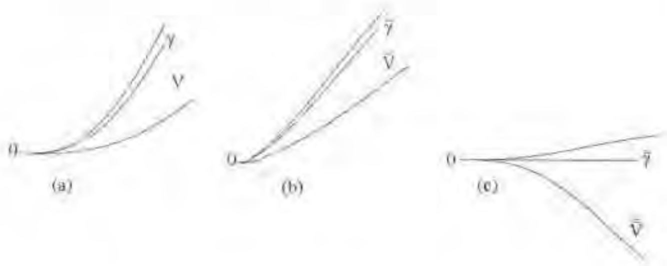

Figure 4.2: Reduction of a non-regular curve to a straight line in three steps (illustration for the proof of theorem 191): (a) a non-regular curve γ inside the subanalytic set V; (b) blow-up of V and γ into $\bar{V}$ and $\bar{\gamma}$ respectively; (c) rectification of $\bar{\gamma}$ into $\bar{\bar{\gamma}}$

Clearly, ζ is not regular at $\varepsilon = 0$. Therefore, we perform the following blow up:

$$\lambda = \bar{\lambda}^3, \mu = \bar{\mu}$$

The subanalytic set V and the curve γ are blown up to the subanalytic set $\bar{V}$ and the curve $\bar{\gamma}$, with regular parametrisation $\bar{\zeta}$:

$$\begin{cases} \bar{V} = \left\{ \left(\bar{\lambda}, \bar{\mu}\right) \in \mathbb{R}^2 : \bar{\lambda}^2 < e^{\bar{\mu}} - 1 < 3\bar{\lambda}^2 \right\} \\ \bar{\gamma} = \left\{ \left(\bar{\lambda}, \bar{\mu}\right) \in \mathbb{R}^2 : e^{\bar{\mu}} - 1 = 2\bar{\lambda}^2, \bar{\lambda}, \bar{\mu} \geq 0 \right\} \\ \bar{\zeta}\left(\varepsilon\right) = \left(\varepsilon, \log\left(1 + 2\varepsilon^2\right)\right), \varepsilon \downarrow 0 \end{cases}$$

Performing the analytic diffeomorphism $(\bar{\bar{\lambda}}, \bar{\bar{\mu}}) = h\left(\bar{\lambda}, \bar{\mu}\right) = (\bar{\lambda}, \bar{\mu} - \log\left(1 + 2\bar{\lambda}^2\right))$, the curve $\bar{\gamma}$ is transformed into the straight line $\bar{\bar{\gamma}}$ (the $\bar{\bar{\lambda}}$-axis), with parametrisation $\bar{\bar{\zeta}}$, and $\bar{V}$ is transformed into $\bar{\bar{V}}$:

$$\begin{cases} \bar{\bar{V}} = \left\{ (\bar{\bar{\lambda}}, \bar{\bar{\mu}}) \in \mathbb{R}^2 : \log(\frac{1+\bar{\bar{\lambda}}^2}{1+2\bar{\bar{\lambda}}^2}) < \bar{\bar{\mu}} < \log(\frac{1+3\bar{\bar{\lambda}}^2}{1+2\bar{\bar{\lambda}}^2}) \right\} \\ \bar{\bar{\gamma}} = \left\{ (\bar{\bar{\lambda}}, 0) \in \mathbb{R}^2 : \bar{\bar{\lambda}} \geq 0 \right\} \\ \bar{\bar{\zeta}}\left(\varepsilon\right) = \left(\varepsilon, 0\right), \varepsilon \downarrow 0 \end{cases}$$

It then follows that $C_3 \subset \bar{\bar{V}}$, since by Taylor's theorem, we can write:

$$\begin{cases} \log(\frac{1+\bar{\bar{\lambda}}^2}{1+2\bar{\bar{\lambda}}^2}) = -\bar{\bar{\lambda}}^2 + O(\bar{\bar{\lambda}}^3), \bar{\bar{\lambda}} \to 0 \\ \log(\frac{1+3\bar{\bar{\lambda}}^2}{1+2\bar{\bar{\lambda}}^2}) = \bar{\bar{\lambda}}^2 + O(\bar{\bar{\lambda}}^3), \bar{\bar{\lambda}} \to 0 \end{cases}$$

As a consequence, for every analytic curve $\hat{\zeta}$ with $j_5(\zeta - \hat{\zeta})_0 = \bar{0}$, there exists a constant $E > 0$ such that $\hat{\zeta}\left(\varepsilon\right) \in V, \forall 0 < \varepsilon < E$.

4.6.2 Algebraic mcc and mmc

Now we turn back to analytic families $(X_\lambda)_\lambda$ of planar vector fields, with an associated analytic family of displacement maps $(\delta_\lambda)_\lambda$. The goal of this section is to prove the existence of algebraic mcc's (respectively mmc's) in certain circumstances. By theorem 190, it suffices to construct an open subanalytic set V with the property: if $\zeta : [0,1] \to \mathbb{R}^p$ is an analytic curve with $\zeta(0) = \lambda^0, \zeta(]0,1]) \subset V$, then ζ is an mcc (respectively mmc).

The construction of such a set is not that evident, as is explained in section 4.6.2. Then, in section 4.6.2, we give a precise definition of a family of vector fields having a stratum of maximal cyclicity (multiplicity) with non-empty interior at λ^0. This property is a sufficient condition to guarantee the existence of an algebraic mcc (respectively mmc), as is proven in section 4.6.2. Finally, in section 4.6.2, we show that if ζ is an mcc (respectively mmc), then the curves ξ_i $(1 \le i \le n)$ are continuous, where $\xi_i(\varepsilon)$ is a zero of $\delta_{\zeta(\varepsilon)}, \forall 1 \le i \le n$.

Situation of the problem

Suppose that $\mathrm{Cycl}\left(X_\lambda, \left(\Gamma, \lambda^0\right)\right) = n$ and let $M > 0$ and let W be a neighbourhood of λ^0 such that δ_λ has at most n zeroes in $[s_0 - M, s_0 + M]$. Then we can define the subanalytic set Z_M^W by

$$\begin{aligned} Z_M^W = \{\lambda \in W : \exists \xi_1, \ldots, \xi_n \in]s_0 - M, s_0 + M[: \xi_1 < \ldots < \xi_n \\ \text{and } \delta_\lambda(\xi_i) = 0, \forall i = 1, \ldots, n\} \end{aligned} \tag{4.22}$$

Let us denote the interior of Z_M^W by $\mathring{Z}_M^W$. Remark that the chosen neighbourhood W in the definition of Z_M^W does not play a very important role: if we take a smaller neighbourhood W_1, then the set $Z_M^{W_1} = Z_M^W \cap W_1$ is just the intersection of Z_M^W with W_1. For fixed $M > 0$, the germs of the sets Z_M^W at λ^0, remain the same for every neighbourhood $W$of λ^0. For this reason, we will often omit the dependence on W in our notation Z_M^W, writing Z_M meaning Z_M^W, for a certain neighbourhood W of λ^0.

If $\mathring{Z}_M \neq \emptyset$ accumulates on λ^0, then (by theorem 190) there exists an algebraic curve $\zeta : [0,1] \to \mathbb{R}^p$ with $\zeta(0) = \lambda^0$. However, this curve is not necessarily an mcc. To situate exactly the problem, let us recall the definition of an mcc: the analytic curve $\zeta : I \subset \mathbb{R} \to \mathbb{R}^p$ with $\zeta(0) = \lambda^0$ is an mcc if and only if

1. $\forall \varepsilon \in I$, the map $\delta_{\zeta(\varepsilon)}$ has exactly n zeroes in $[s_0 - M, s_0 + M]$, say

$$\xi_1(\varepsilon) < \ldots < \xi_n(\varepsilon)$$

2. For $\varepsilon \downarrow 0$,

$$\lim_{\varepsilon \downarrow 0} \xi_i(\varepsilon) = s_0, \forall i = 1, \ldots, n.$$

In fact, last property only needs to be verified for a discrete sequence $(\varepsilon_n)_{n\in\mathbb{N}}$ with $\varepsilon_n \downarrow 0$, which reveals to be an interesting observation for the proof of theorem 201 (proving the existence of an algebraic mcc).

Since ζ lies inside Z_M, the first condition is certainly satisfied. However, we also need to control the positions of the corresponding zeroes $\xi_i(\varepsilon), 1 \le i \le n$. It is possible that some of these zeroes do not tend to s_0; in this case, the second condition for ζ to be an mcc is not satisfied. In fact, $\forall 0 < M' \le M$, there must exist an $E(M') > 0$ such that

$$\forall 0 < \varepsilon < E(M') : \zeta(\varepsilon) \in Z_{M'}.$$

In other words, we need to find an analytic curve ζ, such that its germ at λ^0 lies in the germs of the sets $Z_{M'} \cup \{\lambda^0\}, \forall M' \downarrow 0$. Hence, we are left with the limiting set $Z_0 = \bigcap_{M'\downarrow 0} Z_{M'}$. If this set is open and subanalytic, then there exists an algebraic mcc (by theorem 190). A priori, it is not clear at all whether Z_0 is open or subanalytic; it is even not clear whether Z_0 is non-empty, but this is not essential.

Of course, we have $Z_{M'} \subset Z_M, \forall M' \le M$. It is possible that the sequence of the germs of Z_M at λ^0 stabilizes, i.e. there exists an $M > 0$ such that for every $0 < M' < M$, there exists a neighbourhood $V_{M'}$ of λ^0 with

$$Z_{M'} \cap V_{M'} = Z_M \cap V_{M'}.$$

Under this condition, it is clear how theorem 190 can be applied in a straightforward way to $\mathring{Z}_M$ to show the existence of an algebraic mcc.

It applies for instance to the family of planar vector fields $X_{(\lambda_1,\lambda_2)}$ of type (1.2) with

$$\delta(s,\lambda) = \lambda_1 (s - s_0)^2 - \lambda_1^3 + \lambda_1\lambda_2^2, \qquad \text{for } \lambda = (\lambda_1,\lambda_2) \in \mathbb{R}^2 \tag{4.23}$$

where $s_0 \in \mathbb{R}$ is fixed. Let be $\Gamma = \{(x,y) : x^2 + y^2 = s_0^2\}$ and $\lambda^0 = (0,0)$, then

$$\operatorname{Cycl}(X_\lambda, (\Gamma, \lambda^0)) = 2$$

and $\forall M > 0$:

$$Z_M = \{(\lambda_1,\lambda_2) \in W : 0 < \lambda_1^2 - \lambda_2^2 < M^2\},$$

where W is any neighbourhood of λ^0 in $\mathbb{R}^2$. If we write $Z = \{(\lambda_1,\lambda_2) \in W : \lambda_1^2 > \lambda_2^2\}$, then the germs of Z and Z_M at λ^0 are equal, $\forall M > 0$; hence, the germs at λ^0 stabilize. Notice however that $Z_0 = \bigcap_{M\downarrow 0} Z_M$ is empty and that $\zeta(\varepsilon) = (\varepsilon, 0)$ is an algebraic (even a linear) mcc.

However, the sequence of the germs of Z_M at λ^0 does not always stabilize; consider for instance the family $(X_\lambda)_\lambda$ of type (1.2) with

$$\delta(s,\lambda) = (\lambda_1^2 - \lambda_2^3) + \lambda_2^3 s, \qquad \text{for } \lambda = (\lambda_1,\lambda_2) \in \mathbb{R}^2 \tag{4.24}$$

Let Γ be the unit circle,

$$\Gamma = \{(x,y) : x^2 + y^2 = 1\},$$

and $\lambda^0 = (0,0)$, then Cycl $(X_\lambda, (\Gamma, \lambda^0)) = 1$. Limit cycles of X_λ correspond to isolated zeroes of $\delta(\cdot, \lambda)$. It is clear that $\delta(\cdot,\lambda)$ only has a non-isolated zero if $\lambda_2 \neq 0$. If $\lambda_2 \neq 0$, the zero of $\delta(\cdot,\lambda)$ can be written as:

$$s(\lambda_1, \lambda_2) = 1 - \frac{\lambda_1^2}{\lambda_2^3}.$$

As a consequence, $\forall M > 0$ and for every neighbourhood W of $\bar{0}$ in $\mathbb{R}^2$:

$$Z_M^W = \{(\lambda_1,\lambda_2) \in W : |\lambda_1| < \sqrt{M|\lambda_2|^3}\}$$

Hence, $Z_0^W = \bigcap_{M\downarrow 0} Z_M^W = \{(\lambda_1,\lambda_2) \in W : \lambda_1 = 0\}$. Clearly, the curve $\gamma(\varepsilon) = (0,\varepsilon)$ is an algebraic (even a linear) mcc. Moreover, for every analytic curve $\hat{\gamma} : I \subset \mathbb{R} \to \mathbb{R}^2$ with

$$j_1(\gamma - \hat{\gamma})_0 = 0,$$

and for every $M > 0$, there exists $E(M) > 0$ such that

$$\hat{\gamma}(\varepsilon) \in Z_M, \forall 0 < \varepsilon < E(M).$$

Hence, $\hat{\gamma}$ also is an mcc. Notice also that the zero $s(\hat{\gamma}(\varepsilon))$ of $\delta(\cdot, \hat{\gamma}(\varepsilon))$, depends continuously on ε.

These facts hold in general, in case that the family $(X_\lambda)_\lambda$ has an mc-stratum with non-empty interior at λ^0 (theorem 201, corollary 204 and proposition 203).

In example (4.24), the limiting set Z_0 contains an *algebraic* mcc. This is not always the case; a non-empty Z_0^W does not need to contain an algebraic mcc. Consider for instance the family $(\bar{X}_{\bar{\lambda}})_{\bar{\lambda}}$ obtained from (4.24) after application of the analytic coordinate transformation in parameter space

$$(\lambda_1,\lambda_2) \overset{h}{\mapsto} (\bar{\lambda}_1, \bar{\lambda}_2) = (\lambda_1 + \sin\lambda_2, \lambda_2)$$

in example (4.24); more precisely, $\bar{X}_{(\bar{\lambda}_1,\bar{\lambda}_2)} = X_{(\bar{\lambda}_1 - \sin\bar{\lambda}_2, \bar{\lambda}_2)}$. Then, the limiting set Z_0 is given by the graph of sin :

$$\bar{Z}_0^W = \{(\bar{\lambda}_1,\bar{\lambda}_2) \in W : \bar{\lambda}_1 = \sin\bar{\lambda}_2\}$$

The analytic curve $\gamma(\varepsilon) = (\sin\varepsilon, \varepsilon), \varepsilon > 0$ is an mcc, since it is contained in Z_0^W; clearly, there is no algebraic curve inside $\bar{Z}_0^W$. Since for every open neighbourhood W of $\bar{0} \in \mathbb{R}^2$ and $\forall M > 0$, the subanalytic sets

$$\bar{Z}_M^W = \left\{(\bar{\lambda}_1,\bar{\lambda}_2) \in W : |\bar{\lambda}_1 - \sin\bar{\lambda}_2| < \sqrt{M|\bar{\lambda}_2|^3}\right\}$$

are open, the family $(\bar{X}_{\bar{\lambda}})_{\bar{\lambda}}$ has an mc-stratum with non-empty interior at λ^0. It can be checked analoguously as above that the quadratic curve $\zeta(\varepsilon) = (\varepsilon + \varepsilon^2, \varepsilon)$ is an mcc; moreover, every analytic curve $\hat{\zeta}$ with $j_2\left(\zeta - \hat{\zeta}\right)_0 = 0$, is an mcc with

$$\forall M > 0, \exists E(M) > 0 : \forall 0 < \varepsilon < E(M) : \hat{\zeta}(\varepsilon) \in Z_M.$$

It is hence clear that in a non-stabilizing situation we can not prove the existence of an (algebraic) mcc by considering Z_0. In proving the existence of an mcc, we instead work with the subanalytic set

$$\begin{aligned} W_M = \{(\lambda, \xi_1, \ldots, \xi_n) \in Z_M \times [s_0 - M, s_0 + M]^n : \\ \xi_1 < \ldots < \xi_n, \delta_\lambda(\xi_i) = 0, \forall 1 \le i \le n\}. \end{aligned} \tag{4.25}$$

If $\omega = (\zeta, \xi_1, \ldots, \xi_n)$ is an analytic curve in W_M, that starts at $(\lambda^0, s_0, \ldots, s_0)$, then ζ is clearly an mcc. The existence of such a curve ω, and hence the existence of an mcc is ensured by the curve selection lemma applied to the subanalytic set W_M. Since W_M is not open, we cannot apply theorem 190 to W_M, to guarantee the existence of an algebraic mcc.

Investigating the existence of algebraic mmc's, one can make analoguous remarks. Here, we just describe the set ϑ_M^W of parameter values $\lambda \in W$ for which the total multiplicity of the zeroes of δ_λ in a given interval $[s_0 - M, s_0 + M]$ is equal to the multiplicity of the family. Suppose that $\text{Mult}(X_\lambda, (\Gamma, \lambda^0)) = m$ and let $M > 0$ and let W be a neighbourhood of λ^0 such that if $\lambda \in W$ and δ_λ has n zeroes in $[s_0 - M, s_0 + M]$ with respective multiplicities $m_1, \ldots, m_n$, then $\sum_{i=1}^n m_i \le m$. We can define the subanalytic set ϑ_M^W :

$$\begin{aligned} \vartheta_M^W = \{ \lambda \in W | \exists n \in \mathbb{N} : \forall 1 \le i \le n : \exists m_i \in \mathbb{N}^* : \exists \xi_1, \ldots, \xi_n \in]s_0 - M, s_0 + M[: \\ \left. \xi_1 < \ldots < \xi_n \text{ and } \forall i = 1, \ldots, n : 0 \le j < m_i : \delta_\lambda^{(j)}(\xi_i) = 0, \right\} \\ \left. \text{and } \forall i = 1, \ldots, n : \delta_\lambda^{(m_i)}(\xi_i) \neq 0 \right\} \end{aligned} \tag{4.26}$$

Denote the interior of ϑ_M^W by $\mathring{\vartheta}_M^W$. Notice that, again for fixed $M > 0$, shrinking the neighbourhood W of λ^0, has no influence on the germ of the set ϑ_M^W at λ^0. But shrinking $M > 0$, changes the germ of ϑ_M at λ^0, in general. Analoguous examples as above, illustrate that an analytic curve $\zeta : [0,1] \to \mathbb{R}^p$ with $\zeta(0) = \lambda^0$ with $\zeta(\varepsilon) \in Z_M^W, \forall \varepsilon \downarrow 0$, is not necessary an mmc. Let us recall the definition of an mmc: the analytic curve $\zeta : I \subset \mathbb{R} \to \mathbb{R}^p$ with $\zeta(0) = \lambda^0$ is an mmc if and only if

1. $\forall \varepsilon \in I$, the map $\delta_{\zeta(\varepsilon)}$ has exactly n zeroes in $[s_0 - M, s_0 + M]$, say

$$\xi_1(\varepsilon) < \ldots < \xi_n(\varepsilon)$$

with respective multiplicities $m_1, \ldots, m_n$ such that $\sum_{i=1}^n m_i$

2. For $\varepsilon \downarrow 0$,

$$\lim_{\varepsilon \downarrow 0} \xi_i(\varepsilon) = s_0, \forall i = 1, \ldots, n.$$

In fact, last property only needs to be verified for a discrete sequence $(\varepsilon_n)_{n \in \mathbb{N}}$ with $\varepsilon_n \downarrow 0$, which reveals to be an interesting observation for the proof of theorem 202 (proving the existence of an algebraic mmc).

To finish this section, we quickly discuss the problems in proving the existence of algebraic mmc's. The problems are of the same kind as for the algebraic mcc's. In fact, we can use the same examples to describe the problems. In example (4.23), one has that

$$\mathrm{Mult}\left(X_{(\lambda,\mu)}, (\Gamma, \bar{0})\right) = 2$$

and $\forall M > 0 : \vartheta_M = Z_M$ and $\vartheta = Z$. Clearly, the germs of ϑ and ϑ_M are equal at $\bar{0}, \forall M > 0$; hence, the germs at λ^0 stabilize. Notice again that $\vartheta_0 = \bigcap_{M \downarrow 0} \vartheta_M$ is empty and that $\zeta(\varepsilon) = (\varepsilon, 0)$ is a linear mmc.

But in general, the sequence of the germs of ϑ_M at λ^0 does not stabilize, as is illustrated by example (4.24), where

$$\mathrm{Mult}\left(X_{(\lambda,\mu)}, (\Gamma, \bar{0})\right) = 1.$$

In this example the concepts mmc and mcc coincide. Moreover, $\forall M > 0 : \vartheta_M = Z_M$, and

$$\vartheta_0 = \bigcap_{M \downarrow 0} \vartheta_M = Z_0.$$

As we noticed, there is a linear mmc. Moreover, for every analytic curve $\hat{\gamma} : I \subset \mathbb{R} \to \mathbb{R}^2$ with

$$j_1 (\gamma - \hat{\gamma})_0 = 0,$$

and for every $M > 0$, there exists $E(M) > 0$ such that

$$\hat{\gamma}(\varepsilon) \in \vartheta_M, \forall 0 < \varepsilon < E(M).$$

Hence, $\hat{\gamma}$ also is an mmc. Moreover, notice that the zero $s(\hat{\gamma}(\varepsilon))$ of $\delta(\cdot, \hat{\gamma}(\varepsilon))$, depends continuously on ε. These facts hold in general, in case that the family $(X_\lambda)_\lambda$ has an mm-stratum with non-empty interior at λ^0 (theorem 202, corollary 206 and proposition 205).

In example (4.24), the limiting set ϑ_0 contains an *algebraic* mmc. In general, this is not always the case; a non-empty ϑ_0 does not need to contain an algebraic mmc. This fact can again be demonstrated by investigating the family of planar vector fields defined by $\hat{X}_{(\lambda_1,\mu_1)} = X_{(\lambda_1 - \sin\mu_1, \mu_1)}$, since in this example the concepts mcc (respectively Z_M) and mmc (respectively ϑ_M) coincide. It can be checked analoguously as above that the quadratic curve $\zeta(\varepsilon) = \left(\varepsilon + \varepsilon^2, \varepsilon\right)$ is an mmc; moreover, every analytic curve $\hat{\zeta}$ with $j_2(\zeta - \hat{\zeta})_0 = 0$, is an mmc with

$$\forall M > 0, \exists E(M) > 0 : \forall 0 < \varepsilon < E(M) : \hat{\zeta}(\varepsilon) \in \vartheta_M.$$

Hence, again, it is clear that in a non-stabilizing situation, we can not prove the existence of an (algebraic) mmc by considering ϑ_0. Therefore, in proving the existence of an algebraic mmc we need to construct an open subanalytic set with 'good' properties (see below in section 4.6.2).

Sufficient conditions

In this section, we first state a precise definition of the condition 'having a stratum of maximal cyclicity (respectively multiplicity) with non-empty interior at λ^0.' Next, we give examples of families of vector fields for which this condition is not satisfied, with or without an algebraic mcc (mmc).

Definition 199 *Under the same notations as above, we say that the family*

1. $(X_\lambda)_\lambda$ *has a stratum of maximal cyclicity with non-empty interior at* λ^0 *or an mc-stratum with non-empty interior at* λ^0, *if there exist an* $M > 0$ *and a neighbourhood* W *of* λ^0 *such that there exist a sequence* $(\lambda_j)_{j\in\mathbb{N}}$ *in* $\mathring{Z}_M^W$ *with* $\lambda_j \to \lambda^0$, *and such that the corresponding zeroes* $\xi_1(\lambda_j) < \ldots < \xi_n(\lambda_j)$ *tend to* s_0 *(if* $j \to \infty$*).*
2. $(X_\lambda)_\lambda$ *has a stratum of maximal multiplicity with non-empty interior at* λ^0 *or an mm-stratum with non-empty interior at* λ^0, *if there exist an* $M > 0$ *and a neighbourhood* W *of* λ^0 *such that there exist a sequence* $(\lambda_j)_{j\in\mathbb{N}}$ *in* $\mathring{\vartheta}_M^W$ *with* $\lambda_j \to \lambda^0$, *and such that the corresponding zeroes* $\xi_1(\lambda_j) < \ldots < \xi_n(\lambda_j)$ *tend to* s_0 *(if* $j \to \infty$*).*

Example 200 *Suppose that we are given an analytic family* $(X_\lambda)_\lambda$ *of planar vector fields with a regular Bautin ideal* $\mathcal{I}$. *Let* $\varphi_1, \ldots, \varphi_l$ *be a minimal set of generators for* $\mathcal{I}$ *and let* $h_1, \ldots, h_l$ *be a set of analytic factors such that the displacement map takes the form:*

$$\delta(s,\lambda) = \sum_{i=1}^{l} \varphi_i(\lambda)\, h_i(s,\lambda),$$

Suppose furthermore that the corresponding factor functions, defined by $H_i \equiv h_i(\cdot, \lambda^0)$, $1 \le i \le l-1$, *have strictly increasing order at* s_0 *without gaps, i.e.*

$$orderH_i(s_0) = i - 1, \qquad i = 1, \ldots, l. \tag{4.27}$$

Then, there exists an open neighbourhood W *of* λ^0, *and there exists* $M > 0$ *such that* Z_M^W *defined by*

$$Z_M^W = \{\lambda \in W : \delta_\lambda \text{ has exactly } l-1 \text{ zeroes in }]s_0 - M, s_0 + M[\}$$

is a non-empty open subanalytic set, that accumulates at λ^0. *(From (4.27) it follows that* $\{H_1, \ldots, H_l\}$ *is a Chebyshev system; in particular,* $\mathrm{Cycl}(X_\lambda, (\Gamma, \lambda^0)) = \mathrm{Mult}(X_\lambda, (\Gamma, \lambda^0)) = l - 1$, *and therefore the zeroes, that appear in* Z_M^W *are simple (if* M *and* W *are small enough), and then we can use the Implicit Function Theorem to show that* Z_M^W *is open). As a consequence, the family* $(X_\lambda)_\lambda$ *has a stratum of maximal cyclicity (multiplicity) with non-empty interior at* λ^0.

In case the family does not have a mc-stratum (respectively mm-stratum) with non-empty interior at λ^0, we are not yet sure whether an algebraic mcc do exist. For instance, consider again example (4.16) above: the mc-stratum (respectively mm-stratum) does not have a non-empty interior at λ^0, and there does not exist an algebraic mcc (respectively mmc). In the same way, we can construct an algebraic family $(X_\lambda)_\lambda$ of type (1.2) where the mc-stratum (respectively mm-stratum) does not have a non-empty interior at λ^0 :

$$\delta(s,\lambda) = \lambda_2\left((s-s_0)^2 + \left(\lambda_1 - \lambda_2^2\right)^2\right) \tag{4.28}$$

However, in this case there obviously does exist an algebraic mcc (respectively mmc), but not a linear one.

It is not clear at all if this phenomenon is possible in linear families. In most examples encountered in the literature, nearby vector fields, with maximal cyclicity (respectively multiplicity) are structurally stable and hence occur in open subanalytic sets of the parameter space. In the rest of this section, we will now limit to this case, proving the existence of algebraic mcc's (respectively mmc's).

Existence

Theorem 201 *If the stratum of maximal cyclicity has a non-empty interior at λ^0, then there exists an algebraic mcc ζ. Moreover, there exists a positive integer k such that every analytic curve $\hat{\zeta}$ with $j_k(\zeta - \hat{\zeta})_0 = 0$, is an mcc.*

Proof. Without loss of generality, we can suppose that $\lambda^0 = \bar{0} \in \mathbb{R}^p, s_0 = 0 \in \mathbb{R}$. Since the family $(X_\lambda)_\lambda$ has an mc-stratum with non-empty interior at $\bar{0}$, there exist $M > 0$, a neighbourhood W of $\bar{0}$ in $\mathbb{R}^p$ and an analytic curve $\omega = (\zeta, \xi_1, \ldots, \xi_n) : I = [0,1] \to \mathbb{R}^p \times [-M, M]^n$ such that

$$\begin{cases} \zeta(0) = \bar{0} \\ \xi_i(0) = 0, \forall i = 1, \ldots, n \\ \delta_{\zeta(\varepsilon)}(\xi_i(\varepsilon)) = 0, \forall \varepsilon, \forall i = 1, \ldots, n \\ -M < \xi_1(\varepsilon) < \ldots < \xi_n(\varepsilon) < M, \forall \varepsilon > 0 \\ \zeta(\varepsilon) \in \mathring{Z}_M, \forall \varepsilon > 0 \end{cases}$$

where $Z_M \equiv Z_M^W$ is defined in (4.22) above. In this way, the analytic curve ζ is an mcc.

Now we reduce the subanalytic set Z_M to an open subanalytic set Z_M^* in such a way that

$$\zeta(\varepsilon) \in Z_M^*, \varepsilon \downarrow 0, \varepsilon \neq 0 \tag{4.29}$$

and such that any analytic curve $\hat{\zeta}$ with

$$\hat{\zeta}(0) = \bar{0} \text{ and } \hat{\zeta}(\varepsilon) \in Z_M^*, \varepsilon \downarrow 0, \varepsilon \neq 0, \tag{4.30}$$

is necessarily an mcc. From theorem 191, the result then follows.

To obtain the second property (every analytic curve with (4.30) is an mcc), the set Z_M^* will be constructed such that parameter values λ in $\mathring{Z}_M$ are omitted, as soon as some of their corresponding zeroes are situated at relatively large positions. To accomplish this goal, we lift the parameter space in the (λ, s)-space, where we would like to define the set K of all values (λ, s), situated inside some 'cone' around the s-axis and for which s is a zero of δ_λ in $[-M, M]$. Those parameter values $\lambda \in \pi(K)$ are omitted from $\mathring{Z}_M$ to form Z_M^*. In this way, if $\lambda \in Z_M^*$ and s is a zero of δ_λ in $[-M, M]$, then (λ, s) must ly outside a 'cone' around the s-axis; as a consequence, if $(\lambda_m)_m$ is a sequence in Z_M^* tending to $\bar{0}$ for $m \to \infty$ and with corresponding zeroes $s_1^m < \ldots < s_n^m$, then there is a subsequence $(m_p)_{p\in\mathbb{N}}$ tending to ∞ for $p \to \infty$, such that the zeroes $s_j^{m_p}$ $(1 \leq j \leq n)$ shrink to 0 if $p \to \infty$.

Property (4.29) is obtained by performing reparametrisations of the germ of the parameter curve ζ in function of each of its corresponding zeroes $\xi_1 < \ldots < \xi_n$; let us denote these reparametrisations by $\zeta^1, \ldots, \zeta^n$ (such a reparametrisation exists by Puiseux expansion, except for at most one zero if this zero is fixed at the origin (i.e. $\xi_i \equiv 0$). The graphs of $\frac{1}{2}\zeta^i$ are then develloped around the s-axis in $\mathbb{R}^{p+1}$, to form a cone C^i around the s-axis. Then the set K is defined as the intersection of the regions outside these cones.

