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Faculteit Wetenschappen

Wild Quivers : on a Conjecture of Kac and the Ringel-Hall Algebra

Proefschrift voorgelegd tot het behalen van de graad van Doctor in de Wetenschappen, richting Wiskunde, te verdedigen door

Bert SEVENHANT

Promotor : Prof. Dr. M. Van den Bergh

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IN HET CENTRUM VAN DE KENNIS

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Het laatste paragraafje is een speciaal plaatsje. Tien letterkes kunnen niet genoeg uitdrukken, maar toch zal ik het erop houden. Dank je, Ilse*!*

[1] free translation: Thanks to the colleagues I met at conferences.

Introduction

A quiver is an oriented graph. More formally a quiver is a 4-tuple $Q = (Q_0, Q_1, h, t)$ where Q_0 denotes the set of vertices, Q_1 denotes the set of arrows and $t, h : Q_1 \to Q_0$ are the maps which associate to an arrow it starting and ending vertex. We tacitly assume our quivers to be finite: $|Q_0|, |Q_1| < \infty$.

A representation of Q over a base field $\Bbbk$ consists of a family of finite dimensional $\Bbbk$-vector spaces $(V_i)_{i\in Q_0}$ together with a family of $\Bbbk$-linear maps $(V_x : V_{tx} \to V_{hx})_{x\in Q_1}$. The representations of Q form an Abelian category. In particular we may speak of indecomposable representations.

If V is a representation of Q then its dimension vector $\dim V \in \mathbb{N}^{Q_0}$ is defined by $(\dim V)_i = \dim V_i$. If $\alpha \in \mathbb{N}^{Q_0}$ then any representation with dimension vector α is isomorphic to a representation of the form $((\Bbbk^{\alpha_i})_{i\in Q_0}, (\phi_x)_{x\in Q_1})$ where $\phi_x \in \operatorname{Hom}_{\Bbbk}(\Bbbk^{\alpha_{tx}}, \Bbbk^{\alpha_{hx}}) = \operatorname{Mat}^{\alpha_{tx}\times\alpha_{hx}}(\Bbbk)$. Thus if we define $\operatorname{Rep}(Q,\alpha) = \prod_{x\in Q_1} \operatorname{Mat}^{\alpha_{tx}\times\alpha_{hx}}(\Bbbk)$ then every representation with dimension vector α is isomorphic to a representation corresponding to a point in $\operatorname{Rep}(Q,\alpha)$. We refer to $\operatorname{Rep}(Q,\alpha)$ as the representation space of the pair (Q,α).

Define $\operatorname{Gl}(\alpha) = \prod_{i\in Q_0} \operatorname{Gl}(\alpha_i, k)$. The group $\operatorname{Gl}(\alpha)$ acts on by $\operatorname{Rep}(Q,\alpha)$ by conjugation and two elements of $\operatorname{Rep}(Q,\alpha)$ correspond to isomorphic representations if and only if they are in the same $\operatorname{Gl}(\alpha)$-orbit. Thus the problem of classifying isomorphism classes of representations with dimension vector α is equivalent to describing the orbit space $\operatorname{Rep}(Q,\alpha)/\operatorname{Gl}(\alpha)$.

Example 1 *If Q consists of one vertex and one loop then the $\operatorname{Gl}(\alpha)$-orbits in $\operatorname{Rep}(Q,\alpha)$ are characterized by the Jordan normal form of the corresponding matrices. So the representation theory of quivers may be viewed as a natural extension of a classical linear algebra problem.*

Let us now assume that $\Bbbk$ is algebraically closed. A fundamental result is that quivers may be naturally separated into three classes. To explain this let us assume that Q is not the disjoint union of two smaller quivers.

1. If for any α there are only a finite number of indecomposable representations (up to isomorphism) with dimension vector α then we say that Q is of finite representation type. This is equivalent to the underlying unoriented graph of Q being a (simply laced) Dynkin diagram. In this case the isomorphism classes of indecomposable representation of Q may be listed. The simplest example of this type is given by a quiver consisting of one vertex and no arrows.

2. If for any α the indecomposable representations (up to isomorphism) with dimension vector α depend on at most one parameter then we say that Q is of tame type. This is equivalent to the underlying unoriented graph of Q being a (simply laced) extended Dynkin diagram. In this case the indecomposable representations of Q can still be classified. The simplest example of this type is given by the one loop quiver. It is clear that in this case the indecomposable representations are given by the Jordan blocks.

3. A quiver which is neither of finite representation type or tame is said to be wild. The simplest example is given by a quiver consisting of one vertex and two loops.

In the wild case a full classification of indecomposable representations seems to be out of reach. As a first approximation one may consider the case $\mathbb{k} = \mathbb{F}_q$. Since this field is not algebraically closed is natural to consider *absolutely* indecomposable representations, i.e. those representations which remain indecomposable after tensoring with $\overline{\mathbb{k}}$.

Victor Kac [Kac83] proved that the number of orbits $o_{Q,\alpha}(q)$ and absolutely indecomposable orbits $a_{Q,\alpha}(q)$ in $\mathrm{Rep}(Q, \alpha)$ are polynomials in q with integer coefficients. He conjectured that the coefficients of $a_{Q,\alpha}(q)$ are non-negative and that in addition $a_{Q,\alpha}(0)$ is given by the multiplicity of α viewed as a root of a Kac-Moody Lie algebra naturally associated to Q. When Kac made these conjectures he had very little evidence in the wild case. Such evidence was provided by Le Bruyn, Molenberghs and recently Hua [LB88, LBM88, Hua98]. Very recently Crawley Boevey and Van den Bergh succeeded in proving the positivity conjecture for indivisible dimension vectors and the constant term conjecture for dimension vectors which are 1 in some vertex [CBVdB01].

The results in this thesis center around Kac's conjectures. Although we do not succeed in proving them, we find some equivalent statements, and also some related results which we think are interesting in their own right.

After a first introductory chapter containing basic definitions we present in Chapter 2 some more evidence on the positivity conjecture: we prove that it holds for an m-loop quiver with dimension vector up to 5 and m arbitrary.

In Chapter 3, which is based on [SVdB99b] we use the theory of symmetric functions to find a combinatorial reformulation of the constant term conjecture. The final statement amounts to a signed counting of certain words. While it seems quite plausible that such a counting may be carried out, we have not succeeded in doing it. An interesting side result of this chapter is an explicit expression for the Hall-Littlewood polynomial corresponding to a hook in terms of Schur functions. This generalizes a conjecture by Carbonara (proved by MacDonald) [Car98]. It is a tantalizing question if a similar result exists for more general partitions.

The Ringel-Hall algebra of a quiver is an algebra whose basis consists of the isomorphism classes of indecomposable representations. After adding a diagonal part and performing a Drinfeld double construction one obtains a Hopf algebra which looks very much like the quantum enveloping algebra of a Lie algebra [Gre95, Kap97, Xia97]. In Chapter 4 we carry out this construction in detail. As a result we obtain the precise relations which hold in the resulting algebra. These relations are used in the next chapter.

In Chapter 5, based on [SVdB99a], we introduce three isomorphisms between the (double) Ringel-Hall algebras of quivers with the same underlying graph. By composing these isomorphisms we obtain a new construction of Lusztig's braid group action on a quantum enveloping algebra.

In Chapter 6, based on [SVdB01] we show that the Ringel-Hall algebra is a quantized enveloping algebra of a generalized Kac-Moody Lie algebra. Such objects had been introduced by Kang [Kan95] in the generic case. Our methods are different and more general that those of Kang since they also apply in the non-generic case. One of our main results is a Weyl-Borcherds character formula for the Hall algebra. This yields another reformulation of Kac's constant term conjecture, this time in terms of the multiplicities of the imaginary simple roots of the Ringel-Hall algebra.

Our work in Chapters 4, 5, 6 has been the basis for [DX01a, DX01b, DX01c, XY01] where our results are generalized to arbitrary finite dimensional hereditary algebras (using similar methods). The connection with our work is given by the path algebra which is a certain hereditary algebra naturally associated to any quiver.

Inleiding

Een quiver (of pijlkoker) is een georiënteerde graaf. Als men aan elk punt een vectorruimte hecht, en aan elke pijl een lineaire afbeelding, heeft men een representatie van de quiver. Via basisveranderingen werken groepen Gl_n (samen Gl_α) op de verzameling van representaties met een vaste dimensievector. We kunnen directe sommen nemen van representaties, en zodoende onontbindbaarheid definiëren.

We onderscheiden 3 klassen van quivers. Degene met slechts een eindig aantal onontbindbare representaties (op isomorfisme na) hebben als onderliggende graaf een Dynkin diagram. Een tweede klasse bestaat uit de tamme quivers, waarvan de representatie-theorie vergelijkbaar is met die van de 1-lus quiver. Deze quiver staat in feite voor het Jordan probleem. Een representatie is een vectorruimte en een endomorfisme. Elke (absoluut) onontbindbare representatie is isomorf met een Jordan blok. Een ander voorbeeld is de situatie van 2 vectorruimtes en 2 lineaire afbeeldingen tussen deze twee ruimtes. Ondanks dat tamme quivers oneindig veel representaties hebben, zijn deze geklassificieerd. Helaas zijn de meeste quivers wild.

Het kleinste voorbeeld van een wilde quiver is er één met één punt en 2 lussen. Het komt overeen met het probleem om koppels van vierkante matrices the klassificiëren. Dit is al lang een open probleem. Elke quiver waarvan de representatie theorie dit probleem bevat is wild. Het lijkt op dit moment onmogelijk om voor elke klasse (t.o.v. van de actie Gl_a) een canonische representant te vinden, zoals dit kan voor de Jordan blokken. Om iets te kunnen zeggen over het aantal klassen werken we over eindige velden $\mathbb{F}_q$. Kac bewees dat zowel het aantal $o_{Q,\alpha}(q)$ klassen als het aantal absoluut onontbindbare klassen $a_{Q,\alpha}(q)$ een veelterm is in $\mathbb{Z}[q]$. Kac uitte ook de vermoedens dat de coëfficiënten allen positief zijn, en de de constante term $a_{Q,\alpha}(0)$ (als er geen lussen zijn) de dimensie van $\mathfrak{g}_\alpha$ in een Kac-Moody algebra $\mathfrak{g}$ geassocieerd aan de quiver.

Na een inleidend eerste hoofdstuk, waar basisdefinities van de quiver theorie worden gegeven, geven we in hoofdstuk 2 bescheiden aanwijzingen voor het vermoeden dat de coëfficiënten positief zijn. We vermelden dat Kac nauwelijks voorbeelden had van de veeltermen voor wilde quivers. We zullen aantonen dat voor elke m-lus quiver (met één enkele hoekpunt) het vermoeden waar is tot en met dimensie 5.

In het derde hoofdstuk vinden we een combinatorische herformulering van het constante-term-vermoeden. We gebruiken hierbij de theorie van symmetrische functies. We vertalen het vermoeden naar een eigenschap van Gaussiaanse multinomiaal coëfficiënten. Het is goed mogelijk dat dit nieuw vermoeden te bewijzen is met combinatoriek maar we zijn er niet in geslaagd.

De Ringel-Hall algebra van een quiver is een algebra met als basis de isomorfismeklassen van representaties van de quiver. Door een diagonaal gedeelte bij te voegen kunnen we hier een Hopf algebra van maken, en door een Drinfeld double constructie uit te voeren krijgen we een algebra $U(Q)$ die goed lijkt op de quantum omhullende algebra van een Lie algebra. De constructie wordt in hoofdstuk 4 uitgelegd.

In het 5de hoofdstuk definiëren we 3 isomorfismen op de Ringel-Hall algebras. We bewijzen een opmerking van Lusztig dat de Hall algebra onafhankelijk is van de oriëntatie, waarbij we Fourier transformaties gebruiken. Anderzijds breiden we de reflectiefunctoren die op quivers bestaan uit tot de volledige Hall algebra. Door beide isomorfismen te combineren krijgen we automorfismen van $U(Q)$ die, op zijn minst op de samenstellingsalgebra, de vlechtrelaties respecteren.

Ten slotte breiden we in hoofdstuk 6 de quantum omhullende van een Lie algebra uit. Daarmee kunnen we bewijzen dat de Hall algebra als een veralgemeende quantum omhullende algebra van een Kac-Moody Lie algebra kan worden beschouwd en dit leidt tot een nieuwe vertaling naar dit onderzoeksdomein van het constante-term vermoeden.

Chapter 1

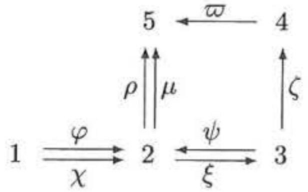
An Introduction to the Representation Theory of Quivers

In the 30 years that have elapsed since Peter Gabriels' stunning paper [Gab72], quivers have studied at from various angles. In this introduction we will follow to some extent the introductory paper of Kraft and Riedtmann [KR86].

1.1 Quiver Conventions

1.1.1 A Quiver and its Representations

A *quiver* $Q = (Q_0, Q_1, h, t)$ is a quadruple consisting of two finite sets and two maps. The first set is the set Q_0 of *vertices*. In most cases this will be either $\{0, 1, ..., n\}$ or $\{1, ..., n\}$. The second set is the set Q_1 of *arrows*. The two maps define the *tail* and *head* points of the arrows: $t, h : Q_1 \longrightarrow Q_0$. Of course these names correspond to the graphical presentation of a quiver.



$Q_0 = \{1, 2, 3, 4, 5\}$
$Q_1 = \{\varphi, \xi, \psi, \chi, \rho, \zeta, \mu, \varpi\}$
e.g. $t\varphi = 1$, $h\varphi = 2$, $h\varpi = 5$

Thus, a quiver is in fact an oriented graph. The corresponding unoriented graph is denoted by $\overline{Q}$.

Below we only will consider *connected* quivers. This means that there is a path of arrows (possibly reversed) between any two vertices.

A quiver is used to encode information about linear maps between vector spaces over a base field $\mathbb{k}$. In this section $\mathbb{k}$ is arbitrary but in the rest of the manuscript it will often be $\mathbb{C}$ or $\mathbb{F}_q$.

A *representation* V over $\mathbb{k}$ of a quiver Q consists of 2 sets: a set of finite dimensional $\mathbb{k}$-vector spaces $\{V_i \mid i \in Q_0\}$ and a set of $\mathbb{k}$-linear maps between them $\{V_\varphi : V_{t\varphi} \longrightarrow V_{h\varphi} \mid \varphi \in Q_1\}$.

$$\begin{array}{ccccc}
 & & V_5 & \xleftarrow{V_\varpi} & V_4 \\
 & & V_\rho \Big\uparrow\Big\uparrow V_\mu & & \Big\uparrow V_\zeta \\
V_1 & \underset{V_\chi}{\overset{V_\varphi}{\rightrightarrows}} & V_2 & \underset{V_\xi}{\overset{V_\psi}{\leftrightarrows}} & V_3
\end{array}$$

The *dimension vector* $\alpha = \dim V = \overrightarrow{V} \in \mathbb{N}^{Q_0}$ of a representation V is defined by $\alpha_i = \dim V_i$.

An important example of a representation is the following: choose a vertex i and suppose that all vector spaces are zero, except the one in vertex i, which is one-dimensional. Let all maps be 0-maps. Then one obtains a simple representation denoted by S_i.

A morphism between 2 representations V and W of a fixed quiver is a set $\{f_i : V_i \longrightarrow W_i \mid i \in Q_0\}$ of linear maps, such that for each arrow $\varphi \in Q_1$ we have $W_\varphi f_{t\varphi} = f_{h\varphi} V_\varphi$, or in other words the following diagram has to commute for each arrow φ.

$$\begin{array}{ccc}
V_{t\varphi} & \xrightarrow{V_\varphi} & V_{h\varphi} \\
{\scriptstyle f_{t\varphi}}\Big\downarrow & & \Big\downarrow{\scriptstyle f_{h\varphi}} \\
W_{t\varphi} & \xrightarrow[W_\varphi]{} & W_{h\varphi}
\end{array}$$

The category of representations of Q is described by $\mathrm{Rep}(Q)$. If all maps f_i are isomorphisms, then f is an isomorphism and V and W are said to be isomorphic. In that case: $\alpha = \overrightarrow{V} = \overrightarrow{W}$.

After choosing bases we get in this case $f_i \in \mathrm{Gl}_{\alpha_i}$. We define $\mathrm{Gl}_\alpha = \prod_{i \in Q_0} \mathrm{Gl}_{\alpha_i}$. Clearly $\mathrm{Gl}_\alpha \ni g$ acts on the set of representations.

$$g \bullet V = g \bullet (V_\varphi)_{\varphi \in Q_1} = (g_{h\varphi} V_\varphi g_{t\varphi}^{-1})_{\varphi \in Q_1}.$$

Given V and W the direct sum $V \oplus W$ is defined as the representation U for which $U_a = V_a \oplus W_a$ and $U_\varphi : V_{t\varphi} \oplus W_{t\varphi} \to V_{h\varphi} \oplus W_{h\varphi} : (v, w) \mapsto (V_\varphi(v), W_\varphi(w))$, or in matrix representation $U_\varphi = \begin{pmatrix} V_\varphi & 0 \\ 0 & W_\varphi \end{pmatrix}$.

A representation is *decomposable* if it is isomorphic to a non trivial direct sum and indecomposable otherwise. From the Krull-Schmidt theorem it follows that the decomposition of a representation into indecomposable representations is unique, up to isomorphism.

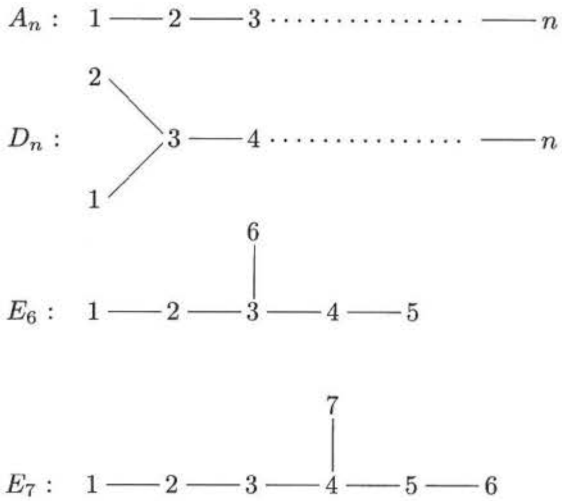
If the field $\mathbb{k}$ is not algebraic closed, it is possible that an indecomposable representation (over $\mathbb{k}$) decomposes over a field extension. If this is not the case then we call the representation *absolute indecomposable.* This will turn out to be a much more fundamental notion than indecomposability.

The affine space of representations of dimension α is denoted by $\text{Rep}(Q, \alpha)$. Since a representation may be viewed as a set of linear maps, and linear maps may be regarded as matrices, we may identify $\text{Rep}(Q, \alpha) = \bigoplus_{\varphi} \text{Mat}^{\alpha_{t\varphi} \times \alpha_{h\varphi}}$, where $\text{Mat}^{a \times b}$ denotes the set of $a \times b$ matrices.

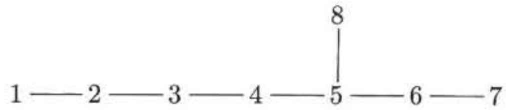
1.1.2 The Three Classes of Quivers

We have to make a distinction between 3 classes of quivers. This distinction can be introduced in several ways. The simplest way is using Dynkin and extended Dynkin diagrams, which are fundamental in Lie theory.

- The *connected* quivers of *finite representation type* are the (simply laced) Dynkin diagrams, when the orientation is omitted. The quivers of *finite representation type* are the disjoint unions of Dynkin diagrams. These quivers have only a finite number of indecomposable representations, up to isomorphism [Gab72]. The 3 types of simply laced Dynkin diagrams are listed below ($Q_0 = \{1, \ldots, n\}$).



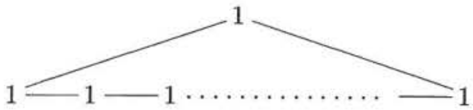
E_8 :



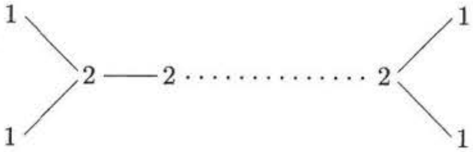
- The connected quivers which we obtain by putting an orientation on an extended Dynkin diagram are called *tame*. The quivers that are disjoint unions of Dynkin and extended Dynkin diagrams, with at least one of the latter are the *tame* quivers. In this case the indecomposables exist in one parameter families [DR76].

  Below is the list of simply laced extended Dynkin diagrams. The index of the diagrams is one less than the numbers of vertices, the numbers on the vertices correspond to the dimension vector δ, which is the smallest dimension vector for which there are more than one indecomposable representation.

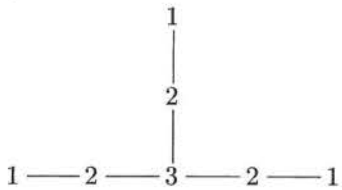
$\tilde{A}_n$:



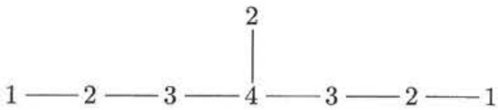
$\tilde{D}_n$:



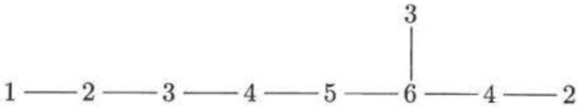
$\tilde{E}_6$:



$\tilde{E}_7$:



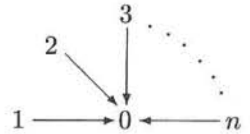
$\tilde{E}_8$:



- All other quivers (and thus most quivers) are called *wild*.

The main difference between the classes may be explained by looking at the representation theory of some easy examples. The quiver A_3 has 6 indecomposable representations. One can easily work this out by performing some base changes. Start with a random representation $1 \overset{V_\varphi}{\to} 2 \overset{V_\psi}{\leftarrow} 3$. First find a basis $\{b_1, \ldots, b_n, c_1, \ldots, c_k, d_1, \ldots, d_l, e_1, \ldots, e_m\}$ of V_2 such that the c_i's span $\operatorname{Im} V_\varphi \cap \operatorname{Im} V_\psi$, the b_i's and c_i's together $\operatorname{Im} V_\varphi$ and the c_i's and d_i's $\operatorname{Im} V_\psi$. After a base change in V_1 and V_3, V_φ becomes $\begin{pmatrix} I_{n+k} & O \\ O & O \end{pmatrix}$, and a similar matrix occurs for V_ψ. Clearly one can decompose a representation in at most 6 distinct representations, which have as dimension vectors respectively 1—0—0 (the representation S_1), 1—1—0 (b_i's), 1—1—1 (c_i), 0—1—0 (e_i, the representation S_2), 0—1—1 (d_i), 0—0—1 (the representation S_3). Note that a slight generalization of this example to a quiver T_n with n arrows pointing from n distinct vertices to the a single central vertex, shows that the problem of the behavior of n subspaces of m dimensional space is encoded in T_n.

T_n :



For $n \geq 5$, this problem is wild.

As an illustration of the representation theory of tame quivers we note that the representation theory of $\tilde{A}_0 = (\{0\}, \{\varphi\}, h, t)$ is nothing else than the Jordan problem.

The "smallest" wild problem is the 2-loop quiver $L_2 = (\{1\}, \{\psi, \varphi\}, h, t)$. The representations are given by couples of $n \times n$-matrices, up to conjugation. It is well known that the corresponding classification problem is hard. A wild quiver is defined as one whose representation theory includes the former problem[1].

1.1.3 Associated Structures

Below we list some structures commonly associated to a quiver Q.

We define a bilinear form on L^{Q_0} (L, depending on the context, being equal to $\mathbb{k}$, $\mathbb{Z}$, $\mathbb{F}_q$, $\mathbb{C}$) via

$$\langle i, i \rangle_Q = 1 - |\{ \text{ loops in } i \}| \tag{1.1}$$

$$\langle i, j \rangle_Q = -|\{\text{arrows from } i \text{ to } j\}|, \quad \text{for } i \neq j \in Q_0. \tag{1.2}$$

[1]The precise definition is: a quiver is wild iff its category of representations contains the category of representations of L_2 as full embedding.

The symmetrization of $\langle .,. \rangle$ is defined as $(x,y)_Q = \langle x,y\rangle_Q + \langle y,x\rangle_Q$. If the quiver is fixed the subscript Q is dropped. Note that the symmetrized bilinear form is independent of the orientation.

The *Tits form* is given by $q_Q(x) = \langle x,x\rangle = \sum_{i\in Q_0} x_i^2 - \sum_{\varphi\in Q_1} x_{t\varphi}x_{h\varphi}$. Note that $(x,y) = q(x+y) - q(x) - q(y)$.

The matrix of the symmetrized bilinear form is called the *Cartan matrix*. It has diagonal entries $C_{ii} = 2 - 2\cdot|\{\text{loops in } i\}|$ while the off diagonal entries are $C_{ij} = -|\{\text{edges linking } i \text{ and } j\}|$.

The classification into finite type, tame type and wild type can be made via Tits form:

- The quiver is of finite representation type if and only if q is positive definite.
- The quiver is tame if and only if q is positive semidefinite. In this case there is a unique $\delta \in \mathbb{N}^{Q_0}$ such that $\{\alpha \mid C\alpha \geq 0\} = \{\alpha \mid C\alpha = 0\} = \mathbb{N}\delta$. The vector δ is indicated on the extended Dynkin diagrams on page 4.
- In all other cases the quiver is wild.

1.2 The Weyl Group

We call a loop free vertex i, considered as element of $\mathbb{Z}^{Q_0}$ *a fundamental root*. We define

$$r_i(\alpha) = \alpha - 2(\alpha, i)i, \tag{1.3}$$

for $i \in Q_0$ and $\alpha \in \mathbb{Z}^{Q_0}$. Note that this defines a reflection on $\mathbb{Z}^{Q_0}$. We call the r_i *the fundamental reflections* of the quiver, and the group generated by these reflections is the *Weyl Group* $W(Q)$. Put $\epsilon(w) = -1$ if w is the product of an odd number of r_i's and $+1$ otherwise. r_i is independent of the orientation and thus W is independent of the orientation.

Consider $\alpha = \sum_{i\in Q_0} \alpha_i i \in \mathbb{N}^{Q_0}$. We call the support $\operatorname{Supp}(\alpha)$ of α the set of those elements i in Q_0 for which $\alpha_i \neq 0$. If α has connected support and for all $j \in Q_0$ we have $(\alpha, j) \leq 0$, then we say that α is in the *fundamental set* M.

All elements of $\mathbb{Z}^{Q_0}$ that can be reached from fundamental root by using fundamental reflections, are called *real roots*:

$$\Delta^{\text{re}}(Q) = \{w(i) | w \in W(Q);\ i \in Q_0 \text{ loop free}\}.$$

The elements reachable from the fundamental set are the *imaginary roots*:

$$\Delta^{\text{im}}(Q) = \bigcup_{w\in W} w(M \cup -M).$$

These two classes of roots form the *root system* $\Delta(Q) = \Delta^{\text{re}} \cup \Delta^{\text{im}}$. If $S \subset \Delta$ then the *positive* elements of S are defined as $S_+ = S \cap \mathbb{N}^{Q_0}$ and the negative elements are defined as $S_- = S \cap (-\mathbb{N}^{Q_0})$. Note that $\Delta = \Delta_+ \cup \Delta_-$ and hence $S = S_+ \cup S_-$. In cases when there is no confusion we will omit the Q, in notations related to roots.

The finite, tame and wild quivers, have different types of root systems, as can be seen from the following results:

- The following statements are equivalent for a quiver Q.
  - Q is of finite type,
  - Q has no imaginary roots,
  - Q has a finite number of roots,
  - the Weyl group of Q is finite.
- A quiver with at least 2 vertices is tame iff there is at least one imaginary root, and in addition all imaginary roots lie on a line. Note that a quiver with a single vertex always has all its roots on a line.
- A quiver is wild iff there is a positive root α with support equal to Q and if in addition $(\alpha, i) < 0$ for all vertices i.

To the bilinear form (,) one may associate a Kac-Moody Lie algebra $\mathfrak{g}$, whose root system coincides with the one defined above. This can be done as follows [Kac80].

Let Γ be the free Abelian group with basis Q_0.

Then there is a unique complex Γ-graded Lie Algebra $\mathfrak{g}(Q) = \bigoplus_{\alpha \in \Gamma} \mathfrak{g}_\alpha$ with the following properties:

1. every graded ideal which intersects $\mathfrak{g}_0$ trivially is zero.
2. $\mathfrak{g}(Q)$ is generated by the elements e_i, f_i, h_i for $i \in Q_0$, such that $\mathfrak{g}_{\alpha_i} = \mathbb{C}e_i$ and $\mathfrak{g}_{-\alpha_i} = \mathbb{C}f_i$. Furthermore the h_i's form a basis for $\mathfrak{g}_0$ and the following relations hold for $i, j \in Q_0$:

$$[h_i, h_j] = 0, \quad [e_i, f_j] = \delta_{ij} h_i \quad [h_i, e_j] = a_{ij} e_j \quad [h_i, f_j] = -a_{ij} f_j$$

An element $\alpha \in \Gamma$ is called a *root* of the Kac-Moody algebra $\mathfrak{g}(Q)$ if $\mathfrak{g}_\alpha \neq 0$, and $m_\alpha = \dim \mathfrak{g}_\alpha$ is the *multiplicity* of the root α.

