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# Geometry and Gevrey asymptotics of two-dimensional turning points

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Peter De Maesschalck,
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# Preface

This thesis deals with singular perturbation problems in the plane, or more generally on 2-manifolds. A differential system (or vector field) with a small parameter $\epsilon$ is called singularly perturbed if the order of the differential system is lower for $\epsilon = 0$ than for $\epsilon \neq 0$. Roughly, the reduced set of differential equations (the equations for $\epsilon = 0$) are easier to study, while the addition of small perturbations causes a significant increase in complexity.

The perturbation can—no matter how small—have a drastic effect on the solutions of the differential equations. Small perturbations can accumulate and have an effect on longer time scales. One says that the differential system exhibits different time scales: a fast time scale, on which small perturbations have little or no effect, and a slow time scale.

This kind of "slow–fast" systems have a large number of applications in both sciences and industry. Applications of singular perturbation theory such as in the study of chemical reactions, specific electric circuits, biological eco-models, ... are well-documented in the literature.

In this thesis, we are interested in the study of so-called "turning points". On the fast time scale the dynamics is determined by a fast attraction towards some stable equilibrum state. The stability of this equilibrum can loose its strength on a slower time scale, and can eventually become unstable at some (turning) point. One expects that this instability will ensure that the solutions that were close to the equilibrum state before the turning point will immediately be repelled from the equilibrum state after passing near the turning point. However, it appears that sometimes the loss of stability is delayed: solutions stay near the equilibrum state for some time before the loss of stability becomes dominant.

The emphasis lies on a geometrical treatment of differential equations in the sense of a dynamical systems approach. We will hence consisder solutions in a phase portrait, and we will refer to solutions of the differential equations as trajectories (or orbits) in this phase plane. The equilibrum state will be seen as a singularity. In the problems that are handled here, curves of singularities appear. Such a curve is called a "critical curve". The small perturbation parameter $\epsilon$ will be seen as a third phase variable and induces a three-dimensional phase space, in which the two-dimensional phase portrait of the unperturbed system is found in the plane $\{\epsilon = 0\}$.

This three-dimensional phase space will play an important role in our study. We gather trajectories in this phase space together to manifolds. Inside the 3-dimensional phase space, these manifolds are seen as curved surfaces, transverse to the plane $\{\epsilon = 0\}$.

We study the behaviour of orbits near the critical curve, and in particular near a turning point. The turning points under consideration are very general and need not be generic. Such a turning point stands for a transition between attracting dynamics and repelling dynamics: before the passage near the turning point the dynamics is governed by an attraction towards the critical curve; after the passage near the turning point the dynamics is governed by a repulsion from the critical curve. In general, the repulsion will be so strong that orbits are repelled from the critical curve immediately after the turning point. However, restricting to small regions in parameter space it is possible that orbits stay near the critical curve for a finite distance after passing near the turning point. This is called a canard solution.

Our aim is to show the existence of these small regions in parameter space, and to give a method on how to determine them. In a first step, we show how to construct control manifolds in parameter space so that along this manifold a “prescibed behaviour” of all orbits is found, i.e. we are able to solve boundary value problems, restricting to parameters on this control manifold. The inverse problem, which is to find out precisely how long orbits stay near the critical curve beyond the turning point, along given control manifolds in parameter space, is also handled.

The study is two-fold. In a first part, turning point problems are approached in a geometric way, with the traditional tools from the dynamical systems theory. The main tools in the study are $C^k$-normal forms, center manifolds and blow up of families of vector fields. These tools allow us to carry out a general study of so-called “canard manifolds”. We also give a characterization of the transit time of orbits near the turning point. The main benefit of this method is the generality in which it can be applied. In this part of the thesis, a canard manifold is simply a union of two center manifolds, restricting parameters to a control manifold in parameter space: we show the existence (for the complete parameter space) of two classes of center manifolds: center manifolds along the attracting branch of the critical curve and center manifolds along the repelling branch. Both classes have an intersecting domain of existence, and in this intersection they are compared. One creates a canard manifold by restricting the parameter space to a manifold where both center manifolds are equal in this intersecting domain.

In a second part, turning point problems are addressed in an asymptotic way. Asymptotic turning point theory in two dimensions has been developed quite extensively, and fairly complete results have been obtained before (both local and global results) using complex analytic techniques. The extension we present is a study of turning points where standard asymptotic methods fail to work, simply because the corresponding canard manifolds lack an asymptotic expansion in terms of the original variables. We show how the geometric blow up tool can be used to find asymptotic expansions in blown up variables, and apply the asymptotic theory to this new context. The benefit over the traditional geometric method is a better characterization of the control curve: in an analytic context we show that the control curve has an asymptotic expansion of Gevrey order. The Gevrey-order of this control curve can be readily obtained from the differential equations, whereas the Gevrey-type of the curve is more difficult to obtain. In the thesis, we show how to give bounds on the Gevrey type of the control curve, by calculating integrals of the divergence of the vector field along well-chosen paths.

Using both tools, it is possible to examine vector fields without a control parameter, and give precise conditions under which canard manifolds exist. The geometric method provides an infinite sequence of necessary conditions that have to be satisfied; this sequence can be calculated by means of Melnikov integrals. The sufficiency of these conditions follows from the Gevrey estimates of the control curves, and the length of the maximal canards is related to the Gevrey type of the control curve; more precisely an upper estimate for the Gevrey type allows to give a lower estimate on the length of the maximal canards. This last part is not done in full generality, but the ideas are shown on an example.

In chapter 1, we give an outline of the main tools that are used in both parts of the study. We give on one hand a review of the notion of "blow up", and on the other hand we state some general definitions of Gevrey-expansions.

In chapter 2, we treat the normally hyperbolic passage along the critical curve, i.e. the passage before and after the turning point. This part is treated in full generality; some efforts have been made to handle non-smooth boundary conditions. The normally hyperbolic passage has an analogue in the second part of the thesis, strongly based on a result of Sibuya.

In chapter 3, we apply the geometric method to the turning point problem. We show that the invariant manifolds that have been obtained in chapter 2 can be extended to the turning point, at least in blow up space. We show how these "center manifolds" can be compared in a family rescaling chart of the blown up turning point, and prove the existence of control curves along which attracting and repelling center manifolds can be "matched". We derive a mechanism to calculate the formal expansion for this control curve. Finally, we provide a link to asymptotic theory, in the sense that if a traditional asymptotic expansion exists for the canard manifolds, then this asymptotic expansion correspond, after blow up, to the asymptotic expansion of the canard manifolds. Under this condition, we show that the canard manifolds can be blown down to a smooth manifold at the turning point (where in general, the canard manifolds are only $C^0$ at the turning point).

In chapter 4, we examine the transition time of orbits inside canard manifolds. We aim on future applications to boundary value problems. More specifically, we study the monotonicity of the transition time, which is important in determining unicity of solutions to boundary value problems.

In chapter 5, our aim is to derive an entry–exit relation along given control curves. The main tool is a study of the divergence integral. We show that any two different canard manifolds are exponentially close to each other, and determine precisely how close.

In chapter 6, the analytic theory is presented. In an analytic framework, it is examined at what steps the results of the previous chapters can be improved. We show the analyticity of canard manifolds in complex sectors in $\mathbf{C}$, and show the analyticity of the control curve in complex sectors. Furthermore, for the control curve Gevrey estimates will be derived, with an estimate for the Gevrey type as well. The technique will be applied to the van der Pol equation, and is general enough to be applied to other singularly perturbed equations.

# Contents

# Chapter 1

# Prerequisites

In this introductory chapter we have collected some known facts in the context of vector fields and singular perturbations; those facts will be useful in further chapters. A notion that is very important throughout my thesis is "family blow up". This technique of desingularizing degenerate singular points of vector fields will form a corner stone in the proof of most results.

## 1.1 Background on blow up

In this section we recall the notion of "blow up" and "family blow up". The digression is kept short; for a complete reference on the subject, we suggest reading [D] and [DR3].

At first, a blow up map can be thought of as a singular change of coordinates, like the polar-coordinate mapping

$$(u, \theta) \mapsto (x, y) = (u\cos\theta, u\sin\theta).$$

Another way to denote this map is $(u, (\overline{x}, \overline{y})) \mapsto (u\overline{x}, u\overline{y})$, where $\overline{x}^2 + \overline{y}^2 = 1$. Now, instead of using coordinates on the circle, we can use charts to rectify parts of the circle. In the above example, near $\overline{x} = 1$, we might as well use the coordinate change $(u, \tilde{y}) \mapsto (x, y) = (u, u\tilde{y})$, with $\tilde{y}$ in a fixed domain. On this new chart, the point $\tilde{y} = \pm\infty$ corresponds to $(\overline{x}, \overline{y}) = (0, \pm 1)$. In order not to exaggerate in notations, one often uses the symbol $\overline{y}$ for $\tilde{y}$, and one says to work in the $\{\overline{x} = 1\}$ chart.

A vector field $X$ can be pulled back under the blow up map, and if the origin is a singularity of the original vector field $X$, then the locus $\{u = 0\}$ will be an invariant set of the pull-back vector field, often even a set of singularities. In the latter case, we desingularize the new vector field by dividing out a positive factor $u^\alpha$.

Although desingularization by means of blow up, as the one above, is practical, it can be made more useful by adapting the exponents to the problem under study.

If instead of the homogeneous blow up we use for example $(x, y) = (u^2\overline{x}, u\overline{y})$, with $\overline{x}^2 + \overline{y}^2 = 1$, or even with $\overline{x}^2 + \overline{y}^4 = 1$ if advantageous, we sometimes get better results. This generalization gives us the possibility of assigning weights to all the variables, and the weights will in practice be chosen in a way to reach the best desingularization.

An extra possibility when blowing up singularly perturbed differential equations is to include $\epsilon$ in the list of variables, coming to the notion of *family blow up.*

If we have a family of 3-dimensional vector fields $X_{\epsilon,\lambda} + 0\frac{\partial}{\partial\epsilon}$, and we want to blow up the origin $(x, y, \epsilon) = (0, 0, 0)$, then we use:

$$\left\{\begin{array}{rcl} x &=& u^p\overline{x} \\ y &=& u^q\overline{y} \\ \epsilon &=& u^m\overline{\varepsilon} \end{array}\right. \tag{1.1}$$

with $(\overline{x}, \overline{y}, \overline{\varepsilon}) \in S^2$, and $u \in \mathbf{R}^+$. The weights $p$, $q$ and $m$ are chosen differently for different systems. The best choice can be evident and found without problem or can be based on the use of Newton polyhedra—see e.g. [D].

On the 2-sphere, we have the relation $\overline{x}^2 + \overline{y}^2 + \overline{\varepsilon}^2 = 1$, often implying the need of working in charts.

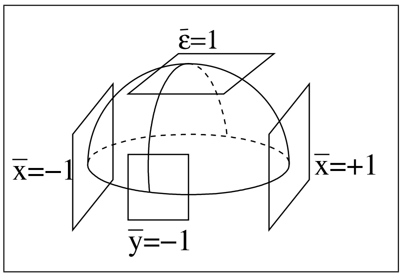


Figure 1.1: Different charts

The dynamics of the vector field can be studied in several regions separately. Keeping $\overline{x}$ near $-1$ on the 2-sphere means that $\overline{\varepsilon}$ is kept small. In traditional coordinates, one investigates the region where $-\epsilon^p/x^m$ is small but positive. One uses the chart

$$\left\{\begin{array}{rcl} x &=& -u^p \\ y &=& u^q\overline{y} \\ \epsilon &=& u^m\overline{\varepsilon} \end{array}\right. \tag{1.2}$$

This chart is valid for $(\overline{y}, \overline{\varepsilon})$ small, and is called a *phase-directional rescaling chart*, or simply *the* $\{\overline{x} = -1\}$ *chart.* Notice that the trivial invariant foliation $d\epsilon = 0$ is replaced here by $d(u^m\overline{\varepsilon}) = 0$.

Orbits of vector fields in this space will respect the foliation $d(u^m\overline{\varepsilon}) = 0$, so as $u$ decreases, $\overline{\varepsilon}$ will increase. Continuing the orbits, we will eventually need to enter a region where $\overline{\varepsilon}$ is no longer small, and where we can bound $|\overline{x}|$ away from 1. This part is visible in "the chart $\{\overline{\varepsilon}=1\}$", commonly known as the *family rescaling chart*, and the formulas to work in this chart are:

$$\left\{\begin{array}{lcl} x & = & u^p\overline{x} \\ y & = & u^q\overline{y} \\ \epsilon & = & u^m \end{array}\right. \tag{1.3}$$

This chart is valid for $(\overline{x},\overline{y})$ in a bounded set. In this chart, $u$ is clearly the singular parameter, and we again have a family of vector fields (since $du=0$). This is the traditional chart where people do "rescaling" in.

Observe that the $\overline{y}$ coordinate in (1.3) is not the same as $\overline{y}$ in (1.2), but intuitively they serve a common purpose, in the sense that they are both a rescaled form of the same $y$ coordinate. It is of course easy to give formulas for the relation between the two expressions of $\overline{y}$.

As $\overline{x}$ gets closer to $+1$, $\overline{\varepsilon}$ gets closer to 0, so we will have to leave the $\overline{\varepsilon}=1$ chart. Like in the first part, we study this section using the chart

$$\left\{\begin{array}{lcl} x & = & +u^p \\ y & = & u^q\overline{y} \\ \epsilon & = & u^m\overline{\varepsilon} \end{array}\right. \tag{1.4}$$

The whole process can be depending on the extra parameters $\lambda$, which we do not blow up. If necessary, one can rescale the extra parameters prior to applying the blow up maps as described above. For a concrete example, see below.

*Important notice: The coordinate changes from one chart to another are analytic in the intersection of their domains of validity!*

## Concrete example

Sometimes it is necessary to include extra parameters in the rescaling to get a good desingularization. Using a simple example, we want to show that this can be done in two ways: one can alter the blow up map to include extra parameters, or one can rescale the extra parameters prior to applying the blow up map.

As a concrete example, that will be used as a reference later on as well, we present a blow up of the family of vector fields

$$X_{\epsilon,a}\colon \left\{\begin{array}{lcl} \dot{x} & = & y-\frac{x^{2n}}{(2n)!}(1+xf(x,y,\epsilon,a)) \\ \dot{y} & = & \epsilon(a-x^{2n-1}) \end{array}\right. \tag{1.5}$$

Let us first consider the subfamily $X_{\epsilon,0}$ as a vector field on $\mathbf{R}^2\times[0,\epsilon_0[$ and use the blow up map

$$\Phi\colon S^2\times\mathbf{R}^+\to\mathbf{R}^3\colon((\overline{x},\overline{y},\overline{\varepsilon}),u)\mapsto(x,y,\epsilon)=(u\overline{x},u^{2n}\overline{y},u^{2n}\overline{\varepsilon}).$$

In the family rescaling chart $\{\overline{\varepsilon} = 1\}$ one finds

$$\left\{ \begin{array}{rcl} u\dot{\overline{x}} & = & u^{2n}(\overline{y} - \frac{\overline{x}^{2n}}{(2n)!}(1 + u\overline{x}f(u\overline{x}, u^{2n}\overline{y}, u^{2n}\overline{\varepsilon}, 0)) \\ u^{2n}\dot{\overline{y}} & = & u^{2n}(0 - u^{2n-1}\overline{x}^{2n-1}) \\ \dot{u} & = & 0 \end{array} \right.$$

After a division by the common factor $u^\alpha := u^{2n-1}$, one gets the blown-up family of vector fields

$$\overline{X}_{u,0}\colon \left\{ \begin{array}{rcl} \dot{\overline{x}} & = & \overline{y} - \frac{\overline{x}^{2n}}{(2n)!} + O(u) \\ \dot{\overline{y}} & = & -\overline{x}^{2n-1} \end{array} \right.$$

When applying the same blow up map (and blow up division) to the complete family $X_{\epsilon,a}$, the blown up vector field yields

$$\overline{X}_{u,a}\colon \left\{ \begin{array}{rcl} \dot{\overline{x}} & = & \overline{y} - \frac{\overline{x}^{2n}}{(2n)!} + O(u) \\ \dot{\overline{y}} & = & \frac{a}{u^{2n-1}} - \overline{x}^{2n-1} \end{array} \right.$$

The lack of regularity at $u = 0$ is a problem that can be countered with two methods: one can write $a = u^{2n-1}A = \epsilon^{(2n-1)/2n}A$, keeping $A$ in a compact set; in that way we restrict parameter space $\{(\epsilon, a)\}$ to $\{(\epsilon, \epsilon^{(2n-1)/2n}A)\colon A \in [-A_0, A_0]\}$. This essentially comes down to altering the blow up map $\Phi$ to

$$\overline{\Phi}\colon S^3 \times \mathbf{R}^+ \to \mathbf{R}^4\colon ((\overline{x}, \overline{y}, \overline{\varepsilon}, \overline{a}), u) \mapsto (x, y, \epsilon, a) = (u\overline{x}, u^{2n}\overline{y}, u^{2n}\overline{\varepsilon}, u^{2n-1}\overline{a}).$$

In the chart $\{\overline{\varepsilon} = 1\}$, one again finds $a = \epsilon^{(2n-1)/2n}\overline{a}$.

Another way of getting a good desingularization is to perform a blow up of the parameter plane prior to blowing up the vector field (this is the method that will be pushed forward throughout the remainder of the thesis): consider again the initial example (1.5), and write

$$(\epsilon, a) = (v^{2n}, v^{2n-1}A). \tag{1.6}$$

The vector field in terms of the parameters $(v, A)$ yields

$$\tilde{X}_{v,A}\colon \left\{ \begin{array}{rcl} \dot{x} & = & y - \frac{x^{2n}}{(2n)!}(1 + xf(x, y, v^{2n}, v^{2n-1}A)) \\ \dot{y} & = & v^{2n}(v^{2n-1} - x^{2n-1}) \end{array} \right.$$

One can blow up this latter family using the blow up map

$$\tilde{\Phi}\colon S^2 \times \mathbf{R}^+\colon ((\overline{x}, \overline{y}, \overline{v}), u) \mapsto (x, y, v) = (u\overline{x}, u^{2n}\overline{y}, u\overline{v}).$$

and it is clear that there is no longer a need to include $A$ in this blow up. Note also that in this last blow up, one can set the weight of the singular parameter $v$ to 1, because the rescaling of this parameter was in fact performed prior to applying the blow up map.

Let us finally remark that the rescaling (1.6) is in fact a blow up of parameter space $S^1 \times \mathbf{R}^+ \to \mathbf{R}^2\colon ((E,A),v) \mapsto (\epsilon, a) = (v^{2n}E, v^{2n-1}A)$, where we have restricted to the chart $\{E=1\}$. To study the complete parameter plane, one also has to consider the charts $\{A=\pm 1\}$. We refer to [DR], where the Van der Pol system is studied this way.