In symbols this construction can be expressed as follows. Since for every $1 \leq i \leq n$, the curve ζ_i is analytic, there are only two possibilities:

$$\xi_i \equiv 0 \tag{4.31}$$

or $\exists 0 < E_i \leq 1$ such that

$$\xi_i(\varepsilon) \neq 0, \forall 0 < \varepsilon < E_i \tag{4.32}$$

Let us define $\mathfrak{I} = \{i : 1 \leq i \leq n \text{ and } \xi_i \not\equiv 0\}$. If $i \in \mathfrak{I}$, then there exist $r_i > 0$, $\nu_i \in \mathbb{N}^*$, $\sigma_i \in \{-1, 1\}$ and an analytic immersion $g_i : I_i \to \mathbb{R}$ where

$$I_i = \{s \in \mathbb{R} : 0 \leq s \leq r_i\}$$

and

$$g_i(s) = s(1 + o(1)), \qquad s \downarrow 0$$

such that

$$\xi_i \circ g_i(s) = \sigma_i s^{\nu_i}, \qquad \forall 0 \leq s \leq r_i \tag{4.33}$$

or, if we denote $S = \min\{r_i^{\nu_i} : i \in \mathfrak{I}\}$, then $\forall i \in \mathfrak{I}$:

$$\xi_i \circ g_i(s^{\frac{1}{\nu_i}}) = \sigma_i s, \qquad \forall 0 \leq s \leq S$$

Then, $\forall i \in \mathfrak{I}$, we define the analytic curve $\zeta^i : [0, S] \to \mathbb{R}^p$, by

$$\zeta^i(s) = \zeta \circ g_i(s^{\frac{1}{\nu_i}}), \qquad \forall 0 \leq s \leq S$$

In this way, the curve ζ^i is a reparametrisation of ζ in function of the i-th zero s (perhaps up to its sign). If $i \in \mathfrak{I}$ (meaning that $\xi_i \equiv 0$), then we define $\zeta^i(s) = s$ and put $\sigma_i = 1, \nu_i = 1$ (after developing this curve around the s-axis in $\mathbb{R}^{p+1}$, we obtain the natural cone). Now, we define the sets K_+ and K_- by the 'cones' around the s-axis:

$$\begin{aligned} K_+ = \{(\lambda, s) \in \mathbb{R}^p \times \mathbb{R} : 0 \le s \le S, \delta_\lambda(s) = 0 \\ \|\lambda\|^2 \le \tfrac{1}{4} \left\|\zeta^i(s)\right\|^2, \forall 1 \le i \le n\Big\} \end{aligned} \tag{4.34}$$

and

$$\begin{aligned} K_- = \{(\lambda, s) \in \mathbb{R}^p \times \mathbb{R} : 0 \le -s \le S, \delta_\lambda(s) = 0 \\ \|\lambda\|^2 \le \tfrac{1}{4} \left\|\zeta^i(-s)\right\|^2, \forall 1 \le i \le n\Big\} \end{aligned} \tag{4.35}$$

Now the set Z_M^* is defined by

$$Z_M^* = \mathring{Z}_M \setminus \left(\pi(K_+) \cup \pi(K_-) \cup \pi(\{(\lambda, s) : \delta_\lambda(s) = 0 \text{ and } s^2 \ge S^2\})\right)$$

where π is the natural projection

$$\pi : \mathbb{R}^p \times \mathbb{R} \to \mathbb{R}^p : (\lambda, s) \mapsto \lambda.$$

Clearly, the set Z_M^* is open. By construction of Z_M^*, and since ζ is an mcc, there exists $E > 0$ such that $\zeta(\varepsilon) \in Z_M^*, \forall 0 < \varepsilon < E$. We are left to prove that the set Z_M^* is subanalytic and that every curve $\hat{\zeta}$ that enters Z_M^* at $\bar{0}$ (i.e. satisfying (4.30)) is necessary an mcc.

Let us start with the last property. Denote the corresponding zeroes by $\hat{\xi}_1(\varepsilon) < \ldots < \hat{\xi}_n(\varepsilon)$. Then we need to prove that there exists a sequence $(\varepsilon_m)_{m\in\mathbb{N}}$ with $\varepsilon_m \downarrow 0$ such that $\forall 1 \le j \le n$:

$$\hat{\xi}_j(\varepsilon_m) \to 0, m \to \infty$$

By the compactness of $[-M, M]$, there exists a sequence $(\varepsilon_m)_{m\in\mathbb{N}}$ with $\varepsilon_m \downarrow 0$ and

$$\forall 1 \le j \le n, \hat{\xi}_j(\varepsilon_m) \to s_j \in [-S, S].$$

We now show that $s_j = 0, \forall 1 \le j \le n$. If $s_j > 0$, then there exists an index N such that $\forall m \ge N$:

$$\hat{\xi}_j(\varepsilon_m) > 0.$$

Since $\hat{\zeta}(\varepsilon_m) \notin \pi(K_+)$, it follows that $\forall m \ge N : \exists 1 \le i(m) \le n$ with

$$\left\|\hat{\zeta}(\varepsilon_m)\right\|^2 \ge \frac{1}{4}\left\|\zeta^{i(m)}\left(\hat{\xi}_j(\varepsilon_m)\right)\right\|^2$$

Since $\{1, 2, \ldots, n\}$ is finite, there is at least one index $1 \le \iota \le n$ that occurs infinitely many times in the sequence $i(m), m \ge N$. Hence, there exists a subsequence $(m_p)_{p\in\mathbb{N}}$ in $\mathbb{N}$ with $m_0 \ge N$ and $m_p \to \infty$ as $p \to \infty$ with $\forall p \in \mathbb{N}$:

$$\left\|\hat{\zeta}(\varepsilon_{m_p})\right\|^2 \ge \frac{1}{4}\left\|\zeta^{\iota}\left(\hat{\xi}_j(\varepsilon_{m_p})\right)\right\|^2$$

Taking the limit for $p \to \infty$, this inequality is reduced to

$$\zeta^{\iota}(s_j) = 0$$

Since g_i is an immersion on I_i, and ζ is the parametrisation of a curve (making no self-intersections), this equality implies that $s_j = 0$. So, our assumption $s_j > 0$ was wrong, implying that $s_j \leq 0$. If $s_j < 0$, we find in an analoguous way that $\exists 1 \leq \iota \leq n$ with

$$\zeta^{\iota}(-s_j) = 0,$$

again implying that $s_j = 0$.

Finally, we prove that the set Z_M^* is subanalytic; it suffices to prove that the sets $\pi(K_+)$ and $\pi(K_-)$ are subanalytic. Therefore, we define $\nu = \nu_1 \cdot \ldots \cdot \nu_n$ and $\forall 1 \leq i \leq n$: the maps

$$\begin{cases} \zeta_{\nu+}^{i}(u) = \zeta^{i}(u^{\nu}), & \forall 0 \leq u \leq S^{\frac{1}{\nu}} \\ \zeta_{\nu-}^{i}(u) = \zeta^{i}(-u^{\nu}), & \forall -S^{\frac{1}{\nu}} \leq u \leq 0 \end{cases}$$

Then the maps $\zeta_{\nu+}^{i}$ and $\zeta_{\nu-}^{i}$ $(1 \leq i \leq n)$ are analytic on respectively $[0, S^{\frac{1}{\nu}}]$ and $[-S^{\frac{1}{\nu}}, 0]$, since, $\forall i \in \mathfrak{I}$:

$$\begin{cases} \zeta_{\nu+}^{i}(u) = \zeta^{i}(u^{\nu}) = \zeta \circ g_i(u^{\frac{\nu}{\nu_i}}), & \forall 0 \leq u \leq S^{\frac{1}{\nu}} \\ \zeta_{\nu-}^{i}(u) = \zeta^{i}(-u^{\nu}) = \zeta \circ g_i(-u^{\frac{\nu}{\nu_i}}), & \forall -S^{\frac{1}{\nu}} \leq u \leq 0 \end{cases}$$

Now we define the sets L_+ and L_- by

$$L_+ = \Big\{(\lambda, u) \in \mathbb{R}^p \times \mathbb{R} : 0 \leq u \leq S^{\frac{1}{\nu}}, \delta_\lambda(u^{\nu}) = 0 \\ \|\lambda\|^2 \leq \tfrac{1}{2}\left\|\zeta_{\nu+}^{i}(u)\right\|^2, \forall 1 \leq i \leq n\Big\},$$

and

$$L_- = \Big\{(\lambda, u) \in \mathbb{R}^p \times \mathbb{R} : -S^{\frac{1}{\nu}} \leq u \leq 0, \delta_\lambda(u^{\nu}) = 0 \\ \|\lambda\|^2 \leq \tfrac{1}{2}\left\|\zeta_{\nu-}^{i}(u^{\nu})\right\|^2, \forall 1 \leq i \leq n\Big\},$$

Clearly, the sets L_+ and L_- are subanalytic; hence, the set Z_M^* is subanalytic since

$$\pi(K_+) = \pi(L_+) \text{ and } \pi(K_-) = \pi(L_-).$$

∎

Notice that in the proof of theorem 201, there is at most one $1 \leq i \leq n$ such that $\xi_i \equiv 0$. Now there are only two possibilies for the set W_M, defined in (4.25) above:

$$W_M^0 \equiv W_M \cap \pi^{-1}(\{\lambda \in W : \delta_\lambda(0) \neq 0\})$$

is empty or not. In case that W_M^0 is empty, then

$$\forall \lambda \in Z_M : \delta_\lambda(0) = 0$$

meaning that one of the n roots of δ_λ is fixed at $0, \forall \lambda \in Z_M$. We also could have deleted the parameters $\lambda \in Z_M$ for which one zero is situated at the origin. Hence, this root surely will not escape through the boundary, and we can replace everywhere n by $n-1$ in the definition of W_M^0, in order to get W_M^0 non-empty. Moreover, we may suppose that $\omega(\varepsilon) \in W_M^0, \forall \varepsilon \in]0,1]$, and continue considering the case in which the original n leads to $W_M \neq \emptyset$, the other case can be treated similarly.

We prefer not to neglect possible zeroes located at 0, because if we do, then the proof cannot be generalised in a straightforward way to the case of a family having a stratum of maximal multiplicity with non-empty interior at λ^0, implying the existence of an algebraic mmc. The reason that we cannot simply neglect the possible zeroes located at 0, is that now we don't only count zeroes, but also the corresponding multiplicities. In particular, it can happen that $s = 0$ appears as zero for δ_λ with different multiplicities for different parameter values λ close to λ^0.

This fact is illustrated by the family of planar vector fields $(X_\lambda)_\lambda$ of type (1.2) with $\lambda = (\lambda_1, \lambda_2, \lambda_3) \in \mathbb{R}^3$ and

$$\delta(s,\lambda) = \lambda_3 \left(s^2+\lambda_1^2\right)\left(s^4+\lambda_2^2\right)\left(\left(s^2-\lambda_2\right)^4+\lambda_1^2+(\lambda_2-\lambda_3)^2\right) \cdot \left((s-\lambda_1)^2+\lambda_2^2+(\lambda_1-\lambda_3)^2\right)$$

Let be $\Gamma = \{(0,0)\}$ and $\lambda^0 = (0,0,0)$. One can easily check that

$$\operatorname{Mult}\left(X_\lambda, \left(\Gamma, \lambda^0\right)\right) = 6$$

For this family, there exist small parameter values for which $s = 0$ is a zero with different multiplicities. Indeed, the curve ζ^1, defined by

$$\zeta^1(\lambda_1) = (\lambda_1, 0, \lambda_1), \lambda_1 \downarrow 0,$$

is a (linear) mmc with corresponding zeroes $\xi_1^1(\lambda_1) = 0$ and $\xi_2^1(\lambda_1) = \lambda_1$, with respective multiplicities 4 and 2; the curve ζ^2, defined by

$$\zeta^2(\lambda_2) = (0, \lambda_2, \lambda_2), \lambda_2 \downarrow 0,$$

also is a (linear) mmc with corresponding zeroes $\xi_1^2(\lambda_1) = 0$ and $\xi_2^2(\lambda_2) = \lambda_2$, with respective multiplicities 2 and 4.

If the family $(X_\lambda)_\lambda$ has a stratum of maximal multiplicity with a non-empty interior at $\lambda^0 = \bar{0}$, then we can take an analytic curve $\omega = (\zeta, \xi_1, \ldots, \xi_n) : I = [0,1] \to \mathbb{R}^p \times [-M, M]^n$ such that

$$\begin{cases} \zeta(0) = \bar{0} \\ \xi_i(0) = 0, \forall 1 \leq i \leq n \\ \delta_{\zeta(\varepsilon)}^{(j)}(\xi_i(\varepsilon)) = 0, \forall \varepsilon, \forall 1 \leq j \leq m_i - 1, \forall 1 \leq i \leq n \\ \delta_{\zeta(\varepsilon)}^{(m_i)}(\xi_i(\varepsilon)) \neq 0 \\ -M < \xi_1(\varepsilon) < \ldots < \xi_n(\varepsilon) < M, \forall \varepsilon > 0 \\ \zeta(\varepsilon) \in \mathring{\vartheta}_M, \forall \varepsilon > 0 \end{cases}$$

where ϑ_M is as defined by (4.26) in section 4.6.2. As such, the analytic curve ζ is an mmc. Analoguously to Z_M^*, we define the set ϑ_M^* by

$$\vartheta_M^* = \mathring{\vartheta}_M \setminus \left(\pi\left(K_+\right) \cup \pi\left(K_-\right) \cup \pi(\{(\lambda, s) : \delta_\lambda\left(s\right) = 0 \text{ and } s^2 \geq S^2\})\right)$$

where $\pi : \mathbb{R}^p \times \mathbb{R} \to \mathbb{R}^p$ is the natural projection and K_+ and K_- are defined in (4.34) and (4.35). Again, in this way, the parameter values λ are omitted from ϑ_M for which some of the corresponding zeroes are situated at relatively large positions. Analoguously to the proof of theorem 201, one shows that

1. the set ϑ_M^* is open and subanalytic,
2. $\zeta\left(\varepsilon\right) \in \vartheta_M^*, \varepsilon \downarrow 0, \varepsilon \neq 0,$
3. every analytic curve $\hat{\zeta}$ with

$$\hat{\zeta}\left(0\right) = \bar{0} \text{ and } \hat{\zeta}\left(\varepsilon\right) \in \vartheta_M^*, \varepsilon \downarrow 0, \varepsilon \neq 0$$

   is necessarily an mmc.

As a consequence, we have the following result:

Theorem 202 *If the family $(X_\lambda)_\lambda$ has an mm-stratum with non-empty interior at λ^0, then there exists an algebraic mmc ζ. Moreover, there exists a positive integer k such that every analytic curve $\hat{\zeta}$ with $j_k(\zeta - \hat{\zeta})_0 = 0$, is an mmc.*

Continuity of the zeroes

Proposition 203 *Suppose that $\gamma : [0,1] \to \mathbb{R}^p$ is an mcc. Denote the zeroes of $\delta_{\gamma(\varepsilon)}$ by*

$$\xi_1\left(\varepsilon\right) < \ldots < \xi_n\left(\varepsilon\right).$$

Then, there exists an $0 < E \leq 1$ such that the curves $\xi_i : [0, E] \to \mathbb{R}^p$ are continuous, $\forall 1 \leq i \leq n$.

Proof. Consider the subanalytic set

$$W_\gamma = W_M \cap \pi^{-1}\left(\gamma\left([0,1]\right)\right),$$

where W_M is the subanalytic set defined in (4.25) and π is the natural projection:

$$\pi : \mathbb{R}^p \times \mathbb{R}^n \to \mathbb{R}^p : (\lambda, s_1, \ldots, s_n) \to \lambda.$$

By the curve selection lemma, there exists an analytic curve

$$\omega : [0,1] \to \mathbb{R}^p \times \mathbb{R}^n : \tau \mapsto \left(\zeta\left(\tau\right), \omega_1\left(\tau\right), \ldots, \omega_n\left(\tau\right)\right)$$

with

$$\forall 0 < \tau \leq 1 : \omega(\tau) \in W_\gamma \text{ and } \omega(0) = \left(\lambda^0, s_0, \ldots, s_0\right). \tag{4.36}$$

From (4.36), it follows that

$$\zeta(\tau) \in \gamma([0,1]), \forall 0 \leq \tau \leq 1. \tag{4.37}$$

By (4.37), the analytic set

$$A = \left\{(\tau, \varepsilon) \in [0,1]^2 : \gamma(\varepsilon) = \zeta(\tau)\right\}$$

accumulates at $(0,0)$ since $\gamma(0) = \zeta(0) = \lambda^0$. The curve selection lemma then ensures the existence of an analytic curve

$$h : [0,1] \to [0,1]^2 : \chi \to (h_1(\chi), h_2(\chi))$$

with $h_1(0) = h_2(0) = 0$ and $h(\chi) \in A, \forall \chi \in [0,1]$. Hence,

$$\forall \chi \in [0,1] : \gamma(h_1(\chi)) = \zeta(h_2(\chi)); \tag{4.38}$$

In particular, it follows that $h_1(\chi) > 0$ if $\chi > 0$. As a consequence, we can write for a certain positive integer r and a constant $a > 0$:

$$\begin{aligned} h_1(\chi) &= a\chi^r + o(\chi^r), \chi \downarrow 0 \\ &= \kappa(\chi)^r \end{aligned}$$

where κ is a local analytic diffeomorphism at $\chi = 0$ with

$$\kappa(\chi) = a^{1/r}\chi \cdot (1 + o(1)), \chi \downarrow 0$$

Hence, it follows from (4.38) that $\exists 0 < E \leq 1$ such that $\forall 0 \leq \varepsilon \leq E$:

$$\gamma(\varepsilon) = \zeta \circ h_2 \circ \kappa^{-1}\left(\varepsilon^{1/r}\right).$$

Therefore, by (4.36), the zeroes of $\delta_{\gamma(\varepsilon)}$ are given by

$$\xi_i(\varepsilon) = \omega_i \circ h_2 \circ \kappa^{-1}\left(\varepsilon^{1/r}\right), \quad i = 1, \ldots, n$$

Since the right-hand side of this equality clearly is continuous in ε, it follows that $\forall 1 \leq i \leq n$: the zero ξ_i depends continuously on $\varepsilon \in [0, E]$. ■

Combining theorem 201 and proposition 203, we get the following corollary:

Corollary 204 *If the family (X_λ) has a stratum of maximal cyclicity with non-empty interior at λ^0, then there exist an analytic curve $\zeta : [0,1] \to \mathbb{R}^p$, a constant $M > 0$ and a positive integer k such that, if $\hat{\zeta} : [0,1] \to \mathbb{R}^p$ is an analytic curve with $j_k(\zeta - \hat{\zeta})_0 = 0$, then*

1. *$\hat{\zeta}$ is an mcc;*

2. *furthermore, for every $0 < M_1 < M$, there exists an $E(M_1) > 0$ such that*

$$\forall 0 < \varepsilon < E(M_1): \ \hat{\zeta}(\varepsilon) \in Z_{M_1}.$$

Clearly, the proofs of proposition 203 and corollary 204 can be modified in a natural way to prove the following analogues:

Proposition 205 *Suppose that $\gamma : [0,1] \to \mathbb{R}^p$ is an mmc. Denote the zeroes of $\delta_{\gamma(\varepsilon)}$ by*

$$\xi_1(\varepsilon) < \ldots < \xi_n(\varepsilon).$$

Then, there exists an $0 < E \leq 1$ such that the curves $\xi_i : [0,E] \to \mathbb{R}^p$ are continuous, $\forall 1 \leq i \leq n$.

Corollary 206 *If the family $(X_\lambda)_\lambda$ has a stratum of maximal multiplicity with non-empty interior at λ^0, then there exist an analytic curve $\zeta : [0,1] \to \mathbb{R}^p$, a constant $M > 0$ and a positive integer k such that, if $\hat{\zeta} : [0,1] \to \mathbb{R}^p$ is an analytic curve with $j_k(\zeta - \hat{\zeta})_0 = 0$, then*

1. *$\hat{\zeta}$ is an mmc;*

2. *furthermore, for every $0 < M_1 < M$, there exists an $E(M_1) > 0$ such that*

$$\forall 0 < \varepsilon < E(M_1): \ \hat{\zeta}(\varepsilon) \in \vartheta_{M_1}.$$

Remark 207 *Notice that propositions 203 and 205 hold in general, not only in case of an mc-stratum (respectively mm-stratum) with non-empty interior at λ^0.*

4.7 Final remarks and open problems

In the previous sections of this chapter we have treated a number of questions concerning the bifurcation diagram of limit cycles, with respect to the stratum of maximal cyclicity (respectively multiplicity). Our main attention was focused on the detection of the structure of this set by means of algebraic 1-parameter subfamilies. In view of making precise calculations in specific problems, the interest to the question grows if we restrict our attention to algebraic p-parameter families of the form (4.3) and if we restrict our attention to the neighbourhood of some a priori chosen closed orbit Γ of X_0. Here we introduce some more interesting notions, related to the results in this chapter, and we state a few more open problems concerning these notions. Consider again example (4.12), where a Hopf-Takens bifurcation occurs. We have seen that there is no linear mcc (or mmc), however it is clear that there exists a quadratic mcc (and mmc). We could now ask whether in general for analytic families there does exist a uniform bound n, inducing an algebraic mcc ζ (respectively mmc) of degree n? The answer also is negative; a counter-example is provided by a simple adaptation of example (4.12) : replace λ_2 by λ_2^k, with $2k > n$. However, under rather generic conditions, there always exists an algebraic mcc ζ (respectively mmc) without (uniform) limitation on the degree, as we have seen in section 4.6.

Definition 208 *The "detectibility degree of maximal cyclicity" (respectively multiplicity) is the minimal degree of an algebraic mcc (respectively mmc), shortly denoted by "ddmc" (respectively "ddmm"). In case no algebraic mcc (respectively mmc) exists, we say that* $ddmc(X_\lambda) = \infty$ *(respectively* $ddmm(X_\lambda) = \infty$*).*

Problem 209 *Does there exist a uniform upperbound for the ddmc (respectively ddmm) depending on the degree N of the algebraic families given in (4.3)?*

In section 4.3.1 we have already observed that $\mathrm{ddmm}(X_\lambda) > 1$ and $\mathrm{ddmc}(X_\lambda) > 1$ in case X_λ linearly depends on λ.

Definition 210 *The "conic degree of maximal cyclicity" (respectively multiplicity) is the minimal value of $n \in \mathbb{N}_1$ such that there exists an mcc γ (respectively mmc) with the property that an analytic curve $\hat{\gamma}$ is an mcc (respectively mmc), if* $j_n(\gamma - \hat{\gamma})_0 = 0$. *It is shortly denoted by cdmc (respectively cdmm).*

Problem 211 *Does there exist a uniform upperbound on the cdmc (respectively cdmm), depending on the degree N of the family (4.3)?*

We have two general theorems that provide a starting point in answering these questions. First we will give an auxiliary lemma, that is a consequence of proposition 196. Roughly speaking, this lemma says that upperbounds for cdmc are preserved by diffeomorphisms that keep the origin on site.

Lemma 212 *Let $h : V \to W$ be a C^ω diffeomorphism with $h(\bar{0}) = \bar{0}$, where V is an open set in $\mathbb{R}^p$ with $\bar{0} \in V$, and $W = h(V) \subset \mathbb{R}^p$. Suppose that there is an algebraic curve γ_W of degree at most n, with $\gamma_W(0) = \bar{0}$, with $\gamma_W(\varepsilon) \in W, \forall \varepsilon \downarrow 0$. Moreover, we suppose that there exists an integer $k \geq n$ such that for every analytic curve $\hat{\gamma}_W$ the following property holds:*

$$\text{if } j_k(\gamma_W - \hat{\gamma}_W)_0 = \bar{0}, \text{ then } \hat{\gamma}_W(\varepsilon) \in W, \forall \varepsilon \downarrow 0.$$

Then there exists an algebraic curve γ_V of degree at most k with $\gamma_V(0) = \bar{0}$ and $\gamma_V(\varepsilon) \in V, \forall \varepsilon \downarrow 0$ such that for every analytic curve $\hat{\gamma}_V$ the following property holds:

$$\text{if } j_k(\gamma_V - \hat{\gamma}_V)_0 = \bar{0}, \text{ then } \hat{\gamma}_V(\varepsilon) \in V, \forall \varepsilon \downarrow 0.$$

Proof. Denote $\rho = h^{-1} \circ \gamma_W$, and define the algebraic curve of degree at most k by

$$\gamma_V = j_k(\rho)_0 . \tag{4.39}$$

Suppose that $\hat{\gamma}_V$ is an analytic curve with

$$j_k\left(\gamma_V - \hat{\gamma}_V\right)_0 = \bar{0}. \tag{4.40}$$

Then we are left to prove that

$$\hat{\gamma}_V\left(\varepsilon\right) \in V, \forall \varepsilon \downarrow 0. \tag{4.41}$$

From (4.39) and (4.40), we also have that

$$j_k\left(\rho - \hat{\gamma}_V\right)_0 = \bar{0}.$$

By proposition 196, it then follows that $j_k\left(h \circ \rho - h \circ \hat{\gamma}_V\right)_0 = \bar{0}$. As a consequence: $h \circ \hat{\gamma}_V\left(\varepsilon\right) \in W, \forall \varepsilon \downarrow 0$, now easily follows from (4.41). ■

Proposition 213 *Suppose that $\left(X_\alpha\right)_\alpha$ is an analytic family of planar vector fields with an associated family of displacement maps of the form:*

$$S^l + \alpha_1 S^{l-1} + \ldots + \alpha_{l-1} S + \alpha_l, \text{ where } \alpha = \left(\alpha_1, \ldots, \alpha_l\right) \tag{4.42}$$

respectively

$$\alpha_0 S^l + \alpha_1 S^{l-1} + \ldots + \alpha_{l-1} S + \alpha_l \text{ where } \alpha = \left(\alpha_0, \alpha_1, \ldots, \alpha_l\right). \tag{4.43}$$

for $S = s - s_0$ and $s^2 = x^2 + y^2, s_0 > 0$. Then,

1. *for $\Gamma = \left\{(x,y) : x^2 + y^2 = s_0^2\right\}$ and $\alpha_0 = \bar{0} \in \mathbb{R}^l$ (respectively $\mathbb{R}^{l+1}$),*

$$\operatorname{Cycl}\left(X_\alpha, \left(\Gamma, \alpha_0\right)\right) = \operatorname{Mult}\left(X_\alpha, \left(\Gamma, \alpha_0\right)\right) = l$$

2. *there exists an algebraic mcc γ of degree smaller than $\left[\frac{l}{2}\right]$, respectively $\left[\frac{l}{2}\right] + 1$, owning the following property: if $k = \left[\frac{l+1}{2}\right]$ and $\hat{\gamma}$ is an analytic curve with $j_k\left(\gamma - \hat{\gamma}\right)_0 = \bar{0}$, then $\hat{\gamma}$ is an mcc. As a consequence,*

   (a)
$$ddmc \leq \left[\frac{l}{2}\right], \text{ respectively } \leq \left[\frac{l}{2}\right] + 1. \tag{4.44}$$

   (b)
$$cdmc \leq \left[\frac{l+1}{2}\right], \text{ respectively } \left[\frac{l+1}{2}\right] + 1. \tag{4.45}$$

3. *statement 2. remains valid if mc is replaced everywhere by mm.*

Proof. We only prove the statements concerning the cyclicity. The statements concerning the multiplicity then easily follow, since in this case an mcc also is an mmc.

Moreover, we only have to prove these statements in case the displacement map looks like (4.42). The other case then follows. Indeed, part (4.44) follows from the following observation: if $\gamma = (\gamma_1, \ldots, \gamma_l)$ is an mcc in case of (4.42), then the analytic curve

$$\varepsilon \mapsto (\varepsilon, \varepsilon\gamma_1(\varepsilon), \ldots, \varepsilon\gamma_l(\varepsilon))$$

is an mcc in case of (4.43). For part (4.45), we notice that, if

$$j_n(\gamma - \hat{\gamma})_0 = \bar{0}, \hat{\gamma}_0(\varepsilon) = \varepsilon + O\left(\varepsilon^{n+1}\right), \varepsilon \to 0,$$

then $j_n\left(\gamma - \hat{\gamma} \circ \hat{\gamma}_0^{-1}\right)_0 = \bar{0}, \hat{\gamma}_0 \circ \gamma_0^{-1}(\varepsilon) = \varepsilon.$

To prove the second statement in case (4.42), we construct an algebraic mcc with the required conical contact, subsequently in the folllowing three cases: $l = 2$, l is even and l is odd.

$\boxed{\text{Case } l = 2}$ In this case the displacement maps are wrtten:

$$\delta(S, a, b) = S^2 + aS + b \tag{4.46}$$

Then, $\forall a \in \left[\frac{1}{4}, 1\right], \forall b < 0$, the curve

$$\gamma : [0, 1] \to \mathbb{R}^2 : \varepsilon \mapsto \varepsilon(a, b)$$

is a linear mcc. Indeed, first of all, it is clear that $\forall \varepsilon > 0$, the map $\delta(\cdot, \gamma(\varepsilon))$ has exactly two zeroes, say $\xi_1(\varepsilon)$ and $\xi_2(\varepsilon)$. Next, when ε tends to 0, these zeroes also tend to 0; it suffices to check this property for a sequence $(\varepsilon_m)_m$ such that the corresponding sequences of zeroes converge, say

$$\xi_1(\varepsilon_m) \to s_1 \text{ and } \xi_2(\varepsilon_m) \to s_2.$$

Since $\xi_1(\varepsilon_m)$ and $\xi_2(\varepsilon_m)$ are zeroes of the quadratic polynomial $\delta(\cdot, \varepsilon_m a, \varepsilon_m b)$, we find by taking the limit for $m \to \infty$:

$$s_1 + s_2 = \lim_{m \to \infty}(-\varepsilon_m a) = 0 \text{ and } s_1 \cdot s_2 = \lim_{m \to \infty}(\varepsilon_m b) = 0$$

As a consequence, $s_1 = s_2 = 0$.

Analoguously, one shows that every analytic curve $\hat{\gamma} : [0, 1] \to \mathbb{R}^2$ with $j_1(\gamma - \hat{\gamma})_0 = 0$ is an mcc. For later use, we also notice that zeroes of $\delta(\cdot, a, b)$ stay outside the interval $\left[-1, \frac{1}{8}\right]$ if $\frac{1}{4} < a < 1, b \leq -\frac{3}{4}$.