We introduce a formal symbol ρ and we extend the action of W to $\mathbb{C}^{Q_0} \oplus \mathbb{C}\rho$ by $r_i \rho = \rho - 1.i$. The m_α may be computed by the following identity in formal power series:

$$\prod_{\alpha \in \Delta_+} (1 - X^\alpha)^{m_\alpha} = \sum_{w \in W} \epsilon(w) X^{\rho - w(\rho)}. \tag{1.4}$$

1.3 Representations and Reflections

Many of the above notions have a direct interpretation in terms of representations. If V is a representation, we then write $[V]$ for its isomorphism class.

The *Euler form* of $V, W \in \mathrm{Rep}(Q)$ is defined as

$$\langle V, W\rangle_Q = \langle V, W\rangle = \dim \mathrm{Hom}(V, W) - \dim \mathrm{Ext}^1(V, W)$$

One may show that $\langle V, W\rangle = \langle \overrightarrow{V}, \overrightarrow{W}\rangle$ (this is not obvious if Q has a loop). We also put $(V, W) = \langle V, W\rangle + \langle W, V\rangle$. Again $(V, W) = (\overrightarrow{V}, \overrightarrow{W})$

If no arrow starts in $i \in Q_0$ then i is called a *sink* and if no arrow ends in i then it is a *source*.

Let i be a source. A representation V is i-admissible if the combination $\Theta : V_i \longrightarrow \bigoplus_{t\varphi=i} V_{h\varphi}$ of all maps V_φ starting in i is injective. Dually, if j a sink, a representation is j-admissible if $\bigoplus_{h\varphi=j} V_{t\varphi} \longrightarrow V_j$ is surjective. In both cases we denote the full subcategory of i-admissible representations in $\mathrm{Rep}(Q)$ by $\mathrm{Adm}_i(Q)$.

If $I \in Q_0$ is a source then we define the quiver r_iQ as the quiver which is equal to Q, except that the orientation of the edges starting in i changed. Then i becomes a sink of r_iQ. We have $\langle \alpha, \beta\rangle_Q = \langle r_i\alpha, r_i\beta\rangle_{r_iQ}$.

We define a bijection r_i between the isomorphism classes of $\mathrm{Adm}(Q)$ and $\mathrm{Adm}(r_iQ)$. Given an i-admissible representation $V = (\{V_j\}, \{V_\varphi\})$ of Q we define a representation

$$r_iV = W \tag{1.5}$$

of Q' such that if $j \neq i$ then $W_j = V_j$ and $W_i = \mathrm{Coker}\,\Theta$. Since Θ is injective, we may consider V_i as a subset of $\Xi = \bigoplus_{t\varphi=i} V_{h\varphi}$, and W_i as Ξ/V_i. Define Θ' as the canonical projection from Ξ on W_i and for each arrow φ ending at i (in Q') define $W_\varphi : W_{t'\varphi} \to W_i$ as the restriction of Θ' to $W_{t'\varphi} = V_{h\varphi}$.

Note that $\overrightarrow{r_iV} = \overrightarrow{V} - (\overrightarrow{V}, i)i = r_i\overrightarrow{V}$ and $\mathrm{Aut}(V) \cong \mathrm{Aut}(r_iV)$. So we get as well

$$\langle \overrightarrow{r_iV}, \overrightarrow{r_iW}\rangle = \langle \overrightarrow{V}, \overrightarrow{W}\rangle. \tag{1.6}$$

There is an obvious dual construction $r_i : \mathrm{Adm}(r_iQ)/_{\cong} \to \mathrm{Adm}(Q)/_{\cong}$ and clearly $r_ir_i = \mathrm{Id}$.

We finish this section with the following fact. If i is a source or sink then every representation is the sum of an i-admissible representation and a number of copies of the simple representation S_i. Indeed if for example i is a source then $V \cong V/\ker\Theta \oplus \ker\Theta$, where we have identified $\ker\Theta$ with a representation of Q concentrated in i. The decomposition is obtained by choosing a complement of $\ker\Theta$ in V_i.

1.4 The Conjectures of Kac

A decade after the paper of Gabriel [Gab72] it was generally assumed that a similar approach to the wild cases would not exist. One of the problems in the wild case is that the number of absolute indecomposables over an algebraically closed field $\mathbb{k}$ is infinite and not classified by $\mathbb{P}(\mathbb{k})$ as in the tame case.

Victor Kac showed that it was possible to obtain meaningful ideas in the wild case in [Kac80]. This chapter gives only a brief account of the results contained in this startling paper.

Theorem 1.1 (**[Kac80]**) *Fix a quiver Q. If the base field $\mathbb{k}$ is algebraically closed then*

1. *there is an indecomposable representation of dimension α iff α is a positive root,*
2. *moreover if α is a real root this indecomposable representation is unique (up to isomorphism).*
3. *If α is imaginary the set of absolute indecomposable representations (up to isomorphism) with dimension vector α is parameterized by a finite union of algebraic varieties. The maximal dimension of these varieties is $1 - (\alpha, \alpha) > 0$.*

The last part clearly means that the number of iso classes of representations will be infinite, for tame and wild quivers. We would like to be able to count these iso classes so we have to shift our discussion to finite fields.

Put $\mathbb{k} = \mathbb{F}_q$. Denote

$$
\begin{aligned}
o_{Q,\alpha}(q) &= \text{number of iso classes of representations of } Q \text{ over } \mathbb{F}_q, \\
&\quad\ \text{of dimension } \alpha, \\
i_{Q,\alpha}(q) &= \text{number of indecomposable isomorphism classes, of dimension } \alpha, \\
a_{Q,\alpha}(q) &= \text{number of absolute indecomposable isomorphism classes,} \\
&\quad\ \text{of dimension } \alpha.
\end{aligned}
$$

Kac proved

Theorem 1.2 *[Kac80] For a fixed quiver Q and dimension α the functions in q: $o_{Q,\alpha}(q)$, $i_{Q,\alpha}(q)$, $a_{Q,\alpha}(q)$ are polynomials in q independent of the orientation of Q, moreover $a_{Q,\alpha}(q)$ has integer coefficients.*

It is easy to find examples where $i_Q(q)$ has non-integer coefficients. Computing numerous examples gives evidence for

Conjecture 1.1 *[Kac80] $a_{Q,\alpha}(q)$ has positive coefficients.*

From our own research we conjectured that $o_{Q,\alpha}(q)$ has positive coefficients as well. However it turns out that this fact is already implied by conjecture 1.1, as was shown by Hua [Hua98, conjecture 2, page 56].

Kac also with conjectured that the coefficients of polynomials $a_{Q,\alpha}(q)$ would have a geometric meaning. In particular he explicitly conjectured the following:

For instance

Conjecture 1.2 *[Kac83] the constant term of $a_{Q,\alpha}(q)$ is equal to the multiplicity m_α of α in the associated Kac-Moody Lie algebra $\mathfrak{g}$.*

One easily deduces from Theorem 1.1 that this conjecture is correct for real roots and non-roots. So it remains only to be proven for imaginary roots. Thus for Dynkin quivers it is trivially true. For tame quivers the conjecture follows from results in [DR76].

In the wild case the conjecture has not yet been fully proved for even a single quiver; although we computed that for the quiver $\cdot \Rrightarrow \cdot$ the conjecture holds up to dimension $(20, 20)$. A small list of other computer results is included in section 3.5. The computations are based on Theorem 3.2 below.

Chapter 2

Some Evidence for the Conjectures

2.1 Introduction

In this chapter we show that conjecture 1.1 holds for the m-loop quiver L_m for dimension up to 5. It is somewhat remarkable that a computer computation allows us to prove a result valid for an infinite number of quivers.

2.1.1 Restating the m-loop Problem

We put $\Bbbk = F_q$, with $q = p^s$ (with p prime). The group Gl_n of invertible $n \times n$-matrices acts by conjugation on $\mathrm{Mat}_n = \mathrm{Mat}^{n\times n} = \Bbbk^{n\times n}$, the set of all $n \times n$-matrices. This action extends to m-tuples of $n \times n$-matrices and we are interested in the orbits of Mat_n^m under this action. In fact in this way we redefined the representations of L_m.

2.1.2 The Generalized Jordan Problem

One may ask for a method of selecting out a canonical representant for each orbit. If $n = 1$ each orbit consists of one point, so the problem is trivial.

If $m = 1$ the Jordan form gives a representant for each orbit (possible over a field extension). The orbits corresponding to absolutely indecomposable representations are given by Jordan blocks.

It seems very likely that in the remaining cases such a specific answer does not exist and hence one should try to solve a weaker problem.

A possible first step is trying to count the numbers of orbits. Of course this only makes sense over finite fields.

2.1.3 Some Results on the Positivity Conjecture of Kac

We introduce the shorthand $o_{L_m,n} = o_{m,n}$, $i_{L_m,n} = i_{m,n}$ and $a_{L_m,n} = a_{m,n}$.

In this notation conjecture 1.1 becomes:

Conjecture 2.1 $a_{m,n}(q) \in \mathbb{N}[q]$.

As an exercise, the reader may verify the conjecture for n or m equal to 1.

The cases $(m,n) \in \{(*,2),(2,3),(3,3),(2,4),(2,5)\}$; were checked by Le Bruyn in [LB88] and in [LBM88] this result was extended to the cases $(m,n) \in \{(2,6),(2,7),(2,8)\}$. This was done by hand, which is rather amazing. In this thesis we use the computer to check these results, and we carry them further.

Theorem 2.1 *Kac's conjecture 2.1 holds for $n \leq 5$ and all m.*

We will prove the theorem in section 2.3.

2.2 An Amazing Piece of Handwork

This section is based mainly on [LB88]. We note that the formulae introduced in chapter 3 simplify and generalize the results stated here.

We will use examples to illustrate the formulas rather than proving them.

2.2.1 Calculating $o_{m,n}(q)$

Using the Burnside formula one easily deduces:

$$o_{m,n}(q) = \frac{1}{|\,\mathrm{Gl}_n\,|} \sum_{i \in \mathcal{S}_n} \alpha_i(q) q^{im}$$

where $\mathcal{S}_n = \{i \mid \exists \mu : \sum \mu_j^2 = i \text{ and } \sum \mu_j = n\}$ and $\alpha_i(q)$ is the number of elements in Gl_n such that its commutator ring has dimension i. Note that this number does not depend on m, so that the problem of calculating $o_{m,n}(q)$ can be tackled by fixing n but without fixing m.

The idea in [LB88] was to calculate all except 2 of these α_i and then to use the fact that their sum is $|\,\mathrm{Gl}_n\,|$ and that $o_{1,n}(q)$ is already known, to calculate the remaining ones.

2.2.2 From $o_{m,n}(q)$ to $i_{m,n}(q)$

Each representation over $\mathbb{k}$ has a unique decomposition as a direct sum of indecomposable representations over $\mathbb{k}$. This gives us an easy way of getting from $i_{m,n}(q)$ to $o_{m,n}(q)$. Unfortunately we have to go the other way, but using recursion it is possible to do this.

If $V = V_1^{a_1} \oplus \cdots \oplus V_j^{a_2}$ is the decomposition of V into distinct indecomposables then the representation type $\tau = \tau(V)$ of V is defined as the $2 \times j$-matrix with columns $\begin{pmatrix} \dim V_i \\ a_i \end{pmatrix}$.

As example consider $V = V_1^2 \oplus V_2 \oplus V_3$ where V_1 and V_2 are distinct 1-dimensional indecomposable representations and V_3 is a 2-dimensional indecomposable representation. Then $\tau = \begin{pmatrix} 1 & 1 & 2 \\ 2 & 1 & 1 \end{pmatrix}$. Note that the sum of the product of the entries in each column (here $1 \cdot 2 + 1 \cdot 1 + 2 \cdot 1$) is the dimension of the representation (here 5). So the products of the columns form a partition of the dimension of the representation. We will call τ a *product partition* of $\dim V$.

For a given representation the type is unique (up to permutation of the columns). So we can count the number $I_\tau(q)$ of orbits for each type τ, and add them up to get $o_{m,n}(q)$.

In our example we get $I_\tau(q) = i_{m,1}(q)(i_{m,1}(q) - 1)i_{m,2}(q)$ orbits with type τ, since there are $i_{m,1}(q)$ choices for V_1, $i_{m,1}(q) - 1$ choices for V_2 and $i_{m,2}(q)$ possibilities for V_3. In this way we get a formula for $o_{m,n}(q)$ using all $i_{m,k}(q)$ with $k \leq n$. Obviously $i_{m,n}(q)$ appears only once. We get

$$i_{m,n}(q) = o_{m,n}(q) - \sum_{\tau} I_\tau(q), \tag{2.1}$$

where τ runs over all product partitions of n, except $\begin{pmatrix} n \\ 1 \end{pmatrix}$.

Note that the $I_\tau(q)$'s contain all $i_{m,k}(q)$ for $k < n$. So this is a very recursive formula in n. In chapter 3 we will give the precise formula for an arbitrary quiver. Note that m is not really involved in (2.1) (i.e. if I_τ is written out for some τ, m only appears as index of the $i_{m,n}$'s).

2.2.3 From $i_{m,n}(q)$ to $a_{m,n}(q)$

Again the obvious way is to go in the other direction : each indecomposable is, over a large enough field extension, a direct sum of absolute indecomposables, and this yields a recursive formula for $a_{m,n}(q)$.

So an indecomposable may decompose over a field extension. To illustrate this let's take $n = 12$. In which way will absolute indecomposables over field extensions build up to an indecomposable W of dimension 12? Obviously a 12-dimensional absolutely indecomposable over $\mathbb{F}_q$ will also be indecomposable. Its clear that any indecomposable summand (over a field extension) of an indecomposable should have dimension divisible by 12.

Now take a 6-dimensional absolute indecomposable representation V defined over some extension $\mathbb{F}_{q^r}$ (r chosen to be minimal), which is a summand over $W \otimes_{\mathbb{F}_q} F_{q^r}$. Obviously $r \neq 1$. On the other hand r can not be more then 2, since the only way $V \oplus V'$ can be defined over $\mathbb{F}_q$ is when V and V' are conjugate in the Frobenius sense. Hence $r = 2$.

For each pair of conjugate absolute indecomposable representations over $\mathbb{F}_{q^2}$ which are not defined over $\mathbb{F}_q$, we get exactly one indecomposable representation of twice the dimension over $\mathbb{F}_q$. This contributes exactly $\frac{1}{2}(a_{m,6}(q^2) - a_{m,6}(q))$ orbits to $i_{m,12}(q)$.

In the same way we find a contribution of $\frac{1}{3}(a_{m,4}(q^3) - a_{m,4}(q))$ to $i_{m,12}(q)$ coming from 4-dimensional representations (since the Frobenius has period 3).

Analogous we can find as contribution from 4-dimensional representations, since Frobenius works in this case with triples.

In the case of 3-dimensional representation, we consider absolute indecomposable representations over $\mathbb{F}_{q^4}$. This time the contribution is $\frac{1}{4}(a_{m,3}(q^4) - a_{m,3}(q^2))$.

Leaving the 2-dimensional case to the reader, we now consider the restriction of a 1-dimensional absolute indecomposable representation. In this case there are 2 possible ways of disturbing the build up of a 12-dimensional indecomposable from a 1-dimensional representation over $\mathbb{F}_{q^{12}}$. Such a r— will not be indecomposable if it was already defined over $\mathbb{F}_{q^6}$ or $\mathbb{F}_{q^4}$. This seems to yield a contribution of $\frac{1}{12}(a_{m,1}(q^{12}) - a_{m,1}(q^6) - a_{m,1}(q^4))$, but the representations defined over $\mathbb{F}_{q^2}$ are subtracted twice in this sum, so one should additionally add them again.

This is clearly a case of Möbius inversion. The whole calculation is captured in [Kac83]:

$$i_{m,n}(q) = \sum_{d|n} \frac{1}{d} \sum_{e|d} \mu(e) a_{m,\frac{n}{d}}(q)\ q^{\frac{d}{e}},$$

where, for $p(e)$ the number of primes dividing e, $\mu(e) = (-1)^{p(e)}$ whenever e is square free and $\mu(e) = 0$ otherwise.

This leads again to a recursive formula for $a_{m,n}(q)$ in terms of $i_{m,n}(q)$ and $a_{m,k}(q)$, with $k < n$.

Repeatedly substituting this form into itself we obtain that $a_{m,n}(q)$ is linear combination of terms of the form q^{im}

So whereas one usually fixes a quiver (represented here by the number n), and varies the dimension vector (the number m) it is more convenient to fix the dimension vector and vary the quiver.

2.3 Proof of Theorem 2.1

The cases with $n = 1$ (being trivial) and $n = 2$ (done by Le Bruyn) are omitted.

2.3.1 $n = 3$

First we will prove the theorem for $n = 3$. The same idea will be used to treat the cases $n = 4$ and $n = 5$.

Using the computations of Le Bruyn one finds

$$\begin{aligned} a_{m,3}(q) &= \frac{1}{(q^3-1)(q^2-1)q^3} q^{9m} - \frac{1}{(q-1)(q^2-1)q^3} q^{5m} \\ &\quad + \frac{1}{(q^3-1)(q-1)q^2} q^{3m}. \end{aligned} \tag{2.2}$$

To stress the general argumentation we will write

$$a_{m,3}(q) = t_9(q)q^{9m} + t_5(q)q^{5m} + t_3(q)q^{3m}, \tag{2.3}$$

where $t_i(q)$ is a rational function in q whose denominator a divisor of $|\,\mathrm{Gl}_3\,|$, and $i \in \mathcal{S}_3$.

Using a computer it is easy to see that for a fixed m this is a polynomial with positive coefficients, but to prove this for all m is more problematic. The problem is caused by the fact that the $t_i(q)$'s are not polynomials but rational functions. The denominator of this rational functions indicates the severeness of the problem. Both the number of problems (read the number of different terms) as their severeness can be reduced with the following trick.

Let P, Q, R be polynomials in q such that $P(q) = Q(q) - q^d R(q)$ (with $d \in \mathbb{N}$) and such that R and P have positive coefficients, then Q will also have positive coefficients. This simple fact gives us a tool to prove recursively that $a_{m,3}(q)$ has positive coefficients.

We know that $a_{1,3}(q) = q$ has positive coefficients.

Consider

$$\begin{aligned} b_{m,3}(q) &= a_{m+1,3}(q) - q^3 a_{m,3}(q) \\ &= t_9(q)q^{9(m+1)} - q^3 t_9(q)q^{9m} + \cdots + t_3(q)q^{3(m+1)} - q^3 t_3(q)q^{3m} \\ &= (q^6 - 1)t_9(q)q^{9m+3} + (q^2 - 1)t_5(q)q^{5m+3} \end{aligned}$$

So we have lost one term. The severeness of the problems is also become more bearable since certain factors of $q^6 - 1$ (resp. $q^2 - 1$) will cancel parts of the denominator of t_9 (resp. t_5).

The actual computation gives the following result.

$$b_{m,3}(q) = \frac{q^2 - q + 1}{q - 1} q^{9m} - \frac{1}{q - 1} q^{5m} = q^{9m+1} + \frac{q^{9m} - q^{5m}}{q - 1}.$$

While from (2.2) it is not even clear that $a_{m,3}$ is a polynomial for each m, $b_{m,3}(q)$ is obviously a polynomial with positive coefficients.

Using as induction hypothesis that $a_{m,3}(q)$ is a polynomial with positive coefficients it follows that $a_{m+1,3}(q) = b_{m,3}(q) + q^3 a_{m,3}(q)$ also has positive coefficients.

2.3.2 $n = 4$

The problem of getting the method to work for $n = 4$, mainly comes down to the question which power we should use to eliminate a term and to make the other terms easier to handle. Clearly it should be in $\mathcal{S}_4$. To figure out mathematically what is the best choice one should be concerned mainly with what power is best for canceling parts of the factors of $|\,\mathrm{Gl}_4\,|$. But it will be the computer that really answers the question.

To start we define

$$b_{m,4}(q) = a_{m+1,4}(q) - q^4 a_{m,4}(q).$$

The new expression is not easy to handle so we add another step:

$$c_{m,4}(q) = b_{m+1,4}(q) - q^8 b_{m,4}(q).$$

The new expression is still not that nice, but good enough for our purposes. Note that this reduction should not be repeated too often since it would create minus signs! In this case we already have denominator $q-1$. The result is:

$$\begin{aligned} c_{m,4}(q) &= \frac{q^{18} - q^{17} + q^{16} + q^{14} - q^{13} + 2q^{12} - q^{11} + q^{10} + q^8 - q^7 + q^6}{q-1} q^{16m} \\ &\quad + \frac{-q^9 - q^6}{q-1} q^{10m} + \frac{-q^7 - q^6}{q-1} q^{6m}. \end{aligned}$$

One can check that for each m this has positive coefficients by writing it as a sum of terms of the form $\frac{q^u - q^v}{q-1}$ with $u > v$.

2.3.3 $n = 5$

The basic principle remains the same. This time we have :

$$\begin{aligned} b_{m,5}(q) &= a_{m+1,5}(q) - q^5 a_{m,5}(q) \qquad \text{and} \\ c_{m,5}(q) &= b_{m+1,5}(q) - q^9 b_{m,5}(q). \end{aligned}$$

The denominator of $c_{m,n}(q)$ still contains $(q^3 - 1)(q - 1)$.
Define κ_j by

$$c_{m,5}(q) = \sum_j \frac{\kappa_j(q)}{(q^3-1)(q-1)} q^{jm+4}$$

where $j \in \{25, 17, 13, 11, 7\}$.

Following Le Bruyn we denote the polynomials as tuples of coefficients ending with the constant term. For instance in stead of $\kappa_7(q) = q^6 + 2q^5 + 2q^4 + q^3$ we write $\kappa_7(q) = (1, 2, 2, 1, 0, 0, 0)$. For reasons to be explained afterwards we arrange this coefficients in a 3-row matrix, where the exponents are decreasing with 1 along the columns and with 3 along the rows.

We get

$$\kappa_{25}(q) = \begin{pmatrix} 1 & -1 & 1 & -1 & 3 & -2 & 2 & -1 & 1 \\ -1 & 2 & -1 & 2 & -2 & 2 & -1 & 2 & -1 \\ 1 & -1 & 2 & -2 & 3 & -1 & 1 & -1 & 1 \end{pmatrix}$$

$$\kappa_{17}(q) = \begin{pmatrix} -1 & 0 & -2 & 0 & -1 \\ 0 & -2 & 0 & -2 & 0 \\ -1 & 0 & -2 & 0 & -1 \end{pmatrix}$$

$$\kappa_{13}(q) = \begin{pmatrix} 0 & -1 & -2 & 0 & 0 \\ 0 & -2 & 0 & -1 & 1 \\ -1 & -1 & -2 & 1 & 0 \end{pmatrix}$$

$$\kappa_{11}(q) = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \end{pmatrix}$$

$$\kappa_{7}(q) = \begin{pmatrix} 0 & 2 & 0 \\ 0 & 2 & 0 \\ 1 & 1 & 0 \end{pmatrix}$$

Dividing a polynomial written down like this by $(q^3 - 1)$ comes down to writing down the matrix of cumulative totals in the rows. The problem is that we have to concatenate the 5 matrices. For the sake of the argument let m be a multiple of 3 greater than 4. Then the matrix of $(q^3 - 1)(q - 1)c_{m,5}$ would be a matrix formed by the matrix of κ_{25} followed by zeros, then κ_{17} and so on until finally κ_7 and zeroes without any importance. Note that $m > 4$ is needed to avoid that the right row of the matrix of κ_{13} gets mixed up with the left of κ_{11} and the multiple of 3 condition makes sure that the rows are in the right position. Now write down the matrix with cumulative totals in the rows. We get

$$(q-1)c_{m,5}(q) = \begin{pmatrix} 1 & 0 & 1 & \cdots & -2 & -2 & 0 & \cdots \\ -1 & 1 & 0 & \cdots & -2 & -2 & 0 & \cdots \\ 1 & 0 & 2 & \cdots & -2 & -1 & 0 & \cdots \end{pmatrix}$$

Now we can divide by $q - 1$ by taking the cumulative totals down the columns, and taking them back up at the top of the next column. This time we get

$$c_{m,5}(q) = \begin{pmatrix} 1 & 1 & 3 & \cdots & 11 & 5 & 0 & \cdots \\ 0 & 2 & 3 & \cdots & 9 & 3 & 0 & \cdots \\ 1 & 2 & 5 & \cdots & 7 & 1 & 0 & \cdots \end{pmatrix}$$

It is essential that all entries in this matrix are positive. To see this we don't really have to calculate them. Kac already proved that at the end 0's have to appear (since $a_{m,n}(q)$ is polynomial and thus $c_{m,5}(q)$), so the only thing to note is that the cumulative total gets never beneath zero. This follows easy since (except for the second coefficient) all positive coefficients in $(q-1)c_{m,5}(q)$ are in front of the negative ones.

Since the cases $m \leq 4$ are readily checked and the analogous arguments easily carry over to the cases $m = 3k + 1$ and $m = 3k + 2$ (swapping (and shifting) rows in 2 matrices), this ends our proof.

Chapter 3

The Constant Term Conjecture and Gaussian Multinomial Coefficients: A Reformulation

In this chapter we reformulate Kac's Conjecture 1.2 in terms of Gaussian multinomial coefficients. Apart from the fact that this makes calculations easier, as indicated above, the main interest is theoretical. Gaussian multinomial coefficients have many non-trivial combinatorial properties, so it is conceivable that this could be used to gain insight in conjecture 1.2.

First we will deduce a recurrence relation for the $o_{Q,\alpha}(q)$'s.

3.1 A Recurrence Relation for the o's

We fix a quiver Q. For simplicity we omit Q from most of the notations. By the Burnside formula we have:

$$o_\alpha(q) = \frac{1}{|\,\mathrm{Gl}_\alpha\,|} \sum_{V \in \mathrm{Rep}(Q,\alpha)} |\,\mathrm{Aut}(V)|.$$

We will need a modification of this formula in subsequent calculations. Define

$$t_\alpha(q) = \frac{1}{|\,\mathrm{Gl}_\alpha\,|} \sum_{V \in \mathrm{Rep}(Q,\alpha)} |\,\mathrm{End}(V)|.$$

Note that any endomorphism ξ of V can be characterized by an internal decomposition $V = V_{\mathbf{n}} \oplus V_{\mathbf{a}}$, with $\xi = (\xi_{\mathbf{n}}, \xi_{\mathbf{a}})$, such that $\xi_{\mathbf{n}}$ acts nilpotently on $V_{\mathbf{n}}$, and $\xi_{\mathbf{a}}$ is an automorphism of $V_{\mathbf{a}}$. Let $\mathrm{Nil}(V)$ be the set of nilpotent endomorphisms of V and let $\psi_{\beta,\gamma}$ be the number of ways in which one can decompose simultaneously the vector spaces V_i involved in the representation V in spaces of dimensions β_i and γ_i. We obtain

$$\begin{aligned} t_\alpha(q) &= \frac{1}{|\,\mathrm{Gl}_\alpha\,|} \sum_{\beta+\gamma=\alpha} \psi_{\beta,\gamma} \sum_{V_{\mathbf{n}} \in \mathrm{Rep}(Q,\beta)} \sum_{V_{\mathbf{a}} \in \mathrm{Rep}(Q,\gamma)} |\,\mathrm{Nil}(V_{\mathbf{n}})||\,\mathrm{Aut}(V_{\mathbf{a}})| \\ &= \sum_{\beta+\gamma=\alpha} \left(\frac{1}{|\,\mathrm{Gl}_\beta\,|} \sum_{V_{\mathbf{n}}} |\,\mathrm{Nil}(V_{\mathbf{n}})| \right) \cdot \left(\frac{1}{|\,\mathrm{Gl}_\gamma\,|} \sum_{V_{\mathbf{a}}} |\,\mathrm{Aut}(V_{\mathbf{a}})| \right). \end{aligned}$$

To obtain the last equation note that

$$\psi_{\beta,\gamma} = \prod_{i \in Q_0} \psi_{\beta_i,\gamma_i} = \prod_{i \in Q_0} \frac{|\,\mathrm{Gl}_{\alpha_i}\,|}{|\,\mathrm{Gl}_{\beta_i}\,|.|\,\mathrm{Gl}_{\gamma_i}\,|}.$$

So we get

$$t_\alpha(q) = \sum_{\beta+\gamma=\alpha} r_\beta(q) o_\gamma(q), \tag{3.1}$$

where $r_\alpha(q) = \left(\frac{1}{|\,\mathrm{Gl}_\alpha\,|} \sum_{V_{\mathbf{n}} \in \mathrm{Rep}(Q,\alpha)} |\,\mathrm{Nil}(V_{\mathbf{n}})| \right)$. Since adding the identity morphism to a nilpotent morphism gives a unipotent morphism, we may equally write

$$r_\alpha(q) = \frac{1}{|\,\mathrm{Gl}_\alpha\,|} |\{(U,V) \in \mathrm{Gl}_\alpha \times \mathrm{Rep}(Q,\alpha) \mid U \bullet V = V, U \text{unipotent}\}|. \tag{3.2}$$

The point of this exercise is that we may evaluate the function $t_\alpha(q)$ in another way. If we start by taking a representation V, then its contribution $\frac{|\,\mathrm{End}(V)|}{|\,\mathrm{Gl}_\alpha\,|}$ to $t_\alpha(q)$ will be the same as for all representations in its orbit. The number of elements in its orbit is $\frac{|\,\mathrm{Gl}_\alpha\,|}{|\,\mathrm{Aut}(V)|}$. Now taking one representant V_i for each orbit we get:

$$t_\alpha(q) = \sum_{i=1}^{n} \frac{|\,\mathrm{End}(V_i)|}{|\,\mathrm{Aut}(V_i)|}. \tag{3.3}$$

Now let W be a representation, with dimension different from 0, and suppose $W = \bigoplus W_i^{\oplus a_i}$ is a decomposition into distinct indecomposables. Clearly $|\,\mathrm{End}(W)| = \prod_{i,j} |\,\mathrm{Hom}(W_i, W_j)|^{a_i a_j}$, while the automorphisms are given by automorphisms of the $W_i^{a_i}$ and morphisms between the different indecomposables, which leads to

$$|\operatorname{Aut}(W)| = \prod_l |\operatorname{Gl}_{a_l}(\operatorname{End}(W_l))| . \prod_{i \neq j} |\operatorname{Hom}(W_i, W_j)|^{a_i a_j}.$$

Whence

$$\frac{|\operatorname{End}(W)|}{|\operatorname{Aut}(W)|} = \prod_l \frac{|\operatorname{Mat}_{a_l}(\operatorname{End}(W_l)|}{|\operatorname{Gl}_{a_l}(\operatorname{End}(W_l)|}.$$

The residue field $\Bbbk_l$ of $\operatorname{End}(W_l)$ is a finite extension of $\mathbb{F}_q$. Now we consider the unit group $\operatorname{Gl}_{a_l}(\operatorname{End}(W_l))$ of $\operatorname{Mat}_{a_l}(\operatorname{End}(W_l))$. It is the inverse image of $\operatorname{Gl}_{a_l}(\Bbbk_l)$, hence

$$\frac{|\operatorname{Mat}_{a_l}(\operatorname{End}(W_l))|}{|\operatorname{Gl}_{a_l}(\operatorname{End}(W_l))|} = \frac{|\operatorname{Mat}_{a_l}(\Bbbk_l)|}{|\operatorname{Gl}_{a_l}(\Bbbk_l)|}. \tag{3.4}$$

The right-hand side is clearly divisible by q (in $\mathbb{Q}$), so by (3.3) $t_\alpha(q)$ is divisible by q as well, whenever $\alpha \neq 0$. This holds for all powers of p, hence, as a rational function, $t_\alpha(q)$ must have a zero in 0.