## 1.2 Definition of Gevrey asymptotics

In chapter 6, we study the asymptotic expansions of one-dimensional center manifolds in the plane, and of two-dimensional center manifolds and canard manifolds. It is by now well-known that such expansions satisfy Gevrey estimates, when working with analytic vector fields. These Gevrey estimates reflect the fact that an asymptotic expansion defines the actual manifold up to exponentially small terms. In some occasions, the asymptotic expansion is not only Gevrey, but also *summable.* To a summable series a unique resummated manifold can be associated. In the plane, we give some situations where we can expect the formal series to be summable. The definitions of Gevrey series and summability are not uniform throughout the literature. We have inspired ourselves on the well written appendix in the thesis of M. Canalis-Durand for these definitions, which in turn was based on works of Balser, CandelPergher, Malgrange, Martinet, Ramis, Schäfke, Sibuya and Tougeron. We also present a proof of an implicit function theorem for Gevrey functions, which can be found in the literature as well but which we have included for the sake of convenience.

**Definition 1.1** *A formal power series $\hat{a}(\epsilon) = \sum_{n=0}^{\infty} a_n \epsilon^n$ is* Gevrey-$1/\sigma$ *in $\epsilon$ of type $A$, if there exist positive constants $C, \alpha$ such that*

$$|a_n| \leq C A^{n/\sigma} \Gamma(\alpha + n/\sigma),$$

*where $\Gamma$ is the well-known Gamma function. We define $1/\sigma$ as the* Gevrey order *of such a series.*

The set of formal power series satisfying Gevrey-$1/\sigma$ type $A$ estimates is closed under addition, multiplication (Cauchy product) and derivation.

Similar estimates as for Gevrey formal power series can be put on analytic functions in sectors, leading to the notion of a Gevrey function. We recall what is understood under a complex sector:

**Definition 1.2** *A complex sector $S_{r,\alpha,\theta}$ with vertex $0$ is an open subset of $\mathbf{C}$:*

$$S_{r,\alpha,\theta} = \{z \in \mathbf{C}\colon Arg(z) \in (\alpha - \theta, \alpha + \theta), 0 < |z| < r\}.$$

*The* opening angle *of the sector is defined as $2\theta$.*

*A subsector $S_{r',\alpha',\theta'}$ of $S_{r,\alpha,\theta}$ is a sector for which $r' < r$ and $[\alpha' - \theta', \alpha' + \theta'] \subset\, ]\alpha - \theta, \alpha + \theta[$.*

**Definition 1.3** *Let $S$ be such a sector and $a\colon S \to \mathbf{C}$ be an analytic function that is continuously extendable to $\partial S$. The function $a$ is Gevrey-$1/\sigma$ in $\epsilon$ of type $T$, if there exists a sequence $(a_n)_{n\in\mathbf{N}}$, if there exists a positive $\alpha$, and if for every subsector $S'$ there exists a positive constant $C_{S'}$ such that*

$$\left| a(\epsilon) - \sum_{i=0}^{n-1} a_i \epsilon^i \right| \leq C_{S'} T^{n/\sigma} \Gamma(\alpha + n/\sigma) |\epsilon|^n, \qquad \forall n \in \mathbf{N}_1, \forall \epsilon \in S',$$

As with formal power series, the set of functions on $S$ that are Gevrey-$1/\sigma$ of type $T$ is closed under addition, multiplication and derivation. Also with respect to substitution of a Gevrey function inside an analytic function, some results can be shown, see theorem 1.10.

**Lemma 1.4** *A Gevrey-$1/\sigma$ function $a(\epsilon)$ is automatically $C^\infty$ extensible to the origin, and its Taylor coefficients are equal to $(a_n)_{n\in\mathbf{N}}$.*

**Proof** This follows from Cauchy's estimate on complex functions. □

The following theorem is due to Malgrange and Ramis:

**Theorem 1.5** *(Borel-Ritt Gevrey) Let $\hat{f}$ be a formal power series that is Gevrey-$1/\sigma$ of type $A$, and let $S = S_{r,\alpha,\theta}$ be a complex sector with opening angle $2\theta < \pi/\sigma$. Then there exists an analytic and bounded function*

$$f\colon S \to \mathbf{C}$$

*so that $\hat{f}$ is Gevrey-$1/\sigma$ asymptotic to $\hat{f}$ of type $T := A/\cos^\sigma(\sigma\theta)$.*

**Remarks:**

1. If the coefficients of $\hat{f}$ are real, and if $\alpha = 0$ (i.e. the sector is a sector containing the positive real axis), then $f$ can be chosen *real analytic* (because the function can then be obtained as an integral over a compact real path).

2. It is clear that as $\theta \to 0$ the Gevrey type $T$ will approximate the formal Gevrey type $A$.

**Lemma 1.6** *Let $a(\epsilon)$ be a Gevrey-$1/\sigma$ function of type $T$ in $\epsilon$ in a sector $S$, and let $C \neq 0$. Then, $a(C\epsilon)$ is a Gevrey-$1/\sigma$ of type $T'$ in $\epsilon$, with $T' = |C|^\sigma T$. An equivalent statement holds for Gevrey series.*

**Lemma 1.7** *Let $a(\epsilon)$ be a Gevrey-$1/\sigma$ function of type $T$ in $\epsilon$ in a sector $S$. Let $\epsilon = u^m$ for some $m \in \mathbf{N}_1$. Then, $a(u^m)$ is Gevrey-$1/m\sigma$ of type $T$ in $u$ in a sector $S' = S^{1/m}$ (the preimage of $S$ under $u \mapsto u^m$). An equivalent statement holds for Gevrey series.*

**Lemma 1.8** *Let $a(\epsilon)$ be Gevrey-$1/\sigma$ of type $T$ in $\epsilon$ in a sector $S$ with opening angle less than $2\pi/m$. Let $u = \epsilon^m$ for some $m \in \mathbf{N}_1$. If the Taylor coefficients of $a$ only contain powers of $\epsilon^m$, then $a(u^{1/m})$ is a Gevrey-$m/\sigma$ function in $u$ (of type $T$) in a sector $S' = S^m$ (image of $S$ under $\epsilon \mapsto \epsilon^m$). An equivalent statement holds for Gevrey series.*

**Lemma 1.9** *Let $\sigma \in \mathbf{N}_1$, and let $a(\epsilon)$ be a Gevrey-$1/\sigma$ function of type $T$. Then there exist functions $a_{(0)}, \ldots, a_{(\sigma-1)}$ that are Gevrey-1 of type $T$ in their arguments and*

$$a(\epsilon) = \sum_{i=0}^{\sigma-1} a_{(i)}(\epsilon^\sigma)\epsilon^i.$$

*The formal expansions of $a_{(i)}$ are unique. An equivalent statement holds for Gevrey series.*

**Theorem 1.10** *[CRSS] (Gevrey substitution theorem) Consider a domain $D \subset \mathbf{C}$, an open set $M \subset D \times \mathbf{C}^n$ and a sector $S \subset \mathbf{C}$. Suppose that $F\colon M \times S \to \mathbf{C}^\ell$ and $g\colon D \times S \to \mathbf{C}^n$ are holomorphic functions such that $(x, g(x, \epsilon)) \in M$ for all $x \in D$, $\epsilon \in S$. Assume that $F(x, z, \epsilon)$ and $g(x, \epsilon)$ have asymptotic expansions of Gevrey order $1/\sigma$ and some type $T < T'$ as $\epsilon \to 0$, $\epsilon \in S$, uniformly for $(x, z) \in M$ resp. $x \in D$. Then, $v\colon D \times S \to \mathbf{C}^\ell$ defined by*

$$v(x, \epsilon) = F(x, g(x, \epsilon), \epsilon)$$

*has an asymptotic expansion of Gevrey order $1/\sigma$ and type $T'$ as $\epsilon \to 0$, $\epsilon \in S$, uniformly for $x$ in any compact subset of $D$.*

**Corollary 1.11** *(Gevrey curves following real orbits) Let $X_\epsilon$ be a real analytic family of vector fields on $\mathbf{R}^2$, and let $\Gamma$ be a real orbit of the unperturbed vector field $X_0$. Let $(x_0(\epsilon), y_0(\epsilon)) \in \mathbf{R}^2$ stand for a curve of initial conditions with $(x_0(0), y_0(0)) \in \Gamma$. Consider now an analytic section $E$ inside $\mathbf{R}^2 \times [0, \epsilon_0[$ intersecting $\Gamma \times \{0\}$ transversally. If $x_0(\epsilon)$, $y_0(\epsilon)$ are two Gevrey-$1/\sigma$ functions in $S_{r,\alpha,\theta}$ of type $T < T'$, then intersecting the orbits through $(x_0(\epsilon), y_0(\epsilon))$ with the section $E$ gives a graph in $\epsilon$ that is Gevrey-$1/\sigma$ of type $T'$ in a sector $S_{r',\alpha,\theta}$, for some $0 < r' < r$.*

**Proof** The transition map towards the plane $E$ is an analytic mapping

$$F\colon (x_0, y_0, \epsilon) \mapsto F(x_0, y_0, \epsilon),$$

that can be complexified in a neighbourhood of $(x, y, \epsilon) = (x_0(0), y_0(0), 0)$. Calculating the intersection of an orbit at "height" $\epsilon$ with the section $E$ yields a point $F(x_0(\epsilon), y_0(\epsilon), \epsilon)$, which defines a Gevrey-$1/\sigma$ curve (in the sense that in analytic coordinates of $\mathbf{R}^2$ it is a graph in $\epsilon$, and the coefficient functions of this graph are Gevrey-$1/\sigma$), due to the previous theorem. $\square$

The following proposition will be very useful:

**Proposition 1.12** *Let $a(\epsilon)$ be a Gevrey-$1/\sigma$ function of type $T$ in a complex sector $S_{r,\alpha,\theta}$. Assume that $a(\epsilon)$ is asymptotic to 0, then there is a $\rho \in \mathbf{Z}$ so that for all subsectors*

$$|a(\epsilon)| \leq C_{S'}|\epsilon|^{\rho} \exp(-1/T|\epsilon|^{\sigma}), \qquad \forall \epsilon \in S',$$

*for some constant $C_{S'} > 0$. The converse is also true: if the above estimate is true for all subsectors $S'$ of $S$, then $a$ is Gevrey-$1/\sigma$ asymptotic to 0 of type $T$.*

### 1.2.1 Gevrey functions and sectorial coverings

When a function is Gevrey-$1/\sigma$, then this can be characterized using asymptotic expansions, as above. However, there is another characterization, using sectorial coverings and "function chains". Roughly the idea is the following. Have in mind a pinced neighbourhood $B(0,r) \setminus \{0\} \subset \mathbf{C}$ that is covered by a finite number of sectors, and on top of each sector an analytic function. If the difference between any two such functions (in the intersection of their domains) is exponentially small, then all these functions are Gevrey (compare with the situation where the difference is 0, from which would follow that these functions are analytic). Let us now make the idea more explicit.

**Definition 1.13** *A good sectorial covering of the origin in $\mathbf{C}$ is a finite (ordered) number of complex sectors $S_j := S_{r,\alpha_j,\theta_j}$, $j = 1, \ldots, n$, so that the following holds:*

*(i)* $\cup_{j=1}^{n} S_j = B(0,r) \setminus \{0\}$

*(ii)* $2\theta_j < \pi$ *for all $j = 1, \ldots, n$ (i.e. no sector has an opening angle wider than $\pi$).*

*(iii) if $|j-k| = 1$ or $(j,k) = (1,n)$ or $(j,k) = (n,1)$ then and only then is $S_j \cap S_k$ nonempty. In that case, we call $S_j$ and $S_k$ adjacent sectors.*

The following proposition makes it possible for a given Gevrey function $f\colon S \to \mathbf{C}$ on a sector $S$ to analytically "extend" the definition on a full neighbourhood of the origin, making a finite number of at most exponentially small jumps:

**Proposition 1.14** *Let $S$ be a complex sector of the origin and $f\colon S \to \mathbf{C}$ be analytic and Gevrey-$1/\sigma$ asymptotic (of type $T$) to some formal power series $\hat{f}$. Let $\tilde{T} > T$ be fixed. Then:*

*(i) There exists a good covering $(S_j)_{j=1,\ldots,n}$ and a sequence of functions $(f_j)_j$ with $f_j\colon S_j \to \mathbf{C}$ analytic and bounded and $(f_1, S_1) = (f, S)$, and all $f_j$ have the following properties:*

*(ii) The functions $f_j$ are also Gevrey-$1/\sigma$ asymptotic (of type $\tilde{T}$) to $\hat{f}$, and:*

*(iii) There is a $C > 0$ so that for all two adjacent sectors $(S_j, S_k)$ one has*

$$|f_j(\epsilon) - f_k(\epsilon)| \leq C \exp\left(-1/\tilde{T}|\epsilon|^\sigma\right), \qquad \forall \epsilon \in S_j \cap S_k.$$

Of course, there is a formal series version of this proposition, by combining it with the theorem of Borel-Ritt Gevrey (theorem 1.5).

The following is the inverse of this proposition, and is a key result in the proof of many results, such as in the proof of a Gevrey implicit function theorem (see below):

**Theorem 1.15** *(Ramis-Sibuya) Let $(S_j)_{j=1,\ldots,n}$ be a good sectorial covering and let $f_j : S_j \to \mathbf{C}$ be analytic and bounded. Suppose that for all two adjacent sectors $(S_j, S_k)$ we have*

$$|f_j(\epsilon) - f_k(\epsilon)| = O\left(\exp\left(-1/T|\epsilon|^\sigma\right)\right), \qquad \forall \epsilon \in S_j \cap S_k, \qquad \text{as } \epsilon \to 0$$

*for some $T > 0$. Then, all functions $f_j$ are Gevrey-$1/\sigma$ asymptotic of type $T$ inside $S_j$ to a common formal power series $\hat{f}$.*

A proof of this theorem can be found in [RA], [SI1] and [SI2].

It is important to notice that for the theorem of Ramis-Sibuya to apply one need not prove that $f_j$ have an asymptotic expansion as $\epsilon \to 0$. Even if one is not interested in Gevrey asymptotics, this theorem gives a method of proving the $C^\infty$ smoothness of functions, by merely showing the boundedness and forming a chain of functions.

An application of this theorem is a proof of theorem 1.10, following a technique that is basically similar to the one in the proof of theorem 1.16 below.

### 1.2.2 Gevrey implicit function theorem

**Theorem 1.16** *Let $F(x, y, \epsilon)$ be an analytic function $M \times S \to \mathbf{C}^\ell$, with $S$ a sector of the origin, $M = M_x \times M_y$ being a subset of $\mathbf{C}^k \times \mathbf{C}^\ell$ ($M_y$ must be convex), and let $F$ have a continuous extension to $M \times \overline{S}$. Assume that $F$ is uniformly Gevrey-$1/\sigma$ of type $T$ in $\epsilon$. Assume furthermore that*

$$F(x, y_0, 0) = 0, \qquad \left(\det \frac{\partial F}{\partial y}\right)(x, y_0, 0) \neq 0.$$

*Then, there exists a unique function*

$$y = g(x, \epsilon)$$

*that is analytic on $M_x \times S'$ for some proper subsector $S'$ of $S$, and for which*

$$g(x, 0) = y_0, \qquad F(x, g(x, \epsilon), \epsilon) = 0, \quad \forall (x, \epsilon) \in M_x \times S'.$$

*The function $g$ is Gevrey-$1/\sigma$ asymptotic as $\epsilon \to 0$, $\epsilon \in S'$ of type $\tilde{T}$ for any $\tilde{T} > T$.*

**Proof** The existence of such a $g$ and the analyticity inside $S'$ (it is possible that one needs to decrease the radius $r$ of $S$) follows from standard implicit function theorems. We only prove here the properties regarding the asymptotic expansion. Let $S_1 = S$, ..., $S_N$ be a good sectorial covering of the origin, and choose $F_1 = F$, ..., $F_N$ a function chain so that $F_i$ is analytic in $M \times S_i$ and

$$F_i - F_{i+1} = O(\exp(-1/\tilde{T}|\epsilon|^\sigma)), \qquad \text{as } \epsilon \to 0, \epsilon \in S_i \cap S_{i+1}$$

uniformly for $(x, y) \in M$ (we have applied a parametric version of proposition 1.14 for this result). Apply in each $S_i$ the implicit function theorem to find a function $g_i\colon M_x \times S_i' \to \mathbf{C}^\ell$, where $S_i'$ is a subsector of $S_i$ (reduced radius). We show that

$$g_i - g_{i+1} = O(\exp(-1/\tilde{T}|\epsilon|^\sigma)), \qquad \text{as } \epsilon \to 0, \epsilon \in S_i' \cap S_{i+1}',$$

uniformly for $x \in M_x$. Let us prove this for $i = 1$. We restrict $\epsilon$ to $S_1 \cap S_2$. Because

$$0 = F_1(x, g_1(x, \epsilon), \epsilon) - F_2(x, g_2(x, \epsilon), \epsilon),$$

one finds

$$0 = (F_1(x, g_1(x, \epsilon), \epsilon) - F_1(x, g_2(x, \epsilon), \epsilon)) + (F_1 - F_2)(x, g_2(x, \epsilon), \epsilon)$$

One uses the mean value theorem to obtain

$$F_1(x, g_1(x, \epsilon), \epsilon) - F_1(x, g_2(x, \epsilon), \epsilon) = H(x, \epsilon)(g_1(x, \epsilon) - g_2(x, \epsilon)),$$

where

$$H(x, \epsilon) := \int_0^1 \frac{\partial F_1}{\partial y}(x, g_2(x, \epsilon) + s(g_1(x, \epsilon) - g_2(x, \epsilon)), \epsilon) ds.$$

We know that $g_i(x, \epsilon) = y_0 + O(\epsilon)$, and hence $H(x, \epsilon) = \frac{\partial F_1}{\partial y}(x, y_0, 0) + O(\epsilon)$. Since the righthand side is invertible one also has that $H(x, \epsilon)$ is invertible for $|\epsilon|$ small enough. We conclude:

$$g_1(x, \epsilon) - g_2(x, \epsilon) = \frac{1}{H(x, \epsilon)}(F_2 - F_1)(x, g_2(x, \epsilon), \epsilon).$$

Bounding the norm of $1/H(x, \epsilon)$ by a constant, one sees that $g_1 - g_2$ is exponentially small because $F_2 - F_1$ is exponentially small. Finish the proof by applying the theorem of Ramis-Sibuya (theorem 1.15) on the function chain $\{g_j\}$. $\square$

Note that the resulting implicit solution is Gevrey-$1/\sigma$ of type $\tilde{T}$, for any $\tilde{T} > T$. This is not necessarily the same as being of type $T$, but the difference is subtle. In any case, one could define a class of functions that are Gevrey-$1/\sigma$ of any type strictly larger than $T$. This new class will still be closed for addition, multiplication etc.