$\boxed{\text{Case } l \text{ is even and } l \geq 4}$ Write $l = 2n$, with $n \geq 2$. From the quadratic case, it follows that the displacement map

$$\delta(S, \alpha) = \prod_{k=1}^{n}\left(S^2 + \varepsilon a_k S + \varepsilon b_k\right) = S^{2n} + \alpha_1 S^{2n-1} + \ldots + \alpha_{2n} \tag{4.47}$$

has exactly $2n$ disjoint zeroes, $\forall 0 < \varepsilon \leq 1$ and $\forall k = 1, \ldots, n:$

$$\begin{cases} a_k & \in & I_k = \left]\hat{a}_k - 2^{-k-1}, \hat{a}_k + 2^{-k-1}\right[, & \text{where } \hat{a}_k = \frac{2^k-1}{2^k} \\ b_k & \in & I_{n+k} = \left]\hat{b}_k - \frac{1}{4}, \hat{b}_k + \frac{1}{4}\right[, & \text{where } \hat{b}_k = -k \end{cases} \tag{4.48}$$

Denote $I = \prod_{i=1}^{2n} I_k$, and $\hat{a} = (\hat{a}_1, \ldots, \hat{a}_n), \hat{b} = (\hat{b}_1, \ldots, \hat{b}_n)$. From equation (4.47), we get an analytic map

$$g = (g_1, \ldots, g_{2n}) : \mathbb{R}^{2n+1} \to \mathbb{R}^{2n} : (a, b, \varepsilon) \mapsto (\alpha_1, \ldots, \alpha_{2n}),$$

where $(a, b) = (a_1, \ldots, a_n, b_1, \ldots, b_n)$, and $\forall k = 1, \ldots, n:$

$$\begin{cases} g_1(a, b, \varepsilon) & = & \varepsilon \sum_{i=1}^{n} a_i \\ g_{2k}(a, b, \varepsilon) & = & \varepsilon^k \sum_{1 \leq i_1 < \ldots < i_k \leq n} b_{i_1} \ldots b_{i_k} + O\left(\varepsilon^{k+1}\right), \varepsilon \to 0 \\ g_{2k-1}(a, b, \varepsilon) & = & \varepsilon^{k+1} \sum_{i=1}^{n} a_i \sum_{\substack{1 \leq i_1 < \ldots < i_k \leq n \\ i_j \neq i}} b_{i_1} \ldots b_{i_k} \\ & & +O\left(\varepsilon^{k+2}\right), \varepsilon \to 0 \end{cases} \tag{4.49}$$

Remark that, for a parameter $\alpha = g(a, b, \varepsilon)$, the map $\delta(\cdot, \alpha)$ has exactly $2n$ disjoint zeroes, if $(a, b) \in I, 0 < \varepsilon \leq 1$; moreover, these zeroes tend to 0, if $\varepsilon \downarrow 0$. In particular, the analytic curve

$$\gamma : [0, 1] \to \mathbb{R}^{2n} : \varepsilon \mapsto g(\hat{a}, \hat{b}, \varepsilon)$$

is an algebraic mcc of degree n. Now we prove that every analytic curve $\hat{\gamma}$ is an mcc if

$$j_n(\gamma - \hat{\gamma})_0 = \bar{0}. \tag{4.50}$$

Equivalently, we need to prove that there exists an $E_0 > 0$ such that

$$\forall 0 \leq \varepsilon < E_0 : \hat{\gamma}(\varepsilon) \in g(I \times [0, 1]).$$

Therefore, it suffices to prove the existence of an open neighbourhood V of $(\hat{a}, \hat{b}, 0)$ such that $g(V)$ is an open neighbourhood of $g(\hat{a}, \hat{b}, 0)$. To accomplish this goal, we would like to apply the Inverse Function Theorem on the map g at $(\hat{a}, \hat{b}, 0)$.

We cannot apply the Inverse Function Theorem straight on, since g is singular at $(\hat{a}, \hat{b}, 0)$. However, after blowing up of the parameter space, this map g is transformed into a regular map at $(\hat{a}, \hat{b}, 0)$. Let the blow-up map be defined by

$$\Phi\left(\underline{\alpha}_1, \ldots, \underline{\alpha}_j, \ldots \underline{\alpha}_{2n}, \varepsilon\right) = \left(\varepsilon\underline{\alpha}_1, \ldots, \varepsilon^{\left[\frac{j+1}{2}\right]}\underline{\alpha}_j, \ldots, \varepsilon^n \underline{\alpha}_{2n}, \varepsilon\right)$$

Then, the blow-down map is given by its inverse: for $\varepsilon \neq 0$,

$$\Phi^{-1}\left(\alpha_1, \ldots \alpha_j, \ldots, \alpha_{2n}, \varepsilon\right) = \left(\varepsilon^{-1}\alpha_1, \ldots, \varepsilon^{-\left[\frac{j+1}{2}\right]}\alpha_j, \ldots, \varepsilon^{-n}\alpha_{2n}, \varepsilon\right)$$

We define the analytic map $h = (h_1, \ldots, h_{2n+1}) : \mathbb{R}^{2n+1} \to \mathbb{R}^{2n+1}$ by

$$h\left(a, b, \varepsilon\right) = \Phi^{-1}\left(g\left(a, b, \varepsilon\right), \varepsilon\right)$$

In appendix B, it is proven by induction on n that $Dh_{(\hat{a},\hat{b},0)}$ is non-singular; more precisely:

$$\det Dh_{(\hat{a},\hat{b},0)} = (-1)^n (2!)^2 (3!)^2 \ldots ((n-1)!)^2 \neq 0.$$

From the Inverse Function Theorem, it follows that there exists an open neighbourhood V of $(\hat{a}, \hat{b}, 0)$ in $I \times]-1, 1[\subset \mathbb{R}^{2n+1}$ such that

$$h : V \to h(V)$$

is an analytic diffeomorphism. As a consequence, $h(V)$ is a neighbourhood of $(\underline{\gamma}(0), 0)$, and therefore there exist positive constants $C_j > 0$ $(1 \leq j \leq 2n)$, and $E_0 > 0$, such that

$$\prod_{j=1}^{2n} \left]\underline{\gamma}_j(0) - C_j, \underline{\gamma}_j(0) + C_j\right[\times]-E_0, E_0[\subset h(V) \tag{4.51}$$

Let $\hat{\gamma}$ be an analytic curve with a contact of order n with γ, i.e. $\hat{\gamma}$ satisfies (4.50), then we define the analytic curves $\underline{\gamma}$ and $\underline{\hat{\gamma}}$ such that

$$\left(\underline{\gamma}(\varepsilon), \varepsilon\right) = \Phi^{-1}(\gamma(\varepsilon), \varepsilon) \text{ and } \left(\underline{\hat{\gamma}}(\varepsilon), \varepsilon\right) = \Phi^{-1}(\hat{\gamma}(\varepsilon), \varepsilon) \tag{4.52}$$

The contact between γ and $\hat{\gamma}$ at $\varepsilon = 0$ defined in (4.50) implies the following asymptotics for the components of $\underline{\gamma}$ and $\underline{\hat{\gamma}}$ at $\varepsilon = 0$:

$$\forall j = 1, \ldots, 2n : \underline{\hat{\gamma}}_j(\varepsilon) = \underline{\gamma}_j(\varepsilon) + O(\varepsilon^{n-\left[\frac{j+1}{2}\right]+1}), \qquad \varepsilon \downarrow 0.$$

Therefore, $\forall j = 1, \ldots, 2n$, there exist positive constants $M_j > 0, 0 < E_j < 1$ such that

$$\left|\underline{\hat{\gamma}}_j(\varepsilon) - \underline{\gamma}_j(\varepsilon)\right| \leq M_j \left|\varepsilon\right|^{n-\left[\frac{j+1}{2}\right]+1}, \quad \forall \left|\varepsilon\right| < E_j \tag{4.53}$$

Define $E > 0$ by

$$E = \min\left\{E_0, E_j, \frac{C_j}{M_j} : 1 \leq j \leq 2n\right\},$$

then from (4.51) and (4.53), it follows that

$$\forall 0 \leq \varepsilon < E : \left(\underline{\hat{\gamma}}(\varepsilon), \varepsilon\right) \in h(V).$$

Now, the required property follows, since from (4.52), one has

$$\hat{\gamma}(\varepsilon) \in g(V) \subset g(I \times [0, 1]), \forall 0 \leq \varepsilon < E.$$

$\boxed{\text{Case } l \text{ is odd and } l \geq 3}$ Write $l = 2n+1$ for $n \geq 1$. From the quadratic case, it follows that the displacement map

$$\delta(S,\alpha) = \prod_{k=1}^{n} \left(S^2 + \varepsilon a_k S + b_k \varepsilon\right)(S + \varepsilon c) = S^{2n+1} + \alpha_1 S^{2n} + \ldots + \alpha_{2n+1} \quad (4.54)$$

has exactly $2n+1$ disjoint zeroes,$\forall 0 < \varepsilon \leq 1$ where $a_k \in I_k, b_k \in I_{k+n}$ (cfr. (4.48)), and

$$c \in \left]-\tfrac{1}{2}, \tfrac{1}{2}\right[, \quad \hat{c} = 0 .$$

Identification of equal coefficients in (4.54) defines an analytic map

$$\bar{g} = (\bar{g}_1, \ldots, \bar{g}_{2n+1}) : \mathbb{R}^{2n+2} \to \mathbb{R}^{2n+1} : (a,b,c,\varepsilon) \mapsto (\alpha_1, \ldots, \alpha_{2n+1}),$$

where $(a,b,c,\varepsilon) = (a_1, \ldots, a_n, b_1, \ldots, b_n, c, \varepsilon)$. Using the expressions in (4.49), we find $\forall k = 1, \ldots, n:$

$$\begin{cases} \bar{g}_1(a,b,c,\varepsilon) &= g_1(a,b,\varepsilon) \\ \bar{g}_{2k}(a,b,c,\varepsilon) &= g_{2k}(a,b,\varepsilon) + O\left(\varepsilon^{k+1}\right), \varepsilon \to 0 \\ \bar{g}_{2k+1}(a,b,c,\varepsilon) &= \varepsilon^{k+1}(g_{2k+1}(a,b,\varepsilon) + c \displaystyle\sum_{1 \leq i_1 < \ldots < i_k \leq n} b_{i_1} \ldots b_{i_k}) \\ & \quad + O\left(\varepsilon^{k+2}\right), \varepsilon \to 0 \end{cases}$$

Now, we proceed as in the even case. In order to make $\bar{g}$ regular at $(\hat{a}, \hat{b}, \hat{c}, 0)$, we first define the blow up map $\Phi : \mathbb{R}^{2n+2} \to \mathbb{R}^{2n+2}$ by

$$\Phi\left(\underline{\alpha}_1, \ldots, \underline{\alpha}_j, \ldots \underline{\alpha}_{2n+1}, \varepsilon\right) = \left(\varepsilon \underline{\alpha}_1, \ldots, \varepsilon^{\left[\frac{j+1}{2}\right]} \underline{\alpha}_j, \ldots, \varepsilon^{n+1} \underline{\alpha}_{2n+1}, \varepsilon\right)$$

Then we can define the analytic map

$$\bar{h} = \left(\bar{h}_1, \ldots, \bar{h}_{2n+2}\right) : \mathbb{R}^{2n+2} \to \mathbb{R}^{2n+2}$$

by

$$\bar{h}(a,b,c,\varepsilon) = \Phi^{-1}\left(\bar{g}(a,b,c,\varepsilon), \varepsilon\right)$$

It is easy to see that

$$\det D\bar{h}_{\left(\hat{a},\hat{b},0,0\right)} = (-1)^n n! \det Dh_{\left(\hat{a},\hat{b},0\right)} \neq 0.$$

Hence, $D\bar{h}_{\left(\hat{a},\hat{b},0,0\right)}$ is non-singular; the sequel of the proof is similar to the one in the even case. ■

Theorem 214 *Suppose we are given an analytic family (X_λ) of planar vector fields such that the Bautin ideal $\mathcal{I}$ is regular and suppose that there is a minimal set of generators $\{\varphi_1, \ldots, \varphi_l\}$ for $\mathcal{I}$ with a local division of the displacement map:*

$$\delta(s,\lambda) = \sum_{i=1}^{l} \varphi_i(\lambda) h_i(s,\lambda)$$

for analytic functions h_i such that $\forall 1 \leq i \leq l$:

$$h_i(s,\lambda) = c_i(\lambda)(s-s_0)^{n+i-1} + o\left((s-s_0)^{n+i-1}\right), s \to s_0$$

with $c_i(\bar{0}) \neq 0, \forall 1 \leq i \leq l, n \in \mathbb{N}$. Then we have:

1. *there exists a linear mic*
2. $\operatorname{Cycl}(X_\lambda, (\Gamma, \bar{0})) = l - 1$
3. *there exists an algebraic mcc and mmc of degree $\left[\frac{l+2}{2}\right]$; as a consequence,*

$$ddmc, ddmm \leq \left[\frac{l+2}{2}\right]$$

Moreover,

$$cdmc, cdmm \leq \left[\frac{l+2}{2}\right]$$

Proof. Statements 1. and 2. are clear. Hence, we are left to prove statement 3. From example 175, we know that the given family (X_λ) has a stratum of maximal cyclicity (multiplicity) with non-empty interior at λ^0.

Without loss of generality, we can also assume that $n = 0$ and $s_0 = 0$. By the regularity of the Bautin ideal, there exists a local C^ω diffeomorphism $\mathcal{H} : (\mathbb{R}^p, \bar{0}) \to (\mathbb{R}^p, \bar{0})$ at $\bar{\lambda} = \bar{0} \in \mathbb{R}^p$ such that the map $\varphi \equiv (\varphi_1, \ldots, \varphi_l)$ takes the form

$$\varphi \circ \mathcal{H}(\bar{\lambda}_1, \ldots, \bar{\lambda}_p) = (\bar{\lambda}_1, \ldots, \bar{\lambda}_l)$$

(proposition 44). By lemma 212, it suffices to prove statement 3. in the new coordinates $\bar{\lambda}$. In these coordinates $\bar{\lambda} = (\bar{\lambda}_1, \ldots, \bar{\lambda}_p)$, we denote the family of displacement maps by $\bar{\delta}$, and factors by $\bar{h}_i, 1 \leq i \leq l$:

$$\bar{\delta}(s,\bar{\lambda}) = \delta(s, \mathcal{H}(\bar{\lambda})) = \sum_{i=1}^{l} \bar{\lambda}_i \bar{h}_i(s,\bar{\lambda}),$$
$$\bar{h}_i(s,\bar{\lambda}) = h_i(s, \mathcal{H}(\bar{\lambda})), \quad \forall 1 \leq i \leq l$$

By lemma 212, it suffices to prove statement 3. in the new coordinates $\bar{\lambda}$. By Rolle's theorem, to have at least l zeroes, it is necessary that this curve lies in a region of parameter space with $\bar{\lambda}_l = \bar{\zeta}_l(\varepsilon)$ small but non-zero (for $\varepsilon \neq 0$). For $\bar{\lambda}_l \neq 0$, we can write

$$\begin{aligned}\bar{\delta}(s,\bar{\lambda}) &= \bar{\lambda}_l \sum_{i=1}^{l} \frac{\bar{\lambda}_i}{\bar{\lambda}_l} \bar{h}_i(s,\bar{\lambda}) \\ &= \bar{\lambda}_l F\left(s, \frac{\bar{\lambda}_1}{\bar{\lambda}_l}, \ldots, \frac{\bar{\lambda}_{l-1}}{\bar{\lambda}_l}, \bar{\lambda}_l, \ldots, \bar{\lambda}_p\right), \end{aligned} \tag{4.55}$$

where F is the analytic map defined by

$$F(s,\mu)=\sum_{i=1}^{l-1}\mu_i\bar{h}_i(s,\mu_l\mu_1,\ldots,\mu_l\mu_{l-1},\mu_l,\ldots,\mu_p)+\bar{h}_l(s,\mu_l\mu_1,\ldots,\mu_l\mu_{l-1},\mu_l,\ldots,\mu_p),$$

where $\mu=(\mu_1,\ldots,\mu_p)\in(\mathbb{R}^p,\bar{0})$. In fact, μ is the parameter obtained after the following blow up of $\bar{\lambda}$:

$$\forall 1\le i\le l-1:\bar{\lambda}_i=\bar{\lambda}_l\mu_i \text{ and } \forall l\le i\le p:\bar{\lambda}_i=\mu_i \tag{4.56}$$

For $\mu=\bar{0}\in\mathbb{R}^p$, we find a real constant $c\neq 0$ such that

$$F(s,\bar{0})=H_l(s)=cs^{l-1}+o\left(s^{l-1}\right),s\to 0$$

Hence, by the Preparation Theorem (theorem 8), there exist analytic maps Q,Ψ,a with

$$\begin{cases} Q:(\mathbb{R}\times\mathbb{R}^p,(0,\bar{0}))\to\mathbb{R} \text{ with } Q(S,\mu)\neq 0,\\ \Psi:(\mathbb{R}\times\mathbb{R}^p,(0,\bar{0}))\to\mathbb{R} \text{ with } \Psi(s,\mu)=s+o(s),s\to 0\\ a=(a_1,\ldots,a_{l-1}):(\mathbb{R}^p,\bar{0})\to\left(\mathbb{R}^{l-1},\bar{0}\right)\end{cases}$$

such that, up to a non-zero factor, F is a polynomial:

$$\begin{cases} F(s,\mu)=Q(s,\mu)\cdot V(\Psi(s,\mu),a(\mu))\\ \text{where } V(S,a_1,\ldots,a_{l-1})=a_1+a_2S+\ldots+a_{l-1}S^{l-2}+S^{l-1}.\end{cases} \tag{4.57}$$

By identification of coefficients corresponding to equal powers of S in (4.57), we derive the following relations between $(a_1,\ldots,a_{l-1})$ and $\mu=(\mu_1,\ldots,\mu_p)$:

$$\begin{cases} a_1(\mu) &=& d_1^1\mu_1\left(1+f_1^1(\mu_l\mu_1,\ldots,\mu_l\mu_{l-1},\mu_l,\ldots\mu_p)\right)\\ & \vdots & \\ a_{l-1}(\mu) &=& \sum_{j=1}^{l-1}d_{l-j}^{l-1}\mu_{l-j}\left(1+f_{l-j}^{l-1}(\mu_l\mu_1,\ldots,\mu_l\mu_{l-1},\mu_l,\ldots,\mu_p)\right)\end{cases} \tag{4.58}$$

where $d_j^i\in\mathbb{R},d_i^i\neq 0$, and $f_j^i:(\mathbb{R}^p,0)\to\mathbb{R}$ are analytic functions with

$$f_j^i(\bar{0})=0,\forall i=1,\ldots,l-1,\forall 1\le j\le i.$$

The formulas in (4.58) define a local diffeomorphism at $\bar{0}\in\mathbb{R}^p$:

$$(\mathbb{R}^p,\bar{0})\to(\mathbb{R}^p,\bar{0}):(\mu_1,\ldots,\mu_p)\mapsto(a_1(\mu),\ldots,a_{l-1}(\mu),\mu_l,\ldots,\mu_p)$$

By setting, for $\mu=(\mu_1,\ldots,,\mu_p)$,

$$b_i(\mu)=\mu_l a_i(\mu),\forall 1\le i\le l-1 \text{ and } b_l(\mu)=\mu_l \tag{4.59}$$

and substitution of (4.57) in (4.55), we can rewrite the map $\bar{\delta}$, for $\mu=(\mu_1,\ldots,,\mu_p)\in(\mathbb{R}^p,\bar{0})$ and $S\in(\mathbb{R},0)$, as

$$\begin{aligned}\delta(s,\mu_l\mu_1,\ldots,\mu_l\mu_{l-1},\mu_l,\ldots,\mu_p)&=\mu_l\cdot Q(s,\mu)\cdot V(\Psi(s,\mu),a(\mu))\\ &=Q(s,\mu)\cdot P(\Psi(s,\mu),b(\mu))\end{aligned}$$

where $P(S,b) = b_1 + b_2 S + \ldots + b_{l-1} S^{l-2} + b_l S^{l-1}$. From (4.59), we see that zeroes s of $\bar{\delta}(\cdot,\bar{\lambda})$ correspond to zeroes S of $P(\cdot,b)$.

By (4.58), (4.59) and (4.56), we have the following relations between $\bar{\lambda} = (\bar{\lambda}_1,\ldots,\bar{\lambda}_p)$ and $b = (b_1,\ldots,b_l)$:

$$\begin{cases} b_1 &= d_1^1 \bar{\lambda}_1 \left(1 + f_1^1(\bar{\lambda})\right) \\ & \vdots \\ b_{l-1} &= \sum_{j=1}^{l-1} d_{l-j}^{l-1} \bar{\lambda}_{l-j} \left(1 + f_{l-j}^{l-1}(\bar{\lambda})\right) \\ b_l &= \bar{\lambda}_l \end{cases}$$

These relations clearly define a local diffeomorphism

$$h : (\mathbb{R}^p, \bar{0}) \to (\mathbb{R}^p, \bar{0}) : \bar{\lambda} \mapsto \left(b(\bar{\lambda}), \bar{\lambda}_{l+1}, \ldots, \bar{\lambda}_p\right).$$

Hence, from theorem 213 and lemma 212, statement 3. is proven. ■

These problems constitute a good starting point in studying the relation between the structure of the bifurcation set of limit cycles, more precisely the structure of the stratum of maximal cyclicity (respectively multiplicity) and the algebraic nature of the perturbation in p-parameter perturbations from a center.

Chapter 5

2-saddle cycle

This chapter calls attention for the transfer of results obtained by 'linearisation' (first order Melnikov function) in the study of unfoldings of a 2-saddle cycle, breaking only one connection. Bifurcations from a Hamiltonian 2-saddle cycle can produce limit cycles, so-called *'alien limit cycles'*, that are not covered by the 'linearisation', even under certain generic conditions. One can already find an example of this phenomenon in paper [DR], on which the study in this chapter is based.

5.1 Introduction

Throughout this chapter, we deal with C^∞ families of planar vector fields $\left(X_{(\nu,\varepsilon)}\right)_{(\nu,\varepsilon)}$ such that $X_{(\nu,0)}$ is of center type, $\forall \nu \in W$ where W is an open and bounded set in $\mathbb{R}^p$. By proposition 19, we can suppose that the vector fields $X_{(\nu,0)}$ are Hamiltonian; more precisely, there exist C^∞ functions $H : U \times W_0 \subset \mathbb{R}^2 \times \mathbb{R}^p \to \mathbb{R}$, $f, g : U \times W_0 \times]-E, E[\subset \mathbb{R}^2 \times \mathbb{R}^p \to \mathbb{R}$ $(E > 0)$ such that

$$X_{(\nu,\varepsilon)} \leftrightarrow \begin{cases} \dot{x} &= -\frac{\partial H}{\partial y}(x,y) + \varepsilon f(x,y,\nu,\varepsilon) \\ \dot{y} &= \frac{\partial H}{\partial x}(x,y) + \varepsilon g(x,y,\nu,\varepsilon) \end{cases} \tag{5.1}$$

We denote the Hamiltonian vector field by $X_H \equiv X_{(\nu,0)}$ and the family of dual 1-forms of $\left(X_{(\nu,\varepsilon)}\right)_{(\nu,\varepsilon)}$ by $\left(\upsilon_{(\nu,\varepsilon)}\right)_{(\nu,\varepsilon)}$:

$$\upsilon_{(\nu,\varepsilon)} = \mathrm{d}H + \varepsilon\bar{\upsilon}_\nu + o(\varepsilon), \varepsilon \to 0. \tag{5.2}$$

In particular, we suppose that the non-isolated periodic sets of X_H have a limit periodic set Γ of X_H as outer boundary of the period annulus. If Γ lies on the level curve $\{H = h_0\}$, then we denote by Γ_x the periodic orbit of X_H, that belongs to the level curve $\{H = h_0 + x\}$, and we consider the Abelian integral

$$I_\nu(x) = \int_{\Gamma_x} \bar{\upsilon}_\nu. \tag{5.3}$$

For $x > 0$, the Abelian integral I_ν is the first order Melnikov function (perhaps up to its sign). Remark that here, we use the terminology '*Abelian integral*' for any integral I_ν associated to a C^∞ unfolding $\left(X_{(\nu,\varepsilon)}\right)_{(\nu,\varepsilon)}$; originally, this name was reserved for integrals of type (5.3) where H and $\bar{v}_\nu$ are polynomials.

Recall that the results in paper [DR] are concerned with the question 'To what extent the Abelian integral I_ν allows to study the limit cycles which bifurcate from Γ'.

In case that Γ is a periodic orbit, then it follows from catastrophy theory, that results on the zero-set of I_ν can be transferred in a straightforward way to results on the set of limit cycles of $X_{(\nu,\varepsilon)}$ (section 1.2.2).

The transfer is not that evident, in case that Γ is a bounded hyperbolic polycycle, also-called a k-saddle cycle, i.e. Γ is a compact connected curve, made by k hyperbolic saddle points, say $s_1, \ldots, s_k$, and saddle connections of X_H (the eigenvalues at the linear part of X_H at the saddle points s_i $(1 \leq i \leq k)$ have a non-zero real part). In case $k = 1$, then Γ is also called a saddle loop or saddle connection.

However, if Γ is a saddle loop, it is proven in [Mar] using the methods of [R86], that the configuration of the limit cycles of $X_{(\nu,\varepsilon)}$, for ε near 0, is completely analoguous to the configuration of zeroes of the Abelian integral $I_\nu(x)$, for x near 0, under certain genericity conditions on I_ν. One could say that the limit cycles shadow the zeroes of the Abelian integral I_ν. These results are recalled in section 1.5.

In [DR], using results from [DRR] and [Mo], it is seen that it cannot be the same as soon as the number of the saddles in the polycycle is at least 2, if the unfolding breaks more than one connection. In case of generic unfoldings of a 2-saddle cycle, breaking both connections in Γ in the bifurcation for $\varepsilon = 0$, the Abelian integral is a very bad approximation of the displacement map. The Abelian integral has only one zero while there are at least two limit cycles bifurcating from Γ (section 1.6.3). More generally, for generic k-parameter unfoldings of a k-saddle cycle [Mo], breaking more than one connection, the associated Abelian integral has exactly one zero, although there are more than k limit cycles that bifurcate from Γ. As a conclusion, we can say that almost all limit cycles are alien limit cycles, since almost none of the limit cycles can be traced by zeroes of the Abelian integral.

The important point to observe is that, although the bifurcation diagram for the zeroes of I is stable, the bifurcation diagram for the limit cycles is no longer trivial in the ε-direction: it is not a trivial product of the one for I times $[0, \varepsilon_0[$, as it is the case for the generic unfoldings of a regular Hamiltonian periodic orbit or a saddle loop.

In [DR], one finds a way to handle the transfer, although the Abelian integral cannot completely control the limit cycles bifurcating from a 2-saddle cycle Γ, that breaks only one connection. To be more precise, it is shown in [DR], that the cyclicity of $(X_\lambda)_\lambda$ along Γ can be bounded in terms of the Abelian integral, if the codimension of the related Abelian integral is finite. This result is recalled in section 1.6.4. It is striking to notice that this upperbound is strictly bigger than the maximal possible number of zeroes of the Abelian integral, as soon as its codimension is at least 3.

In particular, in [DR], it is proven that there exist generic unfoldings of the 2-

saddle cycle, leaving one connection unbroken, that produce 4 limit cycles bifurcating from Γ, while the related Abelian integral can have at most 3 zeroes. Hence, there is at least one alien limit cycle.

Throughout this chapter, Γ is supposed to be a 2-saddle cycle, and only one connection gets broken at the bifurcation for $\varepsilon = 0$ (figure 5.1).

Figure 5.1: One connection of the 2-saddle cycle Γ remains unbroken by the unfolding

In section 5.2, we investigate the question whether the upperbound given by the finite cyclicity result is optimal. From the example in [DR], it is seen that the upperbound is optimal in case the codimension of the Abelian integral is 3. This finite cyclicity result is obtained by a division-derivation algorithm, applied to an asymptotic expansion of Δ, derived in [DR]. We show here that the coefficients, that appear in this asymptotic expansion of Δ, are not independent. As a consequence, we can conclude that the upperbound is not sharp, as soon as the codimension of the Abelian integral is at least 4.

In section 5.3, we consider a particular subfamily of the unfolding of the 2-saddle cycle Γ, leaving one connection unbroken, in which the normal forms at the saddle points stay linear. Such a subfamily can produce at least $2k-2$ limit cycles, while the related Abelian integral has at most k zeroes. Hence, if $k \geq 2$, then already in this particular subfamily, some limit cycles are not related to zeroes of the Abelian integral. From this result, we can also conjecture that the cyclicity of the generic $(3k-1)$-parameter unfolding is at least $3k-1$, while the codimension of the related Abelian integral is just $2k$ (implying the existence of at least $k-1$ alien limit cycles).

5.2 Difference map

Let us recall again that, throughout this chapter, Γ is supposed to be a 2-saddle cycle, and in the unfolding we keep one connection unbroken (i.e. the breaking parameter associated to one connection identically vanishes). To study limit cycles near a 2-saddle cycle, it is convenient to replace the displacement map δ by the difference map Δ. As in case of the displacement map δ, isolated zeroes of Δ correspond to limit cycles of X_λ, and the linear approximation of Δ with respect to $\varepsilon = 0$, coincides with the Abelian integral; moreover, $\Delta|_{\varepsilon=0} \equiv 0$, and hence, one can write

$$\Delta(x,\nu,\varepsilon) = \varepsilon\bar{\Delta}(x,\nu,\varepsilon);$$

in this way, $\bar{\Delta}(x,\nu,0) = I_{\nu}(x)$.

In section 1.6.4, we recalled two important results from [DR]: theorem 129 recalls the asymptotic expansion for the reduced difference map $\bar{\Delta}$ in the asymptotic scale $\mathcal{W}$ (defined in (1.135)), that permits to perform a division-derivation algorithm with respect to this scale, in order to obtain the finite cyclicity result in theorem 134.

In this finite cyclicity result, it is striking to notice that the upperbound stated for the cyclicity is strictly bigger than the codimension of the Abelian integral I_{ν} (if $\text{codim}_{\mathcal{L}} I_{\nu} \geq 3$), and hence strictly bigger than the maximal possible number of zeroes of the Abelian integral.

In [DR], one defines a generic unfolding of the 2-saddle cycle, breaking only one connection, by putting certain genericity conditions on the coefficients that appear in the normal forms and the related Abelian integral I_{ν}, in case the codimension of the related Abelian integral is 3. By these genericity conditions, one obtains a full unfolding of the difference map $\bar{\Delta}$ of codimension 4 in the scale $\mathcal{W}$, meaning that the first 4 coefficients in this asymptotic expansion of $\bar{\Delta}$, can be seen as independent small parameter variables, and the 5-th coefficient is non-zero. By the division-derivation algorithm, the cyclicity is bounded from above by 4. Now, by use of the implicit function theorem, one proves in [DR] the existence of a quadruple zero (a multiple zero of order 4) x_0, along a path $\zeta(x_0)$ in parameter space with $\zeta(x_0) \to 0$ for $x_0 \to 0$:

$$\frac{\partial^i}{\partial x^i}\bar{\Delta}(x_0,\zeta(x_0)) = 0, \forall 0 \leq i \leq 3.$$

By the genericity conditions and catastrophy theory, this result implies the existence of generic unfoldings of the 2-saddle cycle, breaking only one connection, with cyclicity 4, while the Abelian integral has at most 3 zeroes [DR].

Recall that for $k \in \mathbb{N}$, we say that the difference map $\bar{\Delta}$ admits a generic unfolding of codimension k in the asymptotic scale $\mathcal{W}$, if, at the bifurcation value, the $(k+1)$-th coefficient is the first non-zero coefficient in the expansion of $\bar{\Delta}$, and the first k coefficients appearing in the expansion of $\bar{\Delta}$ with respect to $\mathcal{W}$ can be seen as independent parameter variables. In this case, one expects that the upperbound for the maximal possible zeroes, bifurcating from $x=0$, obtained by the division-derivation algorithm (i.e. k) is optimal. In section 5.2.2, we prove that the coefficients in the asymptotic expansion of $\bar{\Delta}$ with respect to the scale $\mathcal{W}$, can not be seen as independent parameter variables (proposition 223 below). As a consequence, the upperbound in the above mentioned finite cyclicity result is not sharp enough.

An interesting question is whether the gap between the cyclicity and the maximal possible zeroes of the related Abelian integral persists; in other words, whether there exist systematically alien limit cycles, as in the example of codimension 4 in [DR], where there is at least one alien limit cycle.