Now reconsider (3.1). Using induction on α, together with the facts that $o_\alpha(q)$ is a polynomial, and $o_0(q) = 1$, leads that $r_\alpha(q)$ has no pole in zero, which means that we can evaluate r_α in 0. So equation (3.1) becomes

$$0 = t_\alpha(0) = \sum_{\beta+\gamma=\alpha} r_\beta(0) o_\gamma(0), \tag{3.5}$$

when $\alpha \neq 0$. Trivially $1 = t_0(q) = o_0(q) r_0(q)$.

We can summarize this by: $\forall \alpha \in \mathbb{N}^{Q_0} : \delta_{\alpha 0} = \sum_{\beta+\gamma=\alpha} o_\beta(0) r_\gamma(0)$, with β, $\gamma \in \mathbb{N}^{Q_0}$.

A more elegant way to write this is by using generating functions. To this end we need a formal exponential e, satisfying $e(\alpha)e(\beta) = e(\alpha+\beta)$. In this way the relation becomes:

$$\left(\sum_{\beta \in \mathbb{N}^{Q_0}} r_\beta(0) e(\beta) \right) . \left(\sum_{\gamma \in \mathbb{N}^{Q_0}} o_\gamma(0) e(\gamma) \right) = 1. \tag{3.6}$$

The general formula for going from $o_\alpha(q)$ to $i_\beta(q)$'s as indicated on the simple example 2.1 for the m-loop quiver above is :

$$o_\alpha(q) = \sum_{\alpha = \sum r_i \beta_i} \prod_i \binom{i_{\beta_i}(q) + r_i - 1}{r_i}.$$

Again this formula can be expressed more compactly in terms of formal power series.

$$\sum_\alpha o_\alpha(q) e(\alpha) = \frac{1}{\prod_\beta (1 - e(\beta))^{i_\beta(q)}}. \tag{3.7}$$

By [Kac83] $i_\alpha(0) = a_\alpha(0)$, so by 3.6 and 3.7

$$\prod_\alpha (1 - e(\alpha))^{a_\alpha(0)} = \sum_\alpha r_\alpha(0) e(\alpha). \tag{3.8}$$

Comparing this with (1.4) we get the following reformulation of conjecture 1.2

Theorem 3.1 *Conjecture 1.2 is equivalent to*

$$r_\alpha(0) = \begin{cases} \epsilon(w) & \text{if } \alpha = \rho - w(\rho) \\ 0 & \text{otherwise.} \end{cases} \tag{3.9}$$

This result was discovered by us in 1998. We were informed by V. Kac that a similar result also appeared in Hua's thesis [Hua98] and furthermore the function $r_\alpha(q)$ is also present in unpublished work of Stanley. The work of Hua and Stanley is purely combinatorial however.

3.2 Gaussian Multinomial Coefficients

The ordinary Newtonian multinomial coefficient for a, b_1, ..., $b_n \in \mathbb{N}$ counts the cardinality of the set $\mathcal{W}$ of words in the letters u_1, u_2, ..., u_n of length a where the letter u_k appears b_k times, for each k. Thus $|\mathcal{W}| = \frac{a!}{b_1! \cdots b_n!}$. To define Gaussian multinomial coefficients we need the polynomials $\phi_a(t) = (t^a - 1)(t^{a-1} - 1) \cdots (t - 1)$. The Gaussian multinomial coefficient corresponding to a, $b_1, \ldots, b_n$ is then

$$\begin{bmatrix} & a & \\ b_1 & \cdots & b_n \end{bmatrix} = \frac{\phi_a(t)}{\phi_{b_1}(t) \cdots \phi_{b_n}(t)}. \tag{3.10}$$

We will use the short hand $\begin{bmatrix} a \\ b \end{bmatrix}$ for $\begin{bmatrix} & a \\ b & b-a \end{bmatrix}$ and define $\begin{bmatrix} a \\ b \end{bmatrix} = 0$ if $b < 0$.

A function $\zeta : \mathcal{W} \longrightarrow \mathbb{N}$ linking Newtonian and Gaussian multinomial coefficients in the sense of (3.11) below is called a Mahonian statistic [CS97],[FZ90].

$$\sum_{w \in \mathcal{W}} t^{\zeta(w)} = \begin{bmatrix} & a & \\ b_1 & \cdots & b_n \end{bmatrix}. \tag{3.11}$$

An example of a Mahonian statistic is the number of inversions in each word: $\zeta(w) = \zeta(\mathrm{u}_{i_1} \cdots \mathrm{u}_{i_a}) = |\{(r, s) \mid i_r < i_s, r > s\}|$.

After choosing a Mahonian statistic the evaluation of a sum involving Gaussian multinomial coefficients becomes a word counting problem (which may or may not be easier). We will reformulate the constant term conjecture 1.2 in terms of Gaussian multinomial coefficients and thus into a word counting problem.

Now let μ be a partition of n. We define the following short hand.

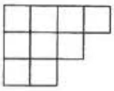
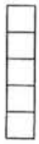
$$[\mu] = \begin{bmatrix} \mu_1 \\ \mu_1 - \mu_2, \mu_2 - \mu_3, \ldots \end{bmatrix}$$

3.3 On Symmetric Functions

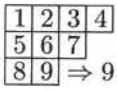
Symmetric functions are a classical area in combinatorics [Mac95]. Two types of functions that appear rather naturally in this context are the Schur functions s_μ and Hall-Littlewood polynomials P_μ. Both are indexed by the set of partitions.

3.3.1 Partitions

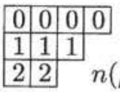
We need to define some extra notations for a partition μ of $n = |\mu|$: $\mu' = (\mu'_i)_i = |\{j \mid \mu_j \geq i\}|$ is the conjugate partition, $l(\mu) = \max\{i \in \mathbb{N} \mid \mu_i \neq 0\} = \mu'_1$ is the length of μ and $n(\mu) = \sum(i-1)\mu_i$ is the content. For instance for $(1^n) = (1,1,1,...,1,0,...)$ we have $l((1^n)) = n$, $n((1^n)) = n(n-1)/2$, $(1^n)' = (n,0,0,...)$. Attach a diagram to a partition in a plane.

$(4,3,2)$:  (1^5) : 

On this diagram we can indicate all structures defined above. The number of boxes is $|\mu|$, $l(\mu)$ is the number of rows, while $n(\mu)$ is the sum of the numbers in the boxes in the right diagram below:

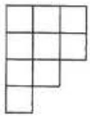
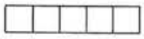


$\Rightarrow 9 = |\mu|$

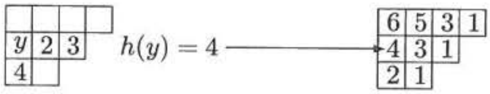


$n(\mu) = 4.0 + 3.1 + 2.2 = 7$

The conjugate partition is obtained as a reflection around the diagonal.

$(4,3,2)'$:  $= (3,3,2,1)$ $(1^5)'$:  $= (5)$

Finally we can define for each cell y of this diagram its hook length $h(y)$, as the total number of cells below and to the right of y.



Given n we define a partial order on the set of partitions as follows. $\lambda \prec \mu$ iff for all $j = 1, \ldots, n : \sum_{i=1}^{j} \lambda_j \geq \sum_{i=1}^{j} \mu_j$.

3.3.2 Symmetric Functions

Now we introduce symmetric functions, following [Mac95]. Consider the ring $\mathbb{Z}[x_1, \ldots, x_n]$, and the action of the symmetric group Σ_n on this ring by permuting the variables. A polynomial is symmetric if it is invariant under this action. The symmetric functions form a subring Λ_n of $\mathbb{Z}[x_1, \ldots, x_n]$.

$$\Lambda_n = \mathbb{Z}[x_1, \ldots, x_n]^{\Sigma_n}$$

Define Λ_n^k the set of all symmetric polynomials of degree k of n variables, together with the zero polynomial. In this way $\Lambda_n = \bigoplus \Lambda_n^k$ is graded.

For each $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$ we denote by x^α the monomial

$$x^\alpha = x_1^{\alpha_1} \ldots x_n^{\alpha_n}.$$

If a partition μ has length $l(\mu) \leq n$ then we put

$$m_\mu(x_1, \ldots, x_n) = \sum x^\alpha$$

summed over all distinct permutations α of $\mu = (\mu_1, \ldots, \mu_n)$. The polynomial m_μ is clearly symmetric, and the m_μ's (as μ runs through all partitions of length $l(\mu) \leq n$) form a $\mathbb{Z}$-basis of Λ_n. This basis respects the grading. The m_μ for which $|\mu| = k$ and $l(\mu) \leq n$ form a basis of Λ_n^k. In particular if $n \geq k$ the m_μ form a basis of Λ_n^k.

In the theory of symmetric functions, the number of variables is usually irrelevant, provided only that it is large enough, and it is often more convenient to work with symmetric functions in infinitely many variables. To make this idea precise, let $m \geq n$ and consider the homomorphism

$$\mathbb{Z}[x_1, \ldots, x_m] \to \mathbb{Z}[x_1, \ldots, x_n]$$

which sends each of $x_{n+1}, \ldots, x_m$ to zero and the other x_i to themselves. On restriction to Λ_m this gives a homomorphism

$$\rho_{m,n} : \Lambda_m \to \Lambda_n.$$

The effect of $\rho_{m,n}$ on the basis elements (m_μ) is easily described; it sends $m_\mu(x_1, \ldots, x_m)$ to $m_\mu(x_1, \ldots, x_n)$ if $l(\mu) \leq n$, and to 0 if $l(\mu) > n$. It follows that $\rho_{m,n}$ is surjective.

The restriction $\rho_{m,n}^k$ of $\rho_{m,n}$ to Λ_m^k defines again a surjective homomorphism

$$\rho_{m,n}^k : \Lambda_m^k \to \Lambda_n^k,$$

which becomes an isomorphism if $m \geq n \geq k$.

We now form the inverse limit

$$\Lambda^k = \lim_{\substack{\longleftarrow \\ n}} \Lambda_n^k$$

of the $\mathbb{Z}$-modules Λ_n^k relative to the homomorphisms $\rho_{m,n}^k$: an element of Λ^k is by definition a sequence $f = (f_n)_n$, where each $f_n \in \Lambda_n^k$, and where, for $m \geq n$, $f_m(x_1, \ldots, x_n, 0, \ldots, 0) = f_n(x_1, \ldots, x_n)$. Since $\rho_{m,n}^k$ is an isomorphism if $m \geq n \geq k$, it follows that the projection

$$\rho_n^k : \Lambda^k \to \Lambda_n^k,$$

which sends f to f_n is a isomorphism for all $n \geq k$. Hence Λ^k has a $\mathbb{Z}$-basis consisting of the *monomial symmetric functions* m_μ (for all partitions μ of k) defined by

$$\rho_n^k(m_\mu) = m_\mu(x_1, \ldots, x_n)$$

for all $n \geq k$. Thus Λ^k is a free $\mathbb{Z}$-module of rank $p(k)$, the number of partitions of k.

Now let $\Lambda = \bigoplus_{k \geq 0} \Lambda^k$, so that Λ is the free $\mathbb{Z}$-module generated by the m_μ for all partitions μ. We have surjective homomorphisms

$$\rho_n = \bigoplus_k \rho_n^k : \Lambda \to \Lambda_n$$

for each $n \geq 0$.

It is clear the Λ has the structure of a graded ring such that ρ_n are ring homomorphisms. The graded ring Λ is called the *ring of symmetric functions* in countable many independent variables.

Now we will define the Schur functions. First we suppose we have a finite number of variables $x_1, \ldots, x_n$. Recall $x^\alpha = x_1^{\alpha_1} \ldots x_n^{\alpha_n}$, and consider the polynomial a_α obtained by anti-symmetrizing x^α:

$$a_\alpha = a_\alpha(x_1, \ldots, x_n) = \sum_{w \in \Sigma_n} \epsilon(w).w(x^\alpha)$$

where $\epsilon(w)$ is the sign of the permutation w. This polynomial is skew-symmetric, i.e. we have $w(a_\alpha) = \epsilon(w)a_\alpha$, for any $w \in \Sigma_n$. In particular a_α vanishes unless $\alpha_i, \ldots, \alpha_n$ are all distinct. Hence we may as well assume that $\alpha_1 > \alpha_2 > \cdots > \alpha_n \geq 0$, and therefore we may write $\alpha = \mu + \delta$, where μ is a partition of length $l(\mu) \leq n$, and $\delta = (n-1, n-2, \ldots, 0)$. Then

$$a_\alpha = a_{\mu+\delta} = \sum_w \epsilon(w).w(x^{\mu+\delta})$$

which can be written as a determinant

$$a_{\mu+\delta} = \det(x_i^{\mu_j+n-j})_{1 \leq i,j \leq n}.$$

This determinant is divisible in $\mathbb{Z}[x_1, \ldots, x_n]$ by each of the differences $x_i - x_j$ $(1 \leq i < j \leq n)$, and hence by their product, which is the Vandermonde determinant

$$\prod_{1 \leq i < j \leq n} (x_i - x_j) = \det(x_i^{n-j}) = a_\delta.$$

So $a_{\mu+\delta}$ is divisible by a_δ in $\mathbb{Z}[x_1, \ldots, x_n]$ and the quotient

$$s_\mu = s_\mu(x_1, \ldots, x_n) = a_{\mu+\delta}/a_\delta$$

is symmetric. So $s_\mu \in \Lambda_n$. It is called the *Schur function* in the variables $x_1, \ldots, x_n$, corresponding to the partition μ (where $l(\mu) \leq n$) and is homogeneous of degree $|\mu|$.

The Schur functions $s_\mu(x_1, \ldots, x_n)$ (where $l(\mu) \leq n$) form a basis of Λ_n.

Now let us consider the effect of increasing the number of variables. If $l(\mu) \leq n$, it is clear that $a_\alpha(x_1, \ldots, x_n, 0) = a_\alpha(x_1, \ldots, x_n)$. Hence

$$\rho_{n+1,n}(s_\mu(x_1, \ldots, x_{n+1})) = s_\mu(x_1, \ldots, x_n).$$

It follows that for each partition μ the polynomials s_μ define a unique element $s_\mu \in \Lambda$, homogeneous of degree $|\mu|$. Thus the s_μ form a $\mathbb{Z}$-basis of Λ, and for each $k \geq 0$ the s_μ such that $|\mu| = k$ form a $\mathbb{Z}$-basis of Λ^k.

Hall-Littlewood polynomials are generalizations of Schur functions depending on an extra parameter. They were introduced by Hall as a means of counting subgroups of Abelian groups.

To explain this idea more precise let $\phi_n(t) = (t-1)...(t^n - 1)$ be as above, for $n > 0$ $(\varphi_0(t) = 1)$ and if $n \geq m \geq 0$ denote $\psi_{n,m} = \dfrac{\varphi_n(t)}{\varphi_m(t)\varphi_{n-m}(t)}$.

For a partition μ set $n_i = \lambda'_i - \lambda'_{i+1}$ and $b_\lambda(t) = \prod_i \varphi_{n_i}$.

If the Schur-function s_λ occurs in the product $s_\mu s_{1^m}$ then $\lambda - \mu$ is a vertical strip of length m. Let $m_i = \lambda'_i - \mu'_i$ (Thus $\sum m_i = m$). In that case set

$$F^\lambda_{\mu 1^m}(t) = \prod_i \psi_{n_i, m_i}(t).$$

If s_λ does not occur in the product set $F^\lambda_{\mu 1^m}(t) = 0$.

The Hall-Littlewood polynomials may be defined inductively as a symmetric functions in $x_1, x_2, \ldots$ with as coefficients polynomials in t, by the following rule:

$$P_0(t) = 1, \quad P_{1^m} = e_m$$
$$P_\lambda = P_\mu P_{1^m} - \sum_{\nu \succ \lambda} F^\nu_{\mu 1^m} P_\nu,$$

where μ is obtained by removing the last column from the diagram of λ and m is the length of that removed column, and e_m is the m-th elementary symmetric function, the sum of all products of m distinct variables.

We get that $P_\lambda(0) = s_\lambda$ and may extend the definition of $F^\lambda_{\mu\nu}(t)$ by

$$P_\mu P_\nu = \sum_\lambda F^\lambda_{\mu\nu}(t) P_\lambda.$$

Now we can come back to Hall's idea. Fix a prime p. An Abelian p-group may be characterized by a partition. Denote by $g^\lambda_{\mu\nu}$ the number of subgroups M of type μ of a p-group G of type λ such that G/M has type ν. The numbers $g^\lambda_{\mu\nu}$ can serve as structure constants of an algebra. In fact denoting $n_\lambda = \sum_i \frac{\lambda'_i(\lambda'_i - 1)}{2}$, we get

$$g^\lambda_{\mu\nu} = p^{n_\lambda - n_\mu - \nu} F^\lambda_{\mu\nu}(1/p)$$

To write the Hall-Littlewood polynomials explicitly, we first define Σ^μ_n as the subgroup of Σ_n consisting of those permutations w such that $\mu_{w(i)} = \mu_i$, for $1 \leq i \leq n$.

The *Hall-Littlewood polynomials* in $n + 1$ variables, corresponding to the partition μ with $l(\mu) \leq n$ may be defined by

$$P_\mu(x_1, \ldots, x_n; t) = \sum_{w \in \Sigma_n / \Sigma^\mu_n} w \left(x_1^{\mu_1} \ldots x_n^{\mu_n} \prod_{\mu_i > \mu_j} \frac{x_i - t x_j}{x_i - x_j} \right)$$

In [Mac95] it is proven that, if $l(\mu) \leq n$,

$$P_\mu(x_1, \ldots, x_n, 0; t) = P_\mu(x_1, \ldots, x_n; t).$$

Hence we can again pass to the limit to define the *Hall-Littlewood function* $P_\mu(x;t)$ to be the limit element of $\Lambda[t]$ whose image in $\Lambda_n[t]$, for each $n \geq l(\mu)$ is $P_\mu(x_1, \ldots, x_n; t)$. Again $P_\mu(x;t)$ is homogeneous of degree $|\mu|$.

The Kostka-Foulkes polynomials are defined by:

$$s_\lambda = \sum_\mu K_{\lambda\mu}(t) P_\mu(x;t) \tag{3.12}$$

Using the partial order on partitions, the matrix is upper-triangular with 1's on the diagonal [Mac95, III.(2.6)]. Hence in particular the Kostka-Foulkes polynomials are invertible. It is a deep theorem that the $K_{\lambda\mu}(q)$ have positive coefficients. The modified Kostka-Foulkes polynomials are defined by

$$\tilde{K}_{\lambda\mu}(q) = q^{n(\mu)} K_{\lambda\mu}(q^{-1}).$$

Below u_μ is an arbitrary unipotent element of Gl_n of type μ, with centralizer $C(u_\mu)$. Put $c_\mu(q) = |C(u_\mu)|$. This is a polynomial in q. According to MacDonald [Mac95, Example III.3.2] we get for $x_q = (q^{-1}, q^{-2}, \ldots)$:

$$P_\mu(x_q; q^{-1}) = \frac{q^{n(\mu)}}{c_\mu(q)}$$

(as formal power series in q^{-1}). A slight variation of another formula in loc. cit. [Mac95, Example I.3.2] gives:

$$s_\lambda(x_q) = \frac{q^{n(\lambda')}}{H_\lambda(q)},$$

where $H_\lambda(q) = \prod(q^{h(y)} - 1)$ (the product runs over the cells of the diagram attached to λ).

Now we look back at (3.12) and conclude that for $q = 0$ we have:

$$\sum_\mu \frac{\tilde{K}_{\lambda\mu}(0)}{c_\mu(0)} = \begin{cases} (-1)^n & \text{if } \lambda = (1^n) \\ 0 & \text{otherwise.} \end{cases} \tag{3.13}$$

Let α be a dimension vector. A *multipartition* λ of $\alpha \in \mathbb{N}^{Q_0}$ is a list of partitions $\lambda = (\lambda_i)_{i \in Q_0}$ such that λ_i is a partition of α_i. We will write $\lambda(j)$ for the dimension vector $(\lambda(j)_i)_{i\in Q_0} = (\lambda_{ij})_{i \in Q_0}$. The multipartition of α consisting of (1^{α_i}) is denoted by (1^α) again.

We put $u_\lambda = (u_{\lambda_i})_{i\in Q_0} \in \mathrm{Gl}_\alpha$, and we define for a multipartition λ of α a function that maps a multipartition into $\mathbb{Z}[q]$

$$f_\lambda(\mu) = \prod_i \tilde{K}_{\lambda_i \mu_i}(q).$$

We will need the following result:

Lemma 3.1 *Let W be a representation of $\mathrm{Gl}_\alpha(\mathbb{F}_p)$ over $\mathbb{F}_p$. Then the function $\mu \mapsto q^{\dim_{\mathbb{F}_p} W^{u_\mu}}$ is a linear combination of the functions f_λ with coefficients in $\mathbb{Z}[q]$.*

Proof Since the $f_\lambda(\mu)$'s form an upper triangular matrix with q-powers on the diagonal it is clear that we can express $q^{\dim_{\mathbb{F}_p} W^{u_\mu}}$ as linear combination of the f_λ's with coefficients in $\mathbb{Z}[q, q^{-1}]$.

Now let q be a fixed power of p and let V be the permutation representation of $W \otimes_{\mathbb{F}_p} \mathbb{F}_q$ (over $\mathbb{C}$). Then $q^{\dim W^{u_\mu}} = \mathrm{Tr}(u_\mu, V)$. By Green's formula for the irreducible characters of $\mathrm{Gl}_\alpha(\mathbb{F}_q)$ we can express the values of the character of V at unipotent elements as a $\mathbb{Q}$-linear combination of products of Green polynomials [Zel81, p.135]. Furthermore the denominators are bounded in terms of α. Expressing these Green polynomials further in terms of the f_λ [Mac95, III.(7.11)] we find, for a fixed q, that $q^{\dim W^{u_\mu}}$ is a linear combination of the $f_\lambda(q)$ with coefficients in $\mathbb{Q}$, whose denominator is still bounded in terms of α.

If we now let q go to ∞ we see that if we consider q as a variable again, the remark of the first paragraph yields that the coefficients must be in $\mathbb{Z}[q]$.

This finishes the proof of the lemma.

3.4 A Reformulation of the Conjecture

It follows from the previous lemma that there are $p_\mu \in \mathbb{Z}[q]$ such that

$$|\operatorname{Rep}(Q,\alpha)^{u_\lambda}| = \sum_\mu p_\mu f_\mu(\lambda). \tag{3.14}$$

From this we can calculate $p_{(1^\alpha)}$ as follows.

$$|\operatorname{Rep}(Q,\alpha)^{u_\lambda}| = \sum_\mu p_\mu \prod_i q^{n(\lambda_i)} K_{\mu_i\lambda_i}(q^{-1}).$$

Now we multiply with $\prod_i q^{-n(\lambda_i)} K^{-1}_{\lambda_i(1^{\alpha_i})}(q^{-1})$ and summing over λ gives entries of the matrix product of K and K^{-1}, resulting in

$$p_{(1^\alpha)} = \sum_\lambda |\operatorname{Rep}(Q,\alpha)^{u_\lambda}| \prod_i q^{-n(\lambda_i)} K^{-1}_{\lambda_i(1^{\alpha_i})}(q^{-1}).$$

Now we finally get back to the r_α-function. It is rather easy to see that

$$\begin{aligned} r_\alpha(q) &= \sum_\lambda \frac{|\operatorname{Rep}(Q,\alpha)^{u_\lambda}|}{c_\lambda(q)} \\ \text{(using 3.14)} &= \sum_{\lambda,\mu} \frac{p_\mu f_\mu(\lambda)}{c_\lambda(q)} \\ &= \sum_\mu p_\mu \prod_{i\in Q_0} \sum_{\lambda_i} \frac{\tilde{K}_{\mu_i,\lambda_i}(q)}{c_{\lambda_i}(q)}. \end{aligned}$$

Evaluation at $q=0$, using 3.13, we get:

$$r_\alpha(0) = (-1)^{\sum \alpha_i} p_{(1^\alpha)}(0). \tag{3.15}$$

So we are interested in

$$(-1)^{\sum \alpha_i} p_{(1^\alpha)} = (-1)^{\sum \alpha_i} \sum_\lambda |\operatorname{Rep}(Q,\alpha)^{u_\lambda}| \prod_i q^{-n(\lambda_i)} K^{-1}_{\lambda_i(1^{\alpha_i})}(q^{-1}).$$

Now we plug in the following formula (see appendix B for the proof)

$$K^{-1}_{\lambda,1^d}(t) = (-1)^{l(\lambda)+d} t^{\sum_{i\geq 2} \frac{\lambda'_i(\lambda'_i+1)}{2}} [\lambda']. \tag{3.16}$$

This gives the following element of $\mathbb{Z}[q]$:

$$\begin{aligned} (-1)^{\sum \alpha_i} p_{(1^\alpha)} &= \sum_\lambda (-1)^{\sum \lambda'_{i1}} q^{\theta_\lambda} \prod_i [\lambda'_i]_{q^{-1}} \\ &= \sum_\lambda q^{-\sum(\lambda'(j),\lambda'(j))/2} \prod_i (-1)^{\lambda'_{i1}} q^{(\lambda'_{i1}(\lambda'_{i1}+1))/2} [\lambda'_i]_{q^{-1}}, \end{aligned}$$

where the crucial steps happen in the exponent of q:

$$\begin{aligned}\theta_\lambda &= \dim \operatorname{Rep}(Q,\alpha)^{u_\lambda} - \sum_i \left(n(\lambda_i) - \sum_{j\geq 2} \frac{\lambda'_{ij}(\lambda'_{ij}+1)}{2} \right) \\ &= \sum_{ik\in Q_0} a_{ik}\lambda'_{ij}\lambda'_{kj} - \sum_i \left((\sum_j \lambda'^2_{ij}) - \frac{\lambda'_{i1}(\lambda'_{i1}+1)}{2} \right) \\ &= -1/2 \sum_j (\lambda'(j), \lambda'(j)) + \sum_i \frac{\lambda'_{i1}(\lambda'_{i1}+1)}{2}.\end{aligned}$$

Now we finally obtain the following reformulation conjecture 1.2. We define the Laurent polynomial

$$\begin{aligned}\Pi_\alpha(t) &= (-1)^{\sum \alpha_i} p_{(1^\alpha)}(t^{-1}) \\ &= \sum_\lambda t^{\sum(\lambda(j),\lambda(j))/2} \prod_i (-1)^{\lambda_{i1}} t^{-(\lambda_{i1}(\lambda_{i1}+1))/2}[\lambda_i].\end{aligned}$$

Clearly $\Pi_\alpha(t) \in \mathbb{Z}[t^{-1}]$, and by equation 3.15 and Theorem 3.1 we get:

Theorem 3.2 *Conjecture 1.2 is equivalent to*

$$\Pi_\alpha(\infty) = \begin{cases} \epsilon(w) & \text{if } \alpha = \rho - w(\rho) \\ 0 & \text{otherwise.} \end{cases} \tag{3.17}$$

3.5 Some Examples of Π_α

To indicate the use of the Laurent polynomials Π_α we first look at the case of a one-vertex quiver without loops. We get for each a that

$$\Pi_a(t) = \sum_{|\mu|=a} (-1)^{\mu_1} t^{\sum_i \mu_i^2 - \mu_1(\mu_1+1)/2}[\mu]$$

is a polynomial in t^{-1}. On the other hand if $a > 1$ we get that for each partition the exponent of t is at least 1, while $[\mu]$ is a polynomial in t. Therefore $\Pi_a(t) \in t\mathbb{Z}[t]$. It follows that $\Pi_a(t)$ is identically zero.