## 1.3 Divergence of vector fields

In this section, we state some basic results regarding the divergence of a vector field. We will use it in chapter 5 to study transition maps in singularly perturbed vector fields.

Throughout this section, $(M, g)$ is an $n$-dimensional Riemannian manifold (with metric $g$), and associated to $(M, g)$ there is a volume form $\Omega$.

**Definition 1.17** *Let $\Omega$ be a volume form on $M$ (associated to a metric $g$). The divergence of $X_\lambda$ with respect to $\Omega$ is the function $M \to \mathbf{R}$ so that*

$$d(X_\lambda \lrcorner \Omega) = \operatorname{div} X_\lambda \Omega.$$

In case $M = \mathbf{R}^n$ and the metric is the standard Euclidian metric, one has

$$\Omega = dx_1 \wedge \cdots \wedge dx_n,$$

and

$$\operatorname{div}\left(\sum_{i=1}^{n} f_i(x) \frac{\partial}{\partial x_i}\right) = \sum_{i=1}^{n} \frac{\partial f_i}{\partial x_i}(x).$$

All our results will depend on the chosen volume form, but this coordinate free way of determining the divergence is a convenient way of showing that integrals of divergence of vector fields depend on the chosen coordinates, but in a nice way:

**Lemma 1.18** *Let $\Omega$ and $\Omega'$ be two equivalent volume forms, i.e. $\Omega' = f(p, \lambda)\Omega$ for some smooth nonzero function $f$ on $M \times \Lambda$. Then,*

$$\operatorname{div}_{\Omega'} X_\lambda = \operatorname{div}_\Omega X_\lambda + \frac{X_\lambda(f)}{f}.$$

*Furthermore, let $J$ be an integral curve of $X_\lambda$ parametrized by the time $t \in [0, \tau]$, then*

$$\int_0^\tau \operatorname{div}_{\Omega'} X_\lambda \, dt = \int_0^\tau \operatorname{div}_\Omega X_\lambda \, dt + \log\left(\frac{f(q)}{f(p)}\right),$$

*where $p$ resp. $q$ is the point of $J$ at time $t = 0$ resp. $t = \tau$. (In particular, along a periodic orbit the integral does not depend on the chosen volume form.)*

**Proof** Elementary. $\square$

**Lemma 1.19** *Let $\Omega$ be a volume form, and let $h(p,\lambda)X_\lambda$ be an equivalent vector field (i.e. $h$ is a strictly positive function). Then,*

$$\operatorname{div}_\Omega(hX_\lambda) = h\operatorname{div}_\Omega X_\lambda + X_\lambda(h).$$

*Furthermore, let $J$ be an integral curve of $X_\lambda$ parametrized by the time $t \in [0,\tau]$, then*

$$\int_0^{\tau'} \operatorname{div}_\Omega(hX_\lambda)\,dt' = \int_0^{\tau} \operatorname{div}_\Omega X_\lambda\,dt + \log\left(\frac{h(q)}{h(p)}\right),$$

*where $p$ resp. $q$ is the points of $J$ at time $t=0$ resp. $t=\tau$, and where $t' = t/h$. (In particular, along a periodic orbit the integral does not depend on the chosen time parametrization.)*

**Proof** Elementary. □

## Geometric significance of the divergence

The following proposition is well-known, but we repeat a proof for the sake of completeness:

**Proposition 1.20** *Let $X$ be a vector field on an open set of $\mathbf{R}^n$. Let $S$, $S'$ be two open sections of $\mathbf{R}^n$, transverse to the flow of $X$. Assume $p \in S$, $q \in S'$ and the orbit through $p$ reaches $q$ in finite time (positive or negative). Let $T\colon S_0 \subset S \to S'$ be the transition map defined in a neighbourhood of $p$. If $\psi_i\colon U_i \to \Sigma_i$ are coordinates for $\Sigma_i$ with $U_i \subset \mathbf{R}^{n-1}$, then*

$$\det(D(\psi_2^{-1}\circ T\circ\psi_1))(s_1) = \frac{\det(D\psi_1(s_1)|X(p))}{\det(D\psi_2(s_2)|X(q))}\exp\left(\int_{O(p,q)}\operatorname{div} X\,dt\right),$$

*where $s_1 = \psi_1^{-1}(p)$, $s_2 = \psi_2^{-1}(q)$, and where $(D\psi_1(s_1)|X(p))$ is a matrix composed of the $n\times(n-1)$ matrix $D\psi_1(s_1)$ and the column vector $X(p)$, and similarly for $(D\psi_2(s_2)|X(q))$. The integral is taken over the orbit $O(p,q)$ from $p$ to $q$ parametrized by $t$.*

**Proof** Let $\varphi(t,x)$ be the flow of $X$ through $x$ at $t=0$. Define

$$\theta(s,s',t) = \varphi(t,\psi_1(s)) - \psi_2(s').$$

If $\tau$ is the transition time from $p$ to $q$, then $\theta$ has a zero at $(s_1,s_2,\tau)$. Furthermore

$$\left\{\begin{array}{rcl} D_s\theta(s,s',t) &=& D_x\varphi(t,\psi_1(s))D\psi_1(s),\\ D_{s'}\theta(s,s',t) &=& -D\psi_2(s'),\\ D_t\theta(s,s',t) &=& X(\varphi(t,\psi_1(s))). \end{array}\right.$$

At $(s_1, s_2, \tau)$ one finds

$$\begin{cases} D_s\theta(s_1, s_2, \tau) &=& D_x\varphi(\tau, p)D\psi_1(s_1) =: A, \\ D_{s'}\theta(s_1, s_2, \tau) &=& -D\psi_2(s_2) =: -B, \\ D_t\theta(s_1, s_2, \tau) &=& X(q). \end{cases}$$

Because of the transversality, $\det(D\psi_2(s_2)|X(q))$ is nonzero, we can apply the implicit function theorem to find a $\zeta(s)$, $\tau(s)$ such that $\zeta_2(s_1) = s_2$, $\tau(s_1) = \tau$, and

$$\theta(s, \zeta(s), \tau(s) = 0, \qquad \text{for all } s \text{ near } s_1.$$

Notice that $\zeta = \psi_2^{-1} \circ T \circ \psi_1$, so we are interested in the derivative of $\zeta$ at $s = s_1$. Deriving the above expression at $(s_1, s_2, T)$ yields:

$$A - BD\zeta(s_1) + X(q)D\tau(s_1) = 0.$$

The $n \times (n-1)$ matrix $A - \tilde{B}$, with $\tilde{B} = BD\zeta(s_1)$ equals $-X(q)D\tau(s_1)$, which is a matrix of rank 1 (all columns are a multiple of $v := X(q)$). Hence the $n \times n$ matrix $(A - \tilde{B}|v)$ has rank 1. A property from linear algebra then shows that

$$\det(A|v) = \det(\tilde{B}|v).$$

Furthermore, it is elementary to prove that $\det(\tilde{B}|v) = \det(B|v)\det D\zeta(s_1)$. This allows us to extract $\det D(\zeta(s_1)$:

$$\det D\zeta(s_1) = \frac{\det(A|v)}{\det(B|v)} = \frac{\det(D_x\varphi(\tau, p)D\psi_1(s_1)|X(q))}{\det(D\psi_2(s_2)|X(q))}$$

Write now

$$(v_1, \ldots, v_{n-1})(t) = D_x\varphi(t, p)D\psi_1(s_1), \quad v_n = X(\varphi(t, p)).$$

We prove that $\det(v_1|\cdots|v_n)(\tau) = \det(D\psi_1(s_1)|X(p))\exp(\int \operatorname{div} X)$ (and by doing that we finish the proof of the proposition). First, observe that

$$(v_1, \ldots, v_n)(0) = (D_x\varphi(0, p)D\psi_1(s_1)|X(p)) = (D\psi_1(s_1)|X(p)).$$

Then, observe that upon writing $V = (v_1, \ldots, v_n)$ we have

$$D_tV(t) = DX_{\varphi(t,p)}V(t),$$

and so if $J = \det V$, we find $D_tJ(t) = D(\det)_{V(t)}DX_{\varphi(t,p)}V(t)$. A last linear algebra property is used to continue: $D(\det)_ABA = \det(A)\operatorname{trace} B$ for all $n \times n$ matrices $A$ and $B$:

$$D_tJ(t) = \operatorname{div} X(\varphi(t, p))J(t).$$

Integrating this equation yields $J(t) = J(0)\exp(\int_0^t \operatorname{div} X)$. □

**Proposition 1.21** *(Manifold version of proposition 1.20.) Let $X$ be a vector field on an $n$-dimensional Riemannian manifold $(M, g)$. Let $S$, $S'$ be two open sections of $M$ transverse to the flow of $X$. Assume $p \in S$, $q \in S'$ and the orbit through $p$ reaches $q$ in finite time (positive or negative). Let $T\colon S_0 \subset S \to S'$ be the transition map defined in a neighbourhood $S_0$ of $p$. If $\psi_i\colon U_i \to \Sigma_i$ are coordinates for $\Sigma_i$ with $U_i \subset \mathbf{R}^{n-1}$, then*

$$\det(D(\psi_2^{-1} \circ T \circ \psi_1))(s_1) = \frac{\langle \Omega(p), D\psi_1(s_1) \times X(p) \rangle}{\langle \Omega(q), D\psi_2(s_2) \times X(q) \rangle} \exp\left( \int_{O(p,q)} \operatorname{div}_\Omega X \, dt \right),$$

*where $s_1 = \psi_1^{-1}(p)$, $s_2 = \psi_2^{-1}(q)$, and $\Omega$ is the volume form associated to $g$, and where $D\psi_1(s_1)$ resp. $D\psi_2(s_2)$ is regarded as a product of $n-1$ vectors in $T_pM$ resp. $T_qM$. The integral is taken over the orbit $O(p,q)$ from $p$ to $q$ parametrized by $t$.*

# Chapter 2

# Normally hyperbolic passage

## 2.1 Introduction

Consider a singularly perturbed vector field on a 2-dimensional manifold, depending on a small parameter $\epsilon$. In this first chapter we will study the passage near normally hyperbolic parts of a "critical curve". Although a thorough study of an invariant foliation was already done by Fenichel ([FE]), we have obtained some refinement in our special case, making use of center manifolds near points of such a critical curve and of $C^k$-normal forms, as in [DR] and [DR2]. We will focus on smoothness properties of invariant manifolds. A study of the transition time inside invariant manifolds will be done in chapter 4. Some results concerning the asymptotic expansion of such invariant manifolds are recalled in chapter 6.

In section 2.5, we will also prove a regularity property on singular integrals, which turns out to be useful in studying specific normal forms of vector fields.

## 2.2 Fundamental notions and statement of results

In this section, we will put some constraints on the vector fields under study. We have tried to write these constraints as much as possible in a coordinate free manner. Before stating precise conditions and results, we quickly review some relevant notions.

**Definition 2.1** *A* critical curve *of a singularly perturbed family of vector fields $X_\epsilon$ on a 2-manifold $M$ is a curve of singularities of the reduced vector field $X_0$. We will regard the critical curve as a curve in the manifold $M \times \{0\} = \{\epsilon = 0\}$ in a 3-dimensional manifold with boundary $M \times [0, \epsilon_0[$ for some $\epsilon_0 > 0$. If the vector field depends on other parameters $\lambda$ we will still call it—with abuse of language—the critical curve.*

If $M$ is the plane $\mathbf{R}^2$ this would mean we consider smooth families of vector fields

$$X_{\epsilon,\lambda}\colon \left\{ \begin{array}{rcl} \dot{x} & = & f(x,y,\epsilon,\lambda) \\ \dot{y} & = & \epsilon g(x,y,\epsilon,\lambda) \end{array} \right. \tag{2.1}$$

with singular parameter $\epsilon \in \mathbf{R}^+$, and where $\lambda \in \Lambda$ ($\Lambda \subset \mathbf{R}^p$). We assume that $X_{0,0,\lambda}$ has a curve of singularities $\gamma$. This curve may depend on $\lambda$, but we will not keep this dependence in the notation of $\gamma$.

**Assumption N1** *The critical curve $\gamma$ is normally attracting at all points of $\gamma$. This means that at any point $p$ of $\gamma$ the reduced vector field $X_0$ has a one-dimensional stable manifold (and a one-dimensional center manifold, namely $\gamma$). Of course, the results in this chapter also extend to normally repelling critical curves.*

To proceed we lift the family of vector fields $X_{\epsilon,\lambda}$ on $M$ to a family $X_\lambda := X_{\epsilon,\lambda} + 0\frac{\partial}{\partial\epsilon}$ on $M \times [0,\epsilon_0[$. Because of the center manifold theorem we can find locally around $p$ (possibly parameter-dependent) 2-dimensional invariant center manifolds $W_p$ (inside $M \times [0,\epsilon_0[$) that are at least $C^1$. Let us consider such a center manifold. Although actually being a manifold with boundary in $M \times [0,\epsilon_0[$, let us—by abuse of language—call it a manifold. Essentially, we want the dynamics on the center manifolds to be topologically equivalent to the dynamics of a model differential equation

$$\epsilon\frac{\partial}{\partial x} + 0\frac{\partial}{\partial \epsilon}. \tag{2.2}$$

More precisely, we want the existence of a $C^1$ embedding

$$\varphi\colon [0,1]^2 \to W_p\colon (x,\epsilon) \mapsto \varphi(x,\epsilon)$$

so that

(i) $\varphi([0,1]^2)$ is a neighbourhood of $p$ inside $W_p$;

(ii) $\varphi([0,1]\times\{0\}) \subset \{\epsilon = 0\}$;

(iii) $\varphi|_{\epsilon=0}$ is orientation-preserving for the standard orientation on the $x$-axis and the chosen orientation on $\gamma$.

(iv) $\varphi$ is a topological equivalence between $X|_{W_p}$ and the model vector field (2.2).

To ensure this model behaviour, it suffices to assume

**Assumption N2** *For any point $p$ of $\gamma$, there exists a sufficiently small neighbourhood $V$ of $p$ in $M \times [0,\epsilon_0[$ so that in $V$ there are no singularities for $\epsilon > 0$. Furthermore, inside center manifolds in $V$, the orientation of the orbits for $\epsilon > 0$ must be compatible to a given orientation on $\gamma$.*

The compatibility of the orientation of orbits inside center manifolds with the orientation of $\gamma$ means the following: if we take two sections $\sigma_1$ and $\sigma_2$, transversally cutting $\gamma$ in points with parameter values $r_1$ and $r_2$, $r_1 < r_2$ (according to the orientation), then the orbits for $\epsilon > 0$ will also be oriented from $\sigma_1$ to $\sigma_2$. Assumption N2 does not depend on the chosen center manifold inside $V$. A choice of orientation will become important in the treatment of turning points, in order to distinguish transitions from attracting to repelling and transitions from repelling to attracting (see chapter 3).

**Definition 2.2** *The basin of attraction of $\gamma$ is the set of points in the manifold $M$ for which the orbit in positive time under the unperturbed vector field $X_{0,\lambda}$ has its $\omega$-set in $\gamma$. (A similar definition holds for the basin of repulsion of a repelling curve $\gamma$, using the $\alpha$-set.)*

**Definition 2.3** *Let $\Sigma$ be a smooth curve in $M \times [0, \epsilon_0[$, possibly depending on some extra parameters. Assume that this curve is a graph in $\epsilon \geq 0$. The saturation of $\Sigma$ is defined as the topological closure of the union of all orbits in positive time (w.r.t. the extended vector field $X_{\epsilon,\lambda} + 0\frac{\partial}{\partial\epsilon}$) of points of $\Sigma$. The need to take a topological closure becomes clear if one considers the limit point of $\Sigma$ as $\epsilon \to 0$: the orbits become singular for this limit point.*

The saturation of $\Sigma$ contains the limit point in $\gamma$ of the orbit of the base point $b := \Sigma \cap \{\epsilon = 0\}$. We will call this limit point a *corner point* of the saturation of $\Sigma$ and denote it by $c$. We need one more definition.

**Definition 2.4** *An admissible entry boundary curve $\Sigma$ is a curve in $M \times [0, \epsilon_0[$-space (possibly $\lambda$-dependent) that is a graph $\epsilon \mapsto s(\epsilon) \in M$. We assume that $s$ is $C^\infty$ for $\epsilon > 0$, and $C^0$ at $\epsilon = 0$. Furthermore, we assume*

$$\forall n \in \mathbf{N}, \exists N \in \mathbf{N} \colon \frac{\partial^n}{\partial\epsilon^n} s(\epsilon) = O(\epsilon^{-N}), \qquad \text{as } \epsilon \to 0.$$

This definition allows graphs that are $C^\infty$ at $\epsilon = 0$, but also graphs like $\epsilon \mapsto \epsilon \log \epsilon$, graphs that are $C^\infty$ in $\epsilon^{1/r}$ for some $r > 0$ etc. An example of what is not admissible would be a curve like

$$\epsilon \mapsto \sin(\exp(1/\epsilon)) \exp(-K/\epsilon),$$

with $K \in \mathbf{N}$, which only satisfies the above condition for $n = 0, \ldots, K-1$, and even for $n = K$, but not for $n > K$.

**Remark** One can characterize an admissible entry boundary curve also as follows: for all $n \in \mathbf{N}$ there is an $M \in \mathbf{N}$ so that $\epsilon^M s(\epsilon)$ is a $C^n$ curve at $\epsilon = 0$.

**Theorem 2.5** *Assume assumptions N1 and N2 are verified for the family of vector fields $X_{\epsilon,\lambda}$ on $M$. The saturation of an admissible entry boundary curve $\Sigma$ is a smooth invariant manifold with boundary along any compact piece of the critical curve $\gamma$, except along the orbit of the base point (intersection with $\{\epsilon = 0\}$) of $\Sigma$ including its $\omega$-limit $c$. At the point $c$, the saturation of $\Sigma$ is continuous.*

If the entry boundary curve is smooth at $\epsilon = 0$, then the saturated manifold will also be smooth at points of the orbit of the base point of $\Sigma$, but will still only be $C^0$ at the corner point $c$.

The proof of the theorem will be given in section 2.4.