Proposition 223, implies that if the codimension of the related Abelian integral is $2k$, the map made by all coefficients in the expansion of $\bar{\Delta}$ up to order $O(x^{k+1}|\omega|^{k+1})$, has rank $3k$ (corollary 224).

From this result, one could conjecture that there exist unfoldings X_λ with at least $3k-1$ limit cycles, bifurcating from the 2-saddle cycle Γ, of which there are at most $2k$ limit cycles covered by zeroes of the related Abelian integral.

To prove this conjecture, we need to find a new useful asymptotic expansion of $\bar{\Delta}$, by regrouping of the terms in the scale $\mathcal{W}$. In section 5.3, we propose a new asymptotic expansion $\bar{\Delta}$ for the particular subfamily of X_λ, leaving the saddles linear at the bifurcation, by the introduction of new compensators. For this subfamily, under certain genericity conditions, one finds that the cyclicity is $2k-1$, while the Abelian integral has at most k zeroes.

Let us finish this discussion by pointing out the differences between the saddle loop and the 2-saddle cycle, in case only one connection gets broken. Because in the study of the 2-saddle cycle only one connection gets broken, it has geometrically a strong resemblance to the saddle loop. Therefore, one could expect to have similar results as in the saddle loop case. However, this is not the case.

The first striking point to notice is that by the occurence of two saddles, in the asymptotic expansion of $\bar{\Delta}$, there are two different compensators that degenerate in the same way for $\varepsilon \to 0$, unlike in the saddle loop case, where we only have to deal with one compensator. Nevertheless, this is not the essential point, since in [DR], one introduces new compensators ω_{2-1} and ω_{21} to make one of the compensators disappearing to higher order terms or at least to hide them behind the principal parts in the asymptotic scale.

The essential point is the following. The Abelian integral expands in the logarithmic scale $\mathcal{L}$, hence the building terms contain only linear terms in $\log x$ (proposition 92). In case of the saddle loop, one can give an expansion of the displacement map $\bar{\delta}$ in a simple asymptotic scale deformation, that for $\varepsilon = 0$, coincides with the one for the Abelian integral. Hence, only the linear terms in $\log x$ play a role, the non-linear terms in $\log x$ are hided behind the principal parts of the building terms in the asymptotic expansion of $\bar{\delta}$. This is not be the case for the 2-saddle cycle. The non-linear terms in $\log x$ also appear as principal parts of the building terms in the expansion of $\bar{\Delta}$. In this way, the map $\bar{\Delta}$ must have an expansion in a larger scale, with non-linear terms in $\log x$. By the gaps that appear in the expansion of $\bar{\Delta}$ for $\varepsilon = 0$, the Abelian integral $I_\nu = \bar{\Delta}\big|_{\varepsilon=0}$ cannot control all limit cycles of X_λ. There are alien limit cycles; as is explained in [DR], these limit cycles correspond to zeroes of the difference map $\bar{\Delta}$, that escape from the domain of validity of the related Abelian integral I_ν. These limit cycles are situated at a distance too close to the 2-saddle cycle.

As a conclusion, we can say that, to study the precise cyclicity near the 2-saddle cycle, in unfoldings in which just one connection gets broken, linear approximation (i.e. the Abelian integral I_ν) of the difference map, with respect to ε, is not sufficient, unlike it was the case for the saddle loop.

Finally, let us remark that the quadratic approximation of the difference map Δ, with respect to ε, was sufficient for unfoldings X_λ of codimension 4 (corresponding to codimension 3 for the related Abelian integral). This fact becomes clear from the proof of the result in [DR], proving the existence of generic unfoldings of a 2-saddle cycle, that leave one connection unbroken, with cyclicity 4. From the formulas in section 5.2.2 for the coefficients in the expansion of $\bar{\Delta}$, it seems that one can study the precise cyclicity along a generic 2-saddle cycle Γ, breaking only one connection, by the $(k+1)$-th order approximation of the difference map Δ, with respect to ε, in case $\operatorname{codim}_{\mathcal{L}} I_\nu = 2k$ or $2k+1$.

5.2.1 Settings

Suppose that the ratio of hyperbolicity at s_1 of X_λ (respectively s_2 of $-X_\lambda$) is given by $1+\varepsilon\alpha^{(1)}$ (respectively $1+\varepsilon\alpha^{(2)}$). Then, near the saddles s_1 and s_2, we can use normalizing coordinates (introduced in [R86]), denoted respectively by (x,y) and (z,w), such that the vector fields X_λ respectively $-X_\lambda$ in these coordinates read as:

$$\left\{\begin{array}{rcl} \dot{y} & = & -y \\ \dot{x} & = & x(1+\varepsilon\alpha^{(1)}(\nu,\varepsilon)+\varepsilon g_1(x,y,\nu,\varepsilon)) \end{array}\right. \tag{5.4}$$

respectively

$$\left\{\begin{array}{rcl} \dot{w} & = & -w \\ \dot{z} & = & z(1+\varepsilon\alpha^{(2)}(\nu,\varepsilon)+\varepsilon g_2(z,w,\nu,\varepsilon)) \end{array}\right. \tag{5.5}$$

for certain C^∞ functions $g_1, g_2 : \left(\mathbb{R}^2\times\mathbb{R}^{p-1}\times\mathbb{R}, \left(0,\nu^0,0\right)\right)\to\mathbb{R}$ such that

$$\left\{\begin{array}{l} j_\infty(g_1(\cdot,\cdot,\nu,\varepsilon))_{(0,0)}(x,y)=\sum_{i=1}^\infty B_i^{(1)}(\nu,\varepsilon)(xy)^i, \\ j_\infty(g_2(\cdot,\cdot,\nu,\varepsilon))_{(0,0)}(z,w)=\sum_{i=1}^\infty B_i^{(2)}(\nu,\varepsilon)(zw), \end{array}\right.$$

for certain C^∞ functions $B_i^{(j)} : \left(\mathbb{R}^{p-1}\times\mathbb{R},\nu^0,0\right)\to\mathbb{R}, i\geq 1, j=1,2.$

In figure 5.2, we represent some transverse sections C_1, C_2, C_3, C_4 corresponding to respectively $\{y=1\}, \{x=1\}, \{w=1\}$ and $\{z=1\}$ in the normalizing coordinates; $\{w=0\}$ is a point on a local stable separatrix of s_2 and $\{x=0\}$ is a point on the unbroken connection.

The difference map Δ is defined as the composition of the transitions D_1 and R_1 (respectively D_2 and R_2) defined by the flow of X_λ (respectively $-X_\lambda$) as follows. The transition map D_1 is the Dulac map at the saddle point s_1 from C_1 to C_2, and the transition map R_1 denotes the regular transition from C_2 to C_4. The transition map D_2 is the Dulac map at the saddle point s_2 from C_3 to C_4, and the map R_2 denotes the regular transition from C_1 to C_3.

Let Δ_1 (respectively Δ_2) be the transition map from C_1 to C_4, defined by the flow of X_λ (respectively $-X_\lambda$), then

$$\Delta_1 = R_1\circ D_1, \Delta_2 = D_2\circ R_2 \text{ and } \Delta = \Delta_2 - \Delta_1.$$

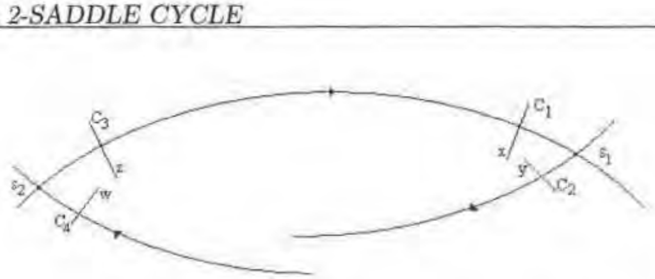

Figure 5.2: Transversal sections C_1, C_2, C_3, C_4 to the unfolding of the 2-saddle cycle Γ, that leaves one connection of Γ unbroken

In the expansion of the Dulac maps D_1 and D_2 respectively, we have to consider the compensators ω_1 and ω_2 associated to $-X_\lambda$ and X_λ, respectively at the saddle points s_1 and s_2 :

$$\omega_1 = \omega_1(x,\nu,\varepsilon) = \frac{x^{\varepsilon\alpha^{(1)}} - 1}{\varepsilon\alpha^{(1)}} \text{ and } \omega_2 = \omega_2(z,\nu,\varepsilon) = \frac{z^{\varepsilon\alpha^{(2)}} - 1}{\varepsilon\alpha^{(2)}}. \tag{5.6}$$

Performing a (parameter dependent) similarity in the coordinates (z, w), we can accomplish that

$$\frac{\partial R_2}{\partial x}(0,\nu,\varepsilon) = 1, \text{ i.e. } R_2(x,\nu,\varepsilon) = x + O\left(x^2\right), x \to 0. \tag{5.7}$$

On the other hand, we can write:

$$R_1(y,\nu,\varepsilon) = \varepsilon u_0 + (1+\varepsilon u)\, y(1+O(y)), y \to 0 \tag{5.8}$$

In accordance to (5.7) and (5.8), we can write expansions for the regular transitions R_1 and R_2 :

$$\begin{array}{l} R_1(y,\nu,\varepsilon) = -\varepsilon\beta + (1+\varepsilon u)\, y\,(1+\varepsilon f_1(y,\nu,\varepsilon)) \\ R_2(x,\nu,\varepsilon) = x\,(1+\varepsilon f_2(x,\nu,\varepsilon)) \end{array} \tag{5.9}$$

where $f_1, f_2 : \left(\mathbb{R}\times\mathbb{R}^{p-1}\times\mathbb{R}, \left(0,\nu^0,0\right)\right) \to \mathbb{R}$ are C^∞ functions. Let us denote

$$\begin{array}{l} j_\infty\left(f_1(\cdot,\nu,\varepsilon)\right)_0(y) = \sum_{i=1}^{\infty} \eta_i^{(1)}(\nu,\varepsilon)\, y^i \\ j_\infty\left(f_2(\cdot,\nu,\varepsilon)\right)_0(x) = \sum_{i=1}^{\infty} \eta_i^{(2)}(\nu,\varepsilon)\, x^i \end{array}$$

for certain C^∞ functions $\eta_i^{(j)} : \left(\mathbb{R}^{p-1}\times\mathbb{R}, \left(\nu^0,0\right)\right) \to \mathbb{R}, i \geq 1, j = 1,2$. From now on, every coefficient we introduce, such as $u, u_0, \alpha^{(1)}, \alpha^{(2)}, \ldots$ are supposed to be smooth functions of the parameter (ν,ε). For $\varepsilon = 0$, the vector field is Hamiltonian, and the Hamiltonian function H is equal to xy and zw respectively in the normalizing coordinates near the saddle points s_1 and s_2. It follows that $\Delta_i - Id$ $(i = 1,2)$ and Δ are divisible by ε; hence, we can write

$$\Delta_i(x,\nu,\varepsilon) = x + \varepsilon\bar{\Delta}_i(x,\nu,\varepsilon), \qquad \forall i = 1,2$$

and

$$\Delta(x,\nu,\varepsilon) = \varepsilon\bar{\Delta}(x,\nu,\varepsilon) = \varepsilon\left(\bar{\Delta}_2(x,\nu,\varepsilon) - \bar{\Delta}_1(x,\nu,\varepsilon)\right)$$

The map $\bar{\Delta}$ is called the reduced difference map.

5.2.2 Coefficients in the asymptotic expansion of $\bar{\Delta}$

In section 1.6.4, we recalled the useful asymptotic expansion for $\bar{\Delta}$, derived in [DR]. By the end of this section, we find recursion formulas for the coefficients in that expansion for $\bar{\Delta}$, in terms of the coefficients arising in the normal forms at the saddle points and the regular transitions. It turns out that there exist relations in between coefficients in this expansion, therefore we cannot use these coefficients as new independent parameter variables. However, they constitute a map of rank $3k$; from this corollary, one can conjecture that there are at least k limit cycles that are not covered by zeroes of the Abelian integral.

The organising of this section is as follows. We start by exploiting the structure of the coefficients in the expansion of the Dulac map, depending on the coefficients in the normal form at the saddle point. Next, we derive recursion formulas for the coefficients in the expansion of $\bar{\Delta}_1$ and $\bar{\Delta}_2$. Finally, by taking the difference, we obtain formulas for the coefficients that appear in the expansion of $\bar{\Delta}$ (recalling relation (1.132), to make one of the compensators to disappear behind the principal parts).

Expansion of the Dulac map

Let us here exploit the coefficients in the expansion of the Dulac map D at the saddle point s (with ratio of hyperbolicity $1+\varepsilon\alpha$) of a planar vector field of type

$$\begin{cases} \dot{y} &= -y \\ \dot{x} &= x(1+\varepsilon\alpha+\varepsilon g(x,y)) \end{cases} \tag{5.10}$$

for a certain function $g:\mathbb{R}^2\to\mathbb{R}$ with

$$j_\infty(g)_{(0,0)}(x,y) = \sum_{i=1}^{\infty} B_i (xy)^i$$

Starting at $(1,y_0)$ in the section $\{x=1\}$ at $t=0$, the orbit arrives at the point $(D(y_0),1)$ in the section $\{y=1\}$ after some time $t=T$, see figure 5.3.

From (5.10), it follows that $T=\log y_0$. To calculate $D(y_0)$, one performs the singular transformation

$$\begin{cases} v &= xy \\ y &= y \end{cases}$$

In these coordinates (v,y) the system (5.10) reads

$$\begin{cases} \dot{v} &= \varepsilon v(\alpha+\bar{g}(v)) \\ \dot{y} &= -y \end{cases}, \tag{5.11}$$

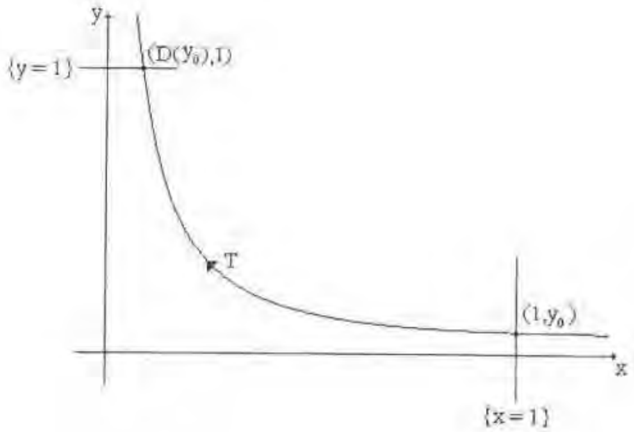

Figure 5.3: Dulac map D

for a certain function $\bar{g} : \mathbb{R} \to \mathbb{R}$ that is C^∞ for $x > 0$ sufficiently small with $j_\infty(\bar{g})_0(v) = \sum_{i=1}^{\infty} B_i v^i$. In this way, the time dependent equation for v in (5.11) is independent of y. If we denote the solution of this equation, with $v(0,x) = x$, by $v(t,x)$, then the Dulac map is given by

$$D(x) = v(\log x, x)$$

since $x = x_0 y_0 = y_0$ and $v(T,x) = x(T)\,y(T) = D(x)$. Therefore, we solve the $\dot{v}$-equation by substitution of the power series in terms of x for v :

$$j_\infty(v(t,\cdot))_0(x) = \sum_{i=0}^{\infty} g_i(t)\,x^i \tag{5.12}$$

Let us introduce the C^∞ function Ω, that coincides with the compensator $\omega_{\varepsilon\alpha}$, for $t = \log x$:

$$\Omega(t, \varepsilon\alpha) = \frac{\mathrm{e}^{\varepsilon\alpha t} - 1}{\varepsilon\alpha}.$$

Let us define the functions $\bar{g}_i, i \geq 1$ by

$$\bar{g}_i(\Omega(t, \varepsilon\alpha)) = g_i(t).$$

Since $\Omega(\log x, \varepsilon\alpha) = \frac{x^{\varepsilon\alpha} - 1}{\varepsilon\alpha} = \omega_{\varepsilon\alpha}$, we have

$$j_\infty(D)_0(x) = \sum_{i=1}^{\infty} \bar{g}_i(\omega_{\varepsilon\alpha})\,x^i \tag{5.13}$$

Proposition 215 *In the notations introduced above, one has*

$$\bar{g}_1(\Omega) = \varepsilon\alpha\Omega + 1 \tag{5.14}$$

and $\forall i \geq 2,$

$$\begin{aligned} \bar{g}_i(\Omega) &= \varepsilon(\varepsilon\alpha\Omega+1)\cdot\left(B_{i-1}\Omega + B_{i-1}\sum_{s=2}^{i-1}\frac{1}{s}\binom{i-2}{s-1}(\varepsilon\alpha)^{s-1}\Omega^s\right) \\ &+ (\varepsilon\alpha\Omega+1)\sum_{s=2}^{i-1} H_{i,s}(B_1,\ldots,B_{i-2})(\varepsilon\Omega)^s \qquad . \end{aligned} \tag{5.15}$$

where $H_{i,s}$ *are multi-variate polynomials of degree* s *in* $(B_1,\ldots,B_{i-2}), \forall 2 \leq s \leq i-1, \forall i \geq 2.$

Proof. Clearly, $g_0(t) \equiv 0, g_1(0) = 1, g_i(0) = 0, \forall i \geq 2$, and the coefficients $g_i, i \geq 1$ satisfy linear differential equations of the form

$$\left\{ \begin{array}{lcl} \frac{d}{dt}g_1 &=& \varepsilon\alpha g_1 \\ \frac{d}{dt}g_{i+1} &=& \varepsilon\alpha g_{i+1} + \varepsilon Q_{i+1}(g_1,g_2,\ldots,g_i) \end{array} \right. \tag{5.16}$$

where Q_{i+1} takes the form

$$Q_{i+1}(g_1,g_2,\ldots,g_i) = \sum_{k=2}^{i+1} B_k \sum_{\substack{i_1+\ldots+i_k=i+1 \\ 1\leq i_s\leq i}} \alpha_{i_1\ldots i_k} g_{i_1}\cdot\ldots\cdot g_{i_k}$$

for certain positive rational numbers $\alpha_{i_1\ldots i_k}$. From (5.16), It follows that

$$g_1(t) = e^{\varepsilon\alpha t}, \tag{5.17}$$

which in turn implies (5.14). From (5.16), we find the following linear differential equations to be satisfied by the coefficients $\bar{g}_i, i \geq 2$:

$$(\varepsilon\alpha\Omega+1)\frac{d}{d\Omega}\bar{g}_i \quad = \quad \varepsilon\alpha\bar{g}_i + \varepsilon Q_i(\bar{g}_1,\bar{g}_2,\ldots,\bar{g}_{i-1}) \tag{5.18}$$

From (5.18) and (5.14), one can deduce that the coefficients $g_i, i \geq 2$ take the form as proposed in (5.15). ■

As a consequence of proposition 215 and relation (5.13), one derives an expansion for the Dulac map D :

Corollary 216 *The Dulac map* D *at the saddle* s *of* (5.10) *is up to terms of order*

$O\left(x^{k+1}\omega^{k+1}\right)$ *given by*

$$\begin{aligned}
D(x) &= (\varepsilon\alpha\omega+1)\,x \\
&+ \varepsilon\omega\,(\varepsilon\alpha\omega+1)\,B_1x^2 \\
&+ \varepsilon\omega\,(\varepsilon\alpha\omega+1)\,(B_2+((B_1)^2+\frac{1}{2}B_2\alpha)\varepsilon\omega)x^3+\ldots \\
&+ (\varepsilon\alpha\omega+1)\cdot\left[(\varepsilon B_{k-1}\omega+\varepsilon B_{k-1}\sum_{i=2}^{k-1}\frac{1}{i}\binom{k-2}{i-1}(\varepsilon\alpha)^{i-1}\omega^i)\right. \\
&\left.+\sum_{i=2}^{k-1}H_{ki}\,(B_1,\ldots,B_{k-2})\,(\varepsilon\omega)^i\right] \\
&+ O\left(x^{k+1}\omega^{k+1}\right), x\to 0
\end{aligned} \tag{5.19}$$

where $\omega=\omega_{\varepsilon\alpha}$ *and* $H_{i,s}$ *are multi-variate polynomials of degree* s *in* $(B_1,\ldots,B_{i-2})$, $\forall 2\le s\le i-1, \forall i\ge 2.$

In particular, one has the following explicit expansion of the Dulac map at the saddle point s up to order $O\left(x^4\omega^4\right), x\to 0$:

$$\begin{aligned}
D(x) &= \varepsilon\alpha x\omega + x \\
&+ \varepsilon^2\alpha B_1x^2\omega^2+\varepsilon B_1x^2\omega \\
&+ \varepsilon^3\alpha[(B_1)^2+\frac{1}{2}\alpha B_2]x^3\omega^3+\varepsilon^2[\frac{3}{2}\alpha B_2+(B_1)^2]x^3\omega^2+\varepsilon B_2x^3\omega \\
&+ O\left(x^4\omega^4\right), x\to 0.
\end{aligned}$$

Notice that the coefficient corresponding to $x\omega$ in the expansion of D is the trace of the linear part of $X_{(\nu,\varepsilon)}$ at s and B_1 is the first saddle quantity of $\frac{1}{\varepsilon}X_{(\nu,\varepsilon)}$ at s.

Expansion of $\bar{\Delta}_1$

Recall that $\Delta_1 = R_1\circ D_1$ and $\Delta_1=\varepsilon\bar{\Delta}_1$. Let us denote the coefficients in the expansion of $\bar{\Delta}_1$ by $G^{(1)}_{i,i-j}$, $0\le j\le i, \forall i\in\mathbb{N}$, then

$$\bar{\Delta}_1(x)=\sum_{i=0}^{k}\sum_{j=0}^{i}G^{(1)}_{i,i-j}x^i\omega_1^{i-j}+O\left(x^{k+1}\omega_1^{k+1}\right), x\to 0. \tag{5.20}$$

An expansion of $\bar{\Delta}_1$, up to terms $O\left(x^2\omega_1^2\right)$, is easily computed by combining (5.9) and (5.19) :

$$\begin{aligned}
\Delta_1(x) &= -\varepsilon\beta+(1+\varepsilon u)\,D_1(x)+O\left(x^2\omega_1^2\right), x\to 0 \\
&= -\varepsilon\beta+(1+\varepsilon u)\left(\varepsilon\alpha^{(1)}\omega_1+1\right)x+O\left(x^2\omega_1^2\right), x\to 0 \\
&= -\varepsilon\beta+(1+\varepsilon u)\,\varepsilon\alpha^{(1)}x\omega_1+(1+\varepsilon u)\,x+O\left(x^2\omega_1^2\right), x\to 0
\end{aligned}$$

Let us now compute the coefficients $G_{k,k-j}^{(1)}$ of $x^k\omega_1^{k-j}$ in $\bar{\Delta}_1, 0 \le j \le k, \forall k \ge 2$, under the assumption that $\eta_j^{(1)} = 0, \forall 1 \le j \le k-2$.

Proposition 217 *Let $k \ge 2$. If $\eta_j^{(1)} = 0, \forall 1 \le j \le k-2$, then the coefficients $G_{k,k-j}^{(1)}, 0 \le j \le k$, in the expansion of $\bar{\Delta}_1$ are given by*

$$G_{k,k}^{(1)} = (1+\varepsilon u) \cdot \left[\left(\varepsilon\alpha^{(1)}\right)^k \eta_{k-1}^{(1)} + \frac{1}{k-1} \left(\varepsilon\alpha^{(1)}\right)^{k-1} B_{k-1}^{(1)} \right.$$
$$\left. + \varepsilon^{k-1}\alpha^{(1)} H_{k,k-1}(B_1^{(1)}, \ldots, B_{k-2}^{(1)}) \right],$$

and $\forall 1 \le j \le k-3$,

$$G_{k,k-j}^{(1)} = (1+\varepsilon u) \cdot$$
$$\left[\left(\varepsilon\alpha^{(1)}\right)^{k-j} \binom{k}{k-j} \eta_{k-1}^{(1)} + \left(\varepsilon\alpha^{(1)}\right)^{k-j-1} \frac{1}{k-1} \binom{k}{k-j} B_{k-1}^{(1)} \right.$$
$$\left. + \varepsilon^{k-j-1} \left[H_{k,k-j}(B_1^{(1)}, \ldots, B_{k-2}^{(1)}) + \alpha^{(1)} H_{k,k-j-1}(B_1^{(1)}, \ldots, B_{k-2}^{(1)}) \right] \right],$$

and

$$G_{k,2}^{(1)} = (1+\varepsilon u) \left[\left(\varepsilon\alpha^{(1)}\right)^2 \binom{k}{2} \eta_{k-1}^{(1)} + \frac{k}{2} \varepsilon\alpha^{(1)} B_{k-1}^{(1)} + \varepsilon H_{k,2}(B_1^{(1)}, \ldots, B_{k-2}^{(1)}) \right],$$
$$G_{k,1}^{(1)} = (1+\varepsilon u) \left[B_{k-1}^{(1)} + \varepsilon\alpha^{(1)} k \eta_{k-1}^{(1)} \right],$$
$$G_{k,0}^{(1)} = (1+\varepsilon u)\, \eta_{k-1}^{(1)}.$$

Proof. If $\eta_j^{(1)} = 0, \forall 1 \le j \le k-2$, then the expansion of the regular transition R_1 in (5.9) reduces to

$$R_1(y) = -\varepsilon\beta + (1+\varepsilon u)\, y + \varepsilon\,(1+\varepsilon u)\,(\eta_{k-1}^{(1)} y^k + O(y^{k+1})), y \to 0$$

Hence,

$$\Delta_1(x) = -\varepsilon\beta + (1+\varepsilon u)\, D_1(x) + \varepsilon\,(1+\varepsilon u)\,(\eta_{k-1}^{(1)} (D_1(x))^k + O(x^{k+1}\omega_1^{k+1})), x \to 0$$

Only the terms $(1+\varepsilon u)\, D_1(x)$ and $\varepsilon\,(1+\varepsilon u)\, \eta_{k-1}^{(1)} (D_1(x))^k$ have a contribution to the coefficient according to $x^k\omega_1^j$ in Δ_1. The coefficient of $x^k\omega_1^j$ in $D_1(x)$ can be found in the k-th order term in x in $D_1(x)$, and the coefficient of $x^k\omega_1^j, 0 \le j \le k$ in $(D_1(x))^k$ in the first order term in x in $D_1(x)$ (see (5.19)). ∎

Expansion of $\bar{\Delta}_2$

Recall that $\Delta_2 = D_2 \circ R_2$ and $\Delta_2 = \varepsilon\bar{\Delta}_2$. Let us denote the coefficients in the expansion of $\bar{\Delta}_2$ by $G^{(2)}_{i,i-j}$, $0 \leq j \leq i, \forall i \in \mathbb{N}$, then

$$\bar{\Delta}_2(x) = \sum_{i=0}^{k}\sum_{j=0}^{i} G^{(2)}_{i,i-j} x^i \omega_1^{i-j} + O\left(x^{k+1}\omega_1^{k+1}\right), x \to 0. \tag{5.21}$$

When substituting $z = R_2(x)$ in $D_2(z)$, we encounter the composition $\omega_2 \circ R_2$. Therefore, we start with the following proposition, that states an expansion for $\omega_2 \circ R_2$, in terms of x^i and $x^i\omega_2$, and whose proof uses the ideas of lemma 111.

Proposition 218 *In the notations introduced above, there exists a C^∞ function $\Psi : \left(\mathbb{R} \times \mathbb{R}^p \times \mathbb{R}, \left(0, \nu^0, 0\right)\right) \to \mathbb{R}$ such that*

$$\omega_2 \circ R_2(x) = \left[1 + \varepsilon\alpha^{(2)}\Psi(x,\nu,\varepsilon)\right] \cdot \omega_2(x) + \Psi(x,\nu,\varepsilon)$$

Moreover, if $j_\infty\left(\Psi(\cdot,\nu,\varepsilon)\right)_0(x,\nu,\varepsilon) = \sum_{i=1}^{\infty}\mu_i(\nu,\varepsilon)x^i$, for certain C^∞ functions $\mu_i : \left(\mathbb{R}^p \times \mathbb{R}, \left(\nu^0, 0\right)\right) \to \mathbb{R}$, then

$$\mu_1 = \eta_1^{(2)} \text{ and } \forall i \geq 2 : \mu_i = \eta_i^{(2)} \operatorname{mod}(\eta_1^{(2)}, \ldots, \eta_{i-1}^{(2)}) \tag{5.22}$$

Proof. By definition of the compensator ω_2 and (5.9), we can write

$$\begin{aligned}
\omega_2 \circ R_2(x) &= \frac{\left(R_2(x)\right)^{\varepsilon\alpha^{(2)}} - 1}{\varepsilon\alpha^{(2)}} \\
&= \frac{x^{\varepsilon\alpha^{(2)}}\left(1 + \varepsilon f_2(x,\nu,\varepsilon)\right)^{\varepsilon\alpha^{(2)}} - 1}{\varepsilon\alpha^{(2)}} \\
&= \left[1 + \varepsilon a^{(2)}\Psi(x,\nu,\varepsilon)\right]\omega_2 + \Psi(x,\nu,\varepsilon)
\end{aligned}$$

where

$$\Psi(x,\nu,\varepsilon) = \begin{cases} \dfrac{\left(1 + \varepsilon f_2(x,\nu,\varepsilon)\right)^{\varepsilon\alpha^{(2)}} - 1}{\varepsilon\alpha^{(2)}} & \text{if } \varepsilon\alpha^{(2)} \neq 0 \\ \log\left(1 + \varepsilon f_2(x,\nu,\varepsilon)\right) & \text{if } \varepsilon\alpha^{(2)} = 0 \end{cases}$$

By Taylor expansion of $(1+X)^s$ at $X = 0$, we find the formulas for the coefficients of Ψ as proposed in (5.22). ■

Now, with the help of proposition 218, an expansion of $\bar{\Delta}_2$, up to terms $O\left(x^2\omega_2^2\right)$, is easily computed by combining (5.9) and (5.19) :

$$\begin{aligned}
\Delta_2(x) &= D_2(x(1 + \varepsilon f_2(x,\eta^{(2)}))) \\
&= x + \varepsilon\alpha^{(2)}x\left[(1 + O(x))\omega_2 + (\eta_1^{(2)}x + O\left(x^2\right))\right] + O\left(x^2\omega_2^2\right), x \to 0 \\
&= \varepsilon\alpha^{(2)}x\omega_2 + x + O\left(x^2\omega_2^2\right), x \to 0
\end{aligned}$$

Let us now compute the coefficients of $x^k\omega_2^{k-j}$ in $\bar{\Delta}_2, 0 \le j \le k, \forall k \ge 2$, under the assumption that $\eta_j^{(2)} = 0, \forall 1 \le j \le k-2$.