The next case we consider is the m-arrow quiver. We get for dimension vector (a, b) the polynomial

$$\Pi_{a,b}(t) = \sum_{\substack{|\mu|=a \\ |\lambda|=b}} (-1)^{\mu_1+\lambda_1} t^{\sum_i \mu_i^2 + \sum_i \lambda_i^2 - m\sum_i \lambda_i\mu_i - \mu_1(\mu_1+1)/2 - \lambda_1(\lambda_1+1)/2}[\mu][\lambda]$$

If we take $a \gg b$ we can repeat the above method to obtain that:

Theorem 3.3 *The Laurent polynomial $\Pi_{a,b}(t)$ is identically zero if $a \gg b$*

In fact we used a computer (see the next section) to obtain the following conjecture.

Conjecture 3.1 *The Laurent polynomial* $\Pi_{a,b}(t)$ *is identically zero if and only if* $a \geq mb + 2$ *or* $b \geq ma + 2$

We include a small computer program in appendix C, which can be used to check conjecture 1.2 and conjecture 3.1 for the m-arrow quiver.

We tested the following cases.

- $b \leq a \leq 20$ and $m = 3$
- $b < 6$ and $a < 21$ for $m = 4$
- $m < 31$, $a = m + 1$ and $b = 1$, which is $\rho - r_1 r_2 \rho$ for the given m.
- $m < 7$, $a = m(m+1)$ and $b = m$, which is $\rho - r_1 r_2 r_1 \rho$ (the calculation for the case $m = 6$ lasted over ten hours on a Pentium II -400).

Chapter 4

Ringel-Hall Algebras

In this chapter we will introduce the notion of a Hopf algebra and its Drinfeld double. From this we will carry on to introduce the double Ringel-Hall algebra $U(Q)$ of a quiver Q.

In the next chapter we will prove the independence of $U(Q)$ of the orientation of Q, by constructing isomorphisms between different $U(Q)$'s. We will also define an automorphism on $U(Q)$, which generalizes the work of Lusztig.

In chapter 6 we will construct another Hopf algebra, the quantized generalized enveloping algebra of a Kac-Moody Lie algebra, which basically is a new quantum group, depending on a certain datum. As it will turn out, both notions will be equivalent if the latter is obtained from a datum connected to the quiver Q.

4.1 A Short Account of Hopf Algebras

4.1.1 The Definition of a Hopf Algebra

An *algebra* A has a multiplication μ and a unit[1] η. By dualizing we get a *coalgebra* C with a comultiplication $\Delta : C \longrightarrow C \otimes C$ and a co-unit ε. It is useful to introduce the following notation, due to Sweedler: $\Delta(a) = \sum a_{(1)} \otimes a_{(2)}$. This notation may become cumbersome in some calculations, so where we will just write $\Delta(a) = a_1 \otimes a_2$, omitting the summation sign and the parentheses. This extends to $\Delta^2(a) = \Delta(\Delta \otimes 1)(a) = \Delta(1 \otimes \Delta)(a) = a_1 \otimes a_2 \otimes a_3$.

A vector space with the structure of an algebra and a coalgebra is a *bialgebra*, if the comultiplication and co-unit are algebra morphisms, or equivalently if the multiplication and the unit are coalgebra morphisms.

Finally a *Hopf Algebra* H is a bialgebra with an antipode S, such that the following three maps are equal.

$$\eta\varepsilon = \mu(S \otimes \mathrm{Id})\Delta = \mu(\mathrm{Id} \otimes S)\Delta.$$

[1]This is a categorical way of saying that there is a 1. The relation between them is as follows: $\eta : \Bbbk \to H : a \mapsto a \cdot 1$.

Thus a Hopf-Algebra over $\Bbbk$ is $\Bbbk$-vector space H together with linear maps

$$\begin{aligned}
\text{(multiplication)} \quad & \mu : H \otimes H \longrightarrow H, \\
\text{(unit)} \quad & \eta : \Bbbk \longrightarrow H, \\
\text{(comultiplication)} \quad & \Delta : H \longrightarrow H \otimes H, \\
\text{(co-unit)} \quad & \varepsilon : H \longrightarrow \Bbbk, \\
\text{(antipode)} \quad & S : H \longrightarrow H,
\end{aligned}$$

satisfying

$$\begin{aligned}
\mu(\mathrm{Id} \otimes \eta) = \mathrm{Id} &= \mu(\eta \otimes \mathrm{Id}), \\
\mu(\mu \otimes \mathrm{Id}) &= \mu(\mathrm{Id} \otimes \mu), \\
(\mathrm{Id} \otimes \varepsilon)\Delta = \mathrm{Id} &= (\varepsilon \otimes \mathrm{Id})\Delta, \\
(\Delta \otimes \mathrm{Id})\Delta &= (\mathrm{Id} \otimes \Delta)\Delta, \\
\eta\varepsilon = \mu(S \otimes \mathrm{Id})\Delta &= \mu(\mathrm{Id} \otimes S)\Delta, \\
\mu(\varepsilon \otimes \varepsilon) &= \varepsilon\mu, \\
\Delta\mu &= (\mu \otimes \mu)\tau_{23}(\Delta \otimes \Delta),
\end{aligned}$$

where $\tau_{ij}(h_1 \otimes \cdots \otimes h_i \otimes \cdots \otimes h_j \otimes \cdots \otimes h_n) = (h_1 \otimes \cdots \otimes h_j \otimes \cdots \otimes h_i \otimes \cdots \otimes h_n)$. Below $\Bbbk = \mathbb{C}$ and all Hopf algebras have invertible antipode[2].

4.1.2 The Drinfeld Double

The following definition is the classical way to introduce the Drinfeld double. Underneath we will give a definition by generators and relations which will be more convenient.

Given two Hopf algebras A and B a *(skew-)Hopf pairing* is a bilinear function $\psi : A \times B \longrightarrow R$ into an integral domain, satisfying the following relations (given $a, a' \in A$ and $b, b' \in B$):

$$\psi(1, b) = \epsilon_B(b), \psi(a, 1) = \epsilon_A(a), \tag{4.1}$$
$$\psi(a, bb') = \psi(\Delta_A(a), b \otimes b') = \psi(a_1, b)\psi(a_2, b'), \tag{4.2}$$
$$\psi(aa', b) = \psi(a \otimes a', \Delta_B^{opp}(b)) = \psi(a, b_2)\psi(a', b_1) \quad \text{and} \tag{4.3}$$
$$\psi(S_A(a), b) = \psi(a, S_B^{-1}(b)). \tag{4.4}$$

Given a skew Hopf pairing ψ on $A \times B$ the *Drinfeld Double D* is defined as the following Hopf structure on $A \otimes B$.

1. The multiplication is characterized by

   (a) $(a \otimes 1)(a' \otimes 1) = aa' \otimes 1$,

[2]All naturally appearing Hopf algebras have invertible antipodes. A Hopf algebra with this property is said to be regular.

(b) $(a \otimes 1)(1 \otimes b) = a \otimes b$,

(c) $(1 \otimes b)(1 \otimes b') = 1 \otimes bb'$ and

(d) $(1 \otimes b)(a \otimes 1) = \psi(a_1, S_B(b_3))(a_2 \otimes b_2)\psi(a_3, b_1)$,

while the unit is $1 \otimes 1$.

2. The coalgebra structure is given by $\epsilon = \epsilon_A \otimes \epsilon_B$ and $\Delta = \tau_{23}(\Delta_A \otimes \Delta_B)$.

3. The antipode is $S(a \otimes b) = (1 \otimes S_B(b))(S_A(a) \otimes 1)$.

The Hopf algebra D contains $A = A \otimes 1$ and $B = 1 \otimes B^{coop}$ as sub-Hopf algebras.

Now we will give an alternative definition.

Theorem 4.1 *D can be defined as the free product $A * B$ divided out by the following relations*

$$(b_1 * a_2)\psi(a_1, b_2) = (a_1 * b_2)\psi(a_2, b_1). \tag{4.5}$$

For the proof we write DD for the algebra defined by (4.5). First note the following equivalent form of (1d).

Lemma 4.1 *(1d) is equivalent to*

$$\psi(a_1, b_2)(1 \otimes b_1)(a_2 \otimes 1) = (a_1 \otimes b_2)\psi(a_2, b_1) \tag{4.6}$$

(4.6) follows from (1d) since

$$\begin{aligned} \psi(a_1, b_2)(1 \otimes b_1)(a_2 \otimes 1) &= \psi(a_2, S(b_3))\psi(a_1, b_4)(a_3 \otimes b_2)\psi(a_4, b_1) \\ &= \psi(a_2 a_1, 1)(a_3 \otimes b_2)\psi(a_4, b_1) \\ &= (\varepsilon(a_2 a_1)a_3 \otimes b_2)\psi(a_4, b_1) \\ &= (a_1 \otimes b_2)\psi(a_2, b_1). \end{aligned}$$

and, on the other hand (1d) follows from (4.6) by

$$\begin{aligned} \psi(a_1, S_B(b_3))(a_2 \otimes b_2)(\psi(a_3, b_1)) &= \psi(a_1, S_B(b_3))\psi(a_2, b_2)(1 \otimes b_1)(a_3 \otimes 1) \\ &= \psi(a_2 a_1, 1)(1 \otimes b)(a_3 \otimes 1) \\ &= (1 \otimes b)(a \otimes 1), \end{aligned}$$

where we used twice that

$$S(b_2)b_3 \otimes b_1 = \eta\varepsilon(b_2 b_3) \otimes b_1 = 1 \otimes \eta\varepsilon(b_2 b_3)b_1 = 1 \otimes b.$$

This finishes the proof of the lemma.

From the lemma it follows that the map from DD to D sending $a * 1$ to $a \otimes 1$ and $1 * b$ to $1 \otimes b$ is well defined. Moreover it is the inverse of the linear map sending $a \otimes b$ to $a * b$. So both maps are isomorphism. Similarly the linear map $a \otimes b \mapsto (b * a)$ is an isomorphism of vector spaces.

This proves that D and DD are isomorphic as algebras, while the Hopf algebra structure is uniquely determined by demanding that $A \otimes 1$ and $1 \otimes B^{coop}$ are sub-Hopf algebras.

We will need the following lemma

Lemma 4.2 *Assume A, B are $\mathbb{N}$-graded. Assume that A is generated over A_0 by a vector space V_A and similarly that B is generated over B_0 by a vector space V_B. Then we only have to impose the relation* (4.5) *for $a \in A_0 \cup V_A$ and $b \in B_0 \cup V_B$*

Let H denote the algebra obtained by imposing the relations as in the lemma. We have to show that (4.5) holds for arbitrary a, b.

We do this by induction on the degree of a and b. So let $b \notin B_0 \cup V_B$ then we can write $b = \sum c_i d_i$ where the c_i's and the d_i's have smaller degree. (4.5) is clearly additive in b, so we may assume $b = cd$. Thus we have to show that the following expression can be reduced to the similar expressions for c and d.

$$(c_1 d_1 * a_2)\psi(a_1, c_2 d_2) = (a_1 * c_2 d_2)\psi(a_2, c_1 d_1)$$

Assuming that for b' of lower degree than b the equation (4.5) is satisfied, we get

$$\begin{aligned}(c_1 d_1 * a_2)\psi(a_1, c_2 d_2) &= c_1(d_1 * a_3)\psi(a_1, c_2)\psi(a_2, d_2) \\ &= c_1 * a_2 * d_2\psi(a_1, c_2)\psi(a_3, d_1) \\ &= a_1 * c_2 * d_2\psi(a_2, c_1)\psi(a_3, d_1) \\ &= (a_1 * c_2 d_2)\psi(a_2, c_1 d_1)\end{aligned}$$

In the same way we can reduce the degree of a until $a \in A_0 \cup V_A$

4.2 The Ringel-Hall Algebra and its Double

4.2.1 The Ringel-Hall Algebra $H(Q)$

Fix a quiver $Q = (Q_0, Q_1, t, h)$, and a finite field $\Bbbk = \mathbb{F}_q$. We put $v = q^{-1/2}$. The Ringel-Hall algebra $H(Q)$ is defined as follows.

As a vector space, $H(Q)$ has a basis given by the isomorphism classes of representations of Q. If A and B are two representations, then the corresponding product is given by

$$[A][B] = v^{-\langle A,B\rangle} \sum_{[C]} g_{AB}^{C}[C], \tag{4.7}$$

where the number g_{AB}^{C} is the cardinality of $\{X \subset C \mid X \cong A,\ C/X \cong B\}$. Note that only for a finite number of isomorphism classes the above set is not empty, leaving a finite sum in the definition of the product.

The algebra obtained here is opposite to the algebra of Ringel. Also note we will need some identities of these structure constants whose proof we postpone to appendix A.

Fixing a dimension vector α, we define $H(Q)_\alpha$ as the subspace with basis the isomorphism classes of α-dimensional representations. This makes $H(Q)$ into a $\mathbb{N}^{Q_0}$-graded algebra.

It is possible to put the structure of a certain type of twisted Hopf algebra [Gre95] on $H(Q)$. To obtain a real Hopf algebra, we have to add a *diagonal* part in degree zero. This is explained in the next section.

4.2.2 Adding a Diagonal Part

Consider $\{K_\alpha | \alpha \in \mathbb{Z}^{Q_0}\}$ as basis of the group algebra of $\mathbb{Z}^{Q_0}$, and consider a crossed product $B(Q)$ with $H(Q)$ having following multiplication:

$$K_\alpha[A] \bullet K_\beta[B] = v^{-(\langle B,A\rangle + (\beta, \overrightarrow{A}))} \sum_{[C]} g_{AB}^C K_{\alpha+\beta}[C].$$

By defining $K_\alpha = K_\alpha[0]$ and $[A] = K_o[A]$, where $o = (0, \ldots, 0)$, we get both $H(Q)$ and $\mathbb{Z}^{Q_0}$ as subalgebras of $B(Q)$. We extend the $\mathbb{N}^{Q_0}$-grading to $B(Q)$ by putting $|[A]| = \overrightarrow{A}$ and $|K_\alpha| = 0$. Here we follow Lusztig's convention [Lus93] of writing $|x|$ for the degree of a homogeneous element.

The algebra $B(Q)$ is made into a Hopf algebra [Kap97, Xia97] by defining Δ, ε and S on the generators as follows.

$$\Delta[A] = \sum_{[B],[C]} v^{-\langle C,B\rangle} g_{BC}^A \frac{|\operatorname{Aut}(B)||\operatorname{Aut}(C)|}{|\operatorname{Aut}(A)|}[B] \otimes K_{\overrightarrow{B}}[C]$$

$$\Delta(K_\alpha) = K_\alpha \otimes K_\alpha$$

$$\varepsilon(K_\alpha[A]) = \delta_{[A],[0]}$$

For the antipode we need to sum over the set $C_{A,n}$ of chains of the form $0 = A_0 \subset A_1 \subset \cdots \subset A_n = A$, where $\forall i : A_i \neq A_{i+1}$. We define

$$S[A] = \sum_{n=0}^{\infty} \sum_{C_{A,n}} \prod_{i=1}^{n} (-1)^n v^{-\langle A_i/A_{i-1}, A_{i-1}\rangle} \frac{\prod_{j=1}^{n} |\operatorname{Aut}(A_j/A_{j-1})| [A_j/A_{j-1}]}{|\operatorname{Aut}(A)|} K_{\overrightarrow{A}}^{-1}$$

and $S(K_\alpha) = K_{-\alpha}$.

4.2.3 The Algebra $U(Q)$

$B(Q)$ has a skew Hopf pairing with itself:

$$\psi(K_\alpha[A], K_\beta[B]) = v^{-(\alpha,\beta)}\psi([A],[B]) = v^{-(\alpha,\beta)} \frac{\delta_{[A],[B]}}{|\operatorname{Aut}(A)|}. \qquad (4.8)$$

So we can define the Drinfeld double structure $D(Q)$ on $B(Q) \otimes B(Q)$.

After dividing out $K_\alpha \otimes 1 - 1 \otimes K_{-\alpha}$, one obtains a new Hopf algebra denoted by $U(Q)$. We put $[A]$, $[A]^-$ and K_α for the class of $[A] \otimes 1$, $1 \otimes [A]$ and $K_\alpha \otimes 1 = 1 \otimes K_{-\alpha}$. In section 4.2.4 we will show that $U(Q)$ is the algebra generated by $[A]$, $[A]^-$ and K_α and satisfying the following relations.

$$
\begin{aligned}
K_o = [0] &= [0]^- = 1, && (4.9)\\
K_\alpha K_\beta &= K_{\alpha+\beta}, && (4.10)\\
[A][B] &= v^{-\langle B,A\rangle} \sum g_{AB}^{C}[C], && (4.11)\\
[A]^-[B]^- &= v^{-\langle B,A\rangle} \sum g_{AB}^{C}[C]^-, && (4.12)\\
[A]K_\alpha &= v^{-(\alpha,\overrightarrow{A})} K_\alpha [A], && (4.13)\\
[A]^-K_\alpha &= v^{(\alpha,\overrightarrow{A})} K_\alpha [A]^- \qquad \text{and} && (4.14)
\end{aligned}
$$

$$
\begin{aligned}
&\sum_{[M],[N]} g_{AB}^{MN} v^{-\langle \overrightarrow{B}-\overrightarrow{N}, \overrightarrow{M}-\overrightarrow{N}\rangle} [M][N]^- K^{-1}_{\overrightarrow{B}-\overrightarrow{N}} \\
&\quad = \sum_{[M],[N]} g_{BA}^{NM} v^{-\langle \overrightarrow{A}-\overrightarrow{M}, \overrightarrow{N}-\overrightarrow{M}\rangle} [N]^-[M] K_{\overrightarrow{A}-\overrightarrow{M}},
\end{aligned}
\tag{4.15}
$$

where for $M, N, A, B \in \operatorname{Rep}(Q)$ we write g_{AB}^{MN} for the number of exact sequences

$$
0 \longrightarrow M \xrightarrow{\ u\ } A \xrightarrow{\ \phi\ } B \xrightarrow{\ v\ } N \longrightarrow 0 \tag{4.16}
$$

divided by $|\operatorname{Aut}(A)||\operatorname{Aut}(B)|$. We refer to appendix A for relations between g_{MN}^{AB}'s and g_{AC}^{B}'s, and other properties of these numbers. For instance, we will use the equation

$$
g_{AB}^{MN} = \frac{|\operatorname{Aut}(M)||\operatorname{Aut}(N)|}{|\operatorname{Aut}(A)||\operatorname{Aut}(B)|} \sum_{[P]} g_{MP}^{A} g_{PN}^{B} |\operatorname{Aut}(P)|. \tag{4.17}
$$

There is a classical decomposition $U(Q) = U(Q)_- \otimes U(Q)_o \otimes U(Q)_+$, where the algebra $U(Q)_-$ is generated by the $[A]^-$'s, the algebra $U(Q)_+$ is generated by the $[A]$'s, and $U(Q)_o$ is generated the K_α's. Note that $U(Q)_- \cong H(Q) \cong U(Q)_+$.

We will need the identity (4.15) in the special case that $B = S_i$. Since N is either a quotient or a subset of S_i it has to be either equal to S_i or 0. In the former case $A \cong M$, in the latter $\overrightarrow{A} - \overrightarrow{M} = i$. Putting the terms where N equals S_i on the lefthand side and the others right we obtain:

$$
\begin{aligned}
[A][S_i]^- - [S_i]^-[A] = &\sum_{[M]} q^{(1/2)\langle M,S_i\rangle} \frac{|\operatorname{Aut}(M)|}{|\operatorname{Aut}(A)|} g_{S_iM}^{A} K_i [M] \\
&- \sum_{[M]} q^{(1/2)\langle S_i,M\rangle} \frac{|\operatorname{Aut}(M)|}{|\operatorname{Aut}(A)|} g_{MS_i}^{A} [M] K_{-i}.
\end{aligned}
\tag{4.18}
$$

Note that if i is a source or sink and in addition if A is admissible one of the two terms on the righthand side of this equation disappears.

Finally it is easy to verify that there is an automorphism ω on $U(Q)$ of order 2 such that $\omega K_\alpha = K_{-\alpha}$ and $\omega[A] = [A]^-$.

4.2.4 Calculations of the Given Relations

From the given equations only the last one does not follow immediately from the construction of $B(Q)$. We will show that (4.15) follows from (1d) on page 33 (or rather from its equivalent form (4.6)) in the case $a = [A]$ and $b = [B]$.

For the lefthand side Λ we get:

$$\begin{aligned}\Lambda &= \psi(a_1, b_2)(1 \otimes b_1)(a_2 \otimes 1)\\ &= \sum_{\substack{[C],[D]\\ [E],[F]}} v^{-\langle F,E\rangle - \langle D,C\rangle} g^A_{CD} g^B_{EF} \frac{|\operatorname{Aut} C||\operatorname{Aut} D||\operatorname{Aut} E||\operatorname{Aut} F|}{|\operatorname{Aut} A||\operatorname{Aut} B|}\\ &\qquad .\psi([C], K_{\overrightarrow{E}}[F])(1 \otimes [E])(K_{\overrightarrow{C}}[D] \otimes 1)\end{aligned}$$

In the non-zero terms we need to have $C \cong F$, since otherwise $\psi([C], K_{\overrightarrow{E}}[F]) = 0$. We also note that $\overrightarrow{C} = \overrightarrow{A} - \overrightarrow{D}$ and $\psi([C], K_{\overrightarrow{E}}[C]) = \dfrac{1}{|\operatorname{Aut} C|}$. Furthermore we have $K_{\overrightarrow{C}}[D] \otimes 1 = (K_{\overrightarrow{C}} \otimes 1)([D] \otimes 1) = K_{\overrightarrow{C}}[D] = v^{(C,D)}[D]K_{\overrightarrow{C}}$. Thus we get

$$\begin{aligned}\Lambda &= \sum_{\substack{[C],[D]\\ [E]}} v^{-\langle C,E\rangle - \langle D,C\rangle} g^A_{CD} g^B_{EC} \frac{|\operatorname{Aut} C||\operatorname{Aut} D||\operatorname{Aut} E||\operatorname{Aut} C|}{|\operatorname{Aut} A||\operatorname{Aut} B|}\\ &\qquad .\psi([C], K_{\overrightarrow{E}}[C])(1 \otimes [E])(K_{\overrightarrow{C}}[D] \otimes 1)\\ &= \sum_{\substack{[C],[D]\\ [E]}} v^{-\langle C,E\rangle + \langle C,D\rangle} g^A_{CD} g^B_{EC} \frac{|\operatorname{Aut} C||\operatorname{Aut} D||\operatorname{Aut} E|}{|\operatorname{Aut} A||\operatorname{Aut} B|} [E]^-[D]K_{\overrightarrow{C}}\\ &= \sum_{[D],[E]} v^{-\langle \overrightarrow{A}-\overrightarrow{D}, \overrightarrow{E}-\overrightarrow{D}\rangle}[E]^-[D]K_{\overrightarrow{A}-\overrightarrow{D}}\\ &\qquad \cdot \frac{|\operatorname{Aut} D||\operatorname{Aut} E|}{|\operatorname{Aut} A||\operatorname{Aut} B|} \sum_{[C]} g^A_{CD} g^B_{EC} |\operatorname{Aut} C|\\ &= \sum_{[D],[E]} g^{ED}_{BA} v^{-\langle \overrightarrow{A}-\overrightarrow{D}, \overrightarrow{E}-\overrightarrow{D}\rangle}[E]^-[D]K_{\overrightarrow{A}-\overrightarrow{D}}.\end{aligned}$$

A similar calculation works for the righthand side. This time $[E]$ is forced to be equal to D leading to:

$$
\begin{aligned}
&(a_1 \otimes b_2)\psi(a_2, b_1) \\
&\quad = \sum_{\substack{[C],[D] \\ [E],[F]}} v^{-\langle F,E\rangle - \langle D,C\rangle} g^A_{CD} g^B_{EF} \frac{|\operatorname{Aut} C||\operatorname{Aut} D||\operatorname{Aut} E||\operatorname{Aut} F|}{|\operatorname{Aut} A||\operatorname{Aut} B|} \\
&\qquad\qquad .([C] \otimes K_{\overrightarrow{E}}[F])\psi(K_{\overrightarrow{C}}[D], [E]) \\
&\quad = \sum_{\substack{[C],[F] \\ [E]}} v^{-\langle F,E\rangle - \langle E,C\rangle} g^A_{CE} g^B_{EF} \frac{|\operatorname{Aut} C||\operatorname{Aut} E||\operatorname{Aut} E||\operatorname{Aut} F|}{|\operatorname{Aut} A||\operatorname{Aut} B|} \\
&\qquad\qquad .[C]K_{-\overrightarrow{E}}[F]^- \psi(K_{\overrightarrow{C}}[E], [E]) \\
&\quad = \sum_{[C],[F]} g^{CF}_{AB} v^{\langle \overrightarrow{B} - \overrightarrow{F}, \overrightarrow{F} - \overrightarrow{C}\rangle} [C][F]^- K_{-\overrightarrow{B} + \overrightarrow{F}}
\end{aligned}
$$

This clearly gives us (4.15).

Now we will prove that the relations (4.9)-(4.15) are sufficient. We need to check that (4.6) holds and by lemma 4.2 is suffices to do this on generators. The above calculation took care of the case $a = [A]$ and $b = [B]$. We check the other cases below.

- In the case $a = K_\alpha$ and $b = K_\beta$ we get for the following trivial equalities:

$$
\begin{aligned}
\psi(K_\alpha, K_\beta)(1 \otimes K_\beta)(K_\alpha \otimes 1) &= v^{-(\alpha,\beta)} K_{-\beta} K_\alpha = \\
K_{\alpha-\beta} v^{-(\alpha,\beta)} &= (K_\alpha \otimes K_\beta)\psi(K_\alpha, K_\beta).
\end{aligned}
$$

- To treat the case $a = K_\alpha$ and $b = [B]$ we put $b_1 = v^{-\langle F,E\rangle} g^B_{EF} \frac{|\operatorname{Aut} E||\operatorname{Aut} F|}{|\operatorname{Aut} B|}[E]$ and $b_2 = K_{\overrightarrow{E}}[F]$, and obtain for the lefthand side

$$
\begin{aligned}
&\psi(a_1, b_2)(1 \otimes b_1)(a_2 \otimes 1) \\
&\quad = \sum_{[E],[F]} v^{-\langle F,E\rangle} g^B_{EF} \frac{|\operatorname{Aut} E||\operatorname{Aut} F|}{|\operatorname{Aut} B|} \psi(K_\alpha, K_{\overrightarrow{E}}[F])(1 \otimes [E])(K_\alpha \otimes 1)
\end{aligned}
$$

  The only term that survives in the evaluation is the one with $[F] = [0]$, and thus $[E] = [B]$. Clearly $g^B_{B0} = 1$. Thus the lefthand term is equal to:

$$
\psi(K_\alpha, K_{\overrightarrow{B}})(1 \otimes [B])(K_\alpha \otimes 1) = v^{-(\alpha, \overrightarrow{B})}[B]^- K_\alpha
$$

For the righthand side we obtain

$$
\begin{aligned}
&(a_1 \otimes b_2)\psi(a_2, b_1) \\
&\quad = \sum_{[E],[F]} v^{-\langle F,E\rangle} g^B_{EF} \frac{|\operatorname{Aut} E||\operatorname{Aut} F|}{|\operatorname{Aut} B|}(K_\alpha \otimes K_{\overrightarrow{E}}[F])\psi(K_\alpha, [E]) \\
&\quad = (K_\alpha \otimes [B])\psi(K_\alpha, K_o) \\
&\quad = K_\alpha [B]^-
\end{aligned}
$$

This leads to equation (4.14).

- It is a similar calculation that leads to (4.13), starting from $a = [A]$ and $b = K_\beta$.

Chapter 5

Reflection Isomorphisms

In this chapter we will introduce three isomorphisms between $U(Q)$'s for different quivers with the same underlying graph.

The first isomorphism was suggested by Lusztig [Lus90]. It uses the notion of Fourier transform to define an isomorphism $F_{Q,Q'}$ between the Ringel-Hall algebras of quivers Q and Q' with the same underlying graph. In other words the Ringel-Hall algebra is independent of the orientation of the quiver.

It turns out (as predicted by Lusztig) that these morphisms are one-to-one and therefore the Ringel-Hall algebra is independent of the orientation of the quiver.

A second isomorphism t_i (defined for a source i) from $U(Q)$ to $U(r_iQ)$ is obtained by extending the reflections r_i (defined by (1.5)), to the double Ringel-Hall Algebra. Note that passing to the double is essential to define t_i on non-admissible representations.

Combining the above isomorphisms one defines an automorphism $\hat{t}_i$ on $U(Q)$ for any vertex i. To this end one introduces two auxiliary quivers with the same underlying graph as Q: (i) Q^i, obtained from Q by making i a source and (ii) $Q_i = r_iQ^i$. We define $\hat{t}_i = F_{Q_i,Q}t_iF_{Q,Q^i}$. We find that on the subalgebra generated by simple representations these automorphism were defined by Lusztig [Lus93]. We have thus given a new definition of these automorphisms, and at the same time we have extended them to $U(Q)$.