## 2.3 Examples

To see the extent in which theorem 2.5 holds, let us briefly discuss two examples

$$X_\epsilon\colon \left\{ \begin{array}{rcl} \dot{x} & = & \epsilon(x^2+\epsilon^2+\lambda^2) \\ \dot{y} & = & -y+\epsilon F(x,y,\epsilon,\lambda), \end{array} \right.$$

where $F$ is $C^\infty$. In that case, given an entry curve of the form $\{x = x_0, y = s(\epsilon,\lambda)\}$, the saturation is a manifold

$$y = \varphi(x,\epsilon,\lambda)$$

that is $C^\infty$ for $x > x_0$ and $\epsilon \geq 0$. Consider also the example

$$X_\epsilon\colon \left\{ \begin{array}{rcl} \dot{x} & = & \epsilon\exp(-x^2/\epsilon) \\ \dot{y} & = & -y+\epsilon F(x,y,\epsilon,\lambda). \end{array} \right.$$

The speed of the slow dynamics, being $O(\epsilon)$ at $x = 0$ or exponentially small at nonzero $x$ is certainly not uniform, nor is the vector field Lipschitz. Nevertheless, saturating entry boundary curves will lead to smooth invariant manifolds.

## 2.4 Proof of theorem 2.5

Let $L$ be a compact piece of the normally attracting critical curve $\gamma$ of $X_{\epsilon,\lambda}$. In this compact set, $L$ is uniformly normally attracting, and at each point of $L$ we can always choose a local $C^k$ center manifold $W$ for the vector field $X_{\epsilon,\lambda}$ in $(x,y,\epsilon)$-space (see [K] or [HPS]).

**Lemma 2.6** *Let $k \in \mathbf{N}_1$ be fixed. In points of $L$, choosing a $C^k$ center manifold for $X_{\epsilon,\lambda}$, the vector field $X_{\epsilon,\lambda}$ is locally $C^k$-equivalent to*

$$\left\{ \begin{array}{rcl} \dot{x} & = & -x \\ \dot{y} & = & g(y,\epsilon,\lambda) \end{array} \right. \tag{2.3}$$

*where $g$ is a positive $C^k$ function for $\epsilon \neq 0$ and $g(y,0,\lambda) = 0$. The chosen center manifold is mapped to $\{x = 0\}$, $L$ is mapped to $\{x = \epsilon = 0\}$ under this equivalence, and the slow dynamics is compatible with the chosen orientation on $\gamma$.*

Note that often, one can consider that $g(y,\epsilon,\lambda) = \epsilon^\sigma$ for some $\sigma \geq 1$, but in general this is not possible without extra blow ups (think for example of a situation where $g(y,\epsilon,\lambda) = \epsilon(\epsilon^2+y^2+\lambda^2)$). We refer also to the example treated in section 4.3. Before proving this lemma, let us recall a theorem, that we will not only use to prove lemma 2.6, but also to prove more complicated statements later on.

**Theorem 2.7 [Bon]** *Let $X_\lambda(x,y,z)$ be a $C^\infty$ family of vector fields on $\mathbf{R}^3$ having the following properties:*

*(i)* $(0,0,0)$ *is a singular point of* $X_\lambda$.

*(ii)* $X_\lambda$ *is tangent to the foliation* $dF(y,z) = 0$ *where* $F(y,z) = y^p z^q$ *for* $(p,q) = (0,1)$ *or* $p, q \in \mathbf{N}_0$ *and relatively prime.*

*(iii)* $DX_\lambda(0,0,0)$ *has exactly one non-zero eigenvalue and the related eigenspace is given by* $y = z = 0$.

*Let $W$ be a $C^k$ center manifold of $X_\lambda$ at $(0,0,0)$ with $k \in \mathbf{N}_0$.*

*Then there exists a local $C^k$ change of coordinates $\varphi$ of the form*

$$(x,y,z) \mapsto (\varphi_1(x,y,z,\lambda), \varphi_2(x,y,z,\lambda), \varphi_3(x,y,z,\lambda))$$

*with*

$$F(\varphi_2(x,y,z,\lambda), \varphi_3(x,y,z,\lambda)) = F(y,z),$$

*and a strictly positive $C^k$ function $h_\lambda(x,y,z)$ such that*

$$[h_\lambda.\varphi_* X](x,y,z) = \pm x\frac{\partial}{\partial x} + Y_\lambda(y,z),$$

*with $Y_\lambda$ of class $C^k$, $Y.F = 0$ and $\varphi(W) = \{x = 0\}$.*

**Proof** (of lemma 2.6) We follow the techniques in [DR]. Take any point of $L$, then a translation will take this point to the origin. Due to the normal hyperbolicity along $L$, a linear change of coordinates will ensure that the linear part of the vector field for $\epsilon = 0$ looks like $\begin{pmatrix} 0 & 0 \\ 0 & \pm 1 \end{pmatrix}$. Apply the theorem now: the vector field $\pm x\frac{\partial}{\partial x} + g(y,\epsilon,\lambda)\frac{\partial}{\partial y}$ is $C^k$-equivalent to the original one. Due to assumption N2, $g$ has no zeroes for $\epsilon > 0$ in a neighbourhood of $(y,\epsilon) = (0,0)$. By changing $x \to -x$ and $t \to -t$ if necessary, we can make $g$ positive, and get to the expression in the lemma. $\square$

**Proof** (of theorem 2.5) For $\epsilon \neq 0$ the saturation of $\Sigma$ is clearly $C^\infty$, since we only deal with regular $C^\infty$ vector fields. The only problem hence deals with the extension for $\epsilon = 0$. It is assumed that the endpoint of $\Sigma$ lies in the basin of attraction of the critical curve $\gamma$. This means that in a neighbourhood along the fast orbit of the endpoint, no singularities appear. Hence, $\epsilon$ is a regular perturbation parameter in that neighbourhood and the saturation of $\Sigma$ will be smooth. We can extend this manifold until we enter a neighbourhood of $c$ where a normal form can be used. The normal form specified in lemma 2.6 can be solved implicitely: given an boundary condition curve $\{x = \gamma(\epsilon,\lambda), y = y_0\}$ the saturation is a graph

$$x(y,\epsilon,\lambda) = \gamma(\epsilon,\lambda)\exp\left(-\int_{y_0}^{y}\frac{ds}{g(s,\epsilon,\lambda)}\right)$$

where $y_0$ is the $y$-coordinate of $c$. We prove the smoothness of this expression as $\epsilon \to 0$ in proposition 2.8 (see the next section). (Note that admissible boundary curve $\gamma$ need not be $C^k$ at $\epsilon \to 0$, but at least there is an $N \in \mathbf{N}$ so that $\epsilon^N \gamma$ is $C^k$ as $\epsilon \to 0$; see the definition of admissible boundary curves. This is a slight obstruction, but it can be removed by applying proposition 2.14, where it is proved that $\exp(-\int_{y_0}^{y} \frac{ds}{g(s,\epsilon,\lambda)})$ is $O(\epsilon^N)$ for all $N$.) Since we can cover the compact $L$ by a finite number of neighbourhoods where a normal form as in lemma 2.6 is valid, the required smoothness of the saturation of $\Sigma$ along $L$ (except in $c$) follows.

We recall that $C^k$-smoothness for all $k$ is enough in our case to conclude $C^\infty$-smoothness, because the domain in which $C^k$-smoothness is proved does not shrink as $k$ increases. □

## 2.5 Some regularity properties

The following properties are useful in the study of the behaviour of vector fields in brought in normal form. It has applications in this chapter, but also in chapters 3 and 4.

**Proposition 2.8** *Let $g$ be a* positive *(not necessarily strictly positive) $C^k$ function on $V \times W \times \Lambda$, where $V$ is a compact interval in $\mathbf{R}$, $W$ is a set of singular parameters (part of a finitely dimensional vector space), and $\Lambda$ is an open set of regular parameters (part of a finitely dimensional vector space). Define for a fixed $y_0 \in V$, and for all $y > y_0$ for which $g(s,\epsilon,\lambda) \neq 0$ on $s \in [y_0, y]$:*

$$w(y,\epsilon,\lambda) := \exp\left(-\int_{y_0}^{y} \frac{ds}{g(s,\epsilon,\lambda)}\right), \qquad \epsilon \in W, \lambda \in \Lambda.$$

*Assume for $y_1 > y_0$ that $(y_1, \epsilon_0, \lambda)$ is in the closure of the domain of $w$. If*

*(a) $g(s,\epsilon_0,\lambda) = 0, \forall s \in [y_0, y_1]$,*

*(b) or if $g(s,\epsilon_0,\lambda)$ is only zero in the end point $s = y_1$ and not in $[y_0, y_1[$, and if $\frac{\partial g}{\partial y}(y_1,\epsilon_0,\lambda) = 0$.*

*Then the function $w$ can be extended in a $C^k$ way to $(y_1,\epsilon_0,\lambda)$, and in this point $w$ and all its derivatives (up to order $k$) are zero.*

**Remark** The proposition is quite elementary if $g = \epsilon^N \tilde{g}$ for some *strictly* positive $\tilde{g}$. In general, the flatness of $g$ can be more complicated, and in extremis writing for example (as in section 4.1.1) something like

$$g(y,\epsilon,\lambda) := \left(1 + \epsilon \sin^2(1/\epsilon^2)\right)\exp(-1/\epsilon),$$

one still gets a smooth function $w$.

**Proof** We will first treat case (a), and then tell how to adapt the proof for case (b). We claim that it suffices to prove that

(P1) for all $N \in \mathbf{N}$: $\displaystyle\lim_{(y,\epsilon)\to(y_1,\epsilon_0)} \frac{w(y,\epsilon,\lambda)}{g(y,\epsilon,\lambda)^N} = 0;$

(P2) for all $K, M \in \mathbf{N}$:

$$\lim_{(y,\epsilon)\to(y_1,\epsilon_0)} w(y,\epsilon,\lambda) g(y,\epsilon,\lambda)^{KM} \left(\int_{y_0}^{y} \frac{ds}{g(s,\epsilon,\lambda)^K}\right)^M = 0,$$

and where the convergence is uniform in $\lambda$. Note that the limits can only be taken in the closure of the set of points $(y,\epsilon,\lambda)$ where $w(y,\epsilon,\lambda)$ is defined properly. Note also that we do not claim anything for $y = y_0$; indeed this point is excluded in the formulation of the proposition. In fact, $w$ is in general not even $C^1$ in the point $(y_0, \epsilon_0, \lambda)$.

The proof of properties (P1) and (P2) will be carried out in two lemma's, but here we will show that those two properties are sufficient to prove the proposition.

To that end, calculate all first-order derivatives of $w$, in the points $(y,\epsilon,\lambda)$ where $w$ is defined:

$$\begin{aligned}
\frac{\partial w}{\partial y}(y,\epsilon,\lambda) &= w(y,\epsilon,\lambda)\frac{-1}{g(y,\epsilon,\lambda)} \\
\frac{\partial w}{\partial \epsilon}(y,\epsilon,\lambda) &= w(y,\epsilon,\lambda)\int_{y_0}^{y} \frac{\frac{\partial g}{\partial \epsilon}(s,\epsilon,\lambda)}{g(s,\epsilon,\lambda)^2} ds \\
\frac{\partial w}{\partial \lambda_\ell}(y,\epsilon,\lambda) &= w(y,\epsilon,\lambda)\int_{y_0}^{y} \frac{\frac{\partial g}{\partial \lambda_\ell}(s,\epsilon,\lambda)}{g(s,\epsilon,\lambda)^2} ds.
\end{aligned}$$

Observing these three equations, one finds that applying a general differential operator $D$ to $w$ results in

$$Dw(y,\epsilon,\lambda) = \sum_i w(y,\epsilon,\lambda) \prod_j \frac{F_{ij}(y,\epsilon,\lambda)}{g(y,\epsilon,\lambda)^{N_{ij}}} \int_{y_0}^{y} \frac{G_{ij}(s,\epsilon,\lambda)}{g(s,\epsilon,\lambda)^{K_{ij}}} ds,$$

with $N_{ij}, K_{ij} \in \mathbf{N}$, $F_{ij}$ and $G_{ij}$ are functions of class $C^{k-|D|}$ and where the sum and products are finite. In order to prove that $Dw(y,\epsilon,\lambda) \to 0$ as $\epsilon \to \epsilon_0$, it is sufficient to prove that each summand of the above expression tends to zero. By raising these expressions to some power, we can distribute the effect of $w(y,\epsilon,\lambda)$ among all types of factors, and we find the following two conditions:

(a) for all $N \in \mathbf{N}$: $\displaystyle\lim_{(y,\epsilon)\to(y_1,\epsilon_0)} \frac{w(y,\epsilon,\lambda)F(y,\epsilon,\lambda)}{g(y,\epsilon,\lambda)^N} = 0;$

(b) for all $K, M \in \mathbf{N}$:

$$\lim_{(y,\epsilon)\to(y_1,\epsilon_0)} w(y,\epsilon,\lambda) g(y,\epsilon,\lambda)^{KM} \left(\int_{y_0}^{y} \frac{G(s,\epsilon,\lambda)}{g(s,\epsilon,\lambda)^K} ds\right)^M = 0.$$

(the factor $g^{KM}$ is included in (b) because we want to do so, and we can: we simply need to increase the $N$ in (a)). Since $F$ and $G$ are at least $C^0$, these properties are true once the two properties (P1) and (P2) are satisfied.

The remainder of the section involves the proof of (P1) and (P2). Let us start with

**Lemma 2.9** *(Under condition (a) in proposition 2.8) For all $K > 0$ there exists a neighbourhood $V$ of $\epsilon = \epsilon_0$ such that*

$$\int_{y_0}^{y} \frac{ds}{g(s,\epsilon,\lambda)} \geq -K(y-y_0)\log g(y,\epsilon,\lambda),$$

*for all $y \in [y_0, y_1]$, $\lambda \in \Lambda$ and $\epsilon \in V$ (only for those $\epsilon$ where $w$ is defined.)*

**Proof** Let

$$F\colon (y,\epsilon,\lambda) \mapsto \int_{y_0}^{y} \frac{ds}{g(s,\epsilon,\lambda)} + K(y-y_0)\log g(y,\epsilon,\lambda).$$

then $F(y_0,\epsilon,\lambda) = 0$, so it remains to prove that $\frac{\partial F}{\partial y}(y,\epsilon,\lambda) \geq 0$ for all $y \in [y_0, y_1]$. In short notation, we have

$$\begin{aligned}\frac{\partial F}{\partial y} &= \frac{1}{g} + K\log g + \frac{K(y-y_0)}{g}\frac{\partial g}{\partial y} \\ &= \frac{1}{g}\left(1 + Kg\log g + K(y-y_0)\frac{\partial g}{\partial y}\right).\end{aligned}$$

The mapping $u \mapsto u\log u$ tends to zero in the origin, so for $\epsilon$ small enough, we may assume that $g\log g \geq -\frac{1}{3K}$. Furthermore, since also $\frac{\partial g}{\partial y}$ tends to zero, we may assume that for $\epsilon$ small enough, $\frac{\partial g}{\partial y} \geq -\frac{1}{3K(y_1-y_0)}$. Applying these inequalities to the equation above, we find

$$\frac{\partial F}{\partial y}(y,\epsilon,\lambda) \geq \frac{1}{g(y,\epsilon,\lambda)}\left(1 + K\frac{-1}{3K} + (y-y_0)K\frac{-1}{3K(y_1-y_0)}\right).$$

As $y - y_0 \leq y_1 - y_0$, we have $\frac{\partial F}{\partial y}(y,\epsilon,\lambda) \geq \frac{1}{3g(y,\epsilon,\lambda)} \geq 0$. □

**Corollary 2.10** *For all $N \in \mathbf{N}$ and for all intervals $[y_0+\delta, y_1]$ on the $y$-axis (with $\delta > 0$), there exists a neighbourhood $V$ of $\epsilon = \epsilon_0$ such that*

$$w(y,\epsilon,\lambda) \leq g(y,\epsilon,\lambda)^{N+1},$$

*for all $y \in [y_0+\delta, y_1]$, $\lambda \in \Lambda$ and $\epsilon \in V$ (only for those $\epsilon$ where $w$ is defined). This proves property (P1) in case (a).*

**Proof** Apply lemma 2.9 with $K = \frac{N+1}{\delta}$, and find

$$w(y,\epsilon,\lambda) \leq g(y,\epsilon,\lambda)^{\frac{N+1}{\delta}y}.$$

The corollary follows from the facts that $\frac{N+1}{\delta}y \geq (N+1)$ and, for $\epsilon$ small enough, $g(y,\epsilon,\lambda) < 1$. □

For the proof of (P2), define

$$F(y,\epsilon,\lambda) = w(y,\epsilon,\lambda)^{1/M} g(y,\epsilon,\lambda)^K \int_{y_0}^{y} \frac{ds}{g(s,\epsilon,\lambda)^K}$$

**Lemma 2.11** *For all $\nu > 0$, there exists a neighbourhood $V$ of $\epsilon = \epsilon_0$ such that*

$$F(y,\epsilon,\lambda) < \nu,$$

*for all $y \in [y_0, y_1]$, $\lambda \in \Lambda$ and $\epsilon \in V$ (only for those $\epsilon$ where $w$ is defined). This proves property (P2) in case (a).*

**Proof** It is sufficient that we prove

$$F(y_0,\epsilon,\lambda) < \nu \qquad \text{and} \qquad \left( F(y,\epsilon,\lambda) \geq \nu \implies \frac{\partial F}{\partial y}(y,\epsilon,\lambda) < 0 \right).$$

The first statement is obvious, since $F(y_0,\epsilon,\lambda) = 0$. To prove the second statement, assume $F(y,\epsilon,\lambda) \geq \nu$. Then (write $g'$ for $\frac{\partial g}{\partial y}$):

$$\begin{aligned}
\frac{\partial F}{\partial y} &= w^{1/M} \left( \frac{-1}{Mg} g^K \int \frac{1}{g^K} + K g^{K-1} g' \int \frac{1}{g^K} + g^K \frac{1}{g^K} \right) \\
&= w^{1/M} + \left( \frac{-1}{Mg} + \frac{Kg'}{g} \right) \left( w^{1/M} g^K \int 1 g^K \right) \\
&= w^{1/M} - \frac{F}{Mg}(1 - KMg').
\end{aligned}$$

Taking $\epsilon$ small enough, we may assume that $g' \leq \frac{1}{2KM}$. Since we also know that $w$ is bounded by 1, we have

$$\frac{\partial F}{\partial y} \leq 1 - \frac{F}{Mg}\left(1 - \frac{1}{2}\right) \leq 1 - \frac{\nu}{2Mg}.$$

As $g$ gets smaller, $\frac{\partial F}{\partial y}$ will turn negative. □

These lemmas prove properties (P1) and (P2) in the case (a) of proposition 2.8, and it was already pointed out that this is enough in view of proving proposition 2.8. Let us now adapt the proof to the case (b). Key element in the proofs of the lemmas was the fact that $g$ and $g'$ becomes zero. In the case (b), we do not have this property uniformly in $[y_0, y_1]$, but only in the end point $y_1$. We will need to be more careful:

**Lemma 2.12** *If we define* $F(y,\epsilon,\lambda) := \int_{y_0}^{y} \frac{ds}{g(s,\epsilon,\lambda)} + K(y-y_0)\log g(y,\epsilon,\lambda)$*, then there exists a* $\delta \in ]y_0, y_1[$ *and a neighbourhood* $V$ *of* $\epsilon = \epsilon_0$ *such that for all* $y \in [\delta, y_1]$*:*

$$F(y,\epsilon,\lambda) \geq F(\delta,\epsilon,\lambda).$$

*for all* $\lambda \in \Lambda$ *and* $\epsilon \in V$ *(only for those* $\epsilon$ *where* $w$ *is defined.)*

**Proof** Completely similar to the proof of lemma 2.9. □

**Corollary 2.13** *For all* $N \in \mathbf{N}$ *there exists a neighbourhood* $V$ *of* $\epsilon = \epsilon_0$*, a* $C > 0$ *and a* $\delta \in ]y_0, y_1[$ *such that*

$$w(y,\epsilon,\lambda) \leq Cg(y,\epsilon,\lambda)^{N+1},$$

*for all* $y \in [\delta, y_1[$*, for all* $\lambda \in \Lambda$ *and* $\epsilon \in V$ *(only for those* $\epsilon$ *where* $w$ *is defined). This proves property (P1) in case (b).*

**Proof** Apply the lemma to $K = \frac{N+1}{\tilde{\delta}}$, where $\tilde{\delta}$ is an arbitrary small number. There exists a $\delta \in [y_0, y_1[$ such that for all $y \in [\delta, y_1[$:

$$w(y,\epsilon,\lambda) \leq g(y,\epsilon,\lambda)^{K(y-y_0)} g(\delta,\epsilon,\lambda)^{-K(\delta-y_0)} w(\delta,\epsilon,\lambda).$$

So for $y \geq \max\{\delta, y_0 + \tilde{\delta}\}$:

$$w(y,\epsilon,\lambda) \leq g(y,\epsilon,\lambda)^{N+1} g(\delta,\epsilon,\lambda)^{-K(\delta-y_0)} w(\delta,\epsilon,\lambda).$$

Since $g(\delta, 0, \lambda) \neq 0$ we find that it can be bounded away from zero, which proves the corollary. □

The proof of (P2) in case (b) goes completely similar as in case (a), by replacing $y_0$ by a $\delta$ close enough to $y_1$.