Proposition 219 *Let $k \ge 2$. If $\eta_j^{(2)} = 0, \forall 1 \le j \le k-2$, then the coefficients $G_{k,k-j}^{(2)}, 0 \le j \le k$, in the expansion* (5.21) *of $\bar{\Delta}_2$, are given by*

$$G_{k,k}^{(2)} = \frac{1}{k-1}\left(\varepsilon\alpha^{(2)}\right)^{k-1} B_{k-1}^{(2)} + \varepsilon^{k-1}\alpha^{(2)} H_{k,k-1}(B_1^{(2)}, \ldots, B_{k-2}^{(2)}),$$

and $\forall 1 \le j \le k-3$:

$$\begin{aligned} G_{k,k-j}^{(2)} &= \left(\varepsilon\alpha^{(2)}\right)^{k-j-1} \frac{1}{k-1}\binom{k}{k-j} B_{k-1}^{(2)} \\ &\quad + \varepsilon^{k-j-1}\left[H_{k,k-j}(B_1^{(2)}, \ldots, B_{k-2}^{(2)}) + \alpha^{(2)} H_{k,k-j-1}(B_1^{(2)}, \ldots, B_{k-2}^{(2)})\right], \end{aligned}$$

and

$$\begin{aligned} G_{k,2}^{(2)} &= \frac{k}{2}\varepsilon\alpha^{(2)} B_{k-1}^{(2)} + \varepsilon H_{k,2}(B_1^{(2)}, \ldots, B_{k-2}^{(2)}), \\ G_{k,1}^{(2)} &= B_{k-1}^{(2)} + \varepsilon\alpha^{(2)}(1 + \varepsilon\alpha^{(2)})\eta_{k-1}^{(2)}, \\ G_{k,0}^{(2)} &= \eta_{k-1}^{(2)}. \end{aligned}$$

Proof. If $\eta_j^{(2)} = 0, \forall 1 \le j \le k-2$, then the expansion of the regular transition R_2 in (5.9) reduces to

$$R_2(x) = x + \varepsilon(\eta_{k-1}^{(2)} x^k + O(x^{k+1})), x \to 0$$

Hence, for $x \to 0$:

$$\begin{aligned} \Delta_2(x) &= (\varepsilon\alpha^{(2)}(\omega_2 \circ R_2) + 1)R_2 \\ &\quad + (\varepsilon\alpha^{(2)}(\omega_2 \circ R_2) + 1)(\varepsilon B_1^{(2)}(\omega_2 \circ R_2) + \ldots)R_2^2 \\ &\quad + \ldots \\ &\quad + \left(\varepsilon\alpha^{(2)}(\omega_2 \circ R_2) + 1\right) \cdot \left[(\varepsilon B_{k-1}\omega + \varepsilon B_{k-1}\sum_{i=2}^{k-1}\frac{1}{i}\binom{k-2}{i-1}(\varepsilon\alpha)^{i-1}\omega^i)\right. \\ &\quad \left. + \sum_{i=2}^{k-1} H_{ki}(B_1, \ldots, B_{k-2})(\varepsilon\omega)^i\right] R_2^k \\ &\quad + O\left(x^{k+1}\omega_2^{k+1}\right) \end{aligned} \tag{5.23}$$

Only the terms $\ldots \cdot R_2$ and $\ldots \cdot R_2^k$ in the right-hand side of this equation for Δ_2 have a contribution to coefficients according to $x^k\omega_1^j, 0 \le j \le k$ in Δ_2, since for $s \ge 2$:

$$(R_2(x))^s = x^s + O\left(x^{s+k}\right), x \to 0$$

and

$$\omega_2 \circ R_2(x) = \left(1 + O\left(x^{k-1}\right)\right)\omega_2 + O\left(x^{k-1}\right), x \to 0. \tag{5.24}$$

The term $\ldots \cdot R_2$ in (5.23) has only a contribution to the coefficients in Δ_2 corresponding to $x^k\omega_2$ and x^k; more explicitly, its contribution to the coefficient of $x^k\omega_2$ in $\bar{\Delta}_2$ is given by

$$\varepsilon\alpha^{(2)}(1 + \varepsilon\alpha^{(2)})\eta_{k-1}^{(2)},$$

and its contribution to the coefficient of x^k in $\bar{\Delta}_2$ is given by

$$\eta_{k-1}^{(2)}.$$

By (5.24), the contribution of the term $\ldots \cdot R_2^k$ in (5.23) to the coefficients of $x^k\omega_2^{k-j}$ is similar to the contribution of D_1^k to Δ_1. ∎

Expansion for $\bar{\Delta}$

Let us now consider the expansion of the reduced difference map $\bar{\Delta}$:

$$\bar{\Delta} = \sum_{i=0}^{k}\sum_{j=0}^{k} G_{i,i-j}F_{i,i-j} + O(x^{k+1}|\omega|^{k+1}), x \to 0, \tag{5.25}$$

where the functions $F_{k,k-j}, 0 \le j \le k$ are defined in (1.135) and $|\omega| = \max\{|\omega_1|, |\omega_2|\}$, and $k \in \mathbb{N}$. In [DR], one already has an explicit expansion of $\bar{\Delta}$ up to terms of order $O\left(x^2|\omega|^2\right), x \to 0$. Therefore, we already have explicit expressions for $G_{k,k-j}, 0 \le j \le k, 0 \le k \le 2$, that we summarize in the following proposition:

Proposition 220 *[DR]The coefficients $G_{k,k-j}, 0 \le j \le k, 0 \le k \le 2$, in the expansion (5.25) of $\bar{\Delta}$ are given by*

$$\begin{aligned} G_{00} &= \beta \\ G_{11} &= \alpha^{(2)} - \alpha^{(1)} \\ G_{10} &= -u \end{aligned}$$

$$\begin{aligned} G_{22} &= \varepsilon(B_1^{(2)}\alpha^{(2)} - B_1^{(1)}\alpha^{(1)}) - \varepsilon^2\alpha^{(1)}(\alpha^{(1)}\eta_1^{(1)} + uB_1^{(1)}) - \varepsilon^3\left(\alpha^{(1)}\right)^2 u\eta_1^{(1)} \\ G_{21} &= (B_1^{(2)} - B_1^{(1)}) + \varepsilon(\alpha^{(2)}\eta_1^{(2)} - 2\eta_1^{(1)}\alpha^{(1)} - uB_1^{(1)}) - 2\varepsilon^2\alpha^{(1)}u\eta_1^{(1)} \\ G_{20} &= (\eta_1^{(2)} - \eta_1^{(1)}) - \varepsilon u\eta_1^{(1)} \end{aligned}$$

Continuing the calculations in a straightforward way, we also find explicit expressions for the coefficients $G_{3,3-j}, 0 \le j \le i$, that are obtained by direct calculations:

Proposition 221 *The coefficients $G_{3,3-j}, 0 \leq j \leq 3$ in the expansion* (5.25) *of $\bar{\Delta}$ are given by*

$$
\begin{array}{rcl}
G_{33} & = & \varepsilon^2(\alpha^{(2)}(B_1^{(2)})^2 - \alpha^{(1)}(B_1^{(1)})^2 + \frac{1}{2}(B_2^{(2)}\left(\alpha^{(2)}\right)^2 - B_2^{(1)}\left(\alpha^{(1)}\right)^2)) \\
& & -\varepsilon^3\alpha^{(1)}(\alpha^{(1)}(2B_1^{(1)}\eta_1^{(1)} + \alpha^{(1)}\eta_2^{(1)}) + \frac{1}{2}uB_2^{(1)}\alpha^{(1)} + u(B_1^{(1)})^2) \\
& & -\varepsilon^4 u\left(\alpha^{(1)}\right)^2(2B_1^{(1)}\eta_1^{(1)} + \alpha^{(1)}\eta_2^{(1)}) \\
G_{32} & = & \varepsilon(\frac{3}{2}B_2^{(2)}\alpha^{(2)} - \frac{3}{2}B_2^{(1)}\alpha^{(1)} + (B_1^{(2)})^2 - (B_1^{(1)})^2) \\
& & +\varepsilon^2(2B_1^{(2)}\alpha^{(2)}\eta_1^{(2)} - 4B_1^{(1)}\alpha^{(1)}\eta_1^{(1)} - 3\left(\alpha^{(1)}\right)^2\eta_2^{(1)} - \frac{3}{2}uB_2^{(1)}\alpha^{(1)} - u(B_1^{(1)})^2) \\
& & -\varepsilon^3\alpha^{(1)}u(4B_1^{(1)}\eta_1^{(1)} + 3\alpha^{(1)}\eta_2^{(1)}) \\
G_{31} & = & (B_2^{(2)} - B_2^{(1)}) + \varepsilon(2B_1^{(2)}\eta_1^{(2)} - 2B_1^{(1)}\eta_1^{(1)} + \alpha^{(2)}\eta_2^{(2)} - 3\alpha^{(1)}\eta_2^{(1)} - uB_2^{(1)}) \\
& & -\varepsilon^2 u(3\alpha^{(1)}\eta_2^{(1)} + 2B_1^{(1)}\eta_1^{(1)}) \\
G_{30} & = & \eta_2^{(2)} - \eta_2^{(1)} - \varepsilon u\eta_2^{(1)}
\end{array}
$$

One can continue the calculations in obtaining $G_{k,k-j}, 0 \leq j \leq k, k \geq 4$; however, the expressions become longer with increasing k, and hence, one looses insight. Instead of writing explicit expressions for the coefficients $G_{k,k-j}, 0 \leq j \leq k, k \geq 4$, it is preferable to compute them modulo some parameter variables.

By (1.132), we can derive structure formulas for the coefficients $G_{k,k-j}, 0 \leq j \leq k$, in the expansion of $\bar{\Delta}$, in case the regular maps take the form $R_1(y) = -\varepsilon\beta + (1+\varepsilon u)y + O(y^k), y \to 0$ and $R_2(x) = x + O(x^k), x \to 0$, for a certain $k \geq 2$:

Proposition 222 *Let $k \geq 2$, and suppose that*

$$\eta_j^{(1)} = \eta_j^{(2)} = 0, \forall 1 \leq j \leq k-2,$$

then the coefficients $G_{k,k-j}, 0 \leq j \leq k$, in the expansion (5.25) *of the reduced difference map $\bar{\Delta} = \bar{\Delta}_2 - \bar{\Delta}_1$, are given by*

$$G_{k,k-j} = G_{k,k-j}^{(2)} - G_{k,k-j}^{(1)}, \forall 0 \leq j \leq k$$

As a consequence of propositions 222, 217 and 219, we can easily derive structure formulas for the coefficients $G_{k,k-j}, 0 \leq j \leq k$, modulo certain coefficients from the expansions of the normal forms, that appear previously in the expansion of $\bar{\Delta}$:

Proposition 223 *If*

$$
\begin{cases}
\beta = 0 \\
\alpha^{(2)} = \alpha^{(1)} = \alpha \\
u = 0 \\
\eta_j^{(1)} = \eta_j^{(2)} = 0, \forall 1 \leq j \leq k-2 \\
B_j^{(2)} = B_j^{(1)} = B_j, \forall 1 \leq j \leq k-2
\end{cases}
$$

then with the notation for $B^{(2-1)}_{k-1}$ defined in (5.27)*, one has*

$$\begin{cases} G_{k,k} &= (\varepsilon\alpha)^k \eta^{(1)}_{k-1} + (\varepsilon\alpha)^{k-1} \frac{1}{k-1} B^{(2-1)}_{k-1} \\ G_{k,k-j} &= (\varepsilon a)^{k-j-1} [\frac{1}{j}\binom{k-2}{j-1} + \frac{1}{j-1}\binom{k-2}{j-2}] B^{(2-1)}_{k-1} \\ & \quad + (\varepsilon a)^{k-j} \binom{k}{k-j} \eta^{(1)}_{k-1}, \qquad \forall 1 \leq j \leq k-2 \\ G_{k,1} &= B^{(2-1)}_{k-1} + \varepsilon\alpha k \eta^{(1)}_{k-1} \\ G_{k,0} &= \eta^{(2)}_{k-1} - \eta^{(1)}_{k-1} \end{cases}$$

Proposition 223 and lemma 9, then imply that for fixed $\varepsilon \neq 0$, the maps

$$(\beta, \tau, u, \eta^{(1)}_{1\ldots k-1}, B^{(2-1)}_{1\ldots k-1}, \eta^{(2)}_{1\ldots k-2}) \mapsto (G_{0,0}, \ldots, G_{k,k}, \ldots G_{k,1})$$

respectively

$$(\beta, \tau, u, \eta^{(1)}_{1\ldots k-1}, B^{(2-1)}_{1\ldots k-1}, \eta^{(2)}_{1\ldots k-1}) \mapsto (G_{0,0}, \ldots, G_{k,k}, \ldots G_{k,0})$$

has rank $3k-1$ respectively $3k$ at 0, where we use the notations

$$\eta^{(2)}_{1\ldots k-2} = (\eta^{(2)}_1, \ldots, \eta^{(2)}_{k-2}) \text{ and } \eta^{(i)}_{1\ldots k-1} = (\eta^{(i)}_1, \ldots, \eta^{(i)}_{k-1}), i = 1, 2 \tag{5.26}$$

and

$$B^{(2-1)}_{1\ldots k-1} = (B^{(2-1)}_1, \ldots, B^{(2-1)}_{k-1}) \text{ and } B^{(2-1)}_j = B^{(2)}_j - B^{(1)}_j, 1 \leq j \leq k-1. \tag{5.27}$$

Corollary 224 *Let $k \in \mathbb{N}, k \geq 2$, then we define the maps*

$$\bar{\Delta}_{2k} : \mathbb{R}^{3k} \to \mathbb{R}^{(k+1)(k+2)/2}$$

respectively

$$\bar{\Delta}_{2k+1} : \mathbb{R}^{3k+1} \to \mathbb{R}^{(k+1)(k+2)/2+1}$$

by the coefficients according to $F_{i,i-j}, 0 \leq j \leq i, i \leq k$ in the expansion of $\bar{\Delta}$:

$$(\beta, \tau, u, \eta^{(1)}_{1\ldots k-1}, B^{(2-1)}_{1\ldots k-1}, \eta^{(2)}_{1\ldots k-2}, \varepsilon) \overset{\bar{\Delta}_{2k}}{\mapsto} (G_{00}, G_{11}, \ldots, G_{kk}, G_{kk-1}, \ldots, G_{k1}, \varepsilon),$$

respectively

$$(\beta, \tau, u, \eta^{(1)}_{1\ldots k-1}, B^{(2-1)}_{1\ldots k-1}, \eta^{(2)}_{1\ldots k-1}, \varepsilon) \overset{\bar{\Delta}_{2k+1}}{\mapsto} (G_{00}, G_{11}, \ldots, G_{kk}, G_{kk-1}, \ldots, G_{k1}, G_{k0}, \varepsilon),$$

where we use the notations in (5.26) *and* (5.27)*. Then, for $\varepsilon_0\alpha_0 \neq 0$,*

1. *the map $\bar{\Delta}_{2k}$ has rank $3k$ at $\left(\beta, \tau, u, B^{2-1}, \eta^{(1)}_{1\ldots k-1}, \eta^{(2)}_{1\ldots k-2}, \varepsilon\right) = (0, \varepsilon_0) \in \mathbb{R}^{3k-1} \times \mathbb{R}$, and*

2. *the map $\bar{\Delta}_{2k+1}$ has rank $3k+1$ at $\left(\beta, \tau, u, B^{2-1}, \eta^{(1)}_{1\ldots k-1}, \eta^{(2)}_{1\ldots k-1}, \varepsilon\right) = (0, \varepsilon_0) \in \mathbb{R}^{3k} \times \mathbb{R}$.*

From corollary 224, one could conjecture that the upperbound for the cyclicity in theorem 134 (i.e. $\text{codim}X_\lambda$) is not optimal. To be more concrete, if $\text{codim}I_\nu = 2k$ (respectively $2k+1$), then we can conjecture that the sharpest upperbound for the cyclicity will be $3k-1$ (respectively $3k$), instead of the upperbound $\text{codim}X_\lambda$ (recall that $\text{codim}X_\lambda = 2k + k(k-1)/2$, respectively $2k+1+k(k+1)/2$). In particular, one can conjecture that there exist 'generic' unfoldings of the 2-saddle cycle leaving one connection unbroken, for which the cyclicity is at least $3k-1$, while the related Abelian integral has at most $2k+1$ zeroes ($\forall k \geq 4$). By this conjecture, there exist unfoldings of the 2-saddle cycle leaving one connection unbroken, for which there are at least $k-2$ limit cycles that are not shadowed by zeroes of the related Abelian integral, the so-called alien limit cycles.

Moreover, from the recursion formulas given in proposition 223, it seems that, the $(k+1)$-th order approximation of Δ with respect to ε will be sufficient to study the exact cyclicity along a 2-saddle cycle, of which one connection remains unbroken, in case $\text{codim}_{\mathcal{L}} I_\nu = 2k$ or $2k+1$. This conjecture is consistent with the observation we made in the example given in [DR], where $\text{codim}_{\mathcal{L}} I_\nu = 3$. In this case, linear approximation was not sufficient to study the precise cyclicity, but quadratic approximation of Δ with respect to ε, is sufficient.

5.3 Particular case

5.3.1 Settings and organising

In this section, we consider a particular subfamily of the unfolding $(X_\lambda)_\lambda$ of a 2-saddle cycle Γ, that keeps one connection unbroken of type (5.1). Suppose again that the normal form of X_λ at the saddle point s_1 (respectively $-X_\lambda$ at the saddle point s_2) is given by (5.4) (respectively (5.5)). In this section, we suppose that the coefficients $\alpha^{(i)}, B_j^{(i)}, j \geq 1$ depend analytically on the parameter $\lambda = (\nu, \varepsilon)$. Then, the subfamily that is subjected to our investigation, is defined by the conditions

$$\alpha^{(1)}(\lambda) = \alpha^{(2)}(\lambda) \equiv \alpha \neq 0, \tag{5.28}$$

for a fixed non-zero value $\alpha \in \mathbb{R}$, and the conditions

$$B_i^{(1)}(\nu, \varepsilon) = B_i^{(2)}(\nu, \varepsilon) = 0, \forall i \geq 1. \tag{5.29}$$

Furthermore, we suppose that the conditions (5.28) and (5.29) define an analytic submanifold M in parameter space. We denote the restrictions of the maps $\bar{\Delta}, I$ to M, by $\bar{\Delta}^M, I^M$ and we suppose that (ν, ε) are analytic coordinates of a local chart for M at 0.

Let us first investigate the conditions (5.28) and (5.29). By condition (5.28), the coefficient $\tau \equiv 0$ according to '$x \log x$' disappears in the expansion of $\bar{\Delta}^M$ (and also from I_ν^M). This condition simplifies our study, since we are left with only 1 compensator $\omega = \omega_{\varepsilon\alpha}$. As a consequence, we don't need the compensators ω_{2-1} and

ω_{21}, as in the more general case. Besides, ω only depends on x and ε, and not on the other parameter variables through α, since α is supposed to be a fixed constant. The condition $\alpha \neq 0$ is necessary, if we want to find alien limit cycles in this particular subfamily. Condition (5.29) restricts the study of $(X_\lambda)_\lambda$ to a submanifold in parameter space for which the normal forms at the saddle points stay linear. As a consequence, we have an explicit expression for the Dulac maps at the saddle points, and so for the reduced difference map $\bar{\Delta}^M$, as we see in section 5.3.2.

In section 5.3.6, we see that for this particular subfamily, the related Abelian integral is C^∞, and hence admits an asymptotic expansion in the Taylor scale. In case the codimension of the related Abelian integral is even (say $2k$), we define the notion of genericity (of codimension $2k-1$) for such a subfamily. For such a family, we expect that the cyclicity would be equal to $2k-1$ (which for sure is an upperbound for the cyclicity).

In section 5.3.5, we show that this subfamily can produce, generically, at least $2k-2$ limit cycles arbitrarily close to Γ, of which at least $k-2$ limit cycles are not related to zeroes of the corresponding Abelian integral I_ν^M. This result is obtained by the introduction of new compensators, leading to a special regrouping of the expansion of $\bar{\Delta}^M$. After a weighted rescaling by ε, the map $\bar{\Delta}^M$ then has an asymptotic expansion in a simple asymptotic scale deformation $\mathcal{W}^*$ of the restricted logarithmic scale $\mathcal{L}^*$ (i.e. the logarithmic scale without the term $x \log x$), that is very similar to the simple asymptotic scale deformation $\mathcal{W}$, defined in (1.109), in case of the saddle loop. The new compensators $\omega_i, i \geq 1$ and the scale $\mathcal{W}^*$ are studied in section 5.3.4.

In other directions of parameter space (i.e. moving the parameter slightly such that $\alpha^{(1)} \neq \alpha^{(2)}$ and such that the coefficients $B_i, 1 \leq i \leq k-1$ become non-zero), the Abelian integral can produce k extra zeroes, bifurcating from $x = 0$. All together, the Abelian integral can produce $2k$ zeroes (by persistence of simple zeroes). One could conjecture that none of these extra k zeroes of the Abelian integral correspond to one of the $k-1$ alien limit cycles, since these limit cycles were created after rescaling by ε, and therefore these limit cycles are not related to zeroes of the Abelian integral (which corresponds to $\bar{\Delta}\big|_{\varepsilon=0}$). This conjecture would imply that a generic unfolding $(X_\lambda)_\lambda$ will produce $3k-1$ limit cycles, for which codimension of the Abelian integral is $2k$; hence, at most $2k$ limit cycles correspond to zeroes of the Abelian integral.

5.3.2 Difference map

By conditions (5.28) and (5.29), the Dulac maps at the saddle points s_1 and s_2 can be solved explicitly:

$$D_1^M(x, \nu, \varepsilon) = x^{1+\varepsilon\alpha} = x(1 + \varepsilon\alpha\omega)$$

and

$$D_2^M(z, \nu, \varepsilon) = z^{1+\varepsilon\alpha} = z(1 + \varepsilon\alpha\omega),$$

where ω is the compensator given by

$$\omega(x) = \omega_{\varepsilon\alpha}(x) = \omega(x, \varepsilon\alpha) = \frac{x^{\varepsilon\alpha} - 1}{\varepsilon\alpha}.$$

Then, the reduced difference map $\bar{\Delta}$ can be written explicitly as:

$$\bar{\Delta}^M(x,\nu,\varepsilon) = \beta - ux^{1+\varepsilon\alpha} + x^{1+\varepsilon\alpha}\left[F_2(x,\nu,\varepsilon) - (1+\varepsilon u) f_1\left(x^{1+\varepsilon\alpha},\nu,\varepsilon\right)\right], \quad (5.30)$$

where the C^∞ function F_2 is defined by the equation

$$(1+\varepsilon f_2(x,\nu,\varepsilon))^{1+\varepsilon\alpha} = 1+\varepsilon F_2(x,\nu,\varepsilon). \quad (5.31)$$

5.3.3 Genericity conditions

For this particular subfamily, the expansion of the Abelian integral reduces to

$$j_\infty\left(I_\nu^M\right)_0(x) = j_\infty\left(\bar{\Delta}^M(\cdot,\nu,0)\right)_0(x) = -\beta + ux + \sum_{i=2}^{\infty}(\eta_{i-1}^{(2)} - \eta_{i-1}^{(1)})x^i \quad (5.32)$$

We notice that for this particular subfamily the map I_ν^M expands in the Taylor scale $\mathcal{T}$. In general, the Abelian integral admits an asymptotic expansion in the logarithmic scale $\mathcal{L}$. Here, the expansion of I_ν^M in $\mathcal{L}$ displays gaps: only pure powers of x show up, the terms containing $x^i \log x$ are missing. This was to be expected. Since the Dulac map now reduces to the identity for $\varepsilon = 0$, only the regular transitions R_1 and R_2 contribute to the Abelian integral I_ν^M, and hence the Abelian integral I_ν^M is C^∞.

Let $k \geq 2$. Recall that the Abelian integral I_ν has codimension $2k$ (denoted by $\mathrm{codim}_{\mathcal{L}} I_\nu^M = 2k$) if and only if

$$\beta(0) = u(0) = (\eta_i^{(2)} - \eta_i^{(1)})(0) = 0, \forall 1 \leq i \leq k-2 \text{ and } (\eta_{k-1}^{(2)} - \eta_{k-1}^{(1)})(0) \neq 0. \quad (5.33)$$

Clearly, if $\mathrm{codim}_{\mathcal{L}} I_\nu^M = 2k$, then the Abelian integral I_ν^M has at most k zeroes.

Let us introduce the notation

$$\eta_{1\ldots j}^{(i)}(\nu,\varepsilon) = (\eta_1^{(i)}(\nu,\varepsilon),\ldots,\eta_j^{(i)}(\nu,\varepsilon)), \qquad \forall i = 1,2, \forall j \geq 1.$$

Definition 225 *Consider an unfolding $(X_\lambda)_\lambda$ of a 2-saddle cycle Γ, that keeps one connection unbroken of type* (5.1). *Consider the particular subfamily $(X_\lambda)_{\lambda\in M}$, defined by the conditions* (5.28) *and* (5.29). *Then, we say that the subfamily $(X_\lambda)_{\lambda\in M}$ is generic of codimension $2k-1$ if the following genericity conditions are satisfied:*

1. $\mathrm{codim}_{\mathcal{L}} I_\nu^M = 2k$, i.e.

   $$\beta(0) = u(0) = (\eta_i^{(2)} - \eta_i^{(1)})(0) = 0, \forall 1 \leq i \leq k-2 \text{ and } (\eta_{k-1}^{(2)} - \eta_{k-1}^{(1)})(0) \neq 0,$$

2. $\eta_{k-1}^{(1)}(0) \neq 0$,

3. the map $Q_1 : \left(\mathbb{R}^{2k-1}, 0\right) \to \left(\mathbb{R}^{2k-1}, 0\right)$:

   $$(\nu,\varepsilon) \mapsto (\beta, u, \eta_{1\ldots k-2}^{(1)}(\nu,\varepsilon), \eta_{1\ldots k-2}^{(2)}(\nu,\varepsilon), \varepsilon) \quad (5.34)$$

   is a local diffeomorphism at 0.

By the third genericity condition, we can suppose without loss of generality that

$$(\nu, \varepsilon) = (\beta, u, \eta^{(1)}_{1\ldots k-2}, \eta^{(2)}_{1\ldots k-2}, \varepsilon).$$

Under these genericity conditions, we show that the particular subfamily, defined in section 5.3.1, can produce $2k-1$ limit cycles. As a consequence, since $\operatorname{codim}_{\mathcal{L}} I^M_\nu = 2k$, there are at least $k-1$ limit cycles, that are not covered by zeroes of the Abelian integral.

Remark 226 *In fact, the third condition can be weakened: we can replace 'diffeomorphism' by 'submersion'. If the map Q_1 (now defined on $(\mathbb{R}^p, 0)$) is merely a local submersion at 0, then we can restrict our study to a submanifold M_1 inside M, such that the map Q_1, defined in the coordinates of a local chart for M_1 at 0, is a local diffeomorphism at 0.*

To simplify the presentation, we introduce new coordinates (ν^1, ε) in parameter space as follows. Write

$$j_\infty \left(F_2\left(\cdot, \nu, \varepsilon\right)\right)_0(x) = \sum_{k=1}^{\infty} \gamma_k(\eta^{(2)}_{1\ldots k}\left(\nu, \varepsilon\right), \varepsilon) \cdot x^k, \tag{5.35}$$

then it is easily verified that, $\forall k \geq 1$:

$$\gamma_k(\eta^{(2)}_{1\ldots k}\left(\nu, \varepsilon\right), \varepsilon) = (1+\varepsilon\alpha)\left(\eta^{(2)}_k\left(\nu, \varepsilon\right) + \Theta_k(\eta^{(2)}_1\left(\nu, \varepsilon\right), \ldots, \eta^{(2)}_{k-1}\left(\nu, \varepsilon\right))\right), \tag{5.36}$$

where Θ_k is a polynomial in the $(k-1)$ variables $\eta^{(2)}_{1\ldots k-1} = (\eta^{(2)}_1, \ldots, \eta^{(2)}_{k-1})$ of degree at most k; in particular, if Θ_k is written as the sum of homogenous polynomials $\Theta_{k,j}$ of degree j, then the polynomial $\Theta_{k,j}$ is divisible by $\alpha\varepsilon^{j-1}, \forall 2 \leq j \leq k, \forall k \geq 2$ (and $\Theta_1 \equiv 0$).

Clearly, by (5.36), if we use the notation

$$\gamma_{1\ldots k-2}(\eta^{(2)}_{1\ldots k-2}, \varepsilon) = (\gamma_1(\eta^{(2)}_1, \varepsilon), \gamma_2(\eta^{(2)}_1, \eta^{(2)}_2, \varepsilon), \ldots, \gamma_{k-2}(\eta^{(2)}_{1\ldots k-2}, \varepsilon)),$$

then the map Q_1, defined in (5.34) is a local diffeomorphism at 0, if and only if the map

$$(\nu, \varepsilon) \mapsto (\beta, u, \eta^{(1)}\left(\nu, \varepsilon\right), \gamma_{1\ldots k-2}(\eta^{(2)}\left(\nu, \varepsilon\right), \varepsilon), \varepsilon) \tag{5.37}$$

is a local diffeomorphism at 0. Let us then denote the inverse of the map, defined in (5.37), by P_1,

$$(\nu^1, \varepsilon) = (\beta, u, \eta^{(1)}_{1\ldots k-2}, \gamma_{1\ldots k-2}, \varepsilon) \overset{P_1}{\mapsto} (\beta, u, \eta^{(1)}_{1\ldots k-2}, \eta^{(2)}_{1\ldots k-2}\left(\gamma_{1\ldots k-2}, \varepsilon\right), \varepsilon) \tag{5.38}$$

where $\eta^{(2)}_{1\ldots k-1}\left(\gamma_{1\ldots k-2}, \varepsilon\right)$ is defined by (5.36), and takes the form:

$$\eta^{(2)}_{1\ldots k-1}\left(\gamma_{1\ldots k-2}, \varepsilon\right) = \left(\eta_1(\gamma_1, \varepsilon), \eta_2(\gamma_1, \gamma_2, \varepsilon), \ldots, \eta_{k-2}(\gamma_{1\ldots k-2}, \varepsilon)\right). \tag{5.39}$$

We denote by $\bar{\Delta}^1$ the reduced difference map $\bar{\Delta}$, expressed in the parameter variable (ν^1, ε) :

$$\bar{\Delta}^1 \left(x, \nu^1, \varepsilon\right) = \bar{\Delta}^M \left(x, P_1 \left(\nu^1, \varepsilon\right)\right).$$

To end this section, let us consider the particular subfamily of $(X_\lambda)_\lambda$, defined by the conditions (5.28) and (5.29), but now we suppose that $\alpha = 0$. Then, the Dulac maps at the saddle points become the identity maps, and hence, the reduced difference map $\bar{\Delta}$ is C^∞ :

$$\begin{aligned}\bar{\Delta}^M (x, \nu, \varepsilon) &= \beta - ux + x \left[f_2 (x, \nu, \varepsilon) - (1 + \varepsilon u) f_1 (x, \nu, \varepsilon)\right] \\ &= \beta - ux + \sum_{i=1}^{\infty} (\eta_i^{(2)} - (1 + \varepsilon u) \eta_i^{(1)}) x^{i+1}.\end{aligned}$$

Hence, if $\alpha = 0$, the map $\bar{\Delta}^M$ can not produce more zeroes than the related Abelian integral. As a consequence, such a subfamily cannot produce alien limit cycles, and, generically, it can be studied by its linear approximation.

5.3.4 New compensators

In section 5.3.5, we will perform some rescalings such that the reduced difference map admits an asymptotic expansion in a sequence $\mathcal{W}^*$, that will be defined in this section. This sequence $\mathcal{W}^*$ looks very similar to the simple asymptotic scale deformation $\mathcal{W}$ of the logarithmic scale, in which the reduced displacement map is expanded, in case of a saddle loop [Mar]. But, it is striking to notice that, in contrast to the saddle loop case, where only one compensator shows up, now a sequence of compensators play a role.