5.1 Fourier Transform

5.1.1 Fourier Transform in General

First recall the definition of finite Fourier transforms. Let X be a finite set, and denote by $F(X)$ the set of functions on X with values in $\mathbb{C}$. Given a map $f : X \to Y$ between finite sets we define $f^* : F(X) \to F(Y)$ by $f^*(g) = g \circ f$.

On $F(X)$ there is an inner product defined by

$$(f,g) = \frac{1}{|X|} \sum_{x \in X} f(x)\overline{g(x)}. \tag{5.1}$$

We will be concerned with a finite set U, and finite dimensional vector spaces V and W, both of dimension n over a finite field $\Bbbk = \mathbb{F}_q$. We also assume there is a non-degenerate pairing $\langle -,- \rangle : V \times W \to \Bbbk$. Further we choose a non trivial additive character $\psi : \Bbbk \to \mathbb{C}^*$. We now define $\hat{-} : F(U \times V) \to F(U \times W)$ and $\check{-} : F(U \times V) \to F(U \times W)$ by.

$$\hat{f}(u,w) = q^{\frac{-n}{2}} \sum_{v \in V} f(u,v)\psi(\langle v,w \rangle).$$
$$\check{f}(u,w) = q^{\frac{-n}{2}} \sum_{v \in V} f(u,v)\overline{\psi(\langle v,w \rangle)},$$

for $f \in F(U \times V)$, $u \in U$, $w \in W$.

Note that

$$\hat{f}(u,-w) = \check{f}(u,w), \tag{5.2}$$
$$\check{\hat{f}} = \hat{\check{f}} = f, \tag{5.3}$$
$$(\hat{f},\hat{g}) = (f,g) = (\check{f},\check{g}). \tag{5.4}$$

These statements are proved as follows.

Using the fact that $\sum_{w \in W} \psi(\langle v,w \rangle) = 0$ except when $v = 0$, in which case the sum is equal to $|W| = q^n$, we get (for $t \in V$ and $u \in U$):

$$\begin{aligned}
\hat{\check{f}}(u,t) &= q^{-n/2} \sum_{w \in W} \check{f}(u,w)\psi(\langle t,w \rangle) \\
&= q^{-n/2} \sum_{w \in W} \left(q^{-n/2} \sum_{v \in V} f(u,v)\psi(\langle v,-w \rangle) \right) \psi(\langle t,w \rangle) \\
&= q^{-n} \sum_{v \in V} f(u,v) \sum_{w \in W} \psi(\langle t,w \rangle - \langle v,w \rangle) \\
&= q^{-n} \sum_{v \in V} f(u,v) \sum_{w \in W} \psi(\langle t-v,w \rangle) \\
&= f(u,t).
\end{aligned}$$

and

$$\begin{aligned}(\hat{f},\hat{g}) =& \frac{1}{|V|}\sum_{w\in W}\hat{f}(w)\overline{\hat{g}(w)}\\ =& \frac{1}{|V|}\sum_{w\in W}\left(q^{-n/2}\sum_{u\in V}f(u)\psi(\langle u,w\rangle)\right)\overline{\left(q^{-n/2}\sum_{v\in V}g(v)\psi(\langle v,w\rangle)\right)}\\ =& \frac{1}{|V|^2}\sum_{u,v\in V}f(u)\overline{g(v)}\sum_{w\in W}\psi(\langle u-v,w\rangle)\\ =& \frac{1}{|V|}\sum_{u\in V}f(u)\overline{g(u)}.\end{aligned}$$

Assuming that U is a H-set, V and W are H-representations and $\langle -,-\rangle$ is H-invariant, one has $\widehat{hf}=h\hat{f}$ for each $h\in H$.

The Fourier transform is *transitive* in the following sense. Assume that there are spaces V_1, V_2, W_1, W_2 equipped with nondegenerate pairings $\langle -,-\rangle_1 : V_1\times W_1\to\mathbb{C}$, and $\langle -,-\rangle_2 : V_2\times W_2\to\mathbb{C}$. We set $V=V_1\oplus V_2$ and $W=W_1\oplus W_2$ and define $\langle -,-\rangle=\langle -,-\rangle_1+\langle -,-\rangle_2$. Then the Fourier transform $F(U\oplus V)\to F(U\oplus W)$ is equal to the composition of the Fourier transforms $F(U\oplus V_1\oplus V_2)\to F(U\oplus V_1\oplus W_2)$ and $F(U\oplus V_1\oplus W_2)\to F(U\oplus W_1\oplus W_2)$. We will call this *transitivity*.

Finally for a map $f: U'\to U$ between finite sets the following diagram is commutative

$$\begin{array}{ccc} F(U'\times V) & \xrightarrow{\widehat{(-)}} & F(U'\times W)\\ {\scriptstyle (f,\mathrm{Id})^*}\downarrow & & \downarrow{\scriptstyle (f,\mathrm{Id})^*}\\ F(U\times V) & \xrightarrow{\widehat{(-)}} & F(U\times W)\end{array} \tag{5.5}$$

It is straightforward to calculate that both compositions yield

$$q^{-n/2}\sum_{v\in V}g(f(u),v)\psi(\langle v,w\rangle).$$

5.1.2 The Fourier Transform for Quiver Representations

For a fixed quiver Q and dimension vector α we put $F(Q,\alpha)=F(\mathrm{Rep}(Q,\alpha))$. We can consider the α-graded part of $H(Q)$ as a the fixed point set of $F(Q,\alpha)$ under the action of Gl_α. This can be done by identifying a representation with the characteristic function of its orbit:

$$H(Q)_\alpha = F(Q,\alpha)^{\mathrm{Gl}_\alpha}. \tag{5.6}$$

This formula suggests the possibility to put a product on the space $F(Q)=\bigoplus F(Q,\alpha)$ which restricts to the product (4.7) on $H(Q)$. It is well known how this should be done. For $f \in F(Q,\alpha)$ and $g \in F(Q,\beta)$ one defines the product fg as

$$(fg)(C) = \frac{q^{\langle \beta,\alpha\rangle/2}}{|\operatorname{Gl}_\alpha||\operatorname{Gl}_\beta|}\sum_{\mathcal{E}_C} f(A)g(B), \tag{5.7}$$

where the sum runs over all exact sequences $\mathcal{E}_C$

$$\mathcal{E}_\mathcal{C} : 0 \longrightarrow A \xrightarrow{\ \varphi\ } C \xrightarrow{\ \theta\ } B \longrightarrow 0.$$

This is an associative product without unit element.

The restriction to $H(Q)_\alpha = F(Q,\alpha)^{\operatorname{Gl}_\alpha}$ of the bilinear form $(-,-)$ defined by (5.1) becomes

$$([A],[B]) = \frac{\delta_{[A],[B]}}{|\operatorname{Aut}(A)|}. \tag{5.8}$$

By linear extension we get a bilinear form $(-,-)$ on $H(Q)$.

Now take a subset $E \subset Q_1$ of edges of the quiver Q and consider the quiver Q' which is obtained from Q by inverting the edges in E. Then

$$\begin{aligned} R(Q,\alpha) &= U_\alpha \times V_\alpha \text{ and} \\ R(Q',\alpha) &= U_\alpha \times W_\alpha, \end{aligned}$$

where

$$\begin{aligned} U_\alpha &= \prod_{\varphi \notin E} \operatorname{Hom}(k^{\alpha_{t\varphi}}, k^{\alpha_{h\varphi}}), \\ V_\alpha &= \prod_{\varphi \in E} \operatorname{Hom}(k^{\alpha_{t\varphi}}, k^{\alpha_{h\varphi}}), \\ W_\alpha &= \prod_{\varphi \in E} \operatorname{Hom}(k^{\alpha_{h\varphi}}, k^{\alpha_{t\varphi}}). \end{aligned}$$

For $u \in U_\alpha$, $v \in V_\alpha$, $w \in W_\alpha$ we denote by $A_{u,v} \in \operatorname{Rep}(Q)$, $A_{u,w} \in \operatorname{Rep}(Q')$ the corresponding representations.

We need to define a pairing on $V_\alpha \times W_\alpha$. To that end we first fix a sign $\sigma_\varphi \in \{-1,+1\}$ for each arrow $\varphi \in Q_1$. Then we define the pairing

$$\langle (A_\varphi)_\varphi, (B_\varphi)_\varphi \rangle = \sum_{\varphi \in E} \sigma_\varphi \operatorname{Tr}(A_\varphi B_\varphi).$$

Given this pairing we may (and will) define a Fourier transform from $F(Q,\alpha)$ to $F(Q',\alpha)$ and extend it linearly to a Fourier transform from $F(Q)$ to $F(Q')$. Then we restrict it to a map from $H(Q)$ to $H(Q')$. We will denote this transform by $F_{Q,Q',\sigma}$. We want the following compatibility relation:

$$F_{Q,Q'',\sigma} = F_{Q',Q'',\varsigma} F_{Q,Q',\sigma},$$

where Q'' is yet another quiver obtained from Q by inverting a set of edges. To do this we must fix for each orientation a sign σ (here for Q) or ς (here for Q') according to the following law

$$\varsigma_\varphi = \begin{cases} -\sigma_\varphi & \text{if } \varphi \in E, \\ \sigma_\varphi & \text{otherwise.} \end{cases}$$

Once these signs are fixed we can drop them from the notation.

5.1.3 $H(Q)$ is Independent of the Orientation

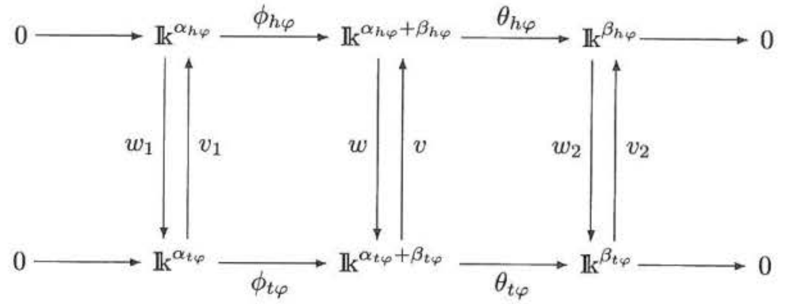
It was stated without proof by Lusztig [Lus90] that

Theorem 5.1 *The Fourier transform defines an algebra isomorphism between $H(Q)$ and $H(Q')$. In particular the Ringel-Hall algebra is independent of the orientation of Q.*

The remainder of this section will be devoted to the proof of this statement.

Given $f \in F(Q,\alpha)$ and $g \in F(Q,\beta)$ we have to show that $\widehat{fg} = \hat{f}\hat{g}$. By transitivity (see page 42) we only have to prove it for the case that there is a single arrow reversed: $E = \{\varphi\}$.

We will first rewrite $\widehat{fg}$, then rewrite $\hat{f}\hat{g}$, and then show that the differences can be eliminated. Before doing this, however, we picture the various maps, related to the arrow φ and its 2 associated vertices, that occur in the calculations.



- Given $u \in U_{\alpha+\beta}$ and $v \in V_{\alpha+\beta}$, we consider exact sequences $\mathcal{F}_{u,v}$ of the form

$$\mathcal{F}_{u,v} : 0 \longrightarrow A_{u_1,v_1} \xrightarrow{\ \phi\ } A_{u,v} \xrightarrow{\ \theta\ } A_{u_2,v_2} \longrightarrow 0, \tag{5.9}$$

  where $u_1 \in U_\alpha$, $u_2 \in U_\beta$, and so on. If we want to indicate that the sum also runs through all v's, we will use $\mathcal{G}_u$, rather than $\mathcal{F}_{u,v}$.

We get:

$$\widehat{fg}(u,w) = q^{(1/2)(-(\alpha_{h\varphi}+\beta_{h\varphi})(\alpha_{t\varphi}+\beta_{t\varphi}))} \sum_{v\in V_{\alpha+\beta}} (fg)(u,v)\psi(\langle v,w\rangle).$$

The product is given by:

$$(fg)(u,v) = \frac{q^{(1/2)(\langle\beta,\alpha\rangle_Q)}}{|\operatorname{Gl}_\alpha||\operatorname{Gl}_\beta|} \sum_{\mathcal{F}_{u,v}} f(u_1,v_1)g(u_2,v_2).$$

Combining this we find:

$$\widehat{fg}(u,w) = \frac{q^{(1/2)(\langle\beta,\alpha\rangle_Q-(\alpha_{h\varphi}+\beta_{h\varphi})(\alpha_{t\varphi}+\beta_{t\varphi}))}}{|\operatorname{Gl}_\alpha||\operatorname{Gl}_\beta|} \times \sum_{\mathcal{G}_u} f(u_1,v_1)g(u_2,v_2)\psi(\langle v,w\rangle). \quad (5.10)$$

- Next we rewrite $\hat{f}\hat{g}$.

  This time we consider the following exact sequences $\mathcal{H}_{u,w}$, for $u\in U_{\alpha+\beta}$ and $w\in W_{\alpha+\beta}$,

$$\mathcal{H}_{u,w}: 0 \longrightarrow A_{u_1,w_1} \xrightarrow{\varphi} A_{u,w} \xrightarrow{\theta} A_{u_2,w_2} \longrightarrow 0. \quad (5.11)$$

This enables us to write out the product in terms of $\hat{f}$ and $\hat{g}$ explicitly :

$$(\hat{f}\hat{g})(u,w) = \frac{q^{(1/2)\langle\beta,\alpha\rangle_{Q'}}}{|\operatorname{Gl}_\alpha||\operatorname{Gl}_\beta|} \sum_{\mathcal{H}_{u,w}} \hat{f}(u_1,w_1)\hat{g}(u_2,w_2).$$

Now we know that:

$$\hat{f}(u_1,w_1) = q^{-\alpha_{h\varphi}\alpha_{t\varphi}/2} \sum_{v_1\in V_1} f(u_1,v_1)\psi(\langle v_1,w_1\rangle) \quad \text{and}$$
$$\hat{g}(u_2,w_2) = q^{-\beta_{h\varphi}\beta_{t\varphi}/2} \sum_{v_2\in V_2} g(u_2,v_2)\psi(\langle v_2,w_2\rangle).$$

To use these two equations, we first note that for each of the $q^{\beta_{t\varphi}\alpha_{h\varphi}}$ maps v compatible with v_1 and v_2 in the diagram above we have $\langle v,w\rangle = \langle v_1,w_2\rangle + \langle v_2,w_2\rangle$. We can rewrite this as:

$$\langle v_1,w_1\rangle + \langle v_2,w_2\rangle = q^{-\beta_{t\varphi}\alpha_{h\varphi}} \sum_{v_1\to v\to v_2} \langle v,w\rangle$$

Putting everything together we get

$$(\hat{f}\hat{g})(u,w) = \frac{q^{(1/2)(\langle\beta,\alpha\rangle_{Q'}-(\alpha_{h\varphi}\alpha_{t\varphi}+\beta_{h\varphi}\beta_{t\varphi}+2\alpha_{h\varphi}\beta_{t\varphi}))}}{|\,\mathrm{Gl}_\alpha\,||\,\mathrm{Gl}_\beta\,|} \times \sum_{\substack{\mathcal{H}_{u,w} \\ v_1\to v\to v_2}} f(u_1,v_1)g(u_2,v_2)\psi(\langle v,w\rangle). \quad (5.12)$$

The expressions (5.10) and (5.12) are equal if the exponents of q are equal and if the summation happens to be over the same data, except for a part that adds up to zero. To prove the former it suffices to check that

$$\langle\beta,\alpha\rangle_Q + \beta_{t\varphi}\alpha_{h\varphi} = \langle\beta,\alpha\rangle_{Q'} + \beta_{h\varphi}\alpha_{t\varphi},$$

which is no problem. To prove the latter we first note that in both cases v has to be compatible with v_1 and v_2. The only remaining difference is that in (5.12) there are maps w_1 and w_2 involved, which are not needed to obtain (5.10). We have to look what happens for w whether or not there exists w_1 and w_2 such that the diagram is commutative.

Let us first look at w for which there exist w_1 and w_2 making the diagram commute. It is pretty straightforward that both summations are equal, in that case. And thus $\widehat{fg}(u,w) = (\hat{f}\hat{g})(u,w)$.

Finally we turn to the w for which there do not exist w_1 and w_2. Clearly the $\mathcal{H}_{u,w}$ is empty in that case, leaving $(\hat{f}\hat{g})(u,w) = 0$. Since $\sum_v \psi(\langle v,w\rangle) = 0$ as well (v compatible with v_1 and v_2), making $\widehat{fg}(u,w) = 0$.

5.2 Reflections from $U(Q)$ to $U(Q')$

Now we return to the reflections of the Weyl group. In the next section we will combine these together with the Fourier transform to generate automorphisms of $U(Q)$. In this section we fix a quiver Q with a source i, and the quiver Q' obtained by inverting the edges starting in i. Consider the reflections r_i defined in (1.3) and (1.5).

We will prove the following result in this section.

Theorem 5.2 *There is a unique algebra isomorphism $t_i : U(Q) \to U(Q')$, satisfying*

$$\begin{aligned} t_i\omega &= \omega t_i, & (5.13)\\ t_i[S_i] &= q^{-1/2}[S_i]^- K_{-i}, & (5.14)\\ t_i[A] &= [r_iA] \qquad \textit{if } A \textit{ is admissible, and} & (5.15)\\ t_iK_\alpha &= K_{r_i\alpha}. & (5.16)\end{aligned}$$

Recall that any representation A of Q is the direct sum of an admissible representation B and some copies of S_i: $A \cong S_i^a \oplus B$ (see chapter 1). We note that from (4.7) we can easily deduce

$$[S_i^{a+1}] = (1+q+...+q^a)^{-1} q^{-a/2} [S_i][S_i^a] \quad \text{and} \tag{5.17}$$

$$[A] = q^{\langle \overrightarrow{B}, i\rangle/2} [S_i^a][B]. \tag{5.18}$$

(5.13-5.16) yield the value of t_i on generators. Hence if t_i exists then it must be unique.

- We first prove that t_i, restricted to $H(Q) \to U(Q')$, is an algebra morphism.

  If t_i exists it has to be compatible with the identity (5.17). By induction we get

  $$t_i[S_i^a] = q^{-a^2/2}[S_i^a]^- K_{-ai}.$$

  On the other hand from (5.18) we obtain (for $A = B \oplus S_i^a$; B admissible)

  $$t_i[A] = q^{-\langle \overrightarrow{B}, i\rangle/2} t_i[S_i^a][r_i B].$$

  Together this defines t_i on the standard basis of $H(Q)$. The definition of t_i is obviously compatible with the relations (5.17) and (5.18). This means that we only have to check compatibility with the Hall product (4.7), in the case A is admissible and B is either of the form S_i^a or admissible.

  If A and B are both admissible and C is such that the following sequence is exact then C is admissible ($S_i \subset C$ is impossible!).

  $$0 \to A \longrightarrow C \longrightarrow B \to 0$$

  Since r_i defines a equivalence between $\mathrm{Adm}(Q)$ and $\mathrm{Adm}(Q')$, we trivially have $g_{AB}^C = g_{r_iAr_iB}^{r_iC}$, while $\langle \overrightarrow{B}, \overrightarrow{A}\rangle_Q = \langle \overrightarrow{r_iB}, \overrightarrow{r_iA}\rangle_{Q'}$ (see (1.6)).

  The only case that still needs to be checked is when A is admissible and $B = S_i$. The product is in this case

  $$\begin{aligned} [A][S_i] &= q^{1/2\langle i, \overrightarrow{A}\rangle}[S_i \oplus A] + q^{1/2\langle i, \overrightarrow{A}\rangle} \sum_{[A'] \in \mathrm{Adm}(Q)} g_{AS_i}^{A'}[A'] \quad (5.19) \\ &= q^{1/2\langle i, \overrightarrow{A}\rangle}[S_i][A] + q^{1/2\langle i, \overrightarrow{A}\rangle} \sum_{[A'] \in \mathrm{Adm}(Q)} g_{AS_i}^{A'}[A'], \quad (5.20) \end{aligned}$$

  where we have used $g_{AS_i}^{A \oplus S_i} = q^{-\langle \overrightarrow{A}, i\rangle}$.

  Now we have to check whether t_i is compatible with (5.20). This means that the following identity has to hold in $U(Q')$.

  $$\begin{aligned} &q^{-1/2}([r_iA][S_i]^- K_{-i} - q^{1/2\langle i, \overrightarrow{A}\rangle}[S_i]^- K_{-i}[r_iA]) \\ &\qquad = q^{1/2\langle i, \overrightarrow{A}\rangle_Q} \sum_{[A'] \in \mathrm{Adm}(Q)} g_{AS_i}^{A'}[r_iA'] \quad (5.21) \end{aligned}$$

Using

$$\begin{aligned} q^{1/2(i,\overrightarrow{A})}[S_i]^- K_{-i}[r_iA] &= q^{1/2((i,\overrightarrow{A})+(i,r_i\overrightarrow{A}))}[S_i]^-[r_iA]K_{-i} \\ &= [S_i]^-[r_iA]K_{-i} \end{aligned}$$

(the exponent vanishes since $(i, r_i\overrightarrow{A}) = (r_i i, \overrightarrow{A}) = -(i, \overrightarrow{A})$), we may rewrite (5.21) as:

$$[r_iA][S_i]^- - [S_i]^-[r_iA] = q^{1/2(\langle i,\overrightarrow{A}\rangle_Q+1)} \sum_{[A']\in \mathrm{Adm}(Q)} g^{A'}_{AS_i}[r_iA']K_i \tag{5.22}$$

We want to rewrite this relation entirely in terms of Q'. To this end we use $\langle i, \overrightarrow{A}\rangle_Q = \langle r_i i, r_i\overrightarrow{A}\rangle_{Q'} = -\langle i, \overrightarrow{r_iA}\rangle_{Q'}$. Invoking the formula (A.1), from the appendix, we have $g^{A'}_{AS_i} = \dfrac{|\operatorname{Aut}(A')|}{|\operatorname{Aut}(A)|} g^{r_iA}_{S_i r_i A'}$. Put $C = r_iA$ and $C' = r_iA'$.

We get

$$[C][S_i]^- - [S_i]^-[C] = q^{1/2(1-\langle i,C\rangle_{Q'})} \sum_{[C']\in \mathrm{Adm}(Q')} g^{C}_{S_iC'} \frac{|\operatorname{Aut}(C')|}{|\operatorname{Aut}(C)|}[C']K_i. \tag{5.23}$$

This identity should hold in $U(Q')$ for all admissible $C \in \operatorname{Rep}(Q')$.
By (4.18) we have

$$[C][S_i]^- - [S_i]^-[C] = \sum_{[M]} q^{(1/2)\langle\overrightarrow{M},i\rangle_{Q'}} \frac{|\operatorname{Aut}(M)|}{|\operatorname{Aut}(C)|} g^{C}_{S_iM} K_i[M]$$

We just have to note that $K_i[M] = q^{-(1/2)(\overrightarrow{M},i)}[M]K_i$ and, whenever $g^C_{S_iM}$ is not zero, $\langle i, \overrightarrow{M}\rangle_{Q'} = \langle i, \overrightarrow{C}\rangle_{Q'} - 1$ to see that this is (5.23).
Thus we have proven that $t_i : U(Q)_+ \to U(Q')$ is an algebra morphism.

- Now we extend t_i to a map $U(Q) \to U(Q')$.

  We first note that t_i acts as it should on K_α's and also the relation (4.13) is satisfied. Thus on $B(Q) = U(Q)_o \otimes U(Q)_+$ t_i is in fact algebra homomorphism. Since t_i is by definition compatible with ω, we also obtain that $t_i : U(Q)_o \otimes U(Q)_+ \to U(Q')$. We only need to check equation (4.15). By lemma 4.2 it suffices to check the equation only in generating cases. So A and B are either admissible or S_i. Leading to 4 cases (in fact 3 distinct cases, see below).

  1. In the case $A = S_i = B$, we may use equation (4.18), which reads

  $$[S_i]^-[S_i] - [S_i][S_i]^- = \frac{K_{-i} - K_i}{q-1}.$$

Applying t_i yields

$$q^{-1}([S_i]K_i[S_i]^- K_{-i} - [S_i]^- K_{-i}[S_i]K_i) = \frac{K_i - K_{-i}}{q-1}.$$

To check this last equation may be checked using the commutation rules.

2. The case where A is admissible and $B = S_i$ may also be proved using equation (4.18). This time we obtain (in $U(Q)$):

$$[A][S_i]^- - [S_i]^-[A] = -\sum_{[M]} q^{(1/2)\langle S_i, M\rangle_Q} \frac{|\operatorname{Aut}(M)|}{|\operatorname{Aut}(A)|} g^A_{MS_i}[M]K_{-i},$$

since $g^A_{S_iM} = 0$. After applying of t_i we obtain

$$q^{-1/2}([r_iA][S_i]K_i - [S_i]K_i[r_iA]) = -\sum_{[M]} q^{(1/2)\langle S_i, M\rangle_Q} \frac{|\operatorname{Aut}(M)|}{|\operatorname{Aut}(A)|} g^A_{MS_i}[r_iM]K_i \quad (5.24)$$

Next we multiply with $q^{1/2}K_{-i}$ and use

$$\langle S_i, M\rangle_Q = \langle i, \overrightarrow{A}\rangle_Q - 1 = -\langle i, \overrightarrow{r_iA}\rangle - 1.$$

We also use again equation (A.1) which implies $\frac{|\operatorname{Aut}(M)|}{|\operatorname{Aut}(A)|} g^A_{MS_i} = g^{r_iM}_{S_ir_iA}$.

The equation (5.24) is transformed into

$$[r_iA][S_i] - q^{-(1/2)(\overrightarrow{r_iA},i)}[S_i][r_iA] = -q^{(1/2)\langle i,\overrightarrow{r_iA}\rangle_{Q'}} \sum_{[M]} g^{r_iM}_{S_ir_iA}[r_iM].$$

Now multiplying with $-q^{(1/2)(D,S_i)}$ and putting $C = r_iA$ and $N = r_iM$, we finally get

$$[S_i][D] - q^{(1/2)(D,S_i)}[D][S_i] = q^{(1/2)\langle D,S_i\rangle} \sum_{[N]\in \operatorname{Adm}(Q')} g^N_{S_iD}[N],$$

which is nothing else then a Q'-analogue of the equation (5.19). This finishes the proof of (4.15).

3. This case may be reduced to the previous one by applying ω.

4. Finally we check the case that both A and B are admissible.

   We will first show that the lefthand side Λ is invariant under the t_i. So we have

$$\Lambda = \sum_{[M],[N]} g^{MN}_{AB} q^{-(1/2)\langle \overrightarrow{B}-\overrightarrow{N}, \overrightarrow{M}-\overrightarrow{N}\rangle}[M][N]^- K^{-1}_{\overrightarrow{B}-\overrightarrow{N}}.$$

We first rewrite this in generators. These are either admissible or equal to S_i. Since M is a subset of A it has to be again admissible. However N may contain copies of S_i. Set $N = N' \oplus S_i^n$ with N' admissible. We note in passing that if i is a sink (like in Q') the situation is exactly opposite. We know that $[N]^- = q^{-(1/2)\langle \overrightarrow{N'}, ni\rangle}[S_i^n]^-[N']^-$. Now we apply t_i to obtain

$$t_i\Lambda = \sum_{[M],[N'],n} g_{MN}^{AB} q^{(1/2)(\langle \overrightarrow{B}-\overrightarrow{N}, \overrightarrow{M}-\overrightarrow{N}\rangle - \langle \overrightarrow{N'}, ni\rangle - n^2)}$$

$$\cdot [r_iM][S_i^a]K_{ai}[r_iN']^- K_{r_i\overrightarrow{N}-r_i\overrightarrow{B}}. \quad (5.25)$$

We rewrite this formula using a number of equalities. First setting $U = r_iM \oplus S_i^a$ we obtain

$$[r_iM][S_i^a] = q^{-(1/2)\langle S_i^a, r_iM\rangle_{Q'}}[r_iM \oplus S_i^a] = q^{(1/2)\langle ai, \overrightarrow{M}\rangle}[U].$$

Commuting K_{ai} with $[r_iN']^-$ gives

$$\begin{aligned} K_{ai}[r_iN']^- K_{r_i\overrightarrow{N'}-r_i\overrightarrow{B}} &= q^{(1/2)\langle ai, \overrightarrow{r_iN'}\rangle_{Q'}}[r_iN']^- K_{ai}K_{r_i\overrightarrow{N}-r_i\overrightarrow{B}} \\ &= q^{-(1/2)\langle ai, \overrightarrow{N'}\rangle_{Q'}}[r_iN']^- K_{r_i\overrightarrow{N'}-r_i\overrightarrow{B}}. \end{aligned}$$

By lemma (A.2) we get

$$g_{AB}^{MN} = q^{\langle ni, \overrightarrow{M}\rangle + \langle \overrightarrow{N'}, ni\rangle} g_{r_iAr_iB}^{Ur_iN'}.$$

This turns expression (5.25) into the expression

$$t_i\Lambda = \sum_{[U],[N']} g_{r_iAr_iB}^{Ur_iN'} q^{\varpi/2}[U][r_iN']K_{\overrightarrow{r_iN'}-\overrightarrow{r_iB}}, \quad (5.26)$$

where

$$\begin{aligned} \varpi = \langle \overrightarrow{B} - \overrightarrow{N}, \overrightarrow{M} - \overrightarrow{N}\rangle - \langle \overrightarrow{N'}, n.i\rangle - n^2 \\ - \langle ai, \overrightarrow{M}\rangle - (ai, \overrightarrow{N'}) + 2\langle ai, \overrightarrow{M}\rangle + 2\langle \overrightarrow{N'}, ai\rangle \\ = \langle \overrightarrow{r_iB} - \overrightarrow{r_iN'}, \overrightarrow{U} - \overrightarrow{r_iN'}\rangle_{Q'} \quad (5.27) \end{aligned}$$

Putting $V = r_iN'$ yields the formula

$$t_i\Lambda = \sum_{[U],[V]} g_{r_iAr_iB}^{UV} q^{(1/2)\langle \overrightarrow{r_iB}-\overrightarrow{V}, \overrightarrow{U}-\overrightarrow{V}\rangle_{Q'}}[U][V]^- K^{-1}_{\overrightarrow{r_iB}-\overrightarrow{V}}.$$

A similar calculation for the righthand side yields

$$\sum_{[U],[V]} g_{r_iBr_iA}^{VU} q^{(1/2)\langle \overrightarrow{r_iA}-\overrightarrow{U}, \overrightarrow{V}-\overrightarrow{U}\rangle_{Q'}}[V]^-[U]K_{\overrightarrow{r_iA}-\overrightarrow{U}}.$$

Clearly these are the lefthand and righthand side of the equation (4.15) applied to $U(Q')$. This finishes the proof that t_i exists and is well defined.