□

A slight generalization is needed. The results of proposition 2.8 remain true if not $g(y,\epsilon,\lambda)$ but $\epsilon^N g(y,\epsilon,\lambda)$ is a $C^k$ function for some $N > 0$:

**Proposition 2.14** *Let* $g$ *be a* positive *(not necessarily strictly positive)* $C^0$ *function on* $V \times W \times \Lambda$*, where* $V$ *is a compact interval in* $\mathbf{R}$*,* $W = [0, \epsilon_0[$ *is a set of singular parameters, and* $\Lambda$ *is a set of regular parameters (part of a finitely dimensional vector space). Assume that* $\epsilon^N g$ *is a* $C^k$ *function for some* $N \geq 0$*. Define for a fixed* $y_0 \in V$*, and for all* $y > y_0$*:*

$$w(y,\epsilon,\lambda) := \exp\left(-\int_{y_0}^{y} \frac{ds}{g(s,\epsilon,\lambda)}\right), \qquad \epsilon \in W, \lambda \in \Lambda.$$

*Assume for* $y_1 > y_0$ *that* $(y_1, 0, \lambda)$ *is in the closure of the domain of* $w$*. If* $g(s, 0, \lambda) = 0, \forall s \in [y_0, y_1]$*, then the function* $w$ *can be extended in a* $C^k$ *way to* $(y_1, 0, \lambda)$*, and in this point* $w$ *and all its derivatives (up to order* $k$*) are zero.*

**Proof** The setting is similar to the setting in proposition 2.8, and the proof can be copied after a slight change: next to the properties (P1) and (P2), we have to prove additionally

$$\text{(P3)} \qquad w(y, \epsilon, \lambda) = O(\epsilon^{\tilde{N}}), \qquad \forall \tilde{N} > 0$$

Let us explain why: to show that the properties (P1) and (P2) are sufficient to prove the smoothness in proposition 2.8, we replaced all differentials of $g$ by constants—this is possible since all differentials are $C^0$. Here, we have to replace the differentials by $C/\epsilon^{\tilde{N}}$, hence (P3) is needed. The proof of (P3) is trivial since $g \leq K\epsilon$ for some constant $K > 0$, and thus $w(y, \epsilon, \lambda) \leq \exp(-\frac{y-y_0}{K\epsilon})$. □

# Chapter 3

# Canards at non-generic turning points

## 3.1 Introduction

Analysis of singularly perturbed vector fields becomes much more complicated when so-called “contact points” appear. These are points on the critical curve where a transition from normal attraction to normal repulsion appears. The competition between slow and fast dynamics is most apparent in points like these. One might expect that in situations where orbits follow the slow movement, the orbits will revert to orbits following fast dynamics immediately after crossing such a contact point, because of the repulsive behaviour after this point. On the contrary, sometimes the orbits will keep on following the critical curve for some time, before the exchange of dominance occurs. Contact points with that property could be called “turning points” to make a distinction with the more common “jump point” which one encounters in relaxation oscillations.

When the family of vector fields has other parameters, we can try—under rather general conditions—to give a regular condition on $\epsilon$ and on the other parameters so that the exchange of stability occurs at exactly the manifold in parameter space given by the condition. This process will create what is commonly known as “overstable solutions”, or “canard solutions”. In the literature, a clear distinction is made between this two names: the term overstable solution is used in the complex setting, whereas real solutions are referred to as canard solutions. In the literature, several successful methods have been brought forward to handle the existence of both overstable solutions and canard solutions. We mention the technique of matching inner and outer solutions, non-standard analysis (Diener et al.) and also complex analytic techniques (Sibuya, Schäfke, Ramis, Canalis-Durand and many others). Lately, serious progress has been made in applying analytic techniques to singularly perturbed problems, and a number of the results that are proved in this work have already been proved using these techniques. But the analytic study does not completely cover the dynamics on the real axis. We present some examples where real canard solutions are created, that cannot be complexified to canard solutions in full complex neighbourhoods (or so called overstable solutions).

The method we use here is based on the construction of center manifolds and the use of $C^k$ normal forms, as in [DR] and [DR2]. The key element is the family blow up—a technique of rescaling variables in a geometrical way. Because of this, the constructed solutions will be smooth in the blow up space. We present conditions under which a blow down of the center manifolds is possible. We extract some consequences from it and relate these to treatments of a different nature like the traditional matching between "inner" (inside the blow up locus) and outer solutions (in original coordinates) or the resummation.

Our major contribution lies in the generality of our results. Where up to now the geometric analysis of turning points (like in [DR]) was restricted to the generic case, we consider here a generalization to non-generic turning points. A specific class of more degenerate systems is precisely described in theorem 3.8. In theorem 3.7 we show that our results can be applied to what we could call the generic turning point. These examples are defined in the plane, but our results apply, as the description shows, to systems on 2-manifolds. All results are valid for vector fields of class $C^r$, $r$ sufficiently large.

A second part of this chapter deals with the angle between center manifolds, defined on different sides of the contact point. In fact, the entire canard phenomenom can be explained geometrically by looking how those two different families of center manifolds intersect. Orbits following the critical curve in positive time, gathered in these invariant (center) manifolds, cross the contact point and intersect with orbits following the critical curve in negative time. Any connection between these manifolds results in canard solutions, so the intersection is crucial in our study. If the manifolds intersect transversally, a straightforward adaption of the techniques in [DR] results in a formula for the angle. We generalize these computations by providing recursive formulas for calculating the first nonzero higher order angle, besides describing its relation with the graph of the control curve.

## 3.2 Fundamental notions and statement of results

This section is a sequel of section 2.2; we hence assume that the notions critical curve, entry boundary curve, saturation, corner point, basin of attraction etc. are known.

Specific in the study of canard solutions we will use a special parameter—denoted by $a$—that will essentially be a parameter breaking the critical manifold in a regular way. In a moment we will give a precise definition of it. If $M$ is the plane $\mathbf{R}^2$ this would mean we consider smooth families of vector fields

$$X_{\epsilon,a,\lambda}\colon \begin{cases} \dot{x} &= f(x,y,\epsilon,a,\lambda) \\ \dot{y} &= \epsilon g(x,y,\epsilon,a,\lambda) \end{cases} \tag{3.1}$$

with singular parameter $\epsilon \in \mathbf{R}^+$, and where $(a,\lambda) \in ]-a_0,a_0[\times\Lambda$ $(\Lambda \subset \mathbf{R}^p)$. We could also work with analytic $X_{\epsilon,a,\lambda}$; properties that are specific to analytic vector fields will be handled in chapter 6. We assume that $X_{0,0,\lambda}$ has a curve of singularities $\gamma$. This curve may depend on $\lambda$, but we will not keep this dependence in the notation of $\gamma$. The dependence on $\lambda$ is not entirely unconditional; obvious bifurcations in the shape of the curve should be avoided. We in fact ask $\gamma = \gamma_\lambda$ to be a trivial $\lambda$-family of simple curves; for a precise statement we refer to a remark after assumption T3.

In this chapter we will deal with simple critical curves with a *single contact point.* By "simple" we mean that the curve can be obtained as an image of a $C^\infty$ embedding of $[0,1]$. It is "critical" since it consists of singularities of the vector field under consideration; $\gamma$ contains a point $p_*$ not lying at its endpoint, with the property that $p_*$ divides $\gamma$ into two parts $\gamma_-$ and $\gamma_+$ with both $\gamma_- \cup \{p_*\}$ and $\gamma_+ \cup \{p_*\}$ simple critical curves. We orient $\gamma$ in a way that $\gamma_+$ comes after $\gamma_-$. The fact that $p_*$ is a simple contact point means that $X_0$ is normally attracting at all points of $\gamma_-$ and normally repelling at all points of $\gamma_+$.

**Assumption T1 (Admissible chart)**

*There exists a (possibly $\lambda$-dependent) chart of $M$ in the neighbourhood of $p_*$ so that in this chart the contact point is the origin $(x,y) = (0,0)$. Writing the vector field in this chart as in (3.1), the critical curve is given by $\gamma = \{\, (x,y) \,|\, f(x,y,0,0,\lambda) = 0 \,\}$ (with $f$ at least $C^1$). The origin divides $\gamma$ in two pieces $\gamma_-$ and $\gamma_+$. Along $\gamma\backslash\{(0,0)\}$ we suppose that both $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ are nonzero.*

A chart where these conditions are met is called an admissible chart.

**Assumption T2 (Normal passage)**

*Keeping $a = 0$, assumptions N1 and N2 (see chapter 2) are verified for the vector field $X_{\epsilon,0,\lambda}$ in $\gamma_-$, whereas the repulsive counterparts of assumptions N1 and N2 are verified in $\gamma_+$.*

Note 1: Assumption T2 gives the basis for a possible transition from attracting to repelling regime. In other situations, where the transition is from repelling to attracting, repelling to repelling or attracting to attracting regime, canard solutions are more likely to occur, and the study is easier. Canard solutions occuring from transitions like these are often called "*faux canards*".

Note 2: In some results we will even allow singularities bifurcating out of $\gamma_- \cup \gamma_+$, for $\epsilon > 0$, but we will limit us to results concerning invariant manifolds consisting of orbits for which in the vicinity of $\gamma_- \cup \gamma_+$ the orientation is compatible with the orientation of $\gamma$.

Assumption T3 will be a regularity condition, and will be described in terms of an "admissible chart" near the contact point. We will assume that after a single *(family) blow up*, the vector field will be desingularized in a nice way. In section 1.1 it is recalled what a family blow up is. In the blow up we still keep $a = 0$ (for now).

**Assumption T3 (Regularity condition)**

> *After blowing up at* $(0,0,0)$ *the* $(x, y, \epsilon)$*-variables — in an admissible chart — of* $X_{\epsilon,0,\lambda}$*, we get the following:*
>
> *The preimages of* $\gamma_-$ *and* $\gamma_+$ *in the blow up space (including the endpoints of* $\gamma_\pm$ *on the blow up locus) are normally hyperbolic. Define* $P_\pm = \gamma_\pm \cap \Sigma$*, where* $\Sigma$ *is the blow up locus, i.e. the preimage of* $(x, y, \epsilon) = (0, 0, 0)$ *under the blow up map.*

*Important Remark:* the regularity condition is restrictive in the following sense: the blow up weights are chosen independent of $\lambda$, hence in order for the blow up procedure to work for all choices of $\lambda$, the order of degeneracy of the critical curve must not depend on $\lambda$. We have in mind for example quadratic contacts where the second order angle changes:

$$\gamma\colon y = \lambda x^2 + O(x^3), \qquad \lambda > \lambda_0 > 0$$

or contacts of order '$2n$', with a fixed $n$, i.e.

$$\gamma\colon y = \lambda x^{2n} + O(x^{2n+1}), \qquad \lambda > \lambda_0 > 0$$

Situations where the order of degeneracy of the critical curve undergoes a bifurcation (for example $\gamma\colon y = \lambda x^2 + x^4$, where $\lambda = 0$ is inside the parameter space) will generally not satisfy assumption T3. The way to proceed in these situations is to include $\lambda$ in the family blow up.

The next assumption is the sequel of assumption T2, but expressed in blow up coordinates as we come to introduce. There must be a way to proceed along the "corners" $P_\pm$. We work in a phase-directional rescaling chart, as in (1.2). Choosing a section $\{u = u_0 > 0\}$ means choosing a section in the neighbourhood of the normally hyperbolic part of the critical curve, whereas choosing a section $\{\bar{\varepsilon} = \bar{\varepsilon}_0 > 0\}$ means choosing a section transversally cutting the blow up locus. As in assumption T2, we have in mind a model vector field to express the dynamics in center manifolds. First, the existence of center manifolds near $P_-$ follows from assumption T3. Choosing a manifold $W$ near $P_-$, then we essentially want the vector field on $W$ to be topologically equivalent to

$$-u\bar{\varepsilon}\frac{\partial}{\partial u} + \bar{\varepsilon}^2\frac{\partial}{\partial \bar{\varepsilon}} \tag{3.2}$$

More precisely:

Let $p = P_-$ be the end point of $\gamma_-$, in a blown up admissible chart, and let $W_p$ be a center manifold of $X$ at $p$. We require the existence of a $C^1$ embedding

$$\varphi\colon [0,1]^2 \to W_p\colon (u, \bar{\varepsilon}) \mapsto \varphi(u, \bar{\varepsilon})$$

so that

(i) $\varphi([0,1]^2)$ is a neighbourhood of $P_-$ inside $W_p$;

(ii) $\varphi([0,1] \times \{0\}) \subset \gamma_- \cup \{P_-\}$;

(iii) $\varphi(\{0\} \times [0,1])$ is inside the blow up locus.

(iv) $\varphi|_{[0,1]\times\{0\}}$ is orientation-preserving for the standard orientation on the $u$-axis and the negative orientation on $\gamma$.

(v) $\varphi$ is a topological equivalence between $X|_{W_p}$ and the model vector field (3.2).

This model behaviour will be ensured by assumption T2 and the next assumption:

**Assumption T4 (Regular corner passage)** *Let $p = P_-$ be the end point of $\gamma_-$ in a blown up admissible chart, then there exists a sufficiently small neighbourhood $V$ of $p$ so that in $V$ there are no singularities for $\bar{\varepsilon} > 0$. A similar requirement is made for the end point $P_+$ of $\gamma_+$.*

**Assumption T5 (Connection condition)**
*Under the conditions of assumption T3, there is a heteroclinic connection $\Gamma$ (for $a = 0$) on the blow up locus $\Sigma$ connecting $P_-$ to $P_+$. We assume that this connection consists of one orbit going from $P_-$ to $P_+$.*

Of course, it would be interesting to see what can happen if we let a parameter tend to the boundary of the parameter set where assumption T5 is satisfied. Possibly, a saddle-node may appear on the connection, or, the connection may be a curve of singularities in the limit. At first, we will focus on the case where there is a connection without singularities, but the techniques are general enough to be used in more degenerate cases, see the examples for such a generalization in the normally hyperbolic passage.

Let us now precisely describe the role of $a$, starting by describing the role of a regular breaking parameter in the case that $\gamma$ remains a critical manifold for the vector fields $X_{\epsilon,a,\lambda}$ with $a \neq 0$. To focus on the specificity of this case let us write $a = A$. In that case the blow up in $(x, y, \epsilon)$-space can not only be applied for $A = 0$ but also for the fields with $A \neq 0$. Due to assumption T3 we recover, for $A = 0$, in the blow up the points $P_\pm$, that will persist as normally hyperbolic points for small values $A \neq 0$ (see eg. [HPS]). Also the invariant manifolds of respectively $P_-$ and $P_+$ inside the blow up locus $\{\epsilon = 0\}$ will persist. We know from assumption T5 that they form a heteroclinic connection for $A = 0$. In the family rescaling chart and inside $\{\epsilon = 0\}$ we can choose a section $\sigma$ transverse to the flow of the blown up vector field. We choose a regular parameter $z$ on $\sigma$ and we denote by $z_\pm(A, \lambda)$ the intersection with $\sigma$ of the invariant manifolds of respectively $P_\pm$. By assumption T5, we know that $z_-(0, \lambda) = z_+(0, \lambda)$.

**Definition 3.1** *We say that $A$ is a* regular breaking parameter *if*

$$\rho(\lambda) := \frac{\partial}{\partial A}(z_- - z_+)(0, \lambda) \neq 0. \tag{3.3}$$

This definition does not depend on the choice of the regular coordinate $z$, nor on the transverse section $\sigma$.

**Assumption T6 (Breaking parameter)** *In the family rescaling chart expressed in (1.3), there exists some $n \in \mathbf{N}$ such that $A := a/u^n$ (hence $a = A\epsilon^{n/m}$) is a regular breaking parameter.*

The idea is that the family $X_{\epsilon,a,\lambda}$ is replaced by a family $X_{\epsilon,A\epsilon^{n/m},\lambda}$ with $A \in ]-A_0, A_0[$, but in this 'subfamily' we know that the critical curve $\gamma$ of $X_{\epsilon,0,\lambda}$ persists to a critical curve of $X_{\epsilon,A\epsilon^{n/m},\lambda}$ with $A \neq 0$.

The presentation in this chapter is adapted to the study of the so called canard solutions. It means that we stay in a region in parameter space outside which no such solutions can exist. This is reflected in the rescaling

$$(a, \epsilon) = (u^n A, u^m),$$

as used to express assumption T6. In fact, if one wants to use family blow up to make a study in a complete neighbourhood of $(0, 0)$ in the $(a, \epsilon)$-plane, the way to proceed is first to make a blow up in the parameter plane by writing

$$(a, \epsilon) = (v^k A, v^\ell E), \tag{3.4}$$

for $A^2 + E^2 = 1$ (or equivalently working with charts $E = +1$ or $A = \pm 1$.) Thereafter one continues with blow up in the $(x, y, v)$-space:

$$(x, y, v) = (u^p \overline{x}, u^q \overline{y}, u\overline{v}),$$

with $\overline{x}^2 + \overline{y}^2 + \overline{v}^2 = 1$ (or equivalently working with charts). It of course leads to the same result under the condition $E = 1$. For a concrete example, we refer to section 1.1.