In this section, we give a precise definition of these new compensators and we show that the sequence $\mathcal{W}^*$ is a simple asymptotic scale deformation of the restricted logarithmic scale $\mathcal{L}^*$ (defined in (1.82) in section 1.4.2). Recall that the definition of simple asymptotic scale deformation can be found in section 1.4.4 in definition 93.

The sequence of compensators $\{\omega_i, i \geq 1\}$ is defined by

$$\omega_i = \omega\left(x, i\varepsilon\alpha\right), \forall i \geq 1. \tag{5.40}$$

Notice that ω_i, for $i \geq 1$, is the compensator associated to a saddle point with ratio of hyperbolicity $1 + i\varepsilon\alpha$, and that ω_i is an unfolding of $\log x$ for $\varepsilon \to 0$. Let us also stress that these new compensators ω_i are not the ones we have encountered before, where the subindex denoted the corresponding saddle point s_i (with ratio of hyperbolicity $r^{(i)} = 1 + \varepsilon\alpha^{(i)}$). Furthermore, notice that ω_1 is in fact the traditional compensator ω.

Let us now give two other characterisations for the compensator $\omega_i, i \geq 1$.

Proposition 227 *Let $i \geq 1$ and let ω_i be the compensator defined by* (5.40). *Then,*

$$\omega_i(x) = \frac{1}{i}\omega\left(x^i\right) \tag{5.41}$$

$$= \omega \cdot \left(1 + P_i\left(\varepsilon\alpha\omega\right)\right) \tag{5.42}$$

where $\omega(x) = \omega(x, \varepsilon\alpha)$ and P_i is a polynomial of degree $i-1$ with $P_i(0) = 0$.

One easily checks the following calculation rules for ∇, when dealing with the compensator $\omega_i, i \geq 1$.

Proposition 228 *Let $\phi \in \mathbb{R}, i, j, k, , r, s \in \mathbb{N}$ and $i, j, k, , r, s \geq 1, m \in \mathbb{Z}, m \neq 0$. Then,*

1. $\nabla\omega_i^m = m\omega_i^m\left(i\varepsilon\alpha + \omega_i^{-1}\right)$; *in particular,*

$$\begin{cases} \nabla\omega_i = \omega_i\left(i\varepsilon\alpha + \omega_i^{-1}\right) \\ \nabla\omega_i^{-1} = -\omega_i^{-1}\left(i\varepsilon\alpha + \omega_i^{-1}\right) \end{cases}$$

2. $\nabla\left(x^\phi\omega_i^m\right) = x^\phi\omega_i^m\left(\phi + im\varepsilon\alpha + m\omega_i^{-1}\right)$; *in particular,*

$$\begin{cases} \nabla\left(x^\phi\omega_i\right) = x^\phi\omega_i\left(\phi + i\varepsilon\alpha + \omega_i^{-1}\right) \\ \nabla\left(x^\phi\omega_i^{-1}\right) = x^\phi\omega_i^{-1}\left(\phi - i\varepsilon\alpha - \omega_i^{-1}\right) \end{cases}$$

3. $\nabla(\omega_r^{-j}\omega_s^{-k}) = -\omega_r^{-j}\omega_s^{-k}\left((ks + jr)\varepsilon\alpha + k\omega_s^{-1} + j\omega_r^{-1}\right)$.

Since the compensator ω_i behaves exactly as the compensator ω, up to some constants (it is ω for α replaced by $i\alpha$), one can prove that $\mathcal{W}^*$ is a simple asymptotic scale deformation of the restricted logarithmic scale $\mathcal{L}^*$, in the same way as we prove in section 1.4.6 that the sequence $1, x\omega, x, x^2\omega, x^2, \ldots$ is a simple asymptotic scale deformation of the logarithmic scale $\mathcal{L}$ (or as in the proof that the sequence $\mathcal{W}_3$ in [DR] is a simple asymptotic scale deformation of the enlarged logarithmic scale $\mathcal{L}^e$ (recalled in proposition 131).

Proposition 229 *The sequence $\mathcal{W}^*$, defined by*

$$1, \varepsilon^{2\varepsilon\alpha}x^{1+\varepsilon\alpha}, \varepsilon^{2\varepsilon\alpha}x^{2+\varepsilon\alpha}\omega_1, \varepsilon^{2\varepsilon\alpha}x^{2+\varepsilon\alpha}, \ldots, \varepsilon^{2\varepsilon\alpha}x^{i+\varepsilon\alpha}\omega_{i-1}, \varepsilon^{2\varepsilon\alpha}x^{i+\varepsilon\alpha}, \ldots; i \geq 1 \tag{5.43}$$

is a simple asymptotic scale deformation of the restricted logarithmic scale $\mathcal{L}^*$, given by

$$1, x, x^2\log x, x^2, x^3\log x, \ldots, x^i\log x, x^i, \ldots, i \geq 1$$

Proof. Notice that $\lim_{\varepsilon\to 0}\varepsilon^{2\varepsilon\alpha} = 1$; hence, the functions in the sequence $\mathcal{W}^*$ are well-defined and C^∞, for $x, \varepsilon > 0$. The factor $\varepsilon^{2\varepsilon\alpha}$ is not important in the division-derivation algorithm, since $\varepsilon^{2\varepsilon\alpha}$ does not depend on x. Moreover, in the first step of the division-derivation algorithm, this factor is cancelled out of the sequence by division.

Using the calculation rules summarised in proposition 228, one can check by induction on $k \geq 1$, that the sequences $\mathcal{W}^*_{2k-1}$ and $\mathcal{W}^*_{2k}$, produced in the $(2k-1)$-th and $(2k)$-th steps of the derivation-division process, are respectively given by:

$$\mathcal{W}^*_{2k-1} = \left\{1, x\omega_k g_1^{2k-1}, x g_2^{2k-1}, \ldots, x^i \omega_{i+k-1} g_{2i-1}^{2k-1}, x^i g_{2i}^{2k-1}, \ldots; i \geq 1\right\}$$

and

$$\mathcal{W}^*_{2k} = \left\{1, \omega_k^{-1} g_1^{2k}, x\omega_{1+k}\omega_k^{-1} g_2^{2k}, \ldots, x^i \omega_{i+k}\omega_k^{-1} g_{2i}^{2k}, x^i \omega_k^{-1} g_{2i+1}^{2k}, \ldots; i \geq 1\right\},$$

where the functions g_i^j $(i \geq 1, j \geq 1)$, appearing in the sequence $\mathcal{W}^*_j, j \geq 1$, are of the following type:

$$g_i^j = \frac{C_i^j + P_i^j\left(\omega_1^{-1}, \ldots, \omega_r^{-1}\right)}{C^j + Q^j\left(\omega_1^{-1}, \ldots, \omega_r^{-1}\right)}$$

for certain constants C_i^j, C^j (meaning that they depend only on the parameter ε in a polynomial way), and for certain multivariate polynomials P_i^j, Q^j with $P_i^j(0) = Q^j(0) = 0$, and $r \leq \left[\frac{i+j+1}{2}\right]$ (where $[u]$ means the integer part of $u \in \mathbb{R}$). ■

5.3.5 Rescaling of the reduced difference map $\bar{\Delta}$

By introduction of new compensators, we can regroup the building terms in the asymptotic expansion (5.30) of $\bar{\Delta}^M$.

Nevertheless, we cannot find extra (i.e. alien) limit cycles by considering merely this expansion. The extra limit cycles can only be found by using the extra terms in the expansion of $\bar{\Delta}^M$, that are not seen in the related Abelian integral.

Since, for $\varepsilon = 0$, the reduced difference map coincides with the Abelian integral, all of these extra terms are divisible by ε. Hence, the considered expansion cannot be generic, since the coefficients in the expansion cannot be seen as independent parameter variables (as soon as the codimension of the Abelian integral is at least 6, i.e. $2k \geq 6$).

In this section, by means of a weighted rescaling of both the phase variable x as well as the parameter variable η, the map $\bar{\Delta}^M$ is reduced to a map Ξ. This map Ξ will serve as the new reduced difference map; in particular, Ξ admits a generic expansion in $\mathcal{W}^*$.

By the rescaling $(\bar{x}, \bar{\nu}, \varepsilon) \mapsto (x, \nu, \varepsilon)$, a 'cone' around the ε-axis, in the product space of the phase variable and the parameter variable, is blown up to a full neighbourhood of $\bar{0}$, in the $(\bar{x}, \bar{\nu}, \varepsilon)$-space, uniformly with respect to ε. Therefore, to study limit cycles, bifurcating from Γ, in such a generic subfamily $(X_\lambda)_{\lambda \in M}$, we can study isolated zeroes of the unfolding Ξ, in a neighbourhood of 0 in the $(\bar{x}, \bar{\nu}, \varepsilon)$-space, that, afterwards, are blown down, to obtain results on a 'cone' around the ε-axis in the (x, ν, ε)-space.

Hence, as for the map $\bar{\Delta}$, the study of limit cycles can be replaced by the study of isolated zeroes of Ξ. However, an important point that we want to stress is that, unlike $\bar{\Delta}^M$, the map Ξ does not coincide with the Abelian integral, for $\varepsilon = 0$. The reason for this alienation, is the weighted rescaling by ε, and it is the key to find zeroes in the unfolding of Ξ, that are not related to zeroes of the Abelian integral.

Let us recall from section 5.3.4 that the simple asymptotic scale deformation $\mathcal{W}^*$ is defined by

$$\mathcal{W}^* = \left\{1, \varepsilon^{2\varepsilon\alpha}\bar{x}^{1+\varepsilon\alpha}, \varepsilon^{2\varepsilon\alpha}\bar{x}^{2+\varepsilon\alpha}\bar{\omega}_1, \ldots, \varepsilon^{2\varepsilon\alpha}\bar{x}^{i+\varepsilon\alpha}\bar{\omega}_{i-1}, \varepsilon^{2\varepsilon\alpha}\bar{x}^{i+\varepsilon\alpha}, \ldots; i \geq 2\right\}.$$

In this section, we prove that, if the particular subfamily $(X_\lambda)_{\lambda\in M}$ is generic of codimension $2k-1$, then the unfolding Ξ admits a generic expansion of order $2k-1$ in $\mathcal{W}^*$. This asymptotic expansion is very similar to the one for the reduced displacement map $\bar{\delta}$ in case of the saddle loop, studied by [Mar]. From this observation, one can conjecture that the cyclicity in such a generic subfamily $(X_\lambda)_{\lambda\in M}$ along the 2-saddle cycle is $2k-1$. Hence, by the weighted rescaling, one has $k-1$ alien limit cycles. This conjecture is discussed in section 5.3.6. We only prove there the existence of at least $k-2$ alien limit cycles.

In next theorem, we will use a C^∞ diffeomorphism $P : (\mathbb{R}^{2k-1}, 0) \to (\mathbb{R}^{2k-1}, 0)$:

$$(\nu^3, \varepsilon) = (\beta, u, \hat{\gamma}^{(1)}_{1\ldots k-2}, \hat{\gamma}^{(2)}_{1\ldots k-2}, \varepsilon) \overset{P}{\mapsto} (\nu, \varepsilon) = (\beta, u, \eta^{(1)}_{1,\ldots k-2}, \eta^{(2)}_{1,\ldots k-2}, \varepsilon), \tag{5.44}$$

where we use again the original coordinates (ν, ε) in parameter space $(\mathbb{R}^{2k-1}, 0)$. In particular, the components of P have the following asymptotics: $\forall 1 \leq i \leq k-2$:

$$\begin{cases} \eta_i^{(1)} &= \dfrac{1}{i\alpha}\hat{\gamma}_i^{(1)} + O(\varepsilon \log \varepsilon), \varepsilon \downarrow 0 \\ \eta_i^{(2)} &= \hat{\gamma}_i^{(2)} + \dfrac{1}{i\alpha}\hat{\gamma}_i^{(1)} + O(\varepsilon \log \varepsilon), \varepsilon \downarrow 0 \end{cases} \tag{5.45}$$

Besides, we will use C^∞ maps $\widehat{\gamma}^{(1)}_{k-1} : (\mathbb{R}^2, 0) \to \mathbb{R}$, $\widehat{\gamma}^{(2)}_{k-1} : (\mathbb{R}^3, 0) \to \mathbb{R}$ with the following asymptotics, for $\varepsilon \downarrow 0$:

$$\begin{cases} \widehat{\gamma}^{(1)}_{k-1}(\widehat{\eta}^{(1)}_{k-1}, \varepsilon) &= \widehat{\eta}^{(1)}_{k-1}\left(1 + (k-1)\,\alpha\varepsilon \log \varepsilon \left(1 + O(\varepsilon \log \varepsilon)\right)\right) \\ \widehat{\gamma}^{(2)}_{k-1}(\widehat{\eta}^{(1)}_{k-1}, \widehat{\eta}^{(2)}_{k-1}, \varepsilon) &= \widehat{\eta}^{(2)}_{k-1} - \widehat{\eta}^{(1)}_{k-1}\varepsilon \log \varepsilon \left(1 + O(\varepsilon \log \varepsilon)\right) \end{cases}. \tag{5.46}$$

Furthermore, we will use a rescaling map $S : (\mathbb{R}^{2k-1}, 0) \to (\mathbb{R}^{2k-2}, 0)$:

$$(\bar{\nu}, \varepsilon) = (\bar{\beta}, \bar{u}, \bar{\eta}^{(1)}_{1\ldots k-2}, \bar{\eta}^{(2)}_{1\ldots k-2}, \varepsilon) \overset{S}{\mapsto} \nu^3 = (\beta, u, \hat{\gamma}^{(1)}_{1\ldots k-2}, \hat{\gamma}^{(2)}_{1\ldots k-2}), \tag{5.47}$$

defined by

$$\begin{cases} \beta &= \varepsilon^{2k}\bar{\beta} \\ u &= \varepsilon^{2k-2}\bar{u} \\ \hat{\gamma}_i^{(1)} &= \varepsilon^{2k-2i-3}\bar{\eta}_i^{(1)}, \forall 1 \leq i \leq k-2 \\ \hat{\gamma}_i^{(2)} &= \varepsilon^{2k-2i-2}\bar{\eta}_i^{(2)}, \forall 1 \leq i \leq k-2 \end{cases}. \tag{5.48}$$

Theorem 230 *Consider an unfolding $(X_\lambda)_\lambda$ of a 2-saddle cycle Γ, that keeps one connection unbroken of type* (5.1). *Suppose that the subfamily of $(X_\lambda)_\lambda$, defined by the conditions* (5.28) *and* (5.29) *is generic of codimension $2k-1$ (as in definition 225), and denote the reduced difference map, associated to this particular subfamily, by $\bar{\Delta}^M$, as we did in* (5.30). *Then, there exist a local C^∞ diffeomorphism P as in* (5.64) *with asymptotics* (5.45), *a C^∞ rescaling map S as defined by* (5.47) *and* (5.48), *C^∞ functions $\hat{\gamma}_{k-1}^{(1)}, \hat{\gamma}_{k-1}^{(2)}$ with asymptotics* (5.46) *and functions $\Xi, \bar{\Psi} : (\mathbb{R} \times \mathbb{R}^{2k-1}, (0,0)) \to \mathbb{R}$, that are C^∞ for $x > 0$, sufficiently small and $\varepsilon \neq 0$, such that the reduced difference map $\bar{\Delta}^M$ is changed into*

$$\bar{\Delta}^M \left(\varepsilon^2 \bar{x}, P\left(S\left(\bar{\nu}, \varepsilon\right), \varepsilon\right)\right) = \varepsilon^{2k} \Xi\left(\bar{x}, \bar{\nu}, \varepsilon\right) \tag{5.49}$$

where

1. *Ξ has an asymptotic expansion of order $2k-1$ in the scale $\mathcal{W}^*$:*

$$\begin{aligned} \Xi\left(\bar{x}, \bar{\nu}, \varepsilon\right) = {} & \bar{\beta} - \bar{u}\varepsilon^{2\varepsilon\alpha}\bar{x}^{1+\varepsilon\alpha} + \varepsilon^{2\varepsilon\alpha}\bar{x}^{1+\varepsilon\alpha}\sum_{i=1}^{k-2}(-\bar{\eta}_i^{(1)}\bar{x}^i\bar{\omega}_i + \bar{\eta}_i^{(2)}\bar{x}^i) \\ & - \varepsilon\hat{\gamma}_{k-1}^{(1)}\varepsilon^{2\varepsilon\alpha}\bar{x}^{k+\varepsilon\alpha}\bar{\omega}_{k-1} + \hat{\gamma}_{k-1}^{(2)}\varepsilon^{2\varepsilon\alpha}\bar{x}^{k+\varepsilon\alpha} \\ & + \varepsilon^{2\varepsilon\alpha}\bar{x}^{1+\varepsilon\alpha-2\delta}\bar{\Psi}_k\left(\bar{x}, \bar{\nu}, \varepsilon\right) \end{aligned} \tag{5.50}$$

2. *$\bar{\Psi}_k$ satisfies the remainder property of order $2k-1$ with respect to $\mathcal{W}^*$. In particular, for any $0 < \delta < 1$:*

$$\bar{\Psi}_k\left(\bar{x}, \nu^3, \varepsilon\right) = O\left(\bar{x}^{k-\delta}\right), \bar{x} \to 0. \tag{5.51}$$

3. *$\hat{\gamma}_{k-1}^{(j)}, j = 1, 2$ depend in a C^∞ way on $(\bar{\nu}, \varepsilon)$; in particular: $\forall j = 1, 2$:*

$$\hat{\gamma}_i^{(j)}\left(0\right) = 0, \forall 1 \leq i \leq k-2 \text{ and } \hat{\gamma}_{k-1}^{(j)}\left(0\right) \neq 0 \tag{5.52}$$

Remark 231 *Notice that the expansion of Ξ, given in* (5.50), *is a generic asymptotic expansion of order $2k-1$ in the simple asymptotic scale deformation $\mathcal{W}^*$, since*

$$\bar{\beta}\left(0\right) = \bar{u}\left(0\right) = \bar{\eta}_i^{(j)}\left(0\right) = 0, \forall 1 \leq i \leq k-2, \forall j = 1, 2 \text{ and } \hat{\gamma}_{k-1}^{(2)}\left(0\right) \neq 0$$

and the map

$$\left(\mathbb{R}^{2k-1}, 0\right) \to \left(\mathbb{R}^{2k-1}, 0\right) : (\bar{\nu}, \varepsilon) \mapsto (\bar{\beta}, -\bar{u}, -\bar{\eta}_1^{(1)}, \bar{\eta}_1^{(2)}, \ldots, -\bar{\eta}_{k-2}^{(1)}, \bar{\eta}_{k-2}^{(2)}, \varepsilon\hat{\gamma}_{k-1}^{(1)}\left(\bar{\nu}, \varepsilon\right))$$

is a local diffeomorphism at 0.

Proof. To simplify the presentation, we will not precisize the domains of definition of the occuring maps since only their existence matters. In this proof, we rather use the notation $(\mathbb{R}^p, 0)$ meaning a neighbourhood of 0 in $\mathbb{R}^p$. The variables $x, \bar{x}, \varepsilon, \nu, \nu^1, \nu^2, \nu^3, \bar{\nu}$ are supposed to be sufficiently small ($x, \bar{x}, \varepsilon$ are supposed to be sufficiently close to $0 \in \mathbb{R}$, and $\nu, \nu^1, \nu^2, \nu^3, \bar{\nu}$ sufficiently close to $0 \in \mathbb{R}^{2k-2}$).

We first perform the coordinate transformation $(\nu, \varepsilon) = P_1(\nu^1, \varepsilon)$, defined (5.38). Rewriting the asymptotic expansion (5.30) of $\bar{\Delta}^M$, we obtain:

$$\begin{aligned}\bar{\Delta}^1\left(x, \nu^1, \varepsilon\right) &= \bar{\Delta}(x, P_1(\nu^1, \varepsilon)) \\ &= \beta - ux^{1+\varepsilon\alpha} \\ &\quad + x^{1+\varepsilon\alpha}\left[\sum_{i=1}^{k-1} x^i(\gamma_i - (1+\varepsilon u)\,\eta_i^{(1)} x^{i\varepsilon\alpha}) + \Psi_k\left(x, \nu^1, \varepsilon\right)\right] \end{aligned} \tag{5.53}$$

where the function Ψ_k is a function of differentiability class C^∞, for $x > 0$ sufficiently small, such that for any $0 < \delta < 1$:

$$\Psi_k\left(x, \nu^1, \varepsilon\right) = O\left(x^{k-\delta}\right), x \downarrow 0. \tag{5.54}$$

Notice that the coefficient γ_{k-1} in the expression (5.53) depends on the parameter $\left(\nu^1, \varepsilon\right)$ by way of (5.36) and (5.39) :

$$\gamma_{k-1} = \gamma_{k-1}(\eta^{(2)}\left(\gamma, \varepsilon\right), \eta_{k-1}^{(2)}\left(P_1\left(\nu^1, \varepsilon\right)\right), \varepsilon) \tag{5.55}$$

Next, by introduction of the compensators $\omega_i, 1 \leq i \leq k-1$, the terms in (5.53) are regrouped, to end up with an expansion in the sequence $\mathcal{W}^*$. To simplify the writing of the expansion, we introduce the following notations: $\forall 1 \leq i \leq k-1$:

$$\left\{\begin{array}{rcl} \hat{\eta}_i^{(1)} &=& i\alpha\left(1+\varepsilon u\right)\eta_i^{(1)} \\ \hat{\eta}_i^{(2)} &=& \gamma_i - \left(1+\varepsilon u\right)\eta_i^{(1)} \end{array}\right. \tag{5.56}$$

and we again perform a coordinate transformation in parameter space

$$(\beta, u, \eta_{1\ldots k-2}^{(1)}, \gamma_{1\ldots k-2}, \varepsilon) \mapsto (\beta, u, \hat{\eta}_{1\ldots k-2}^{(1)}, \hat{\eta}_{1\ldots k-2}^{(2)}, \varepsilon) \tag{5.57}$$

where $\hat{\eta}_{1\ldots k-2}^{(j)} = (\hat{\eta}_1^{(j)}, \ldots, \hat{\eta}_{k-2}^{(j)}), j = 1, 2$. Denote the inverse of the map, defined in (5.57), by P_2 :

$$P_2 : \left(\nu^2, \varepsilon\right) = (\beta, u, \hat{\eta}_{1\ldots k-2}^{(1)}, \hat{\eta}_{1\ldots k-2}^{(2)}, \varepsilon) \mapsto (\beta, u, \eta_{1\ldots k-2}^{(1)}, \gamma_{1\ldots k-2}, \varepsilon) \tag{5.58}$$

where $\eta_{1\ldots k-2}^{(1)} = \eta_{1\ldots k-2}^{(1)}(u, \hat{\eta}_{1\ldots k-2}^{(1)}, \varepsilon)$ and $\gamma_{1\ldots k-2} = \gamma_{1\ldots k-2}(u, \hat{\eta}_{1\ldots k-2}^{(1)}, \hat{\eta}_{1\ldots k-2}^{(2)}, \varepsilon)$ with

$$\left\{\begin{array}{rcl} \eta^{(1)}(u, \hat{\eta}^{(1)}, \varepsilon) &=& (\eta_1^{(1)}(u, \hat{\eta}_1^{(1)}, \varepsilon), \eta_2^{(1)}(u, \hat{\eta}_2^{(1)}, \varepsilon), \ldots, \eta_{k-2}^{(1)}(u, \hat{\eta}_{k-2}^{(1)}, \varepsilon)) \\ \gamma(u, \hat{\eta}^{(1)}, \hat{\eta}^{(2)}, \varepsilon) &=& (\gamma_1(u, \hat{\eta}_1^{(1)}, \hat{\eta}_1^{(2)}, \varepsilon), \gamma_2(u, \hat{\eta}_2^{(1)}, \hat{\eta}_2^{(2)}, \varepsilon), \ldots, \gamma_{k-2}(u, \hat{\eta}_{k-2}^{(1)}, \hat{\eta}_{k-2}^{(2)}, \varepsilon)) \end{array}\right.$$

Let us denote by $\bar{\Delta}^2$ the reduced difference map $\bar{\Delta}^M$, written in the parameter variable (ν^2,ε) :

$$\bar{\Delta}^2\left(x,\nu^2,\varepsilon\right)=\bar{\Delta}^M\left(x,P_1\circ P_2\left(\nu^2,\varepsilon\right)\right).$$

In this way, the asymptotic expansion for the reduced difference map $\bar{\Delta}^M$ in (5.53), for $\nu^2=(\beta,u,\hat{\eta}^{(1)},\hat{\eta}^{(2)})$, can be rewritten as:

$$\bar{\Delta}^2\left(x,\nu^2,\varepsilon\right)=\beta+x^{1+\varepsilon\alpha}\left[-u+\sum_{i=1}^{k-1}(-\varepsilon\widehat{\eta}_i^{(1)}x^i\omega_i+\widehat{\eta}_i^{(2)}x^i)+\Psi_k\left(x,P_2\left(\nu^2,\varepsilon\right)\right)\right], \tag{5.59}$$

where $\hat{\eta}_{k-1}^{(j)}, j=1,2$ depends on $\left(\nu^2,\varepsilon\right)$ as follows:

$$\left\{\begin{array}{lcl}\hat{\eta}_{k-1}^{(1)} & = & \hat{\eta}_{k-1}^{(1)}(u,\eta_{k-1}^{(1)}\left(P_1\circ P_2\left(\nu^2,\varepsilon\right)\right),\varepsilon)\\ \hat{\eta}_{k-1}^{(2)} & = & \hat{\eta}_{k-1}^{(2)}(u,\eta_{k-1}^{(1)}\left(P_1\circ P_2\left(\nu^2,\varepsilon\right)\right),\gamma_{k-1},\varepsilon)\end{array}\right. \tag{5.60}$$

where γ_{k-1} is defined in (5.55) with $\left(\nu^1,\varepsilon\right)=P_2\left(\nu^2,\varepsilon\right)$. Notice that now the genericity conditions 1., 2. and 3. are translated into

$$\widehat{\eta}_i^{(j)}\left(0\right)=0,\forall 1\leq i\leq k-2,\forall j=1,2 \text{ and } \hat{\eta}_{k-1}^{(1)}\left(0\right)\neq 0 \text{ and } \hat{\eta}_{k-1}^{(2)}\left(0\right)\neq 0, \tag{5.61}$$

and the map, defined in (5.57), is a local diffeomorphism at 0.

Notice that the coefficients, according to $x^i\omega_i$ in the expansion of $\bar{\Delta}^2$ in (5.59), all are divisible by ε. This was to be expected, since for $\varepsilon=0$, we obtain the Abelian integral,

$$\left.\bar{\Delta}^2\right|_{\varepsilon=0}=I^M_{P_1\circ P_2(\nu^2,0)},$$

and, as we have seen in (5.32), the logarithmic terms have disappeared from the expansion of the Abelian integral I_ν^M. Therefore, expansion (5.59) is not generic in the sense that the coefficients in this expansion can be seen as independent parameter variables for $\varepsilon=0$. To solve this problem, we rescale the parameter variables with ε, in order to obtain coefficients, all divisible by the same factor of ε.

However, by the genericity conditions (5.61), the coefficients according to $x^k\omega_{k-1}$ and x^k are not small, and therefore these coefficients cannot be rescaled by a factor of ε. To get rid of this problem, we rescale first the variable x :

$$x=\varepsilon^2\bar{x}.$$

By the rescaling in x, also the compensators ω_i are affected. Let us use the notation $\bar{\omega}_i=\omega_i\left(\bar{x},\varepsilon\alpha\right), 1\leq i\leq k-1$, and look at the effect of this rescaling on the compensators ω_i. One has,

$$\begin{aligned}\omega_i&=\frac{\left(\varepsilon^{2\varepsilon i\alpha}-1\right)\bar{x}^{\varepsilon i\alpha}+\bar{x}^{\varepsilon i\alpha}-1}{\varepsilon i\alpha}\\&=\frac{\left(\varepsilon^{\nu_x\varepsilon i\alpha}-1\right)}{\varepsilon i\alpha}\left(1+\varepsilon i\alpha\bar{\omega}_i\right)+\bar{\omega}_i\\&=\left(1+i\alpha\varepsilon\log\varepsilon\left(1+\varphi_i\left(\varepsilon\log\varepsilon\right)\right)\right)\bar{\omega}_i+\log\varepsilon\left(1+\varphi_i\left(\varepsilon\log\varepsilon\right)\right),\end{aligned}$$

where $\varphi_i : \mathbb{R} \to \mathbb{R}$ is the analytic function, with $\varphi_i(0) = 0$, defined by

$$\varepsilon^{2\varepsilon i\alpha} - 1 = i\alpha\varepsilon \log\varepsilon \left(1 + \varphi_i(\varepsilon \log \varepsilon)\right).$$

Again, to simplify the writing of the expansion of the map, obtained from $\bar{\Delta}^2$ after rescaling x, we introduce some notations: $\forall 1 \leq i \leq k-1$,

$$\left\{ \begin{array}{rcl} \widehat{\gamma}_i^{(1)}(\widehat{\eta}_i^{(1)}, \varepsilon) & = & \widehat{\eta}_i^{(1)}\left(1 + i\alpha\varepsilon\log\varepsilon\left(1 + \varphi_i(\varepsilon\log\varepsilon)\right)\right) \\ \widehat{\gamma}_i^{(2)}(\widehat{\eta}_i^{(1)}, \widehat{\eta}_i^{(2)}, \varepsilon) & = & \widehat{\eta}_i^{(2)} - \widehat{\eta}_i^{(1)}\varepsilon\log\varepsilon\left(1 + \varphi_i(\varepsilon\log\varepsilon)\right) \end{array} \right. \tag{5.62}$$

We introduce a last coordinate transformation in parameter space:

$$\left(\nu^2, \varepsilon\right) \mapsto (\beta, u, \hat{\gamma}^{(1)}_{1\ldots k-2}, \hat{\gamma}^{(2)}_{1\ldots k-2}, \varepsilon), \tag{5.63}$$

where $\hat{\gamma}^{(j)}_{1\ldots k-2} = (\hat{\gamma}^{(j)}_1, \ldots, \hat{\gamma}^{(j)}_{k-2}), j = 1, 2$, depends on $\left(\nu^2, \varepsilon\right)$ by means of (5.62). Let us denote the inverse of the map, defined in (5.63), by P_3:

$$\left(\nu^3, \varepsilon\right) = (\beta, u, \hat{\gamma}^{(1)}, \hat{\gamma}^{(2)}, \varepsilon) \overset{P_3}{\mapsto} (\beta, u, \hat{\eta}^{(1)}(\hat{\gamma}^{(1)}, \varepsilon), \hat{\eta}^{(2)}(\hat{\gamma}^{(1)}, \hat{\gamma}^{(2)}, \varepsilon), \varepsilon)$$

where

$$\left\{ \begin{array}{rcl} \hat{\eta}^{(1)}(\hat{\gamma}^{(1)}, \varepsilon) & = & (\hat{\eta}_1^{(1)}(\hat{\gamma}_1^{(1)}, \varepsilon), \hat{\eta}_2^{(1)}(\hat{\gamma}_2^{(1)}, \varepsilon), \ldots, \hat{\eta}_{k-2}^{(1)}(\hat{\gamma}_{k-2}^{(1)}, \varepsilon)) \\ \hat{\eta}^{(2)}(\hat{\gamma}^{(1)}, \hat{\gamma}^{(2)}, \varepsilon) & = & (\hat{\eta}_1^{(2)}(\hat{\gamma}_1^{(1)}, \hat{\gamma}_1^{(2)}, \varepsilon), \hat{\eta}_2^{(2)}(\hat{\gamma}_2^{(1)}, \hat{\gamma}_2^{(2)}, \varepsilon), \ldots, \hat{\eta}_{k-2}^{(2)}(\hat{\gamma}_{k-2}^{(1)}, \hat{\gamma}_{k-2}^{(2)}, \varepsilon)) \end{array} \right.$$

Let us denote the composition of the coordinate transformations in parameter space by P:

$$P = P_1 \circ P_2 \circ P_3 : \left(\mathbb{R}^{2k-1}, 0\right) \to \left(\mathbb{R}^{2k-1}, 0\right). \tag{5.64}$$

Then, in the new coordinates $\left(\nu^3, \varepsilon\right)$ in parameter space (with $\nu^3 = (\beta, u, \hat{\gamma}^{(1)}, \hat{\gamma}^{(2)})$), the expansion (5.59) of the reduced difference map $\bar{\Delta}^M$ reads:

$$\begin{aligned} \bar{\Delta}^3\left(\varepsilon^2\bar{x}, \nu^3, \varepsilon\right) &= \bar{\Delta}^M\left(\varepsilon^2\bar{x}, P\left(\nu^3, \varepsilon\right)\right) \\ &= \beta - u\varepsilon^{2+2\varepsilon\alpha}\bar{x}^{1+\varepsilon\alpha} \\ &\quad + \varepsilon^{2+2\varepsilon\alpha}\bar{x}^{1+\varepsilon\alpha}\sum_{i=1}^{k-1}\left[-\varepsilon^{2i+1}\widehat{\gamma}_i^{(1)}\bar{x}^i\bar{\omega}_i + \varepsilon^{2i}\widehat{\gamma}_i^{(2)}\bar{x}^i\right] \\ &\quad + \varepsilon^{2+2\varepsilon\alpha}\bar{x}^{1+\varepsilon\alpha}\Psi_k\left(\varepsilon^2\bar{x}, P_2 \circ P_3(\nu^3, \varepsilon)\right). \end{aligned}$$

Notice that the coefficients $\hat{\gamma}^{(j)}_{k-1}, j = 1, 2$, depend on the parameter $\left(\nu^3, \varepsilon\right)$, by means of (5.62) as follows:

$$\left\{ \begin{array}{rcl} \hat{\gamma}_{k-1}^{(1)} & = & \hat{\gamma}_{k-1}^{(1)}(\hat{\eta}_{k-1}^{(1)}, \varepsilon) \\ \hat{\gamma}_{k-1}^{(2)} & = & \hat{\gamma}_{k-1}^{(2)}(\hat{\eta}_{k-1}^{(1)}, \hat{\eta}_{k-1}^{(2)}, \varepsilon) \end{array} \right. \tag{5.65}$$

where $\hat{\eta}^{(j)}_{k-1}, j = 1, 2$, is defined by (5.60) with $(\nu^2, \varepsilon) = P_3(\nu^3, \varepsilon)$.