- Finally we show that t_i is invertible by constructing an explicit inverse. Indeed:

$$\begin{aligned} t_i'\omega &= \omega t_i', && (5.28)\\ t_i'[S_i] &= q^{1/2}[S_i]^- K_i, && (5.29)\\ t_i'[A] &= [r_i A] \qquad \text{if } A \text{ is admissible}, && (5.30)\\ t_i' K_\alpha &= K_{r_i\alpha} && (5.31)\end{aligned}$$

defines an inverse of t_i. The proof that t_i' is an algebra morphism is almost identical to the proof for t_i, except that i is a sink in Q', and thus care has to be taken.

To prove that $t_i't_i = \mathrm{Id} = t_it_i'$ the only difficulty rises to check it on $[S_i]$, where

$$\begin{aligned} t_i't_i[S_i] &= t_i'(q^{-1/2}[S_i]^- K_{-i}) = q^{-1/2}t_i'(\omega[S_i]K_{-i})\\ &= q^{-1/2}\omega(q^{1/2}[S_i]^- K_i)K_i = q^{-1/2}q^{1/2}[S_i]K_{-i}K_i = [S_i]\\ &= q^{1/2}q^{-1/2}[S_i]K_iK_{-i} = q^{1/2}\omega(q^{-1/2}[S_i]^- K_{-i})K_{-i}\\ &= q^{1/2}t_i(\omega[S_i])K_{-i} = t_i(q^{1/2}[S_i]^- K_i) = t_it_i'[S_i].\end{aligned}$$

This finishes the proof t_i defines an isomorphism between $U(Q)$ and $U(Q')$.

5.3 An Automorphism of $U(Q)$

5.3.1 Definition

It is time to put both isomorphisms together. First we note that the Fourier transform may be defined on the whole of $U(Q)$ by

$$f(K_\alpha) = K_\alpha, \tag{5.32}$$

$$f([A]) = F_{Q,Q'}([A]), \tag{5.33}$$

$$f([A]^-) = (\overline{F}_{Q,Q'}([A]))^-, \tag{5.34}$$

where $\overline{F}_{Q,Q'}$ is defined as $F_{Q,Q'}$ using the character $\overline{\psi}$ instead of ψ. We will still write $F_{Q,Q'}$ for the extension of $F_{Q,Q'}$ to $U(Q)$. That this extension is well defined, follows from the functoriality of the Drinfeld double construction, the compatibility of the bilinear forms (cfr. (5.8)) and the properties (5.2)-(5.4).

Starting from a vertex i of a quiver Q we change the orientation of certain arrows to obtain a quiver Q^i in which i is a source (we are allowed to change the orientation of arrows not attached to i). Define:

$$\hat{t}_i = F_{Q_i,Q}t_iF_{Q,Q^i} : U(Q) \to U(Q). \tag{5.35}$$

The first question is of course whether this definition depends on the choice of Q^i. Let O^i another choice for Q^i and let $O_i = r_i O^i$. We should have

$$F_{O_i,Q} t_i F_{Q,O^i} = F_{Q_i,Q} t_i F_{Q,Q^i}.$$

Multiplying on the left with F_{Q,O_i} and on the right with $F_{Q^i,Q}$, this is equivalent to

$$t_i F_{Q^i,O^i} = F_{Q_i,O_i} t_i.$$

This follows from the commutativity of the diagram below (for Q^i and O^i quivers containing i as a source and quivers Q_i and O_i obtained from Q^i and O^i by inverting the edges starting in i).

$$\begin{array}{ccc} U(Q^i) & \xrightarrow{t_i} & U(Q_i) \\ {\scriptstyle F_{Q^i,O^i}}\downarrow & & \downarrow{\scriptstyle F_{Q_i,O_i}} \\ U(O^i) & \xrightarrow[t_i]{} & U(O_i) \end{array} \tag{5.36}$$

We only have to check this on $U(Q)_+$ since the commutativity on $U(Q)_o$ is clear and the commutativity of $U(Q)_-$ follows by applying ω and conjugating the character ψ. It is enough to show commutativity on admissible representations and on $[S_i]$. The latter is clear from:

$$\begin{aligned} t_i F_{Q^i,O^i}[S_i] &= t_i[S_i] \\ &= q^{-1/2}[S_i]^- K_{-1} \\ &= F_{Q_i,O_i}(q^{-1/2}[S_i]^- K_{-1}) \\ &= F_{Q_i,O_i} t_i [S_i], \end{aligned}$$

while the former easily follows from (5.5), considering the fact that the respective sets of edges involved in both operations are disjoint.

5.3.2 A Special Case

To shorten the statement of the following theorem we introduce the quantum factorial defined by

$$[n]_v^! = \frac{(v^n - v^{-n})(v^{n-1} - v^{1-n}) \cdots (v - v^{-1})}{(v - v^{-1})^n}. \tag{5.37}$$

We also define $a_v^{(n)} = a^n/[n]_v^!$. Note the $[n]_v^! = [n]_{v^{-1}}^!$.

In the theory of quantum groups it is customary to define v-binomial coefficients differently from combinatorics [Mac95]. In order to avoid confusion we use a different notation.

$$\left\{ \begin{matrix} n \\ t \end{matrix} \right\} = \frac{[n]_v^!}{[t]_v^! [n-t]_v^!}$$

These are Laurent polynomials in v with integer coefficients. Given a Laurent polynomial $\pi(v)$ and a rational number d we use the notation π_d for the Laurent polynomial $\pi(v^d)$.

Theorem 5.3 *For a quiver Q, a vertex i and a representation A of Q, such that $A_i = 0$ we have*

$$\hat{t}_i[A] = \sum_{s+t=-(\overrightarrow{A},i)} (-1)^s q^{s/2} [S_i]^{(s)}_{q^{1/2}} [A] [S_i]^{(t)}_{q^{1/2}}.$$

We may assume, without loss of generality, that i is a source in Q, and that Q' is the quiver obtained by inverting all the arrows starting in i. We put $B = r_i A$. Note that we may also consider A as a representation of Q', since for all non zero maps A_φ, the arrow φ is left unchanged. A similar argument yields $F_{Q',Q}[A] = [A]$. Let C be a quotient of B, and put $b = \dim B_i = -(\overrightarrow{A}, i)$.

For an element u of $U(Q')$, with $|u| = \alpha$ we set

$$\rho_i(u) = [S_i]u - q^{(\alpha,i)/2} u[S_i].$$

For $t \in \{0, \ldots, n\}$ let P_d be the set of Q'-representations D with $\dim D_i = d$ for which $B \twoheadrightarrow D \twoheadrightarrow C$, up to isomorphisms leaving B and C fixed.

We get as in (5.19)

$$\rho_i[C] = q^{(1/2)\langle \overrightarrow{C}, i\rangle} \sum_{[C']\,\mathrm{Adm}} g^{C'}_{S_i C}[C'],$$

and considering (A.1) we have

$$g^{C'}_{S_i C} = \frac{|\operatorname{Aut}(C')|}{|\operatorname{Aut}(C)|} g^{r_i C}_{r_i C' S_i}.$$

This gives us

$$\begin{aligned} \rho_i[C] &= q^{(1/2)\langle \overrightarrow{C}, i\rangle} \sum_{[C'']} \frac{|\operatorname{Aut}(C'')|}{|\operatorname{Aut}(C)|} g^{r_i C}_{C'' S_i}[r_i C''] \\ &= q^{(1/2)\langle \overrightarrow{C}, i\rangle} \sum_{r_i C/C'' \cong S_i} \frac{|\operatorname{Aut}(C'')|}{|\operatorname{Aut}(C)|}[r_i C''] \\ &= q^{(1/2)\langle \overrightarrow{C}, i\rangle} \sum_{C' \in P_{c+1}} \frac{|\operatorname{Aut}(C')|}{|\operatorname{Aut}(C)|}[C'], \end{aligned}$$

where $c = \dim C_i$.

This formula becomes nice if we introduce $\widetilde{[C]} = |\operatorname{Aut}(C)|[C]$:

$$\rho_i \widetilde{[C]} = q^{(1/2)\langle \overrightarrow{C}, i\rangle} \sum_{C' \in P_{c+1}} \widetilde{[C']}.$$

Now we denote by f_b the number of complete flags in a b-dimensional space over $\Bbbk$, which is the number of ways to go from A to B (stepwise), as Q'-representations. This leads to

$$\rho_i^b \widetilde{[A]} = q^{-(1/2)(1+\cdots+b)} f_b \widetilde{[B]}.$$

But

$$f_b = \frac{(q^b-1)\cdots(q-1)}{(q-1)^b} = q^{b(b-1)/4}[b]^!_{q^{1/2}}.$$

We obtain, (using $|\operatorname{Aut}(A)| = |\operatorname{Aut}(B)|$),

$$\rho_i^b[A] = q^{-b(b+1)/4}q^{b(b-1)/4}[b]^!_{q^{1/2}}[B] = q^{-b/2}[b]^!_{q^{1/2}}[B].$$

On $U(Q)$ we define the ρ_i in the same way. We clearly get

$$F_{Q',Q}\circ\rho_i = \rho_i \circ F_{Q',Q}.$$

From $\rho_i[A] = [S_i][A] - q^{(\overrightarrow{A},i)/2}[A][S_i]$ we claim that

$$\rho_i^b[A] = \sum_{s=0}^{b} \nu_s [S_i]^s[A][S_i]^{b-s}, \tag{5.38}$$

where

$$\nu_s = (-1)^s q^{(s-b)/2}\frac{[b]^!_{q^{1/2}}}{[s]^!_{q^{1/2}}[b-s]^!_{q^{1/2}}}.$$

This will be proved below.

This formula allows us to calculate $\hat{t}_i([A])$:

$$\begin{aligned}
\hat{t}_i([A]) &= F_{Q',Q}t_i[A]\\
&= F_{Q',Q}[B]\\
&= F_{Q',Q}\frac{q^{b/2}}{[b]^!_{q^{1/2}}}\rho_i^b[A]\\
&= \frac{q^{b/2}}{[b]^!_{q^{1/2}}}\rho_i^b F_{Q',Q}[A]\\
&= \frac{q^{b/2}}{[b]^!_{q^{1/2}}}\rho_i^b[A]\\
&= \sum_{s+t=-(\overrightarrow{A},i)} (-1)^s q^{s/2}[S_i]^{(s)}_{q^{1/2}}[A][S_i]^{(t)}_{q^{1/2}}
\end{aligned}$$

It remains to prove the claim (5.38).

We define the operators $Lx = [S_i]x$ and $Rx = q^{(|x|,i)/2}x[S_i]$. Clearly we have $\rho_i = L - R$ and $RL = qLR$. So to calculate ρ_i^b we can use the skew binomium formula (B.1). We recall that

$$\begin{bmatrix} b \\ a \end{bmatrix} = \frac{\prod_{k=1}^{b}(t^k-1)}{\prod_{k=1}^{a}(t^k-1)\prod_{k=1}^{b-a}(t^k-1)}.$$

Here $t = q$. Thus

$$\begin{bmatrix} b \\ a \end{bmatrix} = q^{(1/2)(a(b-a))}\frac{[b]^!_{q^{1/2}}}{[a]^!_{q^{1/2}}[b-a]^!_{q^{1/2}}}$$

$$
\begin{aligned}
\rho_i^b[A] &= (L-R)^b[A] \\
&= \sum_{j=0}^{b}(-1)^j \begin{bmatrix} b \\ b-j \end{bmatrix} L^{b-j}R^j[A] \\
&= \sum_{j=0}^{b}(-1)^j \begin{bmatrix} b \\ j \end{bmatrix} L^{b-j}[A][S_i]^j q^{\sum_{k=0}^{j-1}(\overrightarrow{A}+ki,i)/2} \\
&= \sum_{j=0}^{b}(-1)^j \begin{bmatrix} b \\ j \end{bmatrix} [S_i]^{b-j}[A][S_i]^j q^{-bj/2+j(j-1)/2} \\
&= \sum_{j=0}^{b}(-1)^j [S_i]^{b-j}[A][S_i]^j q^{j/2} \frac{[b]^!_{q^{1/2}}}{[j]^!_{q^{1/2}}[b-j]^!_{q^{1/2}}}
\end{aligned}
$$

This finishes the proof.

5.3.3 Comparison with Lusztig's Automorphisms

Q is a again a quiver without loops. The bilinear form $(-,-)$ defines a simply laced Cartan datum with simple roots i. We may [Jan87], associate a quantum enveloping algebra $U_v(\mathfrak{g})$ to this datum. $U_v(\mathfrak{g})$ is the $\mathbb{C}$-algebra with generators E_i, F_i (for $i \in Q_0$) and L_α (for $\alpha \in \mathbb{Z}^{Q_0}$) and with relations:

$$L_o = 1 \tag{5.39}$$

$$L_\alpha L_\beta = L_{\alpha+\beta}, \tag{5.40}$$

$$L_\alpha E_i = v^{(\alpha,i)} E_i L_\alpha, \tag{5.41}$$

$$L_\alpha F_i = v^{-(\alpha,i)} F_i L_\alpha, \tag{5.42}$$

$$E_i F_j - F_j E_i = \delta_{ij} \frac{L_i - L_{-i}}{v - v^{-1}}, \tag{5.43}$$

$$(\text{for } i \neq j) \qquad \sum_{\substack{s+t \\ =1-(i,j)}} (-1)^s (E_i)_v^{(s)} E_j (E_i)_v^{(t)} = 0 \tag{5.44}$$

$$(\text{for } i \neq j) \qquad \sum_{\substack{s+t \\ =1-(i,j)}} (-1)^s (F_i)_v^{(s)} F_j (F_i)_v^{(t)} = 0 \tag{5.45}$$

It is well known that $U_v(\mathfrak{g})$ is a Hopf algebra ([Jan87]).

Let $u(Q)$ be the subalgebra of $U(Q)$ generated by $[S_i]$, $[S_i]^-$ and K_α. We may invoke Theorem 5.3, from the previous section, to obtain the following list

$(i \neq j)$.

$$\begin{aligned}
\hat{t}_i([S_i]) &= q^{-1/2}[S_i]^- K_{-i}, \\
\hat{t}_i([S_i]^-) &= q^{-1/2}[S_i]K_i, \\
\hat{t}_i([S_j]) &= \sum_{s+t=n} (-1)^s q^{s/2} [S_i]^{(s)}_{q^{1/2}} [S_j] [S_i]^{(t)}_{q^{1/2}} \\
\hat{t}_i([S_j]^-) &= \sum_{s+t=n} (-1)^s q^{s/2} ([S_i]^-)^{(s)}_{q^{1/2}} [S_j]^- ([S_i]^-)^{(t)}_{q^{1/2}} \\
\hat{t}_i(K_\alpha) &= K_{r_i \alpha}
\end{aligned}$$

It follows from these formulas that $u(Q)$ is preserved under $\hat{t}_i$. Xiao ([Xia97]) constructed a Hopf algebra isomorphism θ between $U_{q^{-1/2}}(\mathfrak{g})$ and $u(Q)$ sending E_i to $-q^{1/2}[S_i]^-$, F_i to $[S_i]$ and L_i to K_{-i}.

Lusztig ([Lus93]) defined the following automorphism $T''_{i,-1}$ of $U_v(g)$ for $i \in Q_0$.

$$\begin{aligned}
T''_{i,-1}(E_i) &= -F_i L_{-i}, \\
T''_{i,-1}(F_i) &= -L_i E_i, \\
T''_{i,-1}(E_j) &= \sum_{s+t=(i,j)} (-1)^s v^s (E_i)^{(t)}_v E_j (E_i)^{(s)}_v \qquad j \neq i \\
T''_{i,-1}(F_j) &= \sum_{s+t=(i,j)} (-1)^s v^{-s} (F_i)^{(s)}_v F_j (F_i)^{(t)}_v \qquad j \neq i \\
T''_{i,-1}(L_\alpha) &= K_{r_i \alpha}
\end{aligned}$$

A simple verification yields

Theorem 5.4 $\theta T''_{i,-1} = \hat{t}_i \theta$.

Indeed we have for E_i:

$$\begin{aligned}
\theta T''_{i,-1}(E_i) &= \theta(-F_i L_{-i}) = -[S_i]K_i \qquad \text{and} \\
\hat{t}_i \theta(E_i) &= \hat{t}_i(-q^{1/2}[S_i]^-) = -q^{1/2} q^{-1/2} [S_i] K_i
\end{aligned}$$

and for F_i

$$\begin{aligned}
\theta T''_{i,-1}(F_i) &= \theta(-L_i E_i) = +K_{-1} q^{1/2} [S_i]^- \qquad \text{and} \\
\hat{t}_i \theta(F_i) &= \hat{t}_i([S_i]) = q^{-1/2}[S_i]^- K_{-i} = q^{-1/2+(i,-i)/2} K_{-1} [S_i]^-.
\end{aligned}$$

If $j \neq i$ we have for E_j, using $v = q^{-1/2}$:

$$\begin{aligned}
\theta T''_{i,-1}(E_j) &= \theta\left(\sum_{s+t=(i,j)} (-1)^s v^s (E_i)_v^{(t)} E_j (E_i)_v^{(s)}\right) \\
&= \sum (-1)^s v^s (-v^{-(s+t+1)})([S_i]^-)_v^{(t)} [S_j]^- ([S_i]^-)_v^{(s)} \\
&= (-1)v \sum (-1)^t v^{-t} ([S_i]^-)_v^{(t)} [S_j]^- ([S_i]^-)_v^{(s)} \\
&= -q^{1/2} \hat{t}_i [S_j]^- \\
&= \hat{t}_i \theta E_j,
\end{aligned}$$

while for F_j

$$\begin{aligned}
\theta T''_{i,-1}(E_j) &= \theta\left(\sum_{s+t=(i,j)} (-1)^s v^{-s} (F_i)_v^{(s)} F_j (F_i)_v^{(t)}\right) \\
&= \sum (-1)^s v^{-s} ([S_i])_v^{(s)} [S_j] ([S_i])_v^{(t)} \\
&= \hat{t}_i [S_j] \\
&= \hat{t}_i \theta F_j.
\end{aligned}$$

Finally we have the trivial

$$\theta T''_{i,-1} L_\alpha = \theta L_{r_i \alpha} = K_{-r_i \alpha} = \hat{t}_i K_{-\alpha} = \hat{t}_i \theta L_\alpha$$

This result gives a new proof that the automorphisms $T''_{i,-1}$ are indeed automorphisms. It also gives a global definition for them, which was not the case in Lusztig's work.

Remark : Based on our work in chapter 6 Deng and Xiao have given an alternative definition of an isomorphism between $H(Q)$ and $H(r_iQ)$ [DX01a] It is very likely that this isomorphism coincide with the one given by the Fourier transform. In [DX01c] it is shown that the resulting automorphisms of $U(Q)$ satisfy the braid relations, thereby generalizing an important result by Lusztig for quantum enveloping algebras.

Chapter 6

The Quantized Generalized Enveloping Algebra of a Kac-Moody Lie Algebra

In this chapter we introduce the quantized generalized enveloping algebra of a Kac-Moody Lie algebra (QGEA). The aim is to show that the Ringel-Hall algebra can be considered as the positive part of a QGEA (Theorem 6.5), which gives another translation of Kac's constant term conjecture.

In the case of generic parameter the QGEA were introduced by S.J. Kang [Kan95]. Unfortunately in the case of the Hall algebra the parameter is not generic, and since Kang makes essential use of specialization arguments we cannot use his results directly. Luckily it turns out that as long as the defining parameter is not a root of unity one can simply generalize the arguments by Borcherds [Bor88] and Kac [Kac90] to the quantum case. Additionally the proof of Theorem 6.5 uses the positive definiteness of the Green bilinear form on the Hall algebra (see Proposition 6.3). This argument seems to be new.

6.1 Initial Data

First we fix an algebraically closed ground field $\mathbb{k}$. We will need to take rational powers of elements in $\mathbb{k}$. To be able to do this we define a inverse system $(T_a)_{a>0}$, where all $T_a = \mathbb{k}^*$ and the transition maps are $T_{ba} \longrightarrow T_a : x \mapsto x^b$. Fix an element $v \in \mathbb{k}^*$ and let $\mathbf{v} = (v_a)_a$ be a fixed element of the inverse limit such that $v_1 = v$. If $r = c/d \in \mathbb{Q}$ then we define $v^r = v_d^c$. Clearly this is independent of the choice of c and d. We assume in addition that v is not a root of unity, such that $v^r = v^{r'}$ implies $r = r'$.

The initial data are a vector space Y over $\mathbb{k}$, a countable linear independent subset I of Y and a bilinear form. The role of Y will be rather marginal. I

is not necessarily a basis for Y, so Y may be enlarged if needed. Y is equipped with a $\mathbb{Q}$-valued Kac-Moody bilinear form $(-,-)$ [Bor88]. This means

- $(-,-)$ is symmetric,
- for $i \neq j \in I$ we have $(i,j) \leq 0$ and
- if $(i,i) > 0$ then $\frac{2(i,j)}{(i,i)} \in \mathbb{Z}$.

The elements of I will be called *simple roots*. We make a distinction between *real (simple) roots* i for which $(i,i) > 0$, and the *imaginary (simple) roots* j for which $(j,j) \leq 0$. We write $I = I^{re} \cup I^{im}$ for the corresponding decomposition of I.

6.2 Three Algebra Structures

Now we are ready to construct some Hopf algebras associated to this data. These Hopf algebras are generalizations of the quantum enveloping algebras introduced in section 5.3.3.

6.2.1 The Hopf Algebra $\tilde{U}$

First we define the algebra $\tilde{U}$ generated by symbols E_i, F_i, K_i, K_i^{-1} for each $i \in I$ and satisfying

$$\begin{aligned} K_iK_i^{-1} &= 1 = K_i^{-1}K_i & (6.1)\\ K_iK_j &= K_jK_i & (6.2)\\ E_iF_j - F_jE_i &= (K_i - K_i^{-1})\delta_{ij} & (6.3)\\ K_iE_j &= v^{(i,j)}E_jK_i & (6.4)\\ K_iF_j &= v^{-(i,j)}F_jK_i & (6.5) \end{aligned}$$

Note that the K_i's commute. Thus it is safe to define for any $\alpha \in \mathbb{Z}^I$ the symbol $K_\alpha = \prod_i K_i^{\alpha_i}$. In particular $K_i^{-1} = K_{-i}$. We put a Y-grading on $\tilde{U}$ by $\deg K_\alpha = 0$, $\deg E_i = -\deg F_i = i$. As before we write $|x|$ for the degree of a homogeneous element x. From the relations it is clear that any word in the symbols E_i, F_i, K_i, K_i^{-1} can be rewritten as the sum of v-weighted words of the form **ekf** where **e** is a word in the symbols E_i, $\mathbf{k} = K_\alpha$ for some α and **f** a word in the F_i. It is classical that this is an isomorphism (cfr. Kac's argument for Lie algebras). In this way there is a triangular decomposition $\tilde{U} = \tilde{U}_+ \otimes \tilde{U}_o \otimes \tilde{U}_-$, where $\tilde{U}_+$ is generated by the E_i's, $\tilde{U}_o$ generated by the K_α's and $\tilde{U}_-$ is generated by the F_i's.

We may put a Hopf Algebra structure on this algebra by defining the co-unit as $\varepsilon(E_i) = \varepsilon(F_i) = 0$, $\varepsilon(K_i) = 1$, the comultiplication as

$$\begin{aligned}
\Delta(K_i) &= K_i \otimes K_i, \\
\Delta(E_i) &= E_i \otimes K_i + 1 \otimes E_i, \\
\Delta(F_i) &= K_i^{-1} \otimes F_i + F_i \otimes 1
\end{aligned}$$

and, finally, the antipode as

$$\begin{aligned}
S(K_i) &= K_i^{-1}, \\
S(E_i) &= -E_i K_i^{-1}, \\
S(F_i) &= -K_i F_i.
\end{aligned}$$

There is a unique automorphism ω of $\tilde{U}$ satisfying $\omega(E_i) = F_i$, $\omega(F_i) = E_i$, $\omega(K_i) = K_i^{-1}$. It is clear that ω is an anti-automorphism on the level of coalgebras.

6.2.2 Serre Relations and The Hopf Algebra U

For a real simple root i put

$$\begin{aligned}
a_{ij} &= -2\frac{(i,j)}{(i,i)} \quad \text{and} \\
d_i &= \frac{(i,i)}{2}.
\end{aligned}$$

We define the *Quantized Generalized Enveloping Algebra of the Kac-Moody Lie Algebra* U (or shorter the QGEA) to be the quotient of $\tilde{U}$ by the quantum Serre relations.

$$\sum_{p=0}^{a_{ij}+1} (-1)^p \begin{Bmatrix} a_{ij}+1 \\ p \end{Bmatrix}_{d_i} E_i^p E_j E_i^{a_{ij}+1-p} = 0 \qquad \text{if } i \text{ is real} \tag{6.6}$$

$$\sum_{p=0}^{a_{ij}+1} (-1)^p \begin{Bmatrix} a_{ij}+1 \\ p \end{Bmatrix}_{d_i} F_i^p F_j F_i^{a_{ij}+1-p} = 0 \qquad \text{if } i \text{ is real} \tag{6.7}$$

$$E_i E_j - E_j E_i = 0 \qquad \text{if } (i,j) = 0 \tag{6.8}$$

$$F_i F_j - F_j F_i = 0 \qquad \text{if } (i,j) = 0 \tag{6.9}$$

We denote the images of E_i, F_i and K_i in U by the same symbols. The triangular decomposition of $\tilde{U}$ induces a triangular decomposition $U = U_+ \otimes U_o \otimes U_-$ (this may be proved as in [Lus93, Cor. 3.2.5]), where $U_o = \tilde{U}_o$, U_+ is obtained by dividing $\tilde{U}_+$ by the relations (6.6) and (6.8), and U_- is defined by dividing $\tilde{U}_-$ by the relations (6.7) and (6.9).

6.2.3 The Drinfeld Double and The Algebra $\overline{U}$

Put $\tilde{U}_{\geq o} = \tilde{U}_+ \otimes \tilde{U}_o$ and $\tilde{U}_{\leq o} = \tilde{U}_o \otimes \tilde{U}_-$. We define a symmetric pairing $[-,-]$ on $\tilde{U}_{\geq o}$ as follows: on the generators we have $[K_\alpha, K_\beta] = v^{-(\alpha,\beta)}$, $[E_i, E_j] = \delta_{ij}$ and $[K_\alpha, E_i] = [E_i, K_\alpha] = 0$. The form given by linear extension of this is a skew Hopf pairing.

We have to check the conditions (4.1) on the generators. Indeed clearly $[K_i, 1] = v^0 = 1 = \varepsilon(K_i)$ and $[E_i, 1] = 0 = \varepsilon(E_i) = [1, E_i]$.

Furthermore we have to check $[a, bb'] = [\Delta(a), b \otimes b']$. We have, for instance:

$$[K_\alpha, K_\beta K_\gamma] = v^{-(\alpha,\beta+\gamma)} = v^{-(\alpha,\beta)} v^{-(\alpha,\gamma)} = [K_\alpha, K_\beta][K_\alpha, K_\gamma]$$

$$[\Delta(E_i), E_i \otimes 1] = [E_i, E_i][K_i, 1] + [1, E_i][E_i, 1] = [E_i, E_i]v^0 + 0 = [E_i, E_i]$$

From this we can also define a skew Hopf pairing between $\tilde{U}_{\geq o}$ and $\tilde{U}_{\leq o}$, using ω. We define $[a, b]_\omega = [a, \omega(b)]$. From the relations we get, for all $a \in \tilde{U}_{\geq o}$ and $b \in \tilde{U}_{\leq o}$,

$$\sum_{a,b} b_{(2)} a_{(2)} [a_{(1)}, b_{(1)}]_\omega = \sum_{a,b} a_{(1)} b_{(1)} [a_{(2)}, b_{(2)}]_\omega. \tag{6.10}$$

The Serre relations (6.6-6.9) are in the radical of these pairings (when they are defined). Indeed, for instance, let Λ denote the lefthand side of (6.6). Note that, for a homogeneous, $[\Lambda, a] = 0$ is trivial if $|\Lambda| \neq |a|$. Thus it suffices to check $[\Lambda, a] = 0$ for a of the form $E_i^k E_j E_i^{a_{ij}+1-k}$. This follows from a tedious calculation analogue to [Lus93, 1.4.3-6].