In the sections 3.3.1, 3.4 and 3.5 we will hence write system (3.1) as

$$X_{v,A,\lambda} \colon \begin{cases} \dot{x} &= f(x, y, v, A, \lambda) \\ \dot{y} &= vg(x, y, v, A, \lambda) \end{cases} \tag{3.5}$$

to emphasize that we might already have written the original $(a, \epsilon)$ as $(a, \epsilon) = (v^k A, v^\ell)$ for some $(k, \ell) \in \mathbf{N}^2$. We can hence suppose that $X_{0,A,\lambda}$ has a curve of singularities through the origin for all $(A, \lambda)$ under consideration.

As an example, consider the Van der Pol system

$$\begin{cases} \dot{x} &= y - \frac{x^2}{2} - \frac{x^3}{3} \\ \dot{y} &= \epsilon(a - x) \end{cases} \tag{3.6}$$

In this example, $a$ is not a regular breaking parameter, however in [DR] it is shown that $a/\sqrt{\epsilon}$ is a regular breaking parameter. Hence, in (3.6) one can write

$$a = vA, \qquad \epsilon = v^2 E,$$

with $A^2 + E^2 = 1$. Being interested in $A \sim 0$, we can consider the chart $E = 1$, and check the assumptions for the parameters $(v, A)$ instead of $(\epsilon, a)$.

**Important remark:** assumption T2, the second part of assumption T5 and assumption T4 are "open" assumptions, i.e. we could restrict the parameter set $(A, \lambda)$ to an open subset where these conditions are satisfied. It would be interesting to know what happens if $(A, \lambda)$ tends to the boundary of this set, i.e. a singularity could appear on the slow dynamics, or on the heteroclinic connection, or even more degenerate phenomena could occur. In extremis, the connection on the blow up locus could consist out of singular points, with a certain loss of normal hyperbolicity at some point!

To formulate the results, we need some definitions.

**Definition 3.2** *A "simple passage" turning point is a contact point satisfying the properties described in assumptions T1–T6 above.*

The main result is theorem 3.4, but a first preliminary result is the existence of canard solutions, with arbitrary boundary conditions. Two boundary conditions are chosen as follows: take a smooth curve $\Sigma_-$, transverse to the manifold $\{\epsilon = 0\}$ and so that the end point $b_-$ in $\{\epsilon = 0\}$ is inside the basin of attraction of $\gamma_-$, and take a smooth curve $\Sigma_+$ transverse to $\{\epsilon = 0\}$ so that the end point $b_+$ is inside the basin of repulsion of $\gamma_+$. Theorem 3.3 states that we can write the parameter $a$ in terms of $\epsilon$ in a way that the saturation in forward time of $\Sigma_-$ coincides with the saturation in backward time of $\Sigma_+$, thereby creating a manifold of canard solutions for (3.1). The canard solutions are global of nature in this approach.

**Theorem 3.3** *Let $X_{\epsilon,a,\lambda}$ be a vector field on a 2-manifold with a simple passage turning point. Let $\Sigma_\pm$ be admissible entry/exit boundary curves. Then for some $m \in \mathbf{N}_1$ and for $\epsilon \in [0, \epsilon_0[$ with $\epsilon_0 > 0$ sufficiently small, there exists a unique smooth curve $a = \mathcal{A}(\epsilon^{1/m}, \lambda)$ so that $\mathcal{A}(0, \lambda) = 0$ and so that the saturation of $\Sigma_-$ along $X_{\epsilon,\mathcal{A}(\epsilon,\lambda),\lambda}$ forms a manifold with boundary of canard solutions containing $\Sigma_+$ as well. The manifold with boundary is smooth in the blow up space, everywhere except[1] at the two corner points $c_\pm$ defined above. The $\infty$-jet of $\mathcal{A}(u, \lambda)$ w.r.t. $u$ is independent of the chosen admissible entry/exit boundary curves.*

[1] if the boundary curves are nonsmooth at their base points then the smoothness is of course also lost along the orbits of these base points.

Remark 1: A curve $a = \mathcal{A}(\epsilon^{1/m}, \lambda)$, like in the statement of theorem 3.3, is called a *control curve*, or a *canard line*. It depends on the choice of $\Sigma_\pm$. However, two different control curves have an infinite contact at $\epsilon = 0$, uniformly in $\lambda$.

Remark 2: The smoothness of the control curve and of the manifolds will be in terms of the rescaled variables, due to the rescaling. This effect is most visible in the control curve; this curve will in general only be smooth in $\epsilon^{1/m}$ for some $m \in \mathbf{N}$ depending on the blow up construction. In general, the index $m$ is the product of weights in two different rescaling: a first weight from a parameter rescaling as in (3.4), and a second weight from a rescaling as in (1.1). In practice, one of those two weights can always be chosen 1, i.e. one can choose to rescale $\epsilon$ through the first rescaling or through the second rescaling, but one usually avoids rescaling $\epsilon$ twice. Refer also to the example in section 1.1.

Remark 3: This theorem has strong implications on the orbits of points on $\Sigma_-$, for $\epsilon \sim 0$. The corner points are essentially the points where the change of dominance takes place. Following the fast dynamics, a point of $\Sigma_-$ fastly moves towards a small neighbourhood of the critical curve $\gamma_-$, near the corner point $c_-$, then staying close to $\gamma_-$ slowly moves over the contact point towards the repelling part of the critical curve, and near $c_+$ again moves fastly away from $\gamma_+$ finally reaching $\Sigma_+$.

Remark 4: The manifolds with boundary in theorem 3.3, and other manifolds with boundary will be referred to as manifolds.

Remark 5: If two different sets of boundary curves $\Sigma_\pm$ and $\Sigma'_\pm$ are taken, then the two manifolds are infinitely tangent to each other between the corner point $c_-$ or $c'_-$ (whichever is closer to the contact point) and the corner point $c_+$ or $c'_+$. Also the control curves are infinitely tangent to each other.

A second result concerns the possibility of blowing down the invariant manifold and getting smoothness in the original phase space. When blowing down, it is a priori possible to loose differentiability. (Written in polar coordinates, a cone for example is differentiable w.r.t. $(r, \theta)$, but it is not a differentiable object in cartesian coordinates.) In section 3.8.1 we will show an example of such a phenomenom for fields $X_{\epsilon,a,\lambda}$ satisfying all assumptions which we made. So let us ask ourselves the question when the invariant manifolds blow down to differentiable objects. Obviously, a necessary condition is the existence of a Taylor expansion in the origin. The following theorem states that this is in fact also a sufficient condition.

**Theorem 3.4** *Let $X_{\epsilon,a,\lambda}$ be a vector field on a 2-manifold with a simple passage turning point. Because of this, we already know that the blow down of the invariant manifolds from theorem 3.3 are in an admissible chart near the contact point graphs $y = \Psi(x, \epsilon, \lambda)$. Assume that there exist formal power series*

$$\hat{a} = \sum_{n=0}^{\infty} a_n(\lambda)\epsilon^n, \qquad \hat{y} = \sum_{n=0}^{\infty} y_n(x, \lambda)\epsilon^n$$

*so that $y_n$ is smooth in a uniform neighbourhood of $x = 0$, and so that $\hat{y}$ is formally invariant*[2] *under $\hat{X}_{\epsilon,\hat{a},\lambda}$, then $\Psi$ will be smooth in a neighbourhood of the origin. The infinite jet $j_\infty(\Psi)$ will coincide with $\hat{y}$, and also $j_\infty(\mathcal{A})$ will coincide with $\hat{a}$.*

By this theorem, it is also clear that the canard solutions constructed by means of analytic techniques coincide with the invariant manifolds from theorem 3.3. It is well known that such formal solutions are unique, if they exist, under the assumptions that we made.

The above theorem remains valid if one starts with asymptotic series in terms of $\epsilon^{1/m}$ for some $m \in \mathbf{N}_1$; in that case, the blow down manifolds are infinitely smooth w.r.t. $\epsilon^{1/m}$.

The invariant manifolds in theorem 3.3 are constructed by connecting "center manifolds" along $\gamma_-$ to center manifolds along $\gamma_+$. This concept is important in order to understand the next theorem, so let us recall a few notions.

**Definition 3.5** *A "center manifold" for the system (3.1) is the saturation of a local center manifold at a normally hyperbolic point on the critical curve. The saturation may define an invariant manifold up to the contact point, or more specifically, up to a part of the blow up locus. Thus, two classes of center manifolds exist: center manifolds along $\gamma_-$ and along $\gamma_+$. The quotes around center manifolds make it clear that this is* not *a center manifold at the contact point. Important to notice is that the center manifolds depend regularly on $A$, unlike the manifolds of canard solutions, where $A$ has already been expressed in terms of $\epsilon$.*

The idea is that the attracting center manifolds and the repelling center manifolds can be compared when intersecting both with a section transversally cutting the blow up locus. More precisely, in the family rescaling chart we can e.g. take a section $T\colon \{\overline{x} = 0\}$ and look at the intersection of the attracting and repelling center manifolds with $T$. Choosing coordinates $z$ on $\sigma := T \cap \{u = 0\}$, we can use $(z, u)$ as coordinates on $T$. The intersection of the attracting and repelling center manifolds with $T$ are hence graphs

$$z = \zeta_\pm(u, A, \lambda). \tag{3.7}$$

Notice that

$$\zeta_\pm(0, A, \lambda) = z_\pm(A, \lambda),$$

where $z_\pm$ is the graph in (3.3).

Intuitively, the next theorem states that the angle between $\zeta_-$ and $\zeta_+$ is proportional to the angle of the control curve of the canard solutions:

[2]We say that $y = \psi(x, \epsilon)$ is formally invariant under $X_\epsilon\colon \{\dot{x} = f(x, y, \epsilon), \dot{y} = g(x, y, \epsilon)\}$ if the infinite jet of $f(x, \psi(x, \epsilon), \epsilon)\psi'(x, \epsilon) - g(x, \psi(x, \epsilon), \epsilon)$ w.r.t. $\epsilon$ is zero.

**Theorem 3.6** *Under the conditions of theorem 3.3, and with the notations introduced above, we have:*

1. *The first nonzero coefficient of $\epsilon^{1/m}$ in the expansion of $\mathcal{A}$ is related to the first nonzero coefficient in the expansion of $(\zeta_- - \zeta_+)(u, 0, \lambda)$. In fact, they are proportional, and the ratio is precisely $-\rho(\lambda)$, with $\rho(\lambda)$ the nonzero coefficient in (3.3).*

2. *The first nonzero coefficient of $\zeta_- - \zeta_+$ can be calculated, either using the previous item if having a preexisting knowledge of $\mathcal{A}$, or using a Melnikov integral along $\Gamma$ (the heteroclinic connection on the blow up locus from $\zeta_-$ to $\zeta_+$). The integrands can be obtained through a formal recursive process.*

Finally, we present some classes of vector fields having expression (3.1) for which the assumptions of this chapter are satisfied.

For $h = f$ or $g$, we introduce the notation

$$h. := h(0, 0, 0, 0, \lambda), \quad h_x := \frac{\partial h}{\partial x}(0, 0, 0, 0, \lambda),$$

and similar notations for other partial derivatives.

**Theorem 3.7** *If*

*(i)* $f. = 0$, $f_y \neq 0$ *(existence of critical curve by means of implicit function theorem);*

*(ii)* $f_x = 0, f_{xx} \neq 0$ *(at the origin, normal hyperbolicity is lost in the most generic way);*

*(iii)* $g. = 0$, $g_x \neq 0$ *(connection condition);*

*(iv)* $g_x \begin{vmatrix} f_y & f_a \\ f_{xy} & f_{xa} \end{vmatrix} - f_{xx} \begin{vmatrix} f_y & f_a \\ g_y & g_a \end{vmatrix} \neq 0$ *(breaking parameter condition);*

*(v) The product $g_x f_y$ is negative (transition from attracting to repelling).*

*Then, near $(x, y) = (0, 0)$ and for $(a, \epsilon)$ sufficiently small, assumptions T1–T6 are verified for the vector field (3.1), using the blow up $(x, y, \epsilon) = (u\overline{x}, u^2\overline{y}, u^2\overline{\varepsilon})$ and writing $a = uA$. Also, the conditions of theorem 3.4 are satisfied.*

Remark: it may be appropiate to perform some coordinate changes to get the conditions independent of $\lambda$, as required.

The next theorem generalizes the previous one, however, in this case a formal power series solution generally does *not* exist.

**Theorem 3.8** *If*

*(i)* $f_. = 0$, $f_y \neq 0$ *(existence of critical curve by means of implicit function theorem);*

*(ii)* $f_x = 0, f_{xx} = 0, \ldots, f_{x^{2n-1}} = 0, f_{x^{2n}} \neq 0$ *(at the origin, normal hyperbolicity is lost);*

*(iii)* $g_. = 0$, $g_x = 0, \ldots, g_{x^{2n-2}} = 0$, $g_{x^{2n-1}} \neq 0$ *(connection condition);*

*(iv)* $g_{x^{2n-1}} \begin{vmatrix} f_y & f_a \\ f_{xy} & f_{xa} \end{vmatrix} - f_{x^{2n}} \begin{vmatrix} f_y & f_a \\ g_y & g_a \end{vmatrix} \neq 0$ *(breaking parameter condition);*

*(v) The product* $g_{x^{2n-1}} f_y$ *is negative (transition from attracting to repelling).*

*Then, near* $(x, y) = (0, 0)$ *and for* $(a, \epsilon)$ *sufficiently small, assumptions T1–T6 are verified for the vector field (3.1), using the blow up* $(x, y, \epsilon) = (u\overline{x}, u^{2n}\overline{y}, u^{2n}\overline{\varepsilon})$ *and writing* $a = u^{2n-1}A$.

## 3.3 Proof of theorem 3.3

### 3.3.1 Extending manifolds in the blow up space

To extend the center manifolds from section 2.4, we need to blow up the origin. For more information regarding family blow up, we refer to section 1.1. Recall that the parameter plane $(\epsilon, a)$ might have already been rescaled, like in (3.4) with $E = 1$, so that is why this section uses the vector field (3.5) with parameters $(v, A)$. In any case, $v$ plays the role of singular parameter, while $A$ is a regular breaking parameter. We will not blow up $A$, nor $\lambda$. Let

$$\Phi\colon \mathbf{R}^+ \times S^2 \to \mathbf{R}^3\colon (u, (\overline{x}, \overline{y}, \overline{v})) \mapsto (x, y, v) = (u^p\overline{x}, u^q\overline{y}, u^m\overline{v}). \tag{3.8}$$

where $p$, $q$ and $m$ are natural numbers representing weights for the variables $x$, $y$ and $v$. The number $m$ will be especially important, since all objects will be smooth in $u$, and hence smooth in $v^{1/m}$—which might be $\epsilon^{1/mk}$ if a preliminary scaling $\epsilon = v^k$ has taken place, like in (3.4) with $E = 1$ (see also the remark after theorem 3.3).

The preimage of $\gamma_-$ is a subset

$$\overline{\gamma}_- \subset \{\overline{v} = 0\}.$$

So the first place to look at is in a chart $\overline{v} \sim 0$. In other words, we can look in the chart $(\overline{x}, \overline{y}) \in S^1$ ($S^1$ seen as the circle $\{\overline{v} = 0\}$ inside $S^2$). Let $(\overline{x}, \overline{y})$ be represented by an angular value $z$ in the neighbourhood of the endpoint of $\overline{\gamma}_-$, and assume $z = 0$ corresponds to this endpoint.

**Lemma 3.9** *Near $P_-$, and for any $k \in \mathbf{N}$, the vector field $\Phi^*(X)$ is $C^k$ equivalent to*

$$\left\{ \begin{array}{rcl} \dot{u} & = & -u\overline{v}h(u, \overline{v}, A, \lambda) \\ \dot{\overline{v}} & = & m\overline{v}^2 h(u, \overline{v}, A, \lambda) \\ \dot{z} & = & -z, \end{array} \right. \tag{3.9}$$

*for some $C^k$ function $h$. The function $h$ is strictly positive for $\overline{v} > 0$.*

**Proof** Again, we follow the techniques in [DR], and apply theorem 2.7. The necessary conditions can be readily checked: the existence of a $C^{k+2}$-center manifold follows from the general theory in e.g. [HPS]; the presence of the foliation is a result of the blow up: $d(u^m\overline{v}) = 0$ and we also know $dA = 0$, while the partial hyperbolicity is presumed in assumption T3 (normal hyperbolicity at the end point $P_-$).

Hence, a normal form $z\frac{\partial}{\partial z} + Y_\lambda(u, \overline{v}, a)$ is obtained, with $Y$ tangent to the foliation $dA = 0$ and $d(u^m\overline{v}) = 0$. Write

$$Y_\lambda(u, \overline{v}, a) = h_1(u, \overline{v}, A, \lambda)\frac{\partial}{\partial u} + h_2(u, \overline{v}, A, \lambda)\frac{\partial}{\partial \overline{v}}.$$

The set $\{u\overline{v} = 0\}$ is preserved under the normal form with $\{u = 0\}$ as part of the blow up locus, and $\{\overline{v} = 0\}$ outside the blow up locus (corresponding to the manifold $\{\epsilon = 0\}$). In any case, both spaces are invariant under $Y_\lambda$. This invariance implies that $h_1 = O(u)$, and hence

$$h_1(u, \overline{v}, A, \lambda) = -uh_3(u, \overline{v}, A, \lambda),$$

for some function $h_3$. The tangency to $d(u^m\overline{v}) = 0$ then yields

$$h_2(u, \overline{v}, A, \lambda) = m\overline{v}h_3(u, \overline{v}, A, \lambda).$$

The center manifold $\{z = 0\}$ has to include a line of singularities $\{z = 0, \overline{v} = 0\}$. We conclude that $h_3 = O(v)$ and

$$h_3(u, \overline{v}, A, \lambda) = \overline{v}h(u, \overline{v}, A, \lambda)$$

for some function $h$. Since we started with a $C^{k+2}$ normal form, $h$ will at least be $C^k$. The fact that $h$ is nonzero for $\overline{v} > 0$ follows from assumption T4; that it is positive follows from assumption T2 (orbits are compatible to the chosen orientation on the critical curve; hence as $\dot{u}$ must be negative near $\gamma_-$). □

The existence of $C^k$ normal forms for any $k$ enables us to prove $C^\infty$ smoothness of invariant manifolds:

**Proposition 3.10** *The saturation of $\Sigma_-$ forms a smooth invariant manifold in the blow up space, in a neighbourhood of the corner point $P_-$.*

**Proof** We use the normal form in lemma 3.9 to prove the $C^k$-smoothness. We may assume that $\Sigma_-$ is inside the neighbourhood where the normal form is valid, using proposition 2.5 if necessary. We may suppose that the curve is a graph $\{z = \gamma(\overline{v}), u = u_0 > 0\}$, and we suppose that $u_0 > 0$ is chosen such that all $(u, v) \in [0, u_0]^2$ are in the definition domain of the function $h$ of lemma 3.9.