In particular, the genericity conditions 1., 2. and 3. are translated into (5.52) and the map P, defined in (5.64) or (5.44), is a local diffeomorphism at 0. Furthermore, one can easily check that the component functions of P are of type (5.45).

Finally, after rescaling of the parameter variable (ν^3, ε) by the formulas (5.48), the reduced difference map $\bar{\Delta}^3$ is divisible by the factor ε^{2k}. This induces the map Ξ in relation (5.49). The other statements of the theorem follow, if we define the C^k function $\bar{\Psi}_k$ by

$$\Psi_k\left(\varepsilon^2\bar{x}, P_2 \circ P_3\left(\nu^3, \varepsilon\right)\right) = \varepsilon^{2k-2\delta}\bar{\Psi}_k\left(\bar{x}, \bar{\nu}, \varepsilon\right). \tag{5.66}$$

The asymptotics of $\bar{\Psi}_k$, stated in (5.51), follows from (5.54) and (5.66), and implies that the map $\bar{\Psi}_k$ satisfies the remainder property of order $2k-1$ with respect to the simple asymptotic scale deformation $\mathcal{W}^*$ (see corollary 88).

To end, notice that the coefficients $\hat{\gamma}_{k-1}^{(j)}, j = 1, 2$ of $\varepsilon^{2\varepsilon\alpha}\bar{x}^{k+\varepsilon\alpha}\bar{\omega}_{k-1}$ and $\varepsilon^{2\varepsilon\alpha}\bar{x}^{k+\varepsilon\alpha}$ in the expansion (5.50) of Ξ, depend on the parameter variable $(\bar{\nu}, \varepsilon)$ as described in (5.65), by means of the rescaling formulas in (5.48). ■

5.3.6 Maximal cyclicity

In section 5.3.5, the study of the cyclicity of a generic subfamily $(X_\lambda)_{\lambda \in M}$ along Γ can be reduced to the one of the cyclicity of Ξ at 0. Recall that the notion of cyclicity for an unfolding Ξ is defined in definition 78 in section 1.4.1. It is the maximal possible number of isolated zeroes that can bifurcate from $\bar{x} = 0$, at the bifurcation value $(\bar{\nu}, \varepsilon) = (0, 0) \in \mathbb{R}^{2k-1}$.

This reduction was obtained by rescaling the parameter variable ν as well as the phase varibale x with ε. Therefore, the study of isolated zeroes in the unfolding Ξ cannot result in the understanding of the complete bifurcation diagram of limit cycles of $(X_\lambda)_{\lambda \in M}$, near Γ and (ν, ε). By means of Ξ, we can just study the bifurcation diagram of limit cycles in a conical region around the ε-axis.

Nevertheless, we have good reasons to expect that, one can find the maximal cyclicity (i.e. $2k-1$) in this region. At the end of this section, we show that, for a generic subfamily $(X_\lambda)_{\lambda \in M}$ of codimension $2k-1$ (as defined by definition 225 in section 5.3.3), the cyclicity is at least $2k-2$. As a consequence, since the related Abelian integral has at most k zeroes, there are at least $k-2$ alien limit cycles. As a consequence, already in this simple particular subfamily, there occur alien limit cycles.

Let us now compare the asymptotic expansions of Ξ and $\bar{\delta}$ with respect to the simple asymptotic scale $\mathcal{W}^*$ of the restricted logarithmic scale $\mathcal{L}^*$ and $\mathcal{W}$ of the logarithmic scale $\mathcal{L}$ respectively, given in (5.50) and (1.108) respectively. The unfolding Ξ expands in a simple asymptotic scale deformation of the restricted logarithmic scale, i.e. the logarithmic scale without the $\bar{x}\log\bar{x}$-term.

The reason lies in the fact that, for this particular subfamily $(X_\lambda)_{\lambda \in M}$, the parameter $\tau \equiv 0$ (by condition (5.28)). However, this point of difference is not essential. Besides, the non-identically vanishing of τ could produce only one extra zero; moreover, this zero would not correspond to an alien limit cycle, since τ is attached to the $x\log x$-term, and this term then would not disappear from the expansion of the Abelian integral.

Recall that, in case of the saddle loop, all limit cycles near the saddle loop, are covered by zeroes of the related Abelian integral. Therefore, at first sight, the resemblance with the saddle loop, seems to be in contradiction with the aims of this chapter.

More important to observe is that, in case of the saddle loop, the bifurcation diagram of limit cycles, near the saddle loop is studied by the reduced displacement map $\bar{\delta}$ itself, that, for $\varepsilon=0$, corresponds to the Abelian integral: $I_\nu \equiv \bar{\delta}\big|_{\varepsilon=0}$. Here, in case of the particular unfolding of the 2-saddle cycle, we perform a rescaling on the reduced difference map $\bar{\Delta}$, defining the map Ξ. By this rescaling, the origin of the terms, with respect to $\varepsilon=0$, is destroyed, in such a way that Ξ, for $\varepsilon=0$, has no affection anymore with the related Abelian integral.

Indeed, for $\varepsilon=0$, the expansion in (5.50) reduces to

$$\Xi(\bar{x},\bar{\nu},0)=\bar{\beta}-\bar{u}\bar{x}+\sum_{i=1}^{k-2}(-\bar{\eta}_i^{(1)}\bar{x}^{i+1}\log\bar{x}+\bar{\eta}_i^{(2)}\bar{x}^{i+1})+\bar{\gamma}_{k-1}^{(2)}(\bar{\nu},0)\,\bar{x}^k+\bar{x}^{1-2\delta}\bar{\Psi}_k(\bar{x},\bar{\nu},0).$$

From formulas (5.45) and (5.48), we can trace the origin of the coefficients, that appear in this expansion. The coefficient $\bar{\eta}_i^{(2)}$ of $\bar{x}^{i+1}$ is related to $\eta_i^{(2)}-\eta_i^{(1)}, \forall 1\leq i\leq k-2$, and is found in expansion (5.32) of the Abelian integral I_ν. Hence, this coefficient does not contribute to the creation of alien limit cycles. Also the coefficient $\bar{\gamma}_{k-1}^{(2)}$ of $\bar{x}^k$ is related to the corresponding one in the Abelian integral: $\eta_{k-1}^{(2)}-\eta_{k-1}^{(1)}$, and is non-zero.

The coefficient $\bar{\eta}_i^{(1)}$ of $\bar{x}^{i+1}\log\bar{x}$ is related to $\eta_i^{(1)}, \forall 1\leq i\leq k-2$. In the expansion of $\bar{\Delta}$, this coefficient is accompanied by a factor ε, and hence, the term $x^i\log x$ does not appear in the expansion of the Abelian integral. As a consequence, this coefficient can have a contribution in the creation of alien limit cycles.

From these observations, it is reasonable to expect that this particular unfolding of the 2-saddle cycle would create at least $k-2$ alien limit cycles. Nevertheless, one can expect the existence of one extra alien limit cycle, by use of the parameter variable ε in the $\bar{x}^k\bar{\omega}_{k-1}$-term, that appears in expansion (5.50) of Ξ. This was the key to find an alien limit cycle in the codimension 4 example in [DR].

Let us now end this section by proving that the cyclicity in a generic subfamily of codimension $2k-1$ is at least $2k-2$.

By theorem 230, the restriction of Ξ to $\varepsilon=0$, $\Xi|_{\varepsilon=0}$, has a generic expansion of codimension $l\equiv 2k-2$ in the restricted logarithmic scale (i.e. logarithmic scale without the $\bar{x}\log\bar{x}$- and the $\bar{x}^k\log\bar{x}$-terms). Therefore, by theorem 124 (using the remark below definition 122, that states that this restricted logarithmic scale has the Chebychev property), the cyclicity of $\Xi|_{\varepsilon=0}$ at $(\bar{x},\bar{\nu})=(0,0)$,is equal to l. As a trivial consequence, the map Ξ has cyclicity at least l at $\bar{x}=0$ and $(\bar{\nu},\varepsilon)=(0,0)$ (since the cyclicity is equal to l for the subfamily, induced by $\varepsilon=0$). However, it is not that trivial, to transfer this result, obtained for $\varepsilon=0$, to a result on the cyclicity for the particular subfamily $(X_\lambda)_{\lambda\in M}$.

Let us explain the difficulty in this transfer more precisely. We can find sequences $(\bar{x}_n^i)_{n\in\mathbb{N}}, 1\leq i\leq l$ and $(\bar{\nu}_n)_{n\in\mathbb{N}}$ such that

$$\forall 1\leq i\leq l: \lim_{n\to\infty}\bar{x}_n^i\to 0 \text{ and } \lim_{n\to\infty}\bar{\nu}_n\to 0$$

and $0<\bar{x}_n^1<\ldots<\bar{x}_n^l$ are simple zeroes of $\Xi(\cdot,\bar{\nu}_n,0)$, i.e. $\forall n\in\mathbb{N},\forall 1\leq i\leq l:$

$$\begin{aligned}\Xi\left(\bar{x}_n^i,\bar{\nu}_n,0\right)&=0\\ \frac{\partial}{\partial\bar{x}}\Xi\left(\bar{x}_n^i,\bar{\nu}_n,0\right)&\neq 0\end{aligned}$$

To translate this result into a result for $\bar{\Delta}$, we need to blow down to the (x,ν,ε)-space by means of (5.48) and $x=\varepsilon^2\bar{x}$. By this blow down, the sequence of parameter values $((\bar{\nu}_n,0))_{n\in\mathbb{N}}$ reduces to the constant sequence $((0,0))_{n\in\mathbb{N}}$; moreover, the corresponding zeroes $\bar{x}_n^i$ reduce all to 0 $(1\leq i\leq l,\in\mathbb{N})$. Clearly, this result says nothing about the cyclicity of the particular subfamily.

From this observation, it is seen that we need to be able to lift the result on cyclicity, that we obtain for the restriction $\Xi|_{\varepsilon=0}$, to a similar result on cyclicity for $\varepsilon>0$. Next, this last result can be translated into a result on the cyclicity of the particular subfamily $(X_\lambda)_{\lambda\in M}$ near Γ and $(\bar{\nu},\varepsilon)=(0,0)$. This lifting is possible by the following proposition, of which the conditions are satisfied by the map Ξ, defined in theorem 230 and $l=2k-2$.

Proposition 232 *Let $R,E>0$, let W be a neighbourhood of $\bar{\nu}=0$ in $\mathbb{R}^l$ and let $\Xi:]0,R[\times W\times[0,E[\to\mathbb{R}$ be a continuous map such that $\frac{\partial\Xi}{\partial\bar{x}}:]0,R[\times W\times[0,E[\to\mathbb{R}$ is a well-defined continuous map. If there exist sequences $(\bar{\nu}_n)_{n\in\mathbb{N}}$ in W and $(\bar{x}_n^i)_{n\in\mathbb{N}}$ in $]0,R[,1\leq i\leq l$ such that*

$$\lim_{n\to\infty}\bar{\nu}_n=0 \text{ and } \forall 1\leq i\leq l: \lim_{n\to\infty}\bar{x}_n^i=0 \tag{5.67}$$

and such that $\forall n\in\mathbb{N}:\bar{x}_n^1<\ldots<\bar{x}_n^l$ are simple zeroes of $\Xi(\cdot,\bar{\nu}_n,0)$, i.e., $\forall 1\leq i\leq l:$

$$\Xi\left(\bar{x}_n^i,\bar{\nu}_n,0\right)=0 \tag{5.68}$$

$$\frac{\partial}{\partial\bar{x}}\Xi\left(\bar{x}_n^i,\bar{\nu}_n,0\right)\neq 0. \tag{5.69}$$

Then, there exist sequences $(\varepsilon_n)_{n\in\mathbb{N}}$ in $]0,E[$ and $(\xi_n^i(\varepsilon_n))_{n\in\mathbb{N}}$ in $]0,R[,1\leq i\leq l$ such that

$$\lim_{n\to\infty}\varepsilon_n=0 \text{ and } \forall 1\leq i\leq l: \lim_{n\to\infty}\xi_n^i(\varepsilon_n)=0$$

and such that $0<\xi_n^1(\varepsilon_n)<\ldots<\xi_n^l(\varepsilon_n)$ are simple zeroes of $\Xi(\cdot,\bar{\nu}_n,\varepsilon_n)$, $\forall n\in\mathbb{N}$.

Proof. By the implicit function theorem (see remarks on the implicit function theorem in [D69] concerning the dependence on parameters), (5.68) and (5.69), we find, $\forall n \in \mathbb{N}$, a constant $E_n > 0$ and continuous curves $\xi_n^i, 1 \leq i \leq l$,

$$\xi_n^i : [0, E_n] \to]0, \infty[: \varepsilon \mapsto \xi_n^i(\varepsilon)$$

such that $\forall 1 \leq i \leq l : \xi_n^i(0) = \bar{x}_n^i$, and such that $\forall \varepsilon \in [0, E_n]$:

$$0 < \xi_n^1(\varepsilon) < \ldots < \xi_n^l(\varepsilon)$$

and $\forall 1 \leq i \leq l$

$$\Xi(\xi_n^i(\varepsilon), \bar{\nu}_n, \varepsilon) = 0 \text{ and } \frac{\partial \Xi}{\partial \bar{x}}(\xi_n^i(\varepsilon), \bar{\nu}_n, \varepsilon) \neq 0.$$

Then, for each $n \in \mathbb{N}$, we can take an $\varepsilon_n > 0$ with $\varepsilon_n \downarrow 0$ for $n \to \infty$, such that $\xi_n^i(\varepsilon_n) \to 0$, for $n \to \infty$. In this way, we have found the required sequences. ■

Appendix A

Lyapunov quantities

Here we give an alternative proof of proposition 58, due to C. Christopher and N. Lloyd, according to the proof of lemma 47 given in [S]:

Proposition 233 *Consider the C^γ planar vector field $D = P\frac{\partial}{\partial x} + Q\frac{\partial}{\partial y}$ where*

$$\begin{cases} P(x,y) &= y + \sum_{i=1}^{N} a_{2i}x^{2i} + \sum_{i=k}^{N} a_{2i+1}x^{2i+1} + O\left(\|(x,y)\|^{2N+2}\right) \\ Q(x,y) &= -x + O\left(\|(x,y)\|^{2N+2}\right), \quad (x,y) \to (0,0) \end{cases}$$

with $k \in \mathbb{N}, k \ll N$ and $\gamma \in \mathbb{N}, \gamma \geq 2N+2$ or $\gamma \in \{\infty, \omega\}$. If $\{V_i : i \in \mathbb{N}\}$ is a set of Lyapunov quantities for D at the focus at the origin, then

$$\begin{cases} V_l \equiv 0, \forall 0 \leq l < k \\ V_k = \alpha \cdot a_{2k+1} \end{cases}$$

where $\alpha \in \mathbb{Q}^+ \setminus \{0\}$.

Proof. Recall the notations from [S]:

$$D_i = P_i\frac{\partial}{\partial x} + Q_i\frac{\partial}{\partial y}, \forall 1 \leq i \leq 2N+1$$

where $P_i, Q_i \in \mathbb{R}[x,y]$ are homogeneous polynomials such that

$P(x,y) = \sum_{i=1}^{2N+1} P_i(x,y) + O\left(\|(x,y)\|^{2N+2}\right)$ and
$Q(x,y) = \sum_{i=1}^{2N+1} Q_i(x,y) + O\left(\|(x,y)\|^{2N+2}\right)$, for $(x,y) \to (0,0)$

We will prove this property by induction on k.In particular, we will prove property $\mathcal{P}(k)$ by induction on k, where property $\mathcal{P}(k)$ means:

1. the homogenous polynomials F_{2j+1} $(0 \leq j \leq k)$ are odd with respect to y
2. the homogenous polynomials F_{2j+2} $(0 \leq j \leq k-1)$ are even with respect to y
3. the coefficients corresponding to odd powers of y in F_{2k+2} are equal to a_{2k+1}, up to multiplication by a rational number
4. the k-th Lyapunov quantity V_k is equal to a_{2k+1}, up to multiplication by a rational number different from zero.

For $k = 0$, there is nothing to prove, since

$$F_1 \equiv 0, F_2(x, y) = \frac{x^2 + y^2}{2} \text{ and } V_0 = \frac{1}{2} a_1$$

Induction step We show now that $\mathcal{P}(k-1)$ implies $\mathcal{P}(k)$. By $\mathcal{P}(k-1)$, we know that F_{2l+1} (respectively F_{2l+2}) is an odd (respectively even) polynomial with respect to y, $\forall 0 \leq l \leq k-1$. The coefficients of F_{2k+1} are found from the equation

$$D_1 F_{2k+1} = -\left[D_2 F_{2k} + D_3 F_{2k-1} + D_4 F_{2k-2} + \ldots + D_{2k} F_2\right] \tag{A.1}$$

For $j < 2k+1$ odd, $D_j \equiv 0$; as a consequence, equation $(A.1)$ reduces to

$$D_1 F_{2k+1} = -\sum_{j=1}^{k} D_{2j} F_{2(k+1-j)} \tag{A.2}$$

In the right-hand side of equation $(A.2)$, there only appear homogenous polynomials F_l with an *even* index l, where $2 \leq l \leq 2k$. Therefore, the right-hand side of equation $(A.2)$ contains only terms with even powers of y. If $f_{i,j}$ is defined as the coefficient of F_i corresponding to y^j, then equation $(A.2)$ defines a linear system:

$$\mathcal{F}_{2k+1} \cdot \begin{bmatrix} f_{2k+1,1} \\ f_{2k+1,2} \\ \vdots \\ \vdots \\ f_{2k+1,j} \\ \vdots \\ f_{2k+1,2k+1} \end{bmatrix} = \begin{bmatrix} * \\ 0 \\ * \\ \vdots \\ \vdots \\ * \\ 0 \end{bmatrix}$$

where $\mathcal{F}_{2k+1}$ is the sparse matrix given by

$$\mathcal{F}_{2k+1} = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & \cdots & 0 \\ -(2k+1) & 0 & 2 & 0 & \cdots & \cdots & 0 \\ 0 & -2k & 0 & 3 & \ddots & & \vdots \\ 0 & 0 & -(2k-1) & 0 & \ddots & \ddots & \vdots \\ 0 & & 0 & -(2k-2) & \ddots & 2k & 0 \\ 0 & & & 0 & \ddots & & 2k+1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 \end{bmatrix}$$

(Only the diagonal above and under the main diagonal contain non-zero elements: $1, 2, \ldots 2k+1$ and the negatives of these numbers in the reverse order.) The even-numbered rows imply by backward substitution:

$$f_{2k+1,2j} = 0, \forall 0 \leq j \leq k$$

As a consequence, F_{2k+1} is an odd polynomial with respect to y.

The coefficients of F_{2k+2} are determined by the equation:

$$\begin{aligned} D_1 F_{2k+2} - V_k \left(x^2 + y^2\right)^{k+1} &= -\left[D_2 F_{2k+1} + D_3 F_{2k} + \ldots + D_{2k+1} F_2\right] \\ &= -\sum_{j=1}^{k} D_{2j} F_{2k+3-2j} - a_{2k+1} x^{2k+2} \qquad \text{(A.3)} \end{aligned}$$

where the equation $(A.3)$ follows from the fact that

$$D_{2k+1} F_2 = a_{2k+1} \frac{\partial}{\partial x} F_2 = a_{2k+1} x^{2k+2}$$

Equation $(A.3)$ corresponds to the following non-homogenous linear system in the indeterminates $f_{2k+2,j}, 0 \leq j \leq 2k+2$ and V_k :

$$\mathcal{F}_{2k+2} \cdot \begin{bmatrix} f_{2k+2,1} \\ f_{2k+2,2} \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ f_{2k+2,2k+2} \\ V_k \end{bmatrix} = \begin{bmatrix} -a_{2k+1} \\ * \\ 0 \\ * \\ \vdots \\ \vdots \\ * \\ 0 \end{bmatrix}$$

Appendix B

Algebraic Curves of Maximal Cyclicity

Here, we prove some technical lemma's that are used in chapter 4. More precisely, we prove in lemma 235 that the map $h^n : \mathbb{R}^{2n+1} \to \mathbb{R}^{2n+1}$ is a local diffeomorphism at $\left(\hat{a}, \hat{b}, 0\right)$, where the map h^n corresponds to the map h in the proof of proposition 213, in case $l = 2n$ is even and $l \geq 4$, is a local diffeomorphism at $\left(\hat{a}, \hat{b}, 0\right)$; moreover, we compute $\det Dh_{(\hat{a},\hat{b},0)}$. Next, in lemma 236, we prove that the map $g^n : \mathbb{R}^{2n+2} \to \mathbb{R}^{2n+2}$ is a local diffeomorphism at $\left(\hat{a}, \hat{b}, \hat{c}, 0\right)$, where g^n corresponds to the map $\bar{h}$ in the proof of proposition 213, in case $l = 2n+1$ is odd and $l \geq 5$. First, we recall some notations that are used here, and we give a preliminary lemma, that is useful in the computation of $\det Dh_{(\hat{a},\hat{b},0)}$.

Let $n \in \mathbb{N}_2$. Denote the constants

$$\left\{ \begin{array}{lcll} \hat{a}_i & = & \frac{2^i-1}{2^i}, & i = 1, \ldots, n \\ \hat{b}_i & = & -i, & i = 1, \ldots, n \\ \hat{c} & = & 0 & \end{array} \right. .$$

Write

$$\left\{ \begin{array}{ll} \hat{a} = (\hat{a}_1, \ldots, \hat{a}_n), & \hat{b} = (\hat{b}_1, \ldots, \hat{b}_n) \\ a = (a_1, \ldots, a_n), & b = (b_1, \ldots, b_n) \end{array} \right.$$

Define the polynomials $B_j^n : \mathbb{R}^{2n} \to \mathbb{R}, j = 1, 2, \ldots, 2n$ by

$$\begin{cases} B_1^n(a,b) & = \sum_{i=1}^{n} a_i \\ B_{2j}^n(a,b) & = \sum_{1 \le i_1 < \ldots < i_j \le n} b_{i_1} \ldots b_{i_j}, \qquad j = 1, \ldots, n \\ B_{2j+1}^n(a,b) & = \sum_{i=1}^{n} a_i \sum_{\substack{1 \le i_1 < \ldots < i_j \le n \\ i_k \ne i, \forall k = 1, \ldots, j}} b_{i_1} \ldots b_{i_j}, \quad j = 1, \ldots, n-1 \end{cases}.$$

Define the polynomials $C_j^n : \mathbb{R}^{2n+1} \to \mathbb{R}, j = 1, 2, \ldots, 2n+1$ by

$$\begin{cases} C_1^n(a,b,c) & = B_1^n(a,b) + c \\ C_{2j}^n(a,b,c) & = B_{2j}^n(a,b), \qquad j = 1, \ldots, n \\ C_{2j+1}^n(a,b,c) & = B_{2j+1}^n(a,b) + c \cdot B_{2j}^n(a,b), \quad j = 1, \ldots, n-1 \\ C_{2n+1}^n(a,b,c) & = c \cdot B_{2n}^n(a,b) \end{cases}. \tag{B.1}$$

Define $\forall n \in \mathbb{N}_3 : S(n) = n^{n-1} + \sum_{k=1}^{n-1} n^{n-1-k} B_{2k}^{n-1}(\hat{a}, \hat{b})$.

Lemma 234 $\forall n \in \mathbb{N}_3 : S(n) = (n-1)!$

Proof. We will prove by induction on n that $\forall x \in \mathbb{R}, \forall n \in \mathbb{N}_3$:

$$x^{n-1} + x^{n-2} B_2^{n-1} + x^{n-3} B_4^{n-1} + \ldots + x B_{2n-4}^{n-1} + B_{2n-2}^n = \prod_{j=1}^{n-1} (x + b_j) \tag{B.2}$$

where $B_{2j}^{n-1} = B_{2j}^{n-1}(a,b), \forall j = 1, \ldots, n-1$. By putting $x = n$ in equation $(B.2)$, the required result follows:

$$\begin{aligned} S(n) &= \prod_{j=1}^{n-1} \left(n + \hat{b}_j\right) \\ &= (n-1)! \end{aligned}$$

For $n = 3$, the left-hand side of (B.2) is equal to

$$\begin{aligned} x^2 + x B_2^2 + B_4^2 &= x^2 + (b_1 + b_2) x + b_1 b_2 \\ &= (x + b_1)(x + b_2). \end{aligned}$$

Suppose now that (B.2) holds for $n \in \mathbb{N}_3$. Then,

$$
\begin{aligned}
&x^n + x^{n-1}B_2^n + x^{n-2}B_4^n + \ldots + xB_{2n-2}^n + B_{2n}^n \\
&= x^n + x^{n-1}\left(B_2^{n-1} + b_n\right) + x^{n-2}\left(B_4^{n-1} + b_n B_2^{n-1}\right) + \\
&\ldots + x\left(B_{2n-2}^{n-1} + b_n B_{2n-4}\right) + b_n B_{2n-2}^{n-1} \\
&= x\left(x^{n-1} + x^{n-2}B_2^{n-1} + x^{n-3}B_4^{n-1} + \ldots + xB_{2n-4}^{n-1} + B_{2n-2}^{n-1}\right) \\
&+ b_n\left(x^{n-1} + x^{n-2}B_2^{n-1} + x^{n-3}B_4^{n-1} + \ldots + xB_{2n-4}^{n-1} + B_{2n-2}^{n-1}\right) \\
&= (x + b_n)\left(x^{n-1} + x^{n-2}B_2^{n-1} + x^{n-3}B_4^{n-1} + \ldots + xB_{2n-4}^{n-1} + B_{2n-2}^{n-1}\right) \\
&= (x + b_n)(x + b_{n-1})\ldots(x + b_1)
\end{aligned}
$$

This proves the induction step. ■

Lemma 235 *Let $n \in \mathbb{N}_2$. Suppose that $h^n = (h_1^n, \ldots, h_{2n}^n) : \mathbb{R}^{2n+1} \to \mathbb{R}^{2n+1}$ is differentiable of class C^γ (where $\gamma \in \mathbb{N}_1 \cup \{\infty, \omega\}$) with:*

$$
\begin{cases}
h_k^n(a,b,t) &= B_k^n(a,b) + O(t), \quad t \to 0 \ (\forall k = 1, \ldots, 2n) \\
h_{2n+1}^n(a,b,t) &= t + O(t^2), \quad t \to 0
\end{cases}
$$

Then there exists an open neighbourhood V of $(\hat{a}, \hat{b}, 0)$ in $\mathbb{R}^{2n+1}$ such that

$$h^n : V \to h^n(V)$$

is a C^γ-diffeomorphism.