Therefore the pairings are as well defined on U. We denote the algebra with $\overline{U}$ the Hopf algebra obtained from $\tilde{U}$ (or U) by factoring out the left and right radical of $[-,-]_\omega$, restricted to $U_+ \times U_-$. We again have a triangular decomposition $\overline{U} = \overline{U}_+ \otimes \overline{U}_o \otimes \overline{U}_-$.

6.3 Borcherds Character and $\overline{U} = U$

6.3.1 Casimir and Verma

We fix homogeneous bases $(a_u)_u$, $(b_u)_u$ for $\bar{U}_+$, $\bar{U}_-$ which are dual for the form $[-,-]_\omega$ defined above. Put $\bar{u} = |b_u| = -|a_u|$. For $\alpha = \sum_i \alpha_i i \in \mathbb{N}I$ write $\operatorname{ht} \alpha = \sum \alpha_i$. Let M be a $\bar{U}$-module such that for all $m \in M$ we have that $(\bar{U}_-)_{-\alpha} m = 0$ for $\operatorname{ht} \alpha \gg 0$.

Then the action of the operator

$$\Omega = \sum_u S^{-1}(a_u) K_{\bar{u}} b_u K_{\bar{u}}$$

is well defined on M. Using (6.10), one verifies that it satisfies [Lus93]

$$\begin{aligned} \Omega E_i &= E_i \Omega K_i^2 \\ \Omega K_i &= K_i \Omega \\ \Omega F_i &= K_i^{-2} F_i \Omega. \end{aligned} \tag{6.11}$$

We denote by $\mathcal{O}$ the category of Y-graded $\bar{U}$-modules M satisfying

- K_i acts on M_α by multiplication with $v^{(\alpha,i)}$.
- All M_α are finite dimensional.
- The set of α such that $M_\alpha \neq 0$ is contained in the union of a finite number of sets of the form $\mu + \mathbb{N}I$.

One may use the operator Ω to construct a true Casimir operator on objects in $\mathcal{O}$. This is done as follows (cfr [Lus93]). Enlarging Y if necessary we may assume that there exists $\rho \in Y$ such that $(\rho, i) = \frac{1}{2}(i,i)$ for all $i \in I$. Let $M \in \mathcal{O}$ and let C be the operator which acts on M_α by $v^{-(\alpha,\alpha-2\rho)}\Omega$. Then C commutes with the U-action on M. This follows from the formulas (6.11).

Particular objects in $\mathcal{O}$ are the so-called (lowest weight) Verma modules defined by

$$M(\mu) = \bar{U}/(\sum_i \bar{U}F_i + \sum_i \bar{U}(K_i - v^{(\mu,i)})).$$

As usual we denote by $L(\mu)$ the unique simple quotient of $M(\mu)$ in the category $\mathcal{O}$. It is clear that C acts on $M(\mu)$ and $L(\mu)$ by multiplication with $v^{-(\mu,\mu-2\rho)}$.

6.3.2 The Character Ch

As before, if $i \in I^{re}$ we write $r_i(\chi) = \chi - \frac{2(\chi,i)}{(i,i)} i$. Clearly r_i is a reflection on Y. Again, we denote the group generated by the r_i by W. This is the Weyl group of the root datum $(Y, I, (-,-))$. We define the character $\epsilon : W \to \{-1,+1\}$ by $\epsilon(s_i) = -1$.

For $\alpha \in Y$ we introduce a symbol $e(\alpha)$, in such a way that $e(\alpha+\beta) = e(\alpha)e(\beta)$. We also put $w \cdot e(\alpha) = e(w\alpha)$ for $w \in W$. For a Y-graded vector space V we put

$$\mathrm{Ch}(V) = \sum_{\alpha \in Y} \dim V_\alpha e(\alpha).$$

We will now recall how the Weyl group acts on $\mathrm{Ch}(U_+)$.

For $i \in I^{re}$ let U_i be the subalgebra of U generated by E_i, F_i, K_i. Since $U_v(\mathfrak{sl}_2)$ is simple as a Hopf algebra it follows that $U_i \cong U_v(\mathfrak{sl}_2)$. As usual we can make $\tilde{U}$ and U into U_i-modules using a twisted adjoint action:

$$\begin{aligned}
\mathrm{Ad}^\circ(E_i)(a) &= E_i a - v^{(i,|a|)} a E_i \\
\mathrm{Ad}^\circ(F_i)(a) &= (F_i a - a F_i) K_i \\
\mathrm{Ad}^\circ(K_i)(a) &= v^{(i,|a|)} a.
\end{aligned}$$

For $i, j \in I$, $p \in \mathbb{N}$ put $\tilde{T}_{i,j,p} = \mathrm{Ad}^\circ(E_i)^p(E_j) \in \tilde{U}_+$ and denote by $T_{i,j,p}$ the corresponding images in U. Let $V_i, \tilde{V}_i$ be respectively the subalgebras of $U, \tilde{U}$ generated by all $T_{i,j,p}, \tilde{T}_{i,j,p}$, $j \neq i$.

The following properties are easily established.

1. V_i is $\mathrm{Ad}^\circ U_i$-stable. It is sufficient to check this on the $T_{i,j,p}$. The stability under $\mathrm{Ad}^\circ E_i$ and $\mathrm{Ad}^\circ K_i$ is obvious, while the stability under $\mathrm{Ad}^\circ F_i$ follows from the relations in U_i.

2. The action of U_i on V_i is locally finite. This follows from the fact that the Serre relation between E_i and E_j implies that $T_{i,j,a_{ij}+1} = 0$ and hence V_i is generated by the finite dimensional U_i-representations given by $\sum_p T_{i,j,p} \cdot k$.

3. Every element of U_+ can be written as a sum $\sum_n E_i^n b_n$ with $b_n \in V_i$.

Let us now temporarily define $\tilde{U}_+ * \tilde{U}_+$ as $\tilde{U}_+ \otimes \tilde{U}_+$ with the new multiplication

$$(a * b)(c * d) = v^{-(|b|,|c|)}(ac * bd)$$

(for homogeneous a, b, c, d).

It is easy to see that there is an algebra homomorphism $\delta : \tilde{U}_+ \to \tilde{U}_+ * \tilde{U}_+$ given by $\delta(E_i) = 1 \otimes E_i + E_i \otimes 1$. By induction on degree one shows

$$[a, bc] = \sum_i [a_i, b][a'_i, c] \tag{6.12}$$

if $\delta(a) = \sum_i a_i \otimes a'_i$. We now define linear maps $\delta_i : \tilde{U}_+ \to \tilde{U}_+$ by the properties

$$\delta_i(E_j) = \delta_{ij} \qquad \text{and}$$
$$\delta_i(bc) = \delta_i(b)c + v^{-(i,|b|)} b\delta_i(c)$$

(for homogeneous b). One shows by induction that $[E_i a, b] = [a, \delta_i b]$. Furthermore it is also easy to see that if $a \in \tilde{U}_+$ is such that $\delta_i(a) = 0$ then one also has $\delta_i(\mathrm{Ad}^\circ(E_i)(a)) = 0$. It follows that $\delta_i \mid \tilde{V}_i = 0$. Thus

$$[E_i^m a, E_i^n b] = [E_i^{m-1} a, \delta_i(E_i^n b)] = [E_i^{m-1} a, \delta_i(E_i^n) b] = \lambda [E_i^{m-1} a, E_i^{n-1} b]$$

where λ is some non-zero scalar (using the fact that v is not a root of unity). Now it follows that

4. If $a, b \in \tilde{V}_i$ then $[E_i^m a, E_i^n b] = 0$ unless $m = n$. In that case it is a non-zero multiple of $[a, b]$.

By taking images a similar statement is true for U_+.

Let $\bar{V}_i$ be the image of V_i in $\bar{U}_+$. The corresponding properties hold also for $\bar{V}_i$. Note that it follows from 4 that in $\bar{U}$ the decomposition given by 3 is unique, since $[-,-]$ is non-degenerate.

Let $i \in I^{re}$. Properties 3 and 4 imply that $\mathrm{Ch}(\bar{U}_+) = (\sum_n e(ni)) \, \mathrm{Ch}(\bar{V}_i)$. Hence we find

$$e(-\rho)\,\mathrm{Ch}(\bar{U}_+)^{-1} = (e(-\rho) - e(-\rho + i))\,\mathrm{Ch}(\bar{V}_i)^{-1}$$

Properties 1 and 2 imply that $r_i \,\mathrm{Ch}(\bar{V}_i) = \mathrm{Ch}(\bar{V}_i)$. Since also $r_i(e(-\rho) - e(-\rho + i)) = (e(-\rho + i) - e(-\rho))$, we deduce the following result.

Lemma 6.1 *$e(-\rho)\,\mathrm{Ch}(\bar{U}_+)^{-1}$ is W-skew invariant.*

Using this lemma one can now generalize the Borcherds character formula [Bor88]. Let P_- be the set of all $\mu \in Y$ such that

$$-2(\mu, i)/(i, i) \in \mathbb{N} \quad \text{if } i \in I^{re} \quad \text{and} \quad (\mu, i) \leq 0 \text{ for all } i \in I^{\text{im}}.$$

For $\mu \in P_-$ define $S_\mu = \sum \epsilon(s)e(s)$ where s runs over the sums of elements in I^{im} and $\epsilon(s)$ is $(-1)^n$ if such a sum consists of n distinct pairwise orthogonal terms, each orthogonal to μ. Otherwise $\epsilon(s) = 0$.

Proposition 6.1 *The character of $\bar{U}_+$ is given by*

$$\mathrm{Ch}(\bar{U}_+) = \Big(\sum_w \epsilon(w)e(\rho - w\rho)w(S_0)\Big)^{-1}. \tag{6.13}$$

The character of $L(\mu)$, $\mu \in P_-$ is given by

$$\mathrm{Ch}\, L(\mu) = \Big(\sum_w \epsilon(w)e(\rho + w(\mu - \rho))w(S_\mu)\Big)\, \mathrm{Ch}(\bar{U}_+) \tag{6.14}$$

The proof is a copy of [Bor88] or [Kac90, Chapter 11]. One observes first that if μ is in P_- then $L(\mu)$ is integrable, in the sense that for $i \in I^{re}$, E_i and F_i act locally nilpotently. Hence $\mathrm{Ch}(L(\mu))$ is W-invariant.

Secondly, as in the Lie algebra case [Kac90, Prop 9.8]

$$\mathrm{Ch}(L(\mu)) = \sum_\lambda c_{\mu\lambda}\, \mathrm{Ch}(M(\lambda))$$

for $c_{\mu\lambda} \in \mathbb{Z}$, $c_{\mu\mu} = 1$ where the sum is over the λ such that $\lambda > \mu$ (that is $\lambda - \mu$ is a sum of simple roots) and $(\mu, \mu - 2\rho) = (\lambda, \lambda - 2\rho)$.

From $\mathrm{Ch}(M(\lambda)) = e(\lambda)\,\mathrm{Ch}(\bar{U}_+)$ one obtains

$$\frac{e(-\rho)L(\mu)}{\mathrm{Ch}(\bar{U}_+)} = \sum_\lambda c_{\mu\lambda} e(\lambda - \rho). \tag{6.15}$$

Using lemma 6.1 we find that the lefthand side of the (6.15) is skew invariant under W. Then as in [Bor88] or [Kac90, Chapter 11] it follows that the righthand side of (6.15) is equal to

$$\sum_{w \in W} \epsilon(w)e(w(\mu - \rho))w(S_\mu).$$

This yields (6.14). (6.13) follows from considering $\mu = 0$, and thus proposition 6.1 is proven.

6.3.3 $\overline{U} = U$

The following result is also proved in the same way as in the ordinary Kac-Moody case [Kac90]. See also [Jos95, §4.1.17], where the notion of primitive vector is replaced by U_+-homology.

Proposition 6.2 *The bilinear form $[-,-]$ is non-degenerate on U_+. In particular $\bar{U} = U$.*

To start we need to know something about the minimal generators as a two-sided ideal of $J = \ker(\tilde{U}_+ \to \bar{U}_+)$. We follow the method of [Kac90]. It is easy to see that the degrees of the minimal generators of J are the same as the degrees of the minimal generators of the left $\bar{U}_+$-module $R = \ker(\oplus_{i \in I} \bar{U}_+(-i) \xrightarrow{E_i} \bar{U}_+)$, where $\bar{U}_+(-i)$ is the graded $\bar{U}_+$-module satisfying $(\bar{U}_+(-i))_\alpha = (\bar{U}_+)_{\alpha - i}$.

There is a canonical map of $\bar{U}$-modules $\oplus_{i \in I} M(i) \to M(0)$ with cokernel $L(0)$. It is clear that R is isomorphic as $\bar{U}_+$ module to $\ker(\oplus_{i \in I} M(i) \to M(0))$. In this way R acquires a $\bar{U}$-structure.

Since $R \subset \oplus_{i \in I} M(i)$ it follows in particular that R is in $\mathcal{O}$ and hence by [Kac90, Remark 9.3] R is generated as $\bar{U}_+$ module by its primitive vectors (homogeneous elements which are annihilated by the F_i's modulo a submodule). Let $v \in R$ be such a primitive vector of degree α. Looking at the action of C on v (modulo the submodule) one finds that $(\alpha, \alpha - 2\rho) = (i, i - 2\rho)$, for at least one $i \in I$. This yields $(\alpha, \alpha) = 2(\rho, \alpha)$.

Assume that β has minimal height such that $(\mathrm{rad}[-,-] \cap U_+)_\beta \neq 0$. We claim that for all $i \in I$ we have $(\beta, i) \leq 0$. It is clearly sufficient to check this for $i \in I^{re}$. By properties 3,4 and the minimality of β it follows that $(\mathrm{rad}[-,-] \cap V_i)_\beta \neq 0$.

Since U_i acts on $\mathrm{rad}[-,-] \cap V_i = \ker(V_i \to \bar{V}_i)$ we find that $\mathrm{Ch}(\mathrm{rad}[-,-] \cap V_i)$ is r_i-invariant. In particular $(\mathrm{rad}[-,-] \cap V_i)_{r_i\beta} \neq 0$. Now if $i \in I^{re}$ were such that $(\beta, i) > 0$ then $\mathrm{ht}(r_i\beta) < \mathrm{ht}(\beta)$. This would yield a contradiction with the choice of β.

Choose $0 \neq r \in (\mathrm{rad}[-,-] \cap U_+)_\beta$. Clearly r is the image of a minimal generator of J. Hence β is the degree of a minimal generator of R. As was already said above R is generated by its primitive vectors, and so the degrees of the primitive vectors contain the degrees of the minimal generators. In particular β is also the degree of a primitive vector in R. We now have the following information on β.

1. $(\beta, i) \leq 0$ for all $i \in I$.

2. $(\beta, \beta) = 2(\rho, \beta)$.

As in [Kac90, Lemma 11.13.2] this implies that β is the sum of pairwise orthogonal, not necessarily distinct elements of I^{im}. To finish the proof one shows directly (using (6.8) and (6.9)) that $(\mathrm{rad}[-,-] \cap U_+)_\beta$ is zero for such β.

6.4 The Ringel-Hall-Algebra as a Quantized Generalized Enveloping Algebra of a Kac-Moody Lie Algebra

6.4.1 The Algebra A

For simplicity we will assume in this section that the base field is $\mathbb{R}$. If $v \in \mathbb{R}_+$ and $a \in \mathbb{R}$ then v^a denotes the usual a'th power of v. It is clear that this is compatible with the convention which was in force in the previous section.

Below let e_t the canonical t'th basis element of $\mathbb{Z}^n$. In this section we consider the following data.

1. A $\mathbb{N}^n$-graded $\mathbb{R}$-algebra with the following properties.
   (a) $A_0 = \mathbb{R}$.
   (b) $\dim A_\alpha < \infty$ for all $\alpha \in \mathbb{N}^n$.
   (c) $A_{e_t} \neq 0$ for all $t \in \{1, \dots, n\}$.
2. A symmetric positive definite bilinear form $[-,-] : A \times A \to \mathbb{R}$, which is zero on $A_\alpha \times A_\beta$ if $\alpha \neq \beta$. We also assume $[1,1] = 1$.
3. An element $v \in]0,1[$.
4. A symmetric bilinear form $\mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ such that $(e_i, e_i) > 0$ and such that
$$a_{ij} = 2(e_i, e_j)/(e_i, e_i)$$
is a generalized Cartan matrix in the sense of [Kac90, Chapter I].

We make $A \otimes A$ into an algebra by the rule
$$(a \otimes b)(c \otimes d) = v^{-(\deg b, \deg c)}(ac \otimes bd) \tag{6.16}$$
for homogeneous a, b, c, d.

Throughout we now make the following extra hypotheses.

Let δ be the map $A \to A \otimes A$, which is adjoint under $[-,-]$ to the multiplication. Then δ is an algebra morphism for the algebra structure on $A \otimes A$, defined in (6.16).

Note that the adjointness property implies that δ is coassociative, and also that $\varepsilon = [1,-]$ behaves as a co-unit.

For $\alpha \in \mathbb{N}^n - \{0\}$ define $H_\alpha = (\sum_{\substack{\beta+\gamma=\alpha \\ \beta,\gamma\neq\alpha}} A_\beta A_\gamma)^\perp \subset A_\alpha$. Here $(-)^\perp$ is taken inside A_α with respect to $[-,-]$. Since $[-,-]$ is positive definite we can for each H_α choose an orthonormal basis. Let $(\theta_i)_{i\in I}$ be the union of these bases. We define a bilinear form on $\mathbb{Z}I$ by $(i,j) = (\deg\theta_i, \deg\theta_j)$ for $i, j \in I$.

Lemma 6.2 *We have*
$$\delta(\theta_i) = \theta_i \otimes 1 + 1 \otimes \theta_i$$

Remark This lemma was used by Li and Zhang [LZ00] to determine all δ-primitive elements of $H(Q)$ and to obtain a sufficient and necessary condition on $H(Q)$ for which for the composition algebra is the Hall algebra. (The composition algebra is the algebra generated by the simples $[S_i]$)

Indeed choose a homogeneous orthonormal basis $(f_j)_{j\in J}$ for A and assume that $(\theta_i)_{i\in I} \subset (f_j)_{j\in J}$. We have

$$\delta(\theta_i) = \sum_{j,k} c_{j,k} f_j \otimes f_k$$

for $c_{j,k} \in \mathbb{R}$ and hence we find

$$\begin{aligned}[\theta_i, f_l f_m] &= \sum_{j,k} c_{j,k}[f_j, f_l][f_k, f_m] \\ &= c_{l,m}.\end{aligned}$$

The statement of the lemma now follows from the definition of $(\theta_i)_{i\in I}$.

Proposition 6.3 *1. The bilinear form $(-,-)$ on $\mathbb{Z}I$ satisfies the axioms of [Bor88]. That is for $i, j \in I$ we have*

(a) $(i,j) \leq 0$ if $i \neq j$.

(b) If $(i,i) > 0$ then $2(i,j)/(i,i)$ is an integer.

In addition we have

(c) $(i,i) > 0$ if and only if $\theta_i \in \bigcup_t A_{e_t}$.

(d) $\dim A_{e_t} = 1$ for all t.

2. A is isomorphic to the positive part of a quantized generalized Kac-Moody Lie algebra with simple roots I and bilinear form $(-,-)$. $i \in I$ is a real simple root if and only if $\theta_i \in A_{e_t}$ for some t.

Let us prove (1a) first. We have

$$\begin{aligned}\delta(\theta_i\theta_j) &= (\theta_i \otimes 1 + 1 \otimes \theta_i)(\theta_j \otimes 1 + 1 \otimes \theta_j) \\ &= \theta_i\theta_j \otimes 1 + \theta_i \otimes \theta_j + v^{-(i,j)}\theta_j \otimes \theta_i + 1 \otimes \theta_i\theta_j.\end{aligned}$$

Assume $i \neq j$. We deduce

$$\begin{aligned}[\theta_i\theta_j, \theta_i\theta_j] &= 1 \\ [\theta_i\theta_j, \theta_j\theta_i] &= v^{-(i,j)}.\end{aligned}$$

The positivity of $[-,-]$ implies for all $x, y \in \mathbb{R}$:

$$\begin{aligned}0 &\leq [x\theta_i\theta_j + y\theta_j\theta_i, x\theta_i\theta_j + y\theta_j\theta_i] \\ &= x^2 + 2v^{-(i,j)}xy + y^2.\end{aligned}$$

Hence the determinant of the matrix

$$\begin{pmatrix} 1 & v^{-(i,j)} \\ v^{-(i,j)} & 1 \end{pmatrix}$$

must be positive. Since $v < 1$ this implies $(i,j) \leq 0$ which proves (1a). Since if $\theta_i, \theta_j \in A_{e_t}$ then $(i,j) = (\deg\theta_i, \deg\theta_j) = (e_t, e_t) > 0$, (1d) immediately follows.

Now we prove (1c). By hypotheses it is clear that if $\theta_i \in A_{e_t}$ then $(i,i) = (e_t, e_t) > 0$, hence let us prove the converse. Assume $\theta_i \notin \bigcup_t A_{e_t}$. Since $A_{e_t} \neq 0$ it follows by (a) that $(e_t, \deg\theta_i) \leq 0$ and hence $(\alpha, \deg\theta_i) \leq 0$ for every $\alpha \in \mathbb{N}^n$. Applying this with $\alpha = \deg\theta_i$ yields $(i,i) \leq 0$.

Let us finally prove (1b). Let $(i,i) > 0$. Then $\theta_i \in A_{e_t}$ for certain t. Hence $2(i,j)/(i,i)$ is equal to $2(e_t, \deg\theta_j)/(e_t, e_t)$. The latter is a linear combination of $a_{tu} = 2(e_t, e_u)/(e_t, e_t)$ and since by hypotheses the a_{tu} form a generalized Cartan-matrix, they are integers.

Now we prove 2. Let $\tilde{U}$ be as in the previous section. Sending E_i to θ_i defines a surjective map of $\tilde{U}_+$ to A which is compatible with the bilinear forms (this follows from (6.12)). Since the bilinear form is non-degenerate on A this implies that A is equal to $\bar{U}_+$. By Proposition 6.2 it follows that A is isomorphic to U_+. This ends the proof of the proposition.

If $S, T \subset \{1, \ldots, n\}$ then we will say that S, T are connected to each other if there exists $s \in S$, $t \in T$ such that $a_{st} \neq 0$ (and hence $a_{ts} \neq 0$ by axiom (C3) of [Kac90, §1.1]).

For $\alpha = \sum_t \alpha_t e_t \in \mathbb{N}^n$ write $\operatorname{Supp}\alpha$ for the set $\{1 \leq t \leq n \mid \alpha_t \neq 0\}$.

Lemma 6.3 *Assume that $\alpha_1, \alpha_2 \in \mathbb{N}^n$ are such that $(\alpha_1, \alpha_2) = 0$ and $(i, \alpha_1) \leq 0$, $(i, \alpha_2) \leq 0$ for all $i \in I^{re}$. Then there exist three sets $P, P_1, P_2 \in \mathbb{N}^n$ with the following properties.*

1. *P, P_1, P_2 are not connected to each other.*
2. $\operatorname{Supp}\alpha_1 = P_1 \cup P$, $\operatorname{Supp}\alpha_2 = P_2 \cup P$.
3. *P spans an affine submatrix of $(a_{ij})_{ij}$.*

We prove this by constructing the sets P, P_1 and P_2. Let $T_j = \operatorname{Supp}\alpha_j$ for $j = 1,2$ and put $P_1 = T_1 \setminus T_2$, $P_2 = T_2 \setminus T_1$ and $P = T_1 \cap T_2$. The fact that $(\alpha_1, \alpha_2) = 0$ together with $(i, \alpha_1) \leq 0$ for $i \in I^{re}$ implies that $(i, \alpha_1) = 0$ for $i \in T_2$. If $i \in P_2$ is such that $\{i\}$ is connected to T_1 then clearly $(i, \alpha_1) < 0$. We conclude that P_2 is not connected to T_1. Hence P_2 is not connected to P and also not to P_1. By symmetry the same statement holds with P_1 and P_2 interchanged. This proves 1. and 2. Let $\alpha_1' = \alpha_1 | P$. Then $\operatorname{Supp}\alpha_1' = P$ and for all $i \in P$: $(i, \alpha_1) = (i, \alpha_1') = 0$. This implies that P spans an affine submatrix of $(a_{ij})_{i,j}$.

Theorem 6.4 *Let $\mathfrak{g}$ be the Kac-Moody Lie algebra associated to the generalized Cartan matrix $(a_{ij})_{ij}$. Assume that there is no $P \subset \{1, \ldots, n\}$ such that P spans an affine submatrix of $(a_{ij})_{ij}$. Let $q \in \mathbb{N}$.*

The following are equivalent.

1. *For all* $\alpha \in \mathbb{N}^n$ *we have that* $\dim U_+(\mathfrak{g})_\alpha \cong \dim A_\alpha$ *modulo* q.
2. *For all* $\alpha \in \mathbb{N}^n$ *the cardinality of the set* $\{i \in I^{im} \mid \deg\theta_i = \alpha\}$ *is divisible by* q.

The proof is given in the remainder of this section. The theorem is a consequence of the Borcherds character formula for A, and the Kac-Weyl character formula for $U(\mathfrak{g})$.

Let us write

$$\mathrm{Ch}'(A) = \sum_{\alpha\in\mathbb{N}^n} \dim A_\alpha e(\alpha).$$

Then from Proposition 6.3.2 and (6.13) one deduces that

$$\mathrm{Ch}'(A) = (\sum_w \epsilon(w)e(\rho - w\rho)w(S_0'))^{-1}. \tag{6.17}$$

S_0' is $\sum_\alpha \epsilon(\alpha)e(\alpha)$ where α runs over the sums $\deg\theta_{i_1} + \cdots + \deg\theta_{i_t}$ where the i_p are pairwise orthogonal and distinct elements of I. Finally $\epsilon(\alpha) = (-1)^t$.

Note that we have changed the notations slightly compared to those in (6.13). This is because we are evaluating $\mathrm{Ch}'(A)$ and not $\mathrm{Ch}(A)$ (recall that $\mathrm{Ch}(A)$ would be the character for a much finer grading on A).

From the Weyl character formula it follows that (with the same notations)

$$\mathrm{Ch}(U_+(\mathfrak{g})) = (\sum_w \epsilon(w)e(\rho - w\rho))^{-1}.$$

Comparing these two formulas, it follows that statement 1. of the theorem is equivalent to

$$\sum_w \epsilon(w)w\big(e(-\rho)(S_0' - 1)\big) \cong 0 \qquad \text{modulo } q. \tag{6.18}$$

Now $e(-\rho)(S_0'-1)$ is a sum of terms which lie in the interior of the fundamental chamber of $\mathbb{C}^n \oplus \mathbb{C}\rho$. Since the stabilizer under the Weyl group of a point in the interior of the fundamental chamber is trivial [Kac90, Prop. 3.12, beginning of proof] it follows from (6.18) that $S_0' \cong 1$ modulo q.

Now suppose we have a sum $\alpha = \deg\theta_{i_1} + \cdots + \deg\theta_{i_t}$ where the i_p are pairwise orthogonal and distinct. By the previous lemma and the hypotheses all $\deg\theta_{i_p}$ have disjoint pairwise not connected support.

An easy induction argument now shows that the number of such sums for all α is divisible by q if and only if the multiplicity of the imaginary simple roots is divisible by q.

6.4.2 H(Q) is the Positive Part of a QGEA

We now come back to the Hall algebra.

Theorem 6.5 *The Hall algebra over a finite field of a quiver is the positive part of a QGEA.*

There is as well a link with conjecture 1.2 on page 10, that states that $a_\alpha(0)$ is the multiplicity of the root α in $\mathfrak{g}$. This is given in

Theorem 6.6 *Assume that no subset of Q_0 spans a tame subquiver. Then conjecture 1.2 is equivalent to the following statement. The multiplicities of the imaginary simple roots of the QGEA corresponding to the Hall algebra of Q are divisible by q.*

Let Q be a quiver without loops and let $H(Q)$ be the Hall algebra of Q. We make $H(Q) \otimes H(Q)$ into an algebra by defining

$$([A] \otimes [B])([C] \otimes [D]) = v^{-(\vec{B},\vec{C})}[A][C] \otimes [B][D]$$

Following Green [Gre95] we put

$$\delta[A] = \sum_{[B],[C]} v^{-\langle C,B \rangle} g^A_{BC} \frac{|\operatorname{Aut}(B)| \cdot |\operatorname{Aut}(C)|}{|\operatorname{Aut}(A)|}[B] \otimes [C]$$

and

$$[[A],[B]] = v^{-(\alpha,\beta)} \frac{\delta_{[A],[B]}}{|\operatorname{Aut}(A)|}$$

It was shown by Green that δ is an algebra map $H(Q) \to H(Q) \otimes H(Q)$ and that multiplication and δ are adjoint under $[-,-]$. In other words the pair $H(Q), v$ satisfies the hypotheses put on the pair A, v in the beginning of section 6.4.1. Hence Theorem 6.5 follows from Proposition 6.3.