We can reduce to the case $m = 1$ by writing $\tilde{u} = u^m$:

$$\left\{ \begin{array}{rcl} \dot{\tilde{u}} & = & -\tilde{u}\overline{v}\tilde{h}(\tilde{u}, \overline{v}, A, \lambda) \\ \dot{\overline{v}} & = & \overline{v}^2 \tilde{h}(\tilde{u}, \overline{v}, A, \lambda) \\ \dot{z} & = & -z, \end{array} \right.$$

with

$$\tilde{h}(\tilde{u}, \overline{v}, A, \lambda) := mh(\tilde{u}^{1/m}, \overline{v}, A, \lambda)$$

It seems that we loose differentiability in terms of $\tilde{u}$, but for the remainder of the proof, we just need that $\tilde{u}^k \tilde{h}$ is $C^k$, and this is still the case.

So assume now $m = 1$ (and drop the tildes). Observe that $h$ is strictly positive for $\overline{v} > 0$, so in the domain $u\overline{v} > 0$, (3.9) is equivalent to

$$\left\{ \begin{array}{rcl} \dot{u} & = & -u\overline{v} \\ \dot{\overline{v}} & = & \overline{v}^2 \\ \dot{z} & = & -z/h(u, \overline{v}, A, \lambda), \end{array} \right.$$

Fixing $(u_1, \overline{v}_1)$ the orbit in negative time of $(u_1, \overline{v}_1, z_1)$, for any $z_1$, crosses the plane $u = u_0$ at $\overline{v} = u_1\overline{v}_1/u_0$. So, if we take $(u_0, u_1\overline{v}_1/u_0, \gamma(u_1\overline{v}_1/u_0))$ as initial conditions for $(u, \overline{v}, z)$, and if we follow the orbit for a time $T(u_1, \overline{v}_1) = \frac{u_0 - u_1}{u_1\overline{v}_1}$, then we reach a point $(u_1, \overline{v}_1)$ of the saturation of the chosen curve. This yields a graph $z = z(u_1, \overline{v}_1)$:

$$z(u_1, \overline{v}_1) = \gamma(u_1\overline{v}_1/u_0) \exp \left( -\int_0^{T(u_1, \overline{v}_1)} \frac{dt}{h(u_0 - u_1\overline{v}_1 t, \frac{u_1\overline{v}_1}{u_0 - u_1\overline{v}_1 t}, A, \lambda)} \right)$$

Writing

$$t = T(u_1, \overline{v}_1)s,$$

the above expression yields

$$z(u_1, \overline{v}_1) = \gamma(u_1\overline{v}_1/u_0) \exp \left( -\int_0^1 \frac{ds}{g(s, u_1, \overline{v}_1, A, \lambda)} \right)$$

with

$$g(s, u_1, \overline{v}_1, A, \lambda) := \frac{u_1\overline{v}_1}{u_0 - u_1} h \left( u_0 - (u_0 - u_1)s, \frac{u_1\overline{v}_1}{u_0 - (u_0 - u_1)s}, A, \lambda \right)$$

For $u_1$ sufficiently small (let us say $u_1 < \frac{1}{2}u_0$), and $0 \leq s < 1$ the function $g$ is $C^k$ if $u_1\overline{v}_1 \geq 0$ and positive if $u_1\overline{v}_1 > 0$. A problem could raise at $s = 1$: let us however remark that, in the region under consideration (including $s = 1$)

$$0 < \frac{u_1}{u_0 - (u_0 - u_1)s} \leq 1,$$

while

$$\frac{\partial^n}{\partial u_1^n}\left(\frac{u_1}{u_0 - (u_0 - u_1)s}\right) = (-1)^{n-1}n!\frac{u_0 s^{n-1}(1-s)}{(u_0 - (u_0 - u_1)s)^{n+1}},$$

such that $u_1^n \frac{\partial^n}{\partial u_1^n}(\frac{u_1}{u_0-(u_0-u_1)s})$ is bounded. It hence easily follows that

$$u_1^{k-1} g(s, u_1, \overline{v}_1, A, \lambda)$$

is of class $C^k$. This observation will allow us to use proposition 2.14 in the appendix. From this proposition follows

$$\exp\left(-\int_0^S \frac{ds}{g(s, u_1, \overline{v}_1, A, \lambda)}\right)$$

is $C^k$ for all $0 < S \leq 1$, $u_1\overline{v}_1 \geq 0$. Intersecting with the plane $S = 1$ gives us the smoothness of $z(u_1, \overline{v}_1)$, and we also know that $z(u_1, \overline{v}_1)$ is $C^k$-flat at $u_1\overline{v}_1 = 0$.

We have shown that the saturation is $C^k$ in all points including the boundary $u_1\overline{v}_1 = 0$. The boundary consists of the critical curve up until its intersection with the blow up locus, together with the invariant manifold on the blow up locus. Once we have passed the corner point, no more singularities are expected in a neighbourhood of $A = 0$, $u = 0$ (assumption T4). This means that we can saturate the curve beyond the neighbourhood where the normal form is valid. This implies that the saturation is $C^k$ in a neighbourhood that does not depend on $k$. Since the result holds for all $k$, this proves the $C^\infty$ smoothness in a neighbourhood of $P_-$. □

### 3.3.2 Connecting the center manifolds

The saturation of the section $\Sigma_-$ forms a smooth invariant manifold $W_-$, as above, and by reversing time, so will the saturation of $\Sigma_+$ along $-X_{v,A,\lambda}$. Along this work, we had to reduce the neighbourhood in which $v$ can vary, possibly it has been necessary to restrict $A$ to a small neighbourhood of the origin; on the compact set $\Lambda$ we did not put constraints.

To connect the two manifolds $W_-$ and $W_+$ together (and hence construct canard solutions), we consider a chart on the blow up locus where both manifolds are shown to exist. This is the family rescaling chart, shortly denoted as the chart $\overline{v} = 1$. Here, the variables are rescaled as follows:

$$\begin{cases} x &= u^p \overline{x} \\ y &= u^q \overline{y} \\ v &= u^m. \end{cases}$$

In this chart, $u$ becomes a regular perturbation parameter; we obtain a family of vector fields

$$\overline{X}_{u,a,\lambda}\colon \begin{cases} \dot{\overline{x}} &= \overline{f}(\overline{x},\overline{y},u,A,\lambda) \\ \dot{\overline{y}} &= \overline{g}(\overline{x},\overline{y},u,A,\lambda). \end{cases}$$

We have assumed in assumption T5 (connection condition) that a heteroclinic connection $\Gamma$ exists that connects $\gamma_-$ to $\gamma_+$. Since $\overline{X}$ does not have any singularities in the neighbourhood of any compact piece of $\Gamma$, we can extend the two manifolds $W_-$ and $W_+$ to meet in a transversal section $T$: choose a smooth transverse section (transversally intersecting $\Gamma$, and hence locally transverse to the flow of $\overline{X}$), then $W_-$ intersects $T$ in a smooth curve inside $T$, and so will $W_+$. Denote these curves by $\zeta_-$ and $\zeta_+$.

Let $\sigma$ be the intersection of $T$ with the blow up locus $\{u=0\}$. Choose a coordinate system on $\sigma$, and denote the coordinate by $z$. Then, locally $(z,u)$ is a coordinate system in $T$, and $\zeta_-$ and $\zeta_+$ are graphs in $(u,A,\lambda)$. Let $z_\pm(u,A,\lambda)$ be such graph representations (w.r.t. some fixed coordinate system of $T$), then from assumptions T5 and T6 we know:

(i) $z_-(0,0,\lambda)=z_+(0,0,\lambda)$

(ii) $\frac{\partial z_-}{\partial A}(0,0,\lambda)\neq\frac{\partial z_+}{\partial A}(0,0,\lambda)$.

The first condition is merely the existence of a heteroclinic connection $\Gamma$ on the blow up locus for $A=0$. In proposition 3.19 we derive a formula that could be used to check the above equations.

Anyway, under these assumptions, we can use the implicit function theorem to solve

$$z_-(u,\mathcal{A}(u,\lambda),\lambda)=z_+(u,\mathcal{A}(u,\lambda),\lambda)$$

for $\mathcal{A}$. This proves theorem 3.3.

## 3.4 Proof of theorem 3.4

The smoothness of manifolds in the blow up space does not necessarily imply the smoothness in the standard phase space. But, we can recover a great deal. First, outside the blow up locus the blow up map is a regular diffeomorphism, so outside the origin we *can* prove the smoothness of the constructed invariant manifolds. To show smoothness at the origin, extra arguments are needed.

A second observation is the necessity of the conditions in theorem 3.4. If a formal expansion does not exist at the origin, then the manifolds can never be smooth there—this is because the Taylor expansion of any smooth manifold would agree with these conditions.

A third observation is relevant: sometimes, a formal expansion can be found to be formally invariant under the vector field, up to order $k$. In that case, we are able to prove the $C^k$-smoothness of the invariant manifolds in the origin.

### 3.4.1 Reduction to $(v, A)$ parameters

The formal expansions in theorem 3.4 are expressed in terms of $(\epsilon, a)$, but if necessary, we can replace the expansions by expansions in terms of $(v, A)$. Indeed, if

$$(a, \epsilon) = (v^k A, v^\ell),$$

for some $(k, \ell)$ (following the rescaling in (3.4)), we can translate the formal power series $\hat{a}$ to a series in $\hat{A}$; this yields $\hat{A} = v^{-k}\hat{a}(v^\ell, \lambda)$. In this section we will focus on two results: on one hand that $\hat{A}$ is a genuine power series in $v$ and on the other hand that smoothness in terms of $(v, \lambda)$ of canard manifolds is equivalent to smoothness in terms of $(\epsilon, \lambda)$.

Instead of continuing to work with the formal power series, we realize these series as Taylor series of smooth functions; this is possible due to Borel's theorem (see e.g. [WA]). Choose

$$a = \tilde{a}(\epsilon, \lambda), \qquad y = \tilde{\varphi}(x, \epsilon, \lambda).$$

We can consider $y = \tilde{\varphi}(x, \epsilon, \lambda)$ as a perturbation of a canard manifold $W$, but it is advantageous to see it as a canard manifold of a perturbed vector field. We therefore introduce an error function for $(\tilde{\varphi}, \tilde{A})$:

$$E(x, \epsilon, \lambda) := f(x, \tilde{\varphi}(x, \epsilon), \epsilon, \tilde{a}(\epsilon, \lambda), \lambda)\frac{\partial \tilde{\varphi}}{\partial x}(x, \epsilon) - vg(x, \tilde{\varphi}(x, \epsilon), \epsilon, \tilde{a}(\epsilon, \lambda), \lambda).$$

Due to the formal invariance of $(\hat{a}, \hat{\varphi})$, we know that $E$ is flat in $\epsilon$, uniformly in $(x, \lambda)$. Consider now a slightly altered vector field

$$\tilde{X}_{\epsilon,a,\lambda} : \begin{cases} \dot{x} &= f(x, y, \epsilon, a, \lambda) \\ \dot{y} &= vg(x, y, \epsilon, a, \lambda) + E(x, \epsilon, \lambda) \end{cases} \tag{3.10}$$

Clearly, for the family $\tilde{X}$ the same blow up procedure as for $X$ leads to a good desingularization. Hence, theorem 3.3 applies to $\tilde{X}$ and we can choose a control curve $a = \mathcal{A}(\epsilon, \lambda)$ as a blow down of a $C^\infty$ control curve expressed in $(A, v)$-coordinates. Because the same blow up weights were used, it is clear that $\mathcal{A}(\epsilon, \lambda) = \epsilon^{k/\ell}\mathcal{A}_1(\epsilon^{1/\ell}, \lambda)$. Because $\mathcal{A}$ is infinitely tangent to $\tilde{a}$, it is clear that $\hat{A} = \hat{\mathcal{A}}_1(v, \lambda)$, implying that it is a genuine series in $v$.

We can now safely state that under the conditions of theorem 3.4, there exist

$$\hat{A} = \sum_{n=0}^{\infty} A_n(\lambda)v^n, \qquad \hat{\tilde{y}} = \sum_{n=0}^{\infty} \tilde{y}_n(x, \lambda)v^n,$$

with $\hat{\tilde{y}}$ being a formal canard "solution" of (3.5), and where $\tilde{y}$ is obtained using $\hat{A} = v^{-k}\hat{a}$. Assume for a moment that we can prove that the graph $y = \tilde{\Psi}(x, v, \lambda)$ is smooth in terms of $(x, v, \lambda)$, then we can go back to the original parameters:

$$y = \Psi(x, \epsilon, \lambda) := \tilde{\Psi}(x, \epsilon^{1/\ell}, \lambda),$$

and because the formal expansion of $\Psi$ only has terms in powers of $v^\ell$, the above function will still be smooth in $\epsilon$. Similarly, the smoothness of a control curve in terms of $v$ will imply the smoothness of the control curve in terms of $\epsilon$.

### 3.4.2 Canard solution manifold as a graph

Suppose a formal expansion as in theorem 3.4 exists. From now on, we will interpret theorem 3.4 in terms of $(v, A)$ instead of in terms of $(\epsilon, a)$. Reading this section, one can treat $(v, A)$ as being equal to $(\epsilon, a)$, but for the sake of generality we prefer to make a distinction.

Take a control curve $\mathcal{A}(v, \lambda)$ and a manifold of canard solutions $W$ as in theorem 3.3. The blow down of this manifold $W$ is, when restricted to $v = 0$, at least continuous at the origin (because of the existence of a connection on the blow up locus). Outside the origin, this restriction must coincide with the critical curve, and this critical curve can be written as a graph in $x$ (this is assumed in theorem 3.4). Hence, locally around the origin, we can blow down $W$ to a graph in $(x, v, \lambda)$. Assume now that

$$y = \Psi(x, v, \lambda)$$

is the blow down of $W$. We already know that it is smooth outside $(x, v) = (0, 0)$, for $x$ close to 0. To prove theorem 3.4, it suffices now to prove:

**Proposition 3.11** *For all $n, r \in \mathbf{N}$ we have*

$$\lim_{v\to 0} \frac{1}{n!} \frac{\partial^{n+r}\Psi}{\partial v^n \partial x^r}(x, v, \lambda) = \frac{\partial^r y_n}{\partial x^r}(x, \lambda),$$

*where $y_n$ is defined in the statement of theorem 3.4.*

As before we realize these series as Taylor series of smooth functions

$$A = \tilde{A}(v, \lambda), \qquad y = \tilde{\varphi}(x, v, \lambda).$$

so that the infinite jet of $\tilde{A}$ resp. $\tilde{\varphi}$ coincides with the formal power series $\hat{A}$ resp. $\hat{\varphi}$. In view of proving the proposition, we can now say that it is necessary to prove:

$$\begin{cases} \lim_{v\to 0} \dfrac{\partial^{n+r}\Psi}{\partial v^n \partial x^r}(x, v, \lambda) = \dfrac{\partial^{n+r}\tilde{\varphi}}{\partial v^n \partial x^r}(x, 0, \lambda). \\ \lim_{v\to 0} \dfrac{\partial^n \mathcal{A}}{\partial v^n}(v, \lambda) = \dfrac{\partial^n \tilde{A}}{\partial v^n}(0, \lambda), \end{cases}$$

uniformly in $\lambda$ and for all $(n, r)$. One can look at these expressions in blow up coordinates, in the various charts. In the family rescaling chart ($\overline{v} = 1$) it suffices to check that

$$\begin{cases} \lim_{u\to 0} \dfrac{\Psi(u^p\overline{x}, u^m, \lambda) - \tilde{\varphi}(u^p\overline{x}, u^m, \lambda)}{u^s} = 0, \\ \lim_{u\to 0} \dfrac{\mathcal{A}(u^m, \lambda) - \tilde{A}(u^m, \lambda)}{u^s} = 0, \end{cases} \tag{3.11}$$

uniformly in $\lambda$ and for all $s \in \mathbf{N}$. We do not need to prove similar conditions on $\frac{\partial^{n+r}\Delta}{\partial v^n \partial x^r}$ with $\Delta = \Psi - \tilde{\varphi}$, since the function $\overline{\Delta}(u, \overline{x}, \lambda) := \Delta(u^p \overline{x}, u^m, \lambda)$ is a $C^\infty$ function. The existing relations between the derivatives w.r.t. $(v, x)$ and the derivatives of $\overline{H}$ w.r.t. $(u, \overline{x})$ will imply the necessary conditions.

We must investigate these expressions in a full neighbourhood of $(v, x) = (0, 0)$, so it is necessary to look at the phase directional rescaling as well. There, the denominator will be of the form $(u^r \overline{v}^\ell)$ for arbitrary $(r, \ell)$. Supposing that one needs to look in the directional chart $\{\overline{x} = -1\}$, one finds sufficient conditions to be

$$\lim_{u^m \overline{v} \to 0} \frac{\Psi(-u^p, u^m \overline{v}, \lambda) - \tilde{\varphi}(-u^p, u^m \overline{v}, \lambda)}{u^r \overline{v}^s} = 0. \tag{3.12}$$

(If the above equation is true, then one has that the numerator is $O(u^r \overline{v}^s)$, and, due to the smoothness in terms of $(u, \overline{v})$, that its $n$-th derivative w.r.t. $u$ is $O(u^{r-n} \overline{v}^s)$. Similarly, derivatives w.r.t. $\overline{v}$ and $\lambda$ are treated.)

### 3.4.3 Perturbing the vector field

We can consider $y = \tilde{\varphi}$ as a perturbation of the manifold $W$, but we can also regard it as a manifold of canard solution of a perturbed vector field. We therefore introduce an error function for $(\tilde{\varphi}, \tilde{A})$ :

$$E(x, v, \lambda) := f(x, \tilde{\varphi}(x, v), v, \tilde{A}(v, \lambda), \lambda) \frac{\partial \tilde{\varphi}}{\partial x}(x, v) - v g(x, \tilde{\varphi}(x, v), v, \tilde{A}(v, \lambda), \lambda).$$

Due to the formal invariance of $\hat{A}$ and $\hat{y}$, we know that $E$ is flat in $v$, uniformly in $(x, \lambda)$. Consider now a slightly altered vector field

$$\tilde{X}_{v,A,\lambda} \colon \left\{ \begin{array}{rcl} \dot{x} &=& f(x, y, v, A, \lambda) \\ \dot{y} &=& v g(x, y, v, A, \lambda) + E(x, v, \lambda) \end{array} \right. \tag{3.13}$$

Then, the graph $y = \tilde{\varphi}(x, v, \lambda)$ defines a smooth manifold $\tilde{W}$ and $(\tilde{A}, \tilde{W})$ is a manifold of canard solutions for $\tilde{X}$.