Proof. We prove that the differential of B^n at $(\hat{a}, \hat{b})$, denoted by $dB^n_{(\hat{a},\hat{b})}$, is non-singular. Since $\det dB_{(\hat{a},\hat{b})} = \det dh^n_{(\hat{a},\hat{b},0)}$, the result then follows from the Inverse Function Theorem. The differential of B^n at $(\hat{a}, \hat{b})$, denoted by $dB^n_{(\hat{a},\hat{b})}$, is row-equivalent to:

$$
\begin{bmatrix}
dB_o^n(\hat{a}) & * \\
0 & dB_e^n(\hat{b})
\end{bmatrix},
$$

where $dB_o^n(\hat{a})$ denotes the differential at $\hat{a}$ of the map

$$B_o^{n-1} : \mathbb{R}^n \to \mathbb{R}^n : a = (a_1, \ldots, a_n) \mapsto \left(B_1^n, B_3^n, \ldots, B_{2n-1}^n\right)\left(a, \hat{b}\right)$$

and $dB_e^n(\hat{b})$ denotes the differential at $\hat{b}$ of the map

$$B_e^n : \mathbb{R}^n \to \mathbb{R}^n : b = (b_1, \ldots, b_n) \mapsto \left(B_2^n, B_4^n, \ldots, B_{2n}^n\right)(\hat{a}, b).$$

As a consequence,

$$\det dB^n(\hat{a}, \hat{b}) = (-1)^n \cdot \det dB_o^n(\hat{a}) \cdot \det dB_e^n(\hat{b})$$

By induction on $n \in \mathbb{N}_2$, we prove that $dB_o^n(\hat{a})$ and $dB_e^n(\hat{b})$ are non-singular; moreover,

$$\det dB_o^n(\hat{a}) = \det dB_e^n(\hat{b}) = S(2)\,S(3)\ldots S(n)\,.$$

As a consequence, and by lemma 234, we find that

$$\det dh^n_{(\hat{a},\hat{b},0)} = (-1)^n\left(2!\cdot 3!\cdot\ldots\cdot(n-1)!\right)^2.$$

For $n=2$,

$$\det dB_o^2(\hat{a}) = \det dB_e^2(\hat{b}) = \det\left[\begin{array}{cc} 1 & 1 \\ \hat{b}_2 & \hat{b}_1 \end{array}\right] = 1.$$

Suppose now that $n \in \mathbb{N}_3$ and that

$$\det dB_o^{n-1}(\hat{a}') = \det dB_e^{n-1}(\hat{b}') = S(2)\,S(3)\ldots S(n-1)\,,$$

where

$$\left\{\begin{array}{ll} a' = (a_1,\ldots,a_{n-1})\,, & b' = (b_1,\ldots,b_{n-1}) \\ \hat{a}' = (\hat{a}_1,\ldots,\hat{a}_{n-1})\,, & \hat{b}' = (\hat{b}_1,\ldots,\hat{b}_{n-1}) \end{array}\right. .$$

Notice that $\forall n \in \mathbb{N}_3$:

$$\left\{\begin{array}{lcll} B_1^n & = & B_1^{n-1} + a_n & \\ B_2^n & = & B_2^{n-1} + b_n & \\ B_{2k}^n & = & B_{2k}^{n-1} + b_n B_{2(k-1)}^{n-1} & \forall k = 2,\ldots n-1 \\ B_{2k+1}^n & = & B_{2k+1}^{n-1} + b_n B_{2k-1}^{n-1} + a_n B_{2k}^{n-1} & \forall k = 1,\ldots,n-2 \\ B_{2n-1}^n & = & b_n B_{2n-3}^{n-1} + a_n B_{2n-2}^{n-1} & \\ B_{2n}^n & = & b_n B_{2n-2}^{n-1} & \end{array}\right. \tag{B.3}$$

where the functions B_j^n on the left-hand side are evaluated at (a,b) and the functions on the right-hand side are evaluated at (a',b'). From the recursion-formulas in $(B.3)$, it follows that

$$dB_e^n(\hat{b}) = \left[\begin{array}{ccc|c} \frac{\partial}{\partial b_1}B_2^{n-1} & \cdots & \frac{\partial}{\partial b_{n-1}}B_2^{n-1} & 1 \\ \hline \multicolumn{3}{c|}{} & B_2^{n-1} \\ \multicolumn{3}{c|}{\left(\frac{\partial}{\partial b_j}B_{2k}^{n-1} + b_n\frac{\partial}{\partial b_j}B_{2(k-1)}^{n-1}\right)_{j=1\ldots n-1,k=2\ldots n-1}} & B_4^{n-1} \\ \multicolumn{3}{c|}{} & \vdots \\ \multicolumn{3}{c|}{} & B_{2n-4}^{n-1} \\ \hline b_n\frac{\partial}{\partial b_1}B_{2(n-1)}^{n-1} & \cdots & b_n\frac{\partial}{\partial b_{n-1}}B_{2(n-1)}^{n-1} & B_{2n-2}^{n-1} \end{array}\right],$$

where the functions in the matrix are evaluated at $(\hat{a}',\hat{b}')$. Next, we subsequently apply the following elementary row-operations on the rows $(R_1,\ldots,R_n)$ of $dB_e^n(\hat{b})$:

$$\left\{\begin{array}{lcl} R_1' & = & R_1, \text{ and } \forall k = 1,\ldots,n-1 : \\ R_{k+1}' & = & R_{k+1} - b_n R_k' \end{array}\right. , \tag{B.4}$$

and we obtain the matrix

$$\left[\begin{array}{c|c} dB_e^{n-1}(\hat{b}') & * \\ \hline 0 & S(n) \end{array}\right].$$

Hence, we obtain

$$\det dB_e^n(\hat{b}) = dB_e^{n-1}(\hat{b}') \cdot S(n).$$

Analoguously, from the recursion formulas $(B.3)$, we have:

$$dB_o^n(\hat{a}) = \left[\begin{array}{ccc|c} \frac{\partial}{\partial a_1} B_1^{n-1} & \ldots & \frac{\partial}{\partial a_{n-1}} B_1^{n-1} & 1 \\ \hline & & & B_2^{n-1} \\ & & & B_4^{n-1} \\ \multicolumn{3}{c|}{\left(\frac{\partial}{\partial a_j} B_{2k+1}^{n-1} + b_n \frac{\partial}{\partial a_j} B_{2k-1}^{n-1}\right)_{j=1\ldots n-1, k=1\ldots n-2}} & \vdots \\ & & & B_{2n-4}^{n-1} \\ \hline b_n \frac{\partial}{\partial a_1} B_{2n-3}^{n-1} & \ldots & b_n \frac{\partial}{\partial a_{n-1}} B_{2n-3}^{n-1} & B_{2n-2}^{n-1} \end{array}\right],$$

where the functions in the matrix are evaluated at $\left(\hat{a}', \hat{b}'\right)$. Again, we subsequently apply the elementary row-operations $(B.4)$ on the rows $(R_1, \ldots, R_n)$ of $dB_o^n(\hat{a})$, and we obtain the matrix

$$\left[\begin{array}{c|c} dB_o^{n-1}(\hat{a}') & * \\ \hline 0 & S(n) \end{array}\right].$$

Hence,

$$\det dB_o^n(\hat{a}) = dB_o^{n-1}(\hat{a}') \cdot S(n).$$

This proves the induction step. ■

Lemma 236 *Let $n \in \mathbb{N}_2$. Suppose that $g^n = (g_1^n, \ldots, g_{2n}^n) : \mathbb{R}^{2n+2} \to \mathbb{R}^{2n+2}$ is differentiable of class C^γ (where $\gamma \in \mathbb{N}_1 \cup \{\infty, \omega\}$) with:*

$$\begin{cases} g_k^n(a,b,c,t) &= C_k^n(a,b,c) + O(t), \quad t \to 0 \ (\forall k = 1, \ldots 2n+1) \\ g_{2n+2}^n(a,b,t) &= t + O(t^2), \qquad t \to 0 \end{cases} \tag{B.5}$$

Then, there exists an open neighbourhood V of $(\hat{a}, \hat{b}, \hat{c}, 0)$ in $\mathbb{R}^{2n+2}$ such that

$$g^n : V \to g^n(V)$$

is a C^γ-diffeomorphism.

Proof. From $(B.1)$, it is clear that the differential of C^n at $(\hat{a}, \hat{b}, \hat{c})$ is given by

$$dC^n(\hat{a}, \hat{b}, \hat{c}) = \left[\begin{array}{c|c} dB^n(\hat{a}, \hat{b}) & * \\ \hline 0 & \frac{\partial}{\partial c} C_{2n+1}^n(\hat{a}, \hat{b}, \hat{c}) \end{array}\right],$$

and

$$\frac{\partial}{\partial c} C^n_{2n+1}(\hat{a}, \hat{b}, \hat{c}) = \hat{b}_1 \cdot \hat{b}_2 \cdot \ldots \cdot \hat{b}_n \neq 0.$$

From $(B.5)$, it then follows that

$$\begin{aligned}\det dg^n(\hat{a}, \hat{b}, \hat{c}, 0) &= \det dC^n(\hat{a}, \hat{b}, \hat{c}) \\ &= \det dB^n(\hat{a}, \hat{b}) \cdot \frac{\partial}{\partial c} C^n_{2n+1}(\hat{a}, \hat{b}, \hat{c}).\end{aligned}$$

By lemma 235, it then follows that

$$\begin{aligned}\det dg^n(\hat{a}, \hat{b}, \hat{c}, 0) &= (-1)^n \left(2! \cdot 3! \cdot \ldots \cdot (n-1)!\right)^2 \cdot \hat{b}_1 \cdot \hat{b}_2 \cdot \ldots \cdot \hat{b}_n \\ &= \left(2! \cdot 3! \cdot \ldots \cdot (n-1)!\right)^2 n! \neq 0\end{aligned}$$

The required result now follows from the Inverse Function Theorem. ■

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Index

Limietcycli nabij vectorvelden van het centrumtype

Inleiding

Het onderwerp van deze thesis betreft de cycliciteit en bifurcatiediagrammen van limietcycli in C^∞ en C^ω families $(X_\lambda)_\lambda$ van vlakke vectorvelden.

Als we de paramater λ lichtjes wijzigen (perturberen), treden er mogelijks veranderingen op in het faseportret van X_λ, de zogenaamde bifurcaties. De parameterwaarde λ^0, waarvoor er zo'n bifurcatie optreedt, wordt de bifurcatie- of limietwaarde genoemd, en het bijhorende vectorveld het bifurcatie- of limiet-vectorveld.

In deze thesis, vinden we in het faseportret van het limietvectorveld X_{λ^0} een continue band van periodieke banen of een continue schijf van periodieke banen die een singulariteit omringen; we zeggen dat het limietvectorveld X_{λ^0} van het centrumtype is. Een typisch voorbeeld van zo'n vectorveld is een Hamiltoniaans vectorveld.

Een limietcyclus van een vectorveld X_λ is een geïsoleerde periodieke baan van X_λ. Onze aandacht gaat naar het aantal limietcycli (en hun relatieve ligging) van X_λ, die ontstaan door perturbatie van een zogenaamde limiet-periodieke verzameling Γ van X_{λ^0}. In deze thesis is Γ ofwel een niet-geïsoleerde (reguliere) baan, ofwel een niet-ontaard elliptisch punt, ofwel een 2-zadelcyclus. Een niet-ontaard elliptisch punt dat omringd wordt door niet-geïsoleerde periodieke banen, wordt ook een centrum genoemd.

De studie van het aantal limietcycli en hun bifurcaties wordt gemotiveerd door Hilbert's zestiende probleem, dat essentieel vraagt naar het maximaal aantal limietcycli in vlakke polynomiale vectorvelden. Vanaf graad 2 is het een open probleem of het aantal limietcycli begrensd is in functie van de graad. Een aanpak van dit probleem bestaat er in gekende gewone differentiaalvergelijkingen te perturberen en na te gaan hoeveel limietcycli er op die manier kunnen ontstaan. Aldus wordt het zestiende probleem van Hilbert, een globaal probleem, omgezet in lokale problemen, de zogenaamde problemen van eindige cycliciteit.

Laten we benadrukken dat de studie in deze thesis steeds van lokale aard is. We zoeken naar limietcycli van vectorvelden X_λ, in de buurt van een gegeven limiet-periodieke verzameling Γ, waarbij de parameterwaarde λ dichtbij λ^0 ligt. De cycliciteit van $(X_\lambda)_\lambda$ langsheen Γ voor de parameterwaarde $\lambda = \lambda^0$, is het maximaal aantal limietcycli die vanuit Γ kunnen ontstaan na perturbatie van X_{λ^0}. Dit getal wordt genoteerd door

$$\mathrm{Cycl}\left(X_\lambda, (\Gamma, \lambda^0)\right).$$

In deze thesis komen er drie problemen aan bod, die respectievelijk onderzocht worden in hoofdstukken 2 en 3, hoofdstuk 4 en hoofdstuk 5.

Als eerste probleem werd er hoofdzakelijk gezocht naar uniforme resultaten aangaande Hopf-Takens bifurcaties in de nabijheid van een centrum, die uniform zijn in zowel de fasevariabele alsook de parametervariabele.

Als tweede probleem werd de vraag gesteld hoe 1-parametertechnieken gebruikt kunnen worden om de cycliciteit langsheen een niet-geïsoleerde periodieke baan in een multi-parameterfamilie te bepalen.

Het derde probleem betreft het cycliciteitsprobleem bij een 2-zadelcyclus, waarvan in de ontvouwing slechts één connectie gebroken wordt. Indien ε een ééndimensionale parametervariabele is, die de centra induceert, wordt de overdracht van eindige cycliciteitsresultaten door lineaire benadering in ε in vraag gesteld.

Hoofdstuk 1. Voorkennis en technische eigenschappen

In dit hoofdstuk worden een aantal gekende technieken herhaald, die gebruikt worden in de studie van de cycliciteit en bifurcatiediagrammen van limietcycli. Daarnaast worden er bij deze technieken enkele nieuwe technische resultaten bewezen, en de notie 'het voorkomen van centra in een regulier hypervlak' die gebruikt wordt in hoofdstuk 2 wordt hier ook ingevoerd en bestudeerd.

Traditioneel wordt aan een familie van vlakke vectorvelden $(X_\lambda)_\lambda$, nabij een limiet-periodieke verzameling Γ, een familie van reëelwaardige functies δ_λ in een 1-dimensionale variabele s geassocieerd, de zogenaamde 'verplaatsings-afbeeldingen'. Dit gebeurt op een manier dat configuraties van limietcycli γ van X_λ nabij Γ overeenkomen met configuraties van geïsoleerde nulpunten s van de overeenkomstige verplaatsingsafbeelding δ_λ nabij s_0. Een vectorveld X_λ van het centrumtype wordt vertaald naar het identiek nul zijn van de overeenkomstige verplaatsingsafbeelding δ_λ. Bovendien kan de cycliciteit uitgedrukt worden in termen van $(\delta_\lambda)_\lambda$, als het maximaal aantal nulpunten dat kan ontstaan vanuit s_0 door pertubatie van δ_{λ^0}. Wanneer het aantal nulpunten berekend wordt met de multipliciteit van elk nulpunt meegerekend, spreken we van multipliciteit; dit getal wordt genoteerd door $\mathrm{Mult}\left(X_\lambda, (\Gamma, \lambda^0)\right)$.

Indien de limiet-periodieke verzameling Γ een reguliere periodieke baan of een niet-ontaard elliptisch punt is, erft de familie verplaatsingsafbeeldingen de differentieerbaarheidsklasse (C^∞ of C^ω) van de familie vectorvelden $(X_\lambda)_\lambda$, lokaal rond Γ en λ^0; in dit geval spreken we van de reguliere limiet-periodieke verzamelingen. Indien de limiet-periodieke verzameling Γ een hyperbolische polycyclus is (bijvoorbeeld een zadelconnectie of een 2-zadelcyclus), erft de familie verplaatsingsafbeeldingen de differentieerbaarheidsklasse (C^∞ of C^ω) van de familie vectorvelden $(X_\lambda)_\lambda$, lokaal nabij Γ en λ^0, maar niet in het punt s_0 overeenkomstig met Γ. Daar is de functie δ_λ slechts continu.

Laten we eerst het gaval beschouwen van de reguliere limiet-periodieke verzamelingen. Voor differentieerbare functies kan men voor de analyse van nulpunten gebruikmaken van de impliciete functiestelling, de stelling van Rolle en de preparatiestelling (van Weierstrass of Malgrange). Om een bovengrens voor het aantal nulpunten te bepalen is er het welbekende delings-afleidingsalgoritme, dat gebaseerd is op de stelling van Rolle.

Voordat deze stellingen aangewend kunnen worden, moet eerst de ontaarding, die veroorzaakt wordt doordat het vectorveld X_{λ^0} van het centrumtype is, verwijderd worden. Dit is de achterliggende gedachte van de technieken, waarbij Melnikovfuncties of het Bautin ideaal berekend worden [R98].

Indien we te maken hebben met een ééndimensionale parameter λ, die centra induceert, wordt deze meestal door ε genoteerd. In dit geval, kan de verplaatsingsafbeelding δ_ε bestudeerd worden door de eerste niet-nulle afgeleide naar ε, in $\varepsilon = 0$, zeg $M_k = \left.\frac{\partial^k \delta}{\partial \varepsilon^k}\right|_{\varepsilon=0}$, te berekenen; deze afgeleide M_k wordt de k-de orde Melnikovfunctie genoemd. Deze techniek is praktijkvriendelijk in de zin dat er een algoritme bestaat om deze functies te berekenen ([F96],[P]). Het nadeel van deze techniek is dat ze enkel kan gebruikt worden wanneer de parameter, die centra induceert, ééndimensionaal is. Wanneer deze parameter λ multi-dimensionaal is, kan men gebruikmaken van het Bautin ideaal. Aan de hand van de familie $(\delta_\lambda)_\lambda$ wordt dit ideaal gedefinieerd in de ring van analytische functiekiemen in λ^0. De nulverzameling van dit ideaal bevat de parameterwaarden λ voor de welke het overeenkomstige vectorveld X_λ van het centrumtype is. Daarnaast kan de verplaatsingsafbeelding ook ontbonden worden in termen van het Bautin ideaal $\mathcal{I}$. Bovendien wordt een bovengrens voor de cycliciteit afgeleid in termen van dit ideaal, de zogenaamde index; dit getal wordt genoteerd door $\mathrm{Index}(X_\lambda, (\Gamma, \lambda^0))$. Deze techniek is theoretisch zeer sterk; maar in praktijk bijna nooit te gebruiken, daar we meestal geen expliciete uitdrukking hebben voor de verplaatsingsafbeelding, of het Bautin ideaal. Daarom dat we in hoofdstuk 4 zullen kijken hoe we resultaten over de cycliciteit van 1-parameterfamilies kunnen omzetten naar een resultaat over cycliciteit van de multi-parameterfamilie.

Een andere techniek, die gebruikt wordt in de studie van limietcycli in de nabijheid van een niet-ontaard elliptisch punt, is de berekening van zogenaamde Lyapunov getallen. In dit hoofdstuk wordt een definitie van de Lyapunov getallen gegeven door een algebraïsch lemma, dat een rechtsstreekse veralgemening is van een lemma uit [S]. Het nieuwe resultaat hierin is dat we ook de afhankelijkheid van parameters beschouwen en Γ mag zowel een focus als een centrum zijn voor het lineair deel. Daarnaast wordt hier het verband afgeleid dat er bestaat tussen de Lyapunov getallen, de coëfficiënten uit de normaalvorm en de voortbrengers van het Bautin ideaal. Deze resultaten zijn gepubliceerd in [CD1]. Ook wordt de berekening van Lyapunov getallen geïllustreerd voor een familie van Liénard vergelijkingen.

Deze drie technieken, de techniek van het Bautin ideaal, Melnikovfuncties en Lyapunov getallen, worden ook bestudeerd in het speciale geval dat centra enkel geïnduceerd worden door parameterwaarden die gelegen zijn in een regulier hypervlak, $\varepsilon = 0$. Dit is de situatie die we zullen tegenkomen in hoofdstuk 2.

Het laatste deel van hoofdstuk 1 betreft een grafiek als limiet-periodieke verzameling. In dat geval is de verplaatsingsafbeelding niet langer differentieerbaar in het punt dat overeenkomt met de grafiek. De hoger vernoemde stellingen kunnen dan niet meer gebruikt worden voor de studie van limietcycli. 'Deformaties van een eenvoudige asymptotische schaal', vormen een goed kader voor de beschrijving van het type niet-differentieerbaarheid dat we hier tegenkomen [DR]. Hier wordt ook aangetoond dat de Abelse integraal (eerste orde Melnikovfunctie) over een polycyclus een asymptotische ontwikkeling heeft in de logaritmische schaal. Verder beschrijven we ook het gebruik van deze deformaties in de studie van de cycliciteit en het bifurcatiediagram van limietcycli in het geval van de zadelconnectie [Mar] en de 2-zadelcyclus [DR].

Hoofdstuk 2. Hopf-Takens bifurcaties en centra

Een bekend voorbeeld van een stabiel bifurcatiepatroon van limietcycli is de Andronov-Hopf bifurcatie in de nabijheid van een niet-ontaard elliptisch punt, de zogenoemde Hopf-singulariteit. Dankzij de Impliciete functiestelling weten we dat na kleine perturbaties van X_{λ^0}, deze singulariteit blijft bestaan en geen nieuwe singulariteiten gecreëerd worden. Nochtans is het mogelijk dat de stabliteit van deze singulariteit verandert onder de perturbatie, en dat deze verandering van stabiliteit gepaard gaat met het verdwijnen of verschijnen van een limietcyclus, die deze singulariteit omringt. Dit belangrijke bifurcatiefenomeen wordt de Andronov-Hopf bifurcatie genoemd.

Veralgemeningen van de Andronov-Hopf bifurcatie, die aanleiding geven tot meerdere limietcycli, worden veralgemeende Hopf bifurcaties of Hopf-Takens bifurcaties genoemd. De generische veralgemeende Hopf bifurcaties zijn uitvoerig bestudeerd in [T], in het geval er geen centra voorkomen.

Nochtans duiken vaak perturbaties op vanuit centra, waarbij men voortdurend Hopf-Takens bifurcaties beschouwt. Daarom is het interessant om na te gaan hoe de studie in [T] kan veralgemeend worden naar situaties waarin centra voorkomen. We beschouwen niet de meest algemene situatie, maar we beperken ons tot de situatie waarbij centra geïnduceerd worden door parameterwaarden in een regulier hypervlak. Daarnaast werken we ook nog een voorbeeld uit, waarbij er twee parameters verantwoordelijk zijn voor de centra. Hier komen naast de Hopf-bifurcatie ook nog zogenaamde randbifurcaties voor; dit is het fenomeen waarbij er limietcycli uit het domein ontsnappen door de rand.

In [T] worden nodige en voldoende voorwaarden bepaald om de aanwezigheid van een generische Hopf-Takens bifurcatie te garanderen. In deze thesis drukken we vooreerst deze nodige en voldoende voorwaarden uit in termen van Lyapunov getallen.

Daarnaast veralgemenen we de studie van [T] in het geval dat er centra voorkomen, en deze geïnduceerd worden door een regulier hypervlak. In dit geval geven we, voor elk van de technieken die gebruikt worden in de studie van een Hopf-singulariteit, nodige en voldoende voorwaarden voor de aanwezigheid van centra: i.e. in termen van normaalvormen, Lyapunov getallen en Melnikov functies. Belangrijk op te merken is dat we in onze studie een uniform resultaat bekomen, zowel in de faseruimte alsook in de parameterruimte.

Deze resultaten zijn gepubliceerd in [CD1].

Hoofdstuk 3. Veralgemeende Liénard vergelijkingen

In dit hoofdstuk bestuderen we bifurcaties van limietcycli in de nabijheid van een centrum in zowel klassieke als veralgemeende Liénard families. Onze aandacht gaat naar de cycliciteit en de aanwezigheid van Hopf-Takens bifurcaties. We gebruiken de eenvoud van de Liénard familie ter illustratie van de voordelen van de techniek van het Bautin ideaal, dat voortgebracht wordt door Lyapunov getallen, en ter illustratie van de studie uit het voorgaande hoofdstuk.

Hoofdstuk 4. Algebraïsche krommen van maximale cycliciteit

In dit hoofdstuk beschouwen we analytische families van vlakke vectorvelden $(X_\lambda)_\lambda$. Hier onderzoeken we methoden om de cycliciteit langsheen een niet-geïsoleerde periodieke baan Γ in een multi-parameterfamilie te berekenen, aan de hand van 1-parametersubfamilies. Gebruikmakend van de desingularisatie-theorie van Hironaka, construeerde Roussarie een polynomiale kromme ζ in de parameterruimte, zodat de index in de 1-parametersubfamilie, geïnduceerd door ζ, overeenkomt met de index van de multi-parameterfamilie [R00], i.e.

$$\mathrm{Index}\left(X_\lambda, (\Gamma, \lambda^0)\right) = \mathrm{Index}\left(X_{\zeta(\varepsilon)}, (\Gamma, 0)\right).$$

Een dergelijke kromme wordt een kromme van maximale index genoemd (mic). In de geest van dit resultaat, bewijzen we hier vooreerst dat het multi-parameter cycliciteitsprobleem kan herleid worden naar een 1-parameter cycliciteitsprobleem, in de zin dat er analytische krommen in de parameterruimte bestaan waarlangs de maximale cycliciteit (respectievelijk multipliciteit) wordt bereikt. In dit geval spreken we van een kromme van maximale cycliciteit (mcc) (respectievelijk kromme van maximale multipliciteit (mmc)). Dit resultaat is gepubliceerd in [Cau01] en wordt bewezen door gebruik te maken van de analytische meetkunde, en 'het curve selection lemma' in het bijzonder.

Met het oog op efficiënte algoritmes om de cycliciteit te berekenen, onderzoeken we of het mogelijk is dat er polynomiale of zelfs lineaire krommen van maximale index, cycliciteit of multipliciteit bestaan, afhankelijk van bepaalde eigenschappen van de families of het Bautin ideaal. In ieder geval tonen we aan dat voorzichtigheid is aangewezen, indien men conclusies trekt voor de multi-parameterfamilie op basis van resultaten van 1-parametersubfamilies, die geïnduceerd zijn door rechten in de parameterruimte.

In het geval dat het 'stratum van maximale cycliciteit (respectievelijk multipliciteit)' een niet-leeg inwendige heeft dat ophoopt in λ^0, tonen we aan dat er steeds een polynomiale kromme ζ van maximale cycliciteit (respectievelijk maximale multipliciteit) bestaat. In het bijzonder vinden we een kegel van mcc's (respectievelijk mmc's), die een zeker contact hebben met ζ. Om dit resultaat te bekomen, bewijzen we eerst een specificatie van het 'curve selection lemma' voor open subanalytische verzamelingen, dat steunt op de Łojasiewicz ongelijkheid.

Tenslotte bespreken we aanverwante vragen zoals het bestaan van een minimale detectiegraad en conische graad van maximale cycliciteit (respectievelijk multipliciteit) in een aantal specifieke voorbeelden.

De resultaten in dit hoofdstuk zijn gebundeld in [CD2].

Hoofdstuk 5. Twee-zadelcyclus

In dit hoofdstuk beschouwen we C^∞ families van vlakke vectorvelden $\left(X_{(\nu,\varepsilon)}\right)_{(\nu,\varepsilon)}$, die een Hamiltoniaans vectorveld X_H ontvouwen voor $\varepsilon = 0$, in de nabijheid van een 2-zadelcyclus Γ. Hierbij stelt ε een kleine, ééndimensionale parameter voor. Het onderzoek in dit hoofdstuk richt zich op de lineaire benadering I_ν van de verplaatsingsafbeelding $\delta_{(\nu,\varepsilon)}$ met betrekking tot ε, de zogenaamde Abelse integraal. We onderzoeken in welke mate de lineaire benadering van de verplaatsingsafbeelding $\delta_{(\nu,\varepsilon)}$ met betrekking tot ε, kan bijdragen om de cycliciteit te berekenen.

We weten dat, in het geval dat Γ een periodieke baan, een niet-ontaard elliptisch punt of een zadelconnectie [Mar] is, resultaten aangaande configuraties van nulpunten van de Abelse integraal I_ν op een triviale manier kunnen overgebracht worden naar resultaten aangaande configuraties van limietcycli, tenminste als de Abelse integraal een elementaire catastrofe voorstelt.

Wanneer Γ een hyperbolisch k-zadelcyclus is (met $k \geq 2$), is de overdracht van resultaten van de Abelse integraal niet meer triviaal in de ε-richting. Het bifurcatiediagram van een twee-zadelcyclus is bestudeerd in [DRR], en in het algemeen, van een k-zadelcyclus in [Mo]. Gebruikmakend van resultaten in [DRR] en [Mo], wordt in [DR] aangetoond dat de Abelse integraal een slechte benadering is voor de verplaatsingsafbeelding, van zodra de ontvouwing meer dan één connectie breekt: onder generische voorwaarden kan de ontvouwing k limietcycli produceren, waarvan er hoogstens één limietcyclus kan nagetrokken worden door de Abelse integraal.

Zelfs wanneer de twee-zadelcyclus slechts één connectie breekt, kan het maximaal aantal nulpunten van de Abelse integraal niet op een triviale manier overgedragen worden naar het maximaal aantal limietcycli.

In [DR], is er aangetoond dat er generische ontvouwingen van de 2-zadelcyclus bestaan, die slechts één connectie breken, waarvoor de cycliciteit 4 is, terwijl de Abelse integraal hoogstens 3 nulpunten heeft. Bijgevolg kan deze ontvouwing één limietcyclus creëren, die niet afkomstig is van een nulpunt van de Abelse integraal. Een dergelijke limietcyclus wordt een vreemde ('alien') limietcyclus genoemd.

We starten de studie van de zogenaamde 'verschilfunctie' Δ_λ, die hier fungeert als de verplaastingsafbeelding δ_λ. Het voordeel van de verschilfunctie is dat deze eenvoudiger te bestuderen is dan δ_λ, in het geval van de 2-zadelcyclus. Bovendien heeft Δ_λ dezelfde eigenschappen als δ_λ. Geïsoleerde nulpunten van Δ_λ komen overeen met limietcycli van X_λ. Ook wordt het feit dat het vectorveld $X_{(\nu,\varepsilon)}$ van het centrumtype is voor $\varepsilon = 0$, vertaald naar het feit dat de restrictie van de verschilfunctie tot $\{\varepsilon = 0\}$ identiek nul is. Daarom kunnen we de gereduceerde verschilfunctie $\bar{\Delta}_\lambda$ beschouwen, die gedefinieerd is door

$$\Delta_{(\nu,\varepsilon)} = \varepsilon \bar{\Delta}_{(\nu,\varepsilon)}.$$

Tenslotte valt de lineaire benadering van $\Delta_{(\nu,\varepsilon)}$ met betrekking tot $\varepsilon = 0$, ook samen met de Abelse integraal I_ν :

$$I_\nu = \bar{\Delta}_{(\nu,0)}.$$

Hier bestuderen we de structuur van de coëfficiënten, die voorkomen in de ontwikkeling van $\bar{\Delta}$ zoals die beschouwd werd in [DR], in meer detail. Deze studie doet ons de volgende bewering vermoeden: 'Een generische ontvouwing van een 2-zadelcyclus, waarvan één connectie niet gebroken wordt door de ontvouwing, kan $3k$ (respectievelijk $3k-1$) limietcycli creëren, terwijl de Abelse integraal hoogstens $2k+1$ (respectievelijk $2k$) nulpunten kan hebben'. Dit vermoeden impliceert het bestaan van minstens $k-1$ vreemde limietcycli. In het bijzonder bevestigt dit vermoeden dat de lineaire benadering I_ν niet volstaat om, door triviale overdracht van resultaten aangaande de nulpunten van I_ν, resultaten over limietcycli af te leiden.

Tenslotte beschouwen we een speciale subfamilie van deze ontvouwing van de 2-zadelcyclus, deze waarin de zadels na perturbatie lineair blijven. Door een hergroepering van de termen in de ontwikkeling van $\bar{\Delta}$, kunnen we, mits een herschaling in zowel parameter- als faseruimte, de studie van limietcycli herleiden naar de studie van geïsoleerde nulpunten van een ontvouwing Ξ. Deze afbeelding Ξ heeft een ontwikkeling in een deformatie van een eenvoudige asymptotische schaal, die zeer sterk gelijkt op de deformatie van de logaritmische schaal, die men tegenkwam bij de ontwikkeling van de gereduceerde verplaatsingsafbeelding $\bar{\delta}_\lambda$ in het geval van een zadelconnectie [R98]. Gebruikmakende van de resultaten in [Mar] voor $\bar{\delta}_\lambda$ in het geval van een zadelconnectie, tonen we hier aan dat een dergelijke subfamilie tenminste $k-2$ vreemde limietcycli kan produceren, als de Abelse integraal van codimensie k is.