We already noted that conjecture 1.2 is equivalent to $\dim U_+(\mathfrak{g})_\alpha = \dim H(Q)_\alpha$ for all $\alpha \in \mathbb{N}^{Q_0}$. Thus Theorem 6.6 follows from Theorem 6.4.

Appendix A

Some Identities of Structure Constants

A.1 The Structure Constants in a Source

Let i be a source throughout this appendix.

Lemma A.1 *For 2 admissible representations A and B we have*

$$|\operatorname{Aut}(B)|g^{A}_{BS_i^a} = |\operatorname{Aut}(A)|g^{r_iB}_{S_i^a r_iA}. \tag{A.1}$$

In the case $\overrightarrow{A} \neq \overrightarrow{B} + ai$, both sides of the equation are 0.

Assume $\overrightarrow{A} = \overrightarrow{B} + ai$. Using functoriality the number I of injective maps from A to B is equal to the number of surjective maps from r_iB to r_iA is S.

We have $g^{A}_{BS_i^a} = \dfrac{I}{|\operatorname{Aut}(B)|}$ while $g^{r_iB}_{S_i^a r_iA} = \dfrac{I}{|\operatorname{Aut}(r_iA)|}$.

The lemma follows now from $|\operatorname{Aut}(r_iA)| = |\operatorname{Aut}(A)|$.

Lemma A.2 *For $A, B, C \in \operatorname{Rep}(Q)$ admissible and $a \in \mathbb{N}$, we have*

$$g^{A}_{B\ C\oplus S_i^a} = \sum_{[P]} g^{A}_{PC} g^{P}_{BS_i^a}.$$

Let us denote $D = C \oplus S_i^a$. Consider $X \subset A$ such that $A/X \cong D$ then there is a unique P such that $X \subset P \subset A$ and such that $P/X \cong S_i^a$. In that case $A/P \cong C$. Thus we obtain

$$g^{A}_{BD} = \sum_{\substack{P \subset A \\ P \cong C}} g^{P}_{BS_i^a} = \sum_{[P]} g^{A}_{PC} g^{P}_{BS_i^a}.$$

Lemma A.3 *For the admissible representations A, B, M and N we have*

$$g_{A\ B}^{M\ N\oplus S_i^a} = q^{\langle S_i^a,M\rangle+\langle N,S_i^a\rangle} g_{r_iA\ r_iB}^{r_iM\oplus S_i^a\ r_iN}. \tag{A.2}$$

First recall (4.17):

$$g_{AB}^{MN} = \frac{|\operatorname{Aut}(M)||\operatorname{Aut}(N)|}{|\operatorname{Aut}(A)||\operatorname{Aut}(B)|}\sum_{[P]} g_{MP}^A g_{PN}^B |\operatorname{Aut}(P)|. \tag{A.3}$$

Combined with lemma A.2 we deduce:

$$\begin{aligned} g_{A\ B}^{M\ N\oplus S_i^a} &= \frac{|\operatorname{Aut}(M)||\operatorname{Aut}(N\oplus S_i^a)|}{|\operatorname{Aut}(A)||\operatorname{Aut}(B)|}\sum_{[P]} g_{MP}^A g_{P\ N\oplus S_i^a}^B |\operatorname{Aut}(P)| \\ &= \frac{|\operatorname{Aut}(M)||\operatorname{Aut}(N\oplus S_i^a)|}{|\operatorname{Aut}(A)||\operatorname{Aut}(B)|}\sum_{[P],[Q]} g_{MP}^A g_{QN}^B g_{PS_i^a}^Q |\operatorname{Aut}(P)| \end{aligned}$$

Note that if a term in the last expression is non-zero then $P \subset Q \subset B$. In particular P and Q are admissible. Hence we can invoke lemma A.1 to obtain:

$$g_{A\ B}^{M\ N\oplus S_i^a} = \frac{|\operatorname{Aut}(M)||\operatorname{Aut}(N\oplus S_i^a)|}{|\operatorname{Aut}(A)||\operatorname{Aut}(B)|}\sum_{[P],[Q]\mathrm{adm}} g_{r_iMr_iP}^{r_iA} g_{r_iQr_iN}^{r_iB} g_{S_i^ar_iQ}^{r_iP} |\operatorname{Aut}(Q)|.$$

Now i is a sink, and we use the appropriate version of lemma A.2. We obtain:

$$g_{A\ B}^{M\ N\oplus S_i^a} = \frac{|\operatorname{Aut}(M)||\operatorname{Aut}(N\oplus S_i^a)|}{|\operatorname{Aut}(A)||\operatorname{Aut}(B)|}\sum_{[Q]\mathrm{adm}} g_{r_iQr_iN}^{r_iB} g_{r_iM\oplus S_i^a\ r_iQ}^{r_iA} |\operatorname{Aut}(Q)|.$$

We obtain, by (4.17):

$$g_{A\ B}^{M\ N\oplus S_i^a} = \frac{|\operatorname{Aut}(M)||\operatorname{Aut}(N\oplus S_i^a)|}{|\operatorname{Aut}(r_iM\oplus S_i^a)||\operatorname{Aut}(N)|} g_{r_iA\ r_iB}^{r_iM\oplus S_i^a\ r_iN}.$$

The only remaining problem is to show that the fraction in the above equation is equal to $q^{\langle S_i^a,M\rangle+\langle N,S_i^a\rangle}$. We prove this now.

It is clear that

$$\operatorname{Aut}(N\oplus S_i^a) = \begin{pmatrix} \operatorname{Aut}(N) & \operatorname{Hom}(N,S_i^a) \\ 0 & \mathrm{Gl}_a \end{pmatrix},$$

thus we have

$$|\operatorname{Aut}(N\oplus S_i^a)| = |\operatorname{Aut}(N)||\mathrm{Gl}_a||\operatorname{Hom}(N,S_i^a)| = q^{\langle N,S_i^a\rangle}|\operatorname{Aut}(N)||\mathrm{Gl}_a|.$$

A similar calculation yields

$$|\operatorname{Aut}(r_iM\oplus S_i^a)| = q^{\langle S_i^a,r_iM\rangle}|\operatorname{Aut}(M)||\mathrm{Gl}_a| = q^{-\langle S_i^a,M\rangle}|\operatorname{Aut}(M)||\mathrm{Gl}_a|.$$

This implies what we want.

Appendix B

Combinatorial Side Results

In this appendix we prove (3.16). In fact we have the following more general formula.

Theorem B.1 *If $\mu = (c, 1^d)$ is a hook then*

$$K^{-1}_{\lambda\mu}(t) = (-1)^{l(\lambda)+l(\mu)} t^{\vartheta} [\lambda'] \frac{1 - t^{\lambda'_c}}{1 - t^{\lambda'_1}},$$

where $\vartheta = \sum_{i \geq 2} \frac{\lambda'_i(\lambda'_i + 1)}{2} - \sum_{j=2}^{c} \lambda'_j$.

We were informed by Stanley that in the case $\mu = (1^n)$ this formula was proved by MacDonald.

B.1 An Identity of Gaussian Binomial Coefficients

We will first prove an identity between Gaussian binomial coefficients using non-commutative generating functions.

Lemma B.2 *Let $k \in \mathbb{N}$. Let $\mu \in \mathbb{Z}^k$, with $\mu_1 > 0$. Define $a_i = \mu_{i-1} - \mu_i$ and $b = \sum_{i \geq 2} a_i$. We have*

$$\sum_{\sum r_i = m} (-1)^{r_i} t^{\sum_{i\geq 2}((\mu_i + r_i)(\mu_i + r_i + 1))/2} \begin{bmatrix} b + r_1 \\ r_1 \end{bmatrix} \begin{bmatrix} a_2 \\ r_2 \end{bmatrix} \begin{bmatrix} a_3 \\ r_3 \end{bmatrix} \cdots$$
$$= (-1)^m t^{\sum_{i \geq 2}(\mu_i(\mu_i+1))/2}.$$

Suppose x, y are variables satisfying $yx = txy$. Then it is well known that

$$(x + y)^a = \sum_{u \geq 0} \begin{bmatrix} a \\ u \end{bmatrix} x^u y^{a-u}. \tag{B.1}$$

We can also write this with negative exponents:

$$\begin{aligned}(x+y)^{-a} &= \sum_{u\geq 0}\begin{bmatrix}-a\\u\end{bmatrix}x^u y^{-a-u}\\ &= \sum_{u\geq 0}(-1)^u t^{-au+(u(u-1))/2}\begin{bmatrix}a+u-1\\u\end{bmatrix}x^u y^{-a-u}.\end{aligned}$$

Now we introduce variables $(x_i)_i, (y_i)_i$ satisfying

$$y_i x_i = t x_i y_i \qquad y_i y_j = y_j y_i \qquad x_i x_j = x_j x_i$$

Let $k > 2$ be finite and $a_i \in \mathbb{N}$, for $2 \leq i \leq k$, while $b \in \mathbb{N}$. We calculate

$$\begin{aligned}\kappa &= (x_k+y_k)^{a_k}\cdots(x_2+y_2)^{a_2}(x_1+y_1)^{-b-1}\\ &= \sum_{(r_i)_i}(-1)^{r_1}t^{-(b+1)r_1-(r_1(r_1-1))/2}\begin{bmatrix}b+r_1\\r_1\end{bmatrix}\begin{bmatrix}a_2\\r_2\end{bmatrix}\begin{bmatrix}a_3\\r_3\end{bmatrix}\cdots\\ &\quad \times\cdots x_3^{r_3}y_3^{a_3-r_3}x_2^{r_2}y_2^{a_2-r_2}x_1^{r_1}y_1^{-b-1-r_1}\\ &= \sum_{(r_i)_i}(-1)^{r_1}t^{\vartheta_r}\begin{bmatrix}b+r_1\\r_1\end{bmatrix}\begin{bmatrix}a_2\\r_2\end{bmatrix}\begin{bmatrix}a_3\\r_3\end{bmatrix}\cdots\times x_1^{r_1}x_2^{r_2}x_3^{r_3}\cdots\\ &\quad \times y_1^{-b-1-r_1}y_2^{a_2-r_2}y_3^{a_3-r_3},\end{aligned}$$

where

$$\vartheta_r = -br_1 - \frac{r_1(r_1+1)}{2} + \sum_{1\geq j<i}(a_i-r_i)r_j.$$

Thus setting all $x_i = x$ and $y_i = y$ we get

$$\begin{aligned}&(x+y)^{-b-1+a_2+a_3+\dots}\\ &\quad = \sum_{(r_i)_i}(-1)^{r_i}t^{\vartheta_r}\begin{bmatrix}b+r_1\\r_1\end{bmatrix}\begin{bmatrix}a_2\\r_2\end{bmatrix}\begin{bmatrix}a_3\\r_3\end{bmatrix}\cdots\times x^{\sum r_i}y^{-b-1+\sum r_i+\sum_{i\geq 2}a_i}.\end{aligned}$$

We suppose now that $b = \sum a_i$, and look at the coefficient of $x^m y^{-m-1}$ using $(x+y)^{-1} = \sum_m (-1)^m t^{-(m(m+1))/2}x^m y^{-m-1}$. We see

$$(-1)^m = \sum_{\sum r_i = m} t^{\zeta_r}\begin{bmatrix}b+r_1\\r_1\end{bmatrix}\begin{bmatrix}a_2\\r_2\end{bmatrix}\begin{bmatrix}a_3\\r_3\end{bmatrix}\cdots,$$

where

$$\zeta_r = \vartheta_r + \frac{m(m+1)}{2} = \sum_{j>i\geq 2} r_i a_j + \sum_{i\geq 2}\frac{r_i(r_i+1)}{2}.$$

Now define $\nu_i = \mu_i + r_i$, and suppose $a_i = \mu_{i-1} - \mu_i$ then

$$T_r = \sum_{i\geq 2}\frac{\nu_i(\nu_i+1)}{2} - \sum_{i\geq 2}\frac{\mu_i(\mu_i+1)}{2}.$$

This yields the lemma.

B.2 A Proof of Theorem B.1

By definition of $K^{-1}_{\lambda\mu}(t)$ we have

$$P_\lambda(x;t) = \sum_\mu K^{-1}_{\lambda\mu}(t) s_\mu(x).$$

Recall that e_m denotes again the m-th *elementary symmetric function*. It is the sum of all products of m distinct variables. Note that $s_{1^m} = P_{1^m} = e_m$. Pieri's formula for s_μ and P_μ gives the multiplication by e_m. This will lead to a recursion formula for the entries of K^{-1}. However we first need to introduce some notation for partitions. Let us use $\mu <_m \nu$ to signify that the $\nu - \mu$ is a vertical strip of length m. This means that $|\mu| + m = |\nu|$, and the diagram of μ is included in that of ν, and in addition the difference is at most one on each horizontal. (Example: $(3,1,1) <_3 (4,2,1,1)$, while $(3,1,1) \not<_4 (4,3,2)$).

Pieri's formula for s_μ is classical [Mac95, I(5.17)]

$$s_\mu e_m = \sum_{\mu <_m \nu} s_\nu. \tag{B.2}$$

Pieri's formula for P_μ is similar [Mac95, III.3]

$$P_\mu e_m = \sum_{\mu <_m \nu} f^\nu_{\mu,1^m}(t) P_\nu, \tag{B.3}$$

where

$$f^\nu_{\mu,1^m}(t) = \prod_i \begin{bmatrix} \nu'_i - \nu'_{i+1} \\ \nu'_i - \mu'_i \end{bmatrix}.$$

Substituting in (B.3) the transitions $P_\nu = \sum_{\alpha \le \nu} K^{-1}_{\nu\alpha}(t) s_\alpha$ and $P_\mu = \sum_{\beta \le \mu} K^{-1}_{\mu\beta}(t) s_\beta$ we obtain:

$$\sum_{\beta \le \mu} K^{-1}_{\mu\beta}(t) s_\beta e_m = \sum_{\mu <_m \nu} \sum_{\alpha \le \nu} f^\nu_{\mu,1^m}(t) K^{-1}_{\nu\alpha}(t) s_\alpha.$$

Using (B.2) on the lefthand side, we obtain

$$\sum_{\beta \le \mu} \sum_{\beta <_m \delta} K^{-1}_{\mu\beta}(t) s_\delta = \sum_{\mu <_m \nu} \sum_{\alpha \le \nu} f^\nu_{\mu,1^m}(t) K^{-1}_{\nu\alpha}(t) s_\alpha.$$

We note in passing that the summation runs over the Greek letters except μ. Now we look at the coefficients of the s_δ's in this equation. We get for all partitions μ and δ, for which $|\mu| + m = |\delta|$ that

$$\sum_{\substack{\beta \le \mu \\ \beta <_m \delta}} K^{-1}_{\mu\beta}(t) = \sum_{\substack{\mu <_m \nu \\ \delta \le \nu}} f^\nu_{\mu,1^m}(t) K^{-1}_{\nu\delta}(t). \tag{B.4}$$

This relation determines the $K^{-1}_{\psi\chi}$'s uniquely. Indeed apply (B.4) with $\delta = \chi$ and μ obtained by deleting the last column of ψ. Then $K^{-1}_{\psi\chi}$ can be calculated from $K^{-1}_{\mu\beta}$'s with $|\mu| < |\psi|$ and $K^{-1}_{\nu\chi}$'s with $\nu \prec \psi$. We may assume that these are known by induction.

In the case $\delta = (c, 1^d)$ we find a simplified version of (B.4). Denote $\psi = (c-1, 1^{d-m+1})$ and $\chi = (c, 1^{d-m})$

$$K^{-1}_{\mu,\psi}(t) + K^{-1}_{\mu,\psi}(t) = \sum_{\mu <_m \nu} f^{\nu}_{\mu,1^m} K^{-1}_{\nu(c,1^d)}(t) \tag{B.5}$$

We will now use this formula to prove Theorem B.1 by induction. First note that for $\lambda = 1^n$ Theorem B.1 is obvious. So we only need to show that the formula in Theorem B.1 satisfies the recursion relation (B.5). We will assume that $c > 1$, the case $c = 1$ is similar but requires fewer steps.

So assume $c > 1$. Set $\gamma = \nu'$ and $\eta = \mu'$. Put $b = \eta_1$, $a_i = \eta_{i-1} - \eta_i$, $g_i = \gamma_{i-1} - \gamma_i$ and $r_i = \gamma_i - \eta_i$.

Substituting the formula of Theorem B.1 in (B.5) we get:

$$\begin{aligned}
&\Bigg((-1)^{b+d-m} t^{\sum_{i\geq 2}(\eta_i(\eta_i+1))/2 - \sum_{j=2}^{c-1}\eta_j} \frac{1-t^{\eta_{c-1}}}{1-t^{\eta_1}} + \\
&\quad (-1)^{b+d-m} t^{\sum_{i\geq 2}(\eta_i(\eta_i+1))/2 - \sum_{j=2}^{c}\eta_j} \frac{1-t^{\eta_c}}{1-t^{\eta_1}} \Bigg) \times \begin{bmatrix} & b & \\ a_2 & a_3 & \dots \end{bmatrix} \\
&\qquad = \sum_{\eta' <_m \gamma'} (-1)^{\gamma_1+d+1} t^{\sum_{i\geq 2}(\gamma_i(\gamma_i+1))/2 - \sum_{j=2}^{c}\gamma_j} \\
&\qquad\qquad \times \begin{bmatrix} & \gamma_1 & \\ g_2 & g_3 & \dots \end{bmatrix} \frac{1-t^{\gamma_c}}{1-t^{\gamma_1}} \begin{bmatrix} g_2 \\ r_1 \end{bmatrix} \begin{bmatrix} g_2 \\ r_1 \end{bmatrix} \cdots .
\end{aligned}$$

Using the r_i we can eliminate the γ's and g's and after some manipulation we obtain:

$$\begin{aligned}
&(-1)^{-m+1} t^{\sum_{i\geq 2}(\eta_i(\eta_i+1))/2 - \sum_{j=2}^{c-1}\eta_j} (2 - t^{\eta_{c-1}} - t^{\eta_c}) \\
&\quad = (1 - t^{\eta_c}) \sum_{\sum_i r_i = m} (-1)^r_1 t^{\sum_{i\geq 2}((\eta_i+r_i)(\eta_i+r_i+1))/2 - \sum_{j=2}^{c-1}(\eta_j+r_j)} \\
&\qquad\qquad \times \begin{bmatrix} b+r_1-1 \\ r_1 \end{bmatrix} \begin{bmatrix} a_2 \\ r_2 \end{bmatrix} \begin{bmatrix} a_3 \\ r_3 \end{bmatrix} \cdots \\
&\quad + t^{\eta_c}(1 - t^{a_c}) \sum_{\sum_i r_i = m} (-1)^r_1 t^{\sum_{i\geq 2}((\eta_i+r_i)(\eta_i+r_i+1))/2 - \sum_{j=2}^{c-1}(\eta_j+r_j)} \\
&\qquad\qquad \times \begin{bmatrix} b+r_1-1 \\ r_1 \end{bmatrix} \begin{bmatrix} a_2 \\ r_2 \end{bmatrix} \begin{bmatrix} a_3 \\ r_3 \end{bmatrix} \cdots \begin{bmatrix} a_c - 1 \\ r_c - 1 \end{bmatrix} \cdots .
\end{aligned} \tag{B.6}$$

We will rewrite first the second term of the righthand side. Put

$$\bar{\mu}_i = \begin{cases} \eta_i - 1 & i < c \\ \eta_i & i \geq c \end{cases}, \qquad \bar{r}_i = \begin{cases} r_i - 1 & i = c \\ r_i & i \neq c \end{cases}.$$

We put as well

$$\bar{a}_i = \bar{\mu}_{i-1} - \bar{\mu}_i = \begin{cases} a_i - 1 & i = c \\ a_i & i \neq c \end{cases}, \qquad \bar{b}_i = b - 1.$$

With these notations the second term of the righthand side becomes:

$$t^{\eta_c}(1-t^{a_c}) \sum_{\sum_i \bar{r}_i = m-1} (-1)^{\bar{r}_1} t^{\sum_{i\geq 2}((\bar{\mu}_i+\bar{r}_i)(\bar{\mu}_i+\bar{r}_i+1))/2} \begin{bmatrix} \bar{b}+\bar{r}_1 \\ \bar{r}_1 \end{bmatrix} \begin{bmatrix} \bar{a}_2 \\ \bar{r}_2 \end{bmatrix} \begin{bmatrix} \bar{a}_3 \\ \bar{r}_3 \end{bmatrix} \cdots.$$

Using lemma B.2 this is equal to

$$(-1)^{m-1} t^{\eta_c}(1-t^{a_c}) t^{\sum_{i\geq 2}(\bar{\mu}_i(\bar{\mu}_i+1))/2}.$$

This is equal to

$$(-1)^{m-1} t^{\eta_c}(1-t^{a_c}) t^{\sum_{i\geq 2}(\eta_i(\eta_i+1))/2 - \sum_{j=2}^{c-1}\eta_j}.$$

We now subtract this term from the lefthand side of (B.6). We are left with proving:

$$\begin{aligned} (-1)^m t^{\sum_{i\geq 2}(\eta_i(\eta_i+1))/2 - \sum_{j=2}^{c}\eta_j}(1-t^{\eta_c}) \\ = \sum_{\sum_i r_i = m} (-1)^{r}_1 t^{\sum_{i\geq 2}((\eta_i+r_i)(\eta_i+r_i+1))/2 - \sum_{j=2}^{c-1}(\eta_j+r_j)} \\ \times \begin{bmatrix} b+r_1-1 \\ r_1 \end{bmatrix} \begin{bmatrix} a_2 \\ r_2 \end{bmatrix} \begin{bmatrix} a_3 \\ r_3 \end{bmatrix} \cdots \end{aligned} \tag{B.7}$$

With the same $\bar{\mu}, \bar{a}$ and $\bar{b}$ as before, the righthand side becomes

$$\sum_{\sum_i r_i = m} (-1)^{r_1} t^{\sum_{i\geq 2}((\bar{\mu}_i+r_i)(\bar{\mu}_i+r_i+1))/2 - (\bar{\mu_c}+r_c)} \begin{bmatrix} \bar{b}+r_1 \\ r_1 \end{bmatrix} \begin{bmatrix} \bar{a}_2 \\ r_2 \end{bmatrix} \cdots \begin{bmatrix} \bar{a}_c+1 \\ r_c \end{bmatrix} \cdots.$$

But

$$\begin{bmatrix} \bar{a}_c+1 \\ r_c \end{bmatrix} = \begin{bmatrix} \bar{a}_c \\ r_c-1 \end{bmatrix} + t^{r_c} \begin{bmatrix} \bar{a}_c \\ r_c \end{bmatrix}$$

This leads to

$$\begin{aligned} &\sum_{\sum_i r_i = m} (-1)^{r_1} t^{\sum_{i\geq 2}((\bar{\mu}_i+r_i)(\bar{\mu}_i+r_i+1))/2 - (\bar{\mu_c}+r_c)} \begin{bmatrix} \bar{b}+r_1 \\ r_1 \end{bmatrix} \begin{bmatrix} \bar{a}_2 \\ r_2 \end{bmatrix} \cdots \begin{bmatrix} \bar{a}_c \\ r_c-1 \end{bmatrix} \cdots \\ &\quad + \sum_{\sum_i r_i = m} (-1)^{r_1} t^{\sum_{i\geq 2}((\bar{\mu}_i+r_i)(\bar{\mu}_i+r_i+1))/2 - \bar{\mu}_c} \begin{bmatrix} \bar{b}+r_1 \\ r_1 \end{bmatrix} \begin{bmatrix} \bar{a}_2 \\ r_2 \end{bmatrix} \cdots \begin{bmatrix} \bar{a}_c \\ r_c \end{bmatrix} \cdots. \end{aligned}$$

We can invoke lemma B.2 again on the second part to see it is equal to

$$(-1)^m t^{\sum_{i\geq 2}\eta_i(\eta_i+1)/2 - \sum_{j=2}^{c}\eta_j}. \tag{B.8}$$

The first part we can rewrite with the $\bar{r}_i$'s, and again invoke the lemma, to obtain

$$(-1)^{m-1} t^{\sum_{i\geq 2} \eta_i(\eta_i+1)/2 - \sum_{j=2}^{c-1} \eta_j}. \tag{B.9}$$

Now it is clear that the sum of (B.9) and (B.8) is the lefthand side of (B.7). This finishes the proof of the theorem.

Appendix C

Computer program

Step 1: the Gaussian multinomial coefficients of a partition.

```
WITH(combinat):
prd := b ->PRODUCT(t^a-1,a=1..b):

gmcp:=PROC (part::LIST)
          LOCAL f1,hulp;
          GLOBAL gmca;
*1     hulp := prd(part[NOPS(part)])/prd(part[1]);
       FOR f1 FROM NOPS(part) BY -1 TO 2 DO
*2          hulp := hulp/prd(part[f1]-part[f1-1])
            OD;
       gmc(part):=SIMPLIFY(hulp)
END:
```

This procedure encodes the Gaussian multinomial coefficients $[\mu]$. We first note that maple denotes its partitions increasing ($[1, 1, 3, 4]$), rather than decreasing. The essential lines are the ones indicated with a * and a number. The line *1 assigns to "hulp" for a partition μ of length n (=NOPS(part)) the formula $\phi_{\mu_n}/\phi_{\mu_1}$. Afterwards, in *2, for each two consecutive parts of μ hulp is divided by $\phi_{\mu_{f_1}-\mu_{f_1-1}}$. Clearly that gives the desired coefficient.

Step 2a: pa calculates the Π_a[1].

```
pa := PROC (a)
          LOCAL l1,N,f2,s1, hulps,par,curr,p;
          GLOBAL paf;
     hulps := 0;
     par:=PARTITION(a);
     FOR f2 FROM 1 TO NOPS(par) DO
```

[1]For a loopfree one vertex quiver we calculate Π_a which is encoded in "paf".

```
            curr := par[f2];
            N:=NOPS(curr);
            l1:=curr[N];
            gmcp(curr);
*3          term(f2) := (-1)^(l1) * t^(SUM(curr[p]^2,p=1..N) - l1*(l1+1)/2)* gmc(curr);
            hulps := SIMPLIFY(hulps + term(f2));
            OD;
      paf := hulps
END:
```

The crucial line here is *3 which assigns for each partition to the local variable "term", the term that partition contributes to the entire sum Π_a. The next line adds the term to "hulps", meanwhile simplifying to reduce calculation time.

Step 2b: pab calculates for a given vector a, b and a number of arrows m ="pijl", the polynomial $\Pi_{a,b}$= "pabf".

```
pab := PROC (a,b,pijl)
            LOCAL l1,m1,N,M,f2,f3,s1, hulps,para,parb,curra,currb,p;
            GLOBAL pabf;
      hulps := 0;
      para:=PARTITION(a);
      parb:=PARTITION(b);
      FOR f2 FROM 1 TO NOPS(para) DO
            FOR f3 FROM 1 TO NOPS(parb) DO
                  curra := para[f2];
                  currb := parb[f3];
                  N:=NOPS(curra);
                  M:=NOPS(currb);
                  l1:=curra[N];
                  m1:=currb[M];
*                 hulps := SIMPLIFY(hulps + (-1)^(l1+m1) *
*                           t^(SUM(curra[p]^2,p=1..N)+ SUM(currb[p]^2,p=1..M)
*4                          - l1*(l1+1)/2 - m1*(m1+1)/2
*                           -pijl*(SUM(curra[N-p]*currb[M-p],p=0..(MIN(M,N)-1))))
*                           * gmc(curra) * gmc(currb))
                  OD;
            OD;
      pabf := hulps
end:
```

The crucial part is the multiline expression *4, where for each 2 partition the right term is added to the total "hulps".

Step3: An example of running the programs

As an example we list some of the tests we ran. First of all for each partition we calculate the Gaussian multinomial coefficient.

```
FOR k FROM 1 TO 20 DO
   FOR l FROM 1 TO
      NOPS(partition(k)) DO gmcp(partition(k)[l])
   OD
OD:
```

Next we run the following program for dimension $k \geq s$ and 3 arrows.

```
FOR s FROM 19 TO 20 DO
   FOR k FROM s TO 20 DO
      pab(k,s,3):PRINT(k,s,EVALB(SIMPLIFY(pabf)=0),LIMIT(pabf, t=INFINITY))
   OD
OD:
```

The output gives the results for (19,19),(19,20) and (20,20):

$$19,\ 19,\ \mathit{false},\ 0$$

$$20,\ 19,\ \mathit{false},\ 0$$

$$20,\ 20,\ \mathit{false},\ 0$$

For instance the last result means that for dimension vector $(20, 20)$ the test if $\Pi_{a,b}$ is identically zero answered "false" while the limit for t going to infinity is 0.

We obtain a more interesting output for the dimensions $(12, 4)$, $(13, 4)$ and $(14, 4)$ (again in the 3-arrow case). We note that $(12, 4) = \rho - r_1 r_2 r_1 \rho$ while $14 = 3 \cdot 4 + 2$. This means that these 3 consecutive dimensions, behave differently under the conjectures we test. Note that $\epsilon(r_1 r_2 r_1) = -1$. This coincides with the last number on the output line of (12,4). So the conjecture holds true in this particular case.

$$12,\ 4,\ \mathit{false},\ -1$$

$$13,\ 4,\ \mathit{false},\ 0$$

$$14,\ 4,\ \mathit{true},\ 0$$

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