So, on one hand we have a family of vector fields (3.5), and on the other hand, we have a family of vector fields (3.13) from which we know that it has an invariant manifold $\{A = \tilde{A}(v, \lambda), y = \tilde{\varphi}(x, v, \lambda)\}$. Both families of vector fields are strongly related to each other: the infinite jets with respect to $v$ of both vector fields are the same.

Consider the blow up map introduced for the vector field $X$. If we apply the same blow up map on $\tilde{X}$, then we can compare $X$ with $\tilde{X}$ in blow up coordinates. They are everywhere infinitely tangent to each other along the blow up locus. Looking in the phase directional rescaling charts, we used $C^k$-normal form coordinates as in lemma 3.9, so applying the same transformation to $\overline{\tilde{X}}$, it is not hard to prove that the perturbed vector field looks like

$$\overline{\tilde{X}}: \begin{cases} \dot{u} &= -u\overline{v}\tilde{h}(u,\overline{v},z,A,\lambda) \\ \dot{\overline{v}} &= m\overline{v}^2\tilde{h}(u,\overline{v},z,A,\lambda) \\ \dot{z} &= -z+\tilde{f}(u,\overline{v},z,A,\lambda), \end{cases} \tag{3.14}$$

where $\tilde{h}-h = O(u^r\overline{v}^\ell)$, $\tilde{f} = O(u^r\overline{v}^\ell)$ for $(r,\ell)$ arbitrary high (provided that we choose a $C^k$ normal form with $k \geq r+\ell$ and shrink the neighbourhood of $(u,\overline{v},z) = (0,0,0)$).

The following lemma proves (3.12):

**Lemma 3.12** *Let $z = \tilde{\psi}(u,\overline{v},A,\lambda)$ be invariant under the $C^k$ vector field (3.14) in a neighbourhood of $(u,\overline{v},z) = (0,0,0)$. Then $z = \tilde{\psi}(u,\overline{v},A,\lambda)$ is $O(u^r\overline{v}^\ell)$ flat to $z = 0$, with $r+\ell \leq k$. This asymptotic property is uniform in $\lambda \in \Lambda$ and in $A$ near the origin.*

Before proving this lemma, we show how this lemma can be used to prove (3.11), finishing the proof of theorem 3.4. In the family rescaling chart, these invariant manifolds can be extended until they intersect the section $T$, and we can do the same thing in backward time for manifolds coming from the other side. The lemma states that at some point (near infinity in the family rescaling chart) the invariant manifold of $\overline{\tilde{X}}$ is $O(u^r)$-close to the invariant manifold of $\overline{X}$.

Following the flow of $\overline{X}$ in the family rescaling map will not decrease the order of separation between the two manifolds; this is due to the absence of singularities in a tube around the heteroclinic connection $\Gamma$, and due to the fact that $\dot{u} = 0$ there.

If we apply the implicit function theorem to connect the center manifolds coming from the attracting side and from the repelling side, then no difference is seen between the perturbed invariant manifolds and the actual invariant manifolds up to order $r$. So in the implicit solution we will see no difference between $\mathcal{A}$ and $\tilde{A}$ up to order $r$. This means that $\mathcal{A}$ and $\tilde{A}$ are asymptotic of order $O(u^r)$ for any $r$. Hence, the second part of (3.11) is shown. The first part easily follows now too, since from now on, we can treat $\mathcal{A}$ and $\tilde{A}$ to be the same, and the invariant manifold will then stay $O(u^r)$-close to the perturbed invariant manifold uniformly in any compact subset of the family rescaling chart.

Remains to prove the lemma.

**Proof** (of lemma 3.12) The smoothness of such manifolds has already been proved. So let us first consider the restriction to $u = 0$. This restriction is invariant under

$$\overline{v}^2 h(0,\overline{v},A,\lambda)\frac{\partial}{\partial\overline{v}} + (-z)\frac{\partial}{\partial z}.$$

In the domain $\overline{v} \geq 0$, there is a unique invariant (center) manifold: $z = 0$. Hence $\tilde{\psi} = O(u)$. Writing now $z = uz_1$, we can pullback the vector field and write it in terms of $(u, \overline{v}, z_1)$. This yields

$$\begin{cases} \dot{u} &=& -u\overline{v}\tilde{h}(u, \overline{v}, uz_1, A, \lambda) \\ \dot{\overline{v}} &=& m\overline{v}^2\tilde{h}(u, \overline{v}, uz_1, A, \lambda) \\ \dot{z_1} &=& -(1 - \overline{v}\tilde{h}(u, \overline{v}, uz_1, A, \lambda))z_1 + \frac{1}{u}\tilde{f}(u, \overline{v}, uz_1, A, \lambda) \end{cases}$$

which leads to an equivalent vector field

$$\begin{cases} \dot{u} &=& -u\overline{v}\tilde{h}_1(u, \overline{v}, z, A, \lambda) \\ \dot{\overline{v}} &=& m\overline{v}^2\tilde{h}_1(u, \overline{v}, z, A, \lambda) \\ \dot{z_1} &=& -z_1 + \tilde{f}_1(u, \overline{v}, z_1, A, \lambda), \end{cases}$$

but where $\tilde{f}_1 = O(u^{r-1}\overline{v}^\ell)$ and is a $C^{k-1}$ function. This new vector field has an invariant manifold $z_1 = \psi_1(u, \overline{v}, A, \lambda) := \frac{1}{u}\tilde{\psi}(u, \overline{v}, A, \lambda)$. Remains to prove that $\psi_1$ is $O(u^{r-1}\overline{v}^\ell)$. Continuing this process reduces to the case $r = 0$. So let us now assume $r = 0$. Looking at $\overline{v} = 0$, we find that $z_r = \tilde{\psi}_r(u, 0, A, \lambda)$ is identically 0, since $\tilde{f}$ is $O(\overline{v}^\ell)$, with $\ell \geq 1$. Hence, we can proceed as before by writing $z = \overline{v}z_1$. □

## 3.5 Proof of theorem 3.6

In section 3.3, we have proved the existence of the "center manifolds" coming from both sides of the blow up locus, and meeting somewhere on the blow up locus. In fact, they only meet for $\epsilon = a = 0$. So what happens when $\epsilon \neq 0$, or $a \neq 0$? In these cases, the manifolds are separated. We derive formulas for calculating the separation of these manifolds.

Since the comparison between the attracting and repelling center manifolds is done in the family rescaling charts, we will work most of the time in this chart.

In section 3.5.2 we state some facts on saddle connections in the plane using Melnikov theory, which we will use to prove the second part of theorem 3.6.

### 3.5.1 The relation between angle and the control curve

We focus on the first part of theorem 3.6. We use the notations from theorem 3.6. Let $\mathcal{A}(u, \lambda)$ be a control curve as in theorem 3.3, and define

$$\Delta(u, A, \lambda) = \zeta_-(u, A, \lambda) - \zeta_+(u, A, \lambda)$$

where $\zeta_\pm$ hase been defined in (3.7). $\Delta$ is the separation of the forward and backward center manifolds. Of course, $\Delta$ as well as $\mathcal{A}$ depend on the boundary curves $\Sigma_-$ and $\Sigma_+$ chosen in theorem 3.6, but the asymptotic expansion is unique. Suppose

$$\mathcal{A}(u, \lambda) = a_r(\lambda)u^r + o(u^r),$$

for $r \geq 1$ and with $a_r(\lambda) \neq 0$, then,

$$\Delta(u, \mathcal{A}(u, \lambda), \lambda) \equiv 0.$$

This has an effect on the asymptotic expansion:

$$\Delta(u, 0, \lambda) + \mathcal{A}(u, \lambda)\frac{\partial \Delta}{\partial a}(u, 0, \lambda) = o(\mathcal{A}(u, \lambda)).$$

Hence,

$$\lim_{u \to 0}\left(\frac{\Delta(u, 0, \lambda)}{\mathcal{A}(u, \lambda)} + \frac{\partial \Delta}{\partial a}(0, 0, \lambda)\right) = 0. \tag{3.15}$$

Notice that $\frac{\partial \Delta}{\partial a}(0, 0, \lambda)$ is the nonzero number $\rho(\lambda)$ in (3.3). (An expression for this number is calculated in proposition 3.19.) Conclusion:

$$\left\{\begin{array}{rcl} \mathcal{A}(u, \lambda) & = & a_r(\lambda)u^r + o(u^r) \\ \Delta(u, 0, \lambda) & = & b_r(\lambda)u^r + o(u^r), \end{array}\right. \qquad \text{with } \frac{b_r(\lambda)}{a_r(\lambda)} = -\rho(\lambda) \neq 0. \tag{3.16}$$

Note that (3.15) is slightly more general than (3.16), since it can also be used in the case $\mathcal{A}(u, \lambda)$ is infinitely flat in $u$. But in such a case, we must add the assumption that $\mathcal{A}(u, \lambda) \neq 0$ for $u \neq 0$.

### 3.5.2 Perturbations of regular orbits in the plane

We intend to study the breaking of a heteroclinic connection on the plane, inducing expressions that can be used in the specific problem that we want to investigate. We will consider a 1-dimensional parameter $\mu$, which will be the bifurcation parameter. The whole setting may depend on other parameters $\lambda$, which are "trivial" in the sense that they do not induce bifurcations. We hence will work with a family, depending on $\lambda$, of 3-dimensional situations. We will keep $\mu$ a 1-dimensional parameter although it is not necessary.

Consider a vector field

$$X_\mu \colon \left\{\begin{array}{rcl} \dot{x} & = & f(x, y, \mu) \\ \dot{y} & = & g(x, y, \mu), \end{array}\right.$$

where $f$ and $g$ are smooth on $\mathbf{R}^2 \times P$. We will work most of the time with the extended vector field $X_\mu + 0\frac{\partial}{\partial \mu}$. Let $\varphi(t, (x, y, \mu))$ be the flow for this vector field. Choose now a (fibred) section $T \subset \mathbf{R}^2 \times P$ transverse to the flow of $X_\mu$, with a coordinate mapping

$$\psi \colon (h, \mu) \mapsto (\psi_0(h, \mu), \mu).$$

We try to calculate intersections of heteroclinic connections passing through $T$ in these coordinates. In this section however, instead of a heteroclinic connection, a perturbation of a regular orbit is considered. In the next section, we will see how the results can be maintained if we let the orbit tend to a heteroclinic connection.

Consider the projection on $T$ along the orbits of $X_\mu$:

$$P(x, y, \mu) := \varphi(\tau(x, y, \mu), (x, y, \mu)),$$

where $\tau$ is the transition time to go from $(x, y, \mu)$ to a point in $T$. Of course, $P$ is not defined everywhere, but is certainly defined in an open neighbourhood of the chosen orbit, at least if the chosen orbit is cutting $T$ just one time. One sees that $P$ is constant along orbits of $X_\mu$ and hence

$$H(x, y, \mu) := \langle (1, 0), \psi^{-1}(P(x, y, \mu)) \rangle$$

(the $h$ coordinate of $\psi^{-1}(P(x, y, \mu))$) is a first integral for $X_\mu$ in some open neighbourhood of the orbit.

**Lemma 3.13** *There is an integrating factor $\theta$ defined on the domain of $P$ so that*

$$X_\mu \colon \left\{ \begin{array}{lcl} \dot{x} & = & -\theta(x, y, \mu)^{-1} \frac{\partial H}{\partial y}(x, y, \mu) \\ \dot{y} & = & \theta(x, y, \mu)^{-1} \frac{\partial H}{\partial x}(x, y, \mu). \end{array} \right.$$

**Proof** Because $H$ is invariant along orbits of $X_\mu$, one has $f \frac{\partial H}{\partial x} + g \frac{\partial H}{\partial y} = 0$. So one could define $\theta$ as a factor between 2 colinear nonzero vectors $(f, g)$ and $(-\frac{\partial H}{\partial y}, \frac{\partial H}{\partial x})$. □

The key to finding the angle between the manifolds lies in the study of $\theta$. But first, we will show how to use the lemma to calculate the intersection.

Let $\gamma$ be the chosen orbit for $\mu = 0$, and assume $\gamma$ cuts $T$ transversally and just one time. We study perturbations of $\gamma$ as follows: we consider a *vertical* line segment (segments of the form $\{(x, y, \mu) | \mu \in (\mathbf{R}, 0)\}$, with $(x, y)$ chosen on $\gamma$) and let points of such a line segment flow. It is in a way unnatural to choose line pieces instead of more general curves, but in view of proving the results in this chapter, vertical line segments suffice. The flow of this line segment is intersected with $T$, and compared to $\gamma \cap T$.

The curve $\gamma$ intersects $T$ in a single point, but if we lift the curve vertically ($\pi^{-1}(\gamma)$ if $\pi$ is the projection onto $\{\mu = 0\}$), then the intersection of $\pi^{-1}(\gamma)$ with $T$ is some curve, say parametrized as $h = c(\mu)$. Any point above $\gamma$ meets this curve, if it follows the flow of $X_0$. So consider a point $(p, \mu)$ above $\gamma$ (i.e. $p \in \gamma$), and let it flow along $X_0$ until it meets $T$. Elementary properties of line integrals gives us

$$c(\mu) - H(p, \mu) = \int X_0(H) dt,$$

where the integration is along the curve $\gamma$ (lifted to height $\mu$) from $(p, \mu)$ to $T$, parametrized by the time of $X_0$. (Indeed, one can see that the above integral is equal to the line integral $\int_\gamma \operatorname{grad} H$.)

We could calculate $c(\mu)$, but this is not necessary: let $(q, \mu)$ be another point above $\gamma$ at the same height, then

$$H(p, \mu) - H(q, \mu) = -\int_{\gamma(p,q)} X_0(H)dt, \tag{3.17}$$

where $\gamma(p, q)$ is the piece of $\gamma$ from $p$ to $q$.

In this section, we will only describe $j_k^\mu H(p) - j_k^\mu H(q)$ (writing $j_k^\mu$ for the $k$-jet w.r.t. $\mu$); only in a later section we will proceed to the limit (letting $p$ and $q$ tend to infinity, or in other words, letting $\gamma(p, q)$ tend to a heteroclinic connection).

For fixed $p$ and $q$, we have

$$j_k^\mu H(p) - j_k^\mu H(q) = -\int_{\gamma(p,q)} j_k^\mu X_0(H)dt,$$

Elaborating the integrand yields

$$X_0(H) = f_0 \frac{\partial H}{\partial x} + g_0 \frac{\partial H}{\partial y} = \theta(f_0 g - g_0 f).$$

So, if we are able to calculate $\theta$ up to any order, then we can calculate the contact between the invariant manifolds up to any order.

For calculating $\theta$ up to any order, observe that since $d^2 = 0$ and $dH = \theta\omega_X$, with $\omega_X$ the 1-form associated to $X_\mu$ one finds

$$d\theta \wedge \omega_X + \theta d\omega_X = 0$$

which amounts to saying that

$$X_\mu(\theta) = -\theta \operatorname{div} X_\mu. \tag{3.18}$$

This expression gives $\theta$ as a solution to a differential equation; if we have an initial condition for $\theta$, we can calculate $\theta$ explicitely. An appropiate initial condition is obtained as follows:

**Lemma 3.14** *In points $p \in T$, we have*

$$\theta(p) := \det\left(\partial_h \psi_0(h, \mu) \,|\, X_\mu(p)\right)^{-1},$$

*with $h = H(p)$. (We write $\partial_h \psi_0(h, \mu)$ as the first column in a matrix, and $X_\mu(p)$ as the second column.)*

**Proof** There is a strong relation between $H$ and $\psi$, since

$$H(\psi_0(h, \mu), \mu) = h.$$

Derive this equation with respect to $h$:

$$D_{(x,y)} H(\psi_0(h, \mu), \mu)\partial_h \psi_0(h, \mu) = 1.$$

Let $p = \psi_0(h, \mu)$, then the above equation becomes

$$D_{(x,y)}H(p)\partial_h \psi_0(h, \mu) = 1,$$

or differently written

$$\det \left( \partial_h \psi_0(h, \mu), (-\frac{\partial H}{\partial y}(p), \frac{\partial H}{\partial x}(p)) \right) = 1.$$

Since $\theta f = -\frac{\partial H}{\partial y}$ and $\theta g = \frac{\partial H}{\partial x}$ it gives the result. □

The problem with this result is that the initial condition $\theta$ depends on $\mu$. We will see that this is a problem that we would like to avoid. First, notice that if $\theta_0 = \theta|_{\mu=0}$ then

$$X_0(\theta_0) = -\theta_0 \operatorname{div} X_0,$$

so

$$\theta_0(p) = \theta_0(P(p, 0)) \exp \left( \int_{O(p,0)} \operatorname{div} X_0 dt \right),$$

where $O(p, 0)$ is the orbit along $X_0$ from $p$ to the intersection point $P(p, 0)$ of the orbit with $T$. Assuming we can calculate this integral explicitely, then we can move on to $\theta_1 := \frac{\partial \theta}{\partial \mu}|_{\mu=0}$. However, plugging this into equation (3.18), we see that at some point we need to calculate $X_1(\theta_0)$, with $X_1 = \frac{\partial X}{\partial \mu}|_{\mu=0}$.

This means that we must be able to derive $\theta_0 \circ P$ as well as an integral along orbits of $X_0$. It is however possible to avoid deriving $\theta_0 \circ P$, if we choose the section $T$ in such a way that $\theta_0$ does not depend on $\mu$. Since the choice of the section will not affect implicit results (such as the calculation of the control curve), we have some freedom in the choice of the coordinate system on $T$.

**Lemma 3.15** *On $T$, there exists a coordinate system (and an associated integrating factor $\theta$ on the blow up locus) so that for all points $p \in T$ close enough to the intersection point $\gamma \cap T$, we have*

$$\theta(p) = 1.$$

**Proof** Let $h = \alpha(k, \mu)$ be a regular change of coordinates, so that

$$\psi_\alpha(k, \mu) := (\psi_0(\alpha(k, \mu), \mu), \mu)$$

is a new coordinate function for $T$. We will put conditions on $\alpha$ so that $\theta$ with respect to this new coordinate function has the required property. According to lemma 3.14, we need to solve

$$g(p)\frac{\partial}{\partial k}(\psi_{0x}(\alpha(k, \mu), \mu)) - f(p)\frac{\partial}{\partial k}(\psi_{0y}(\alpha(k, \mu), \mu)) = 1,$$

where $p = \psi_\alpha(k,\mu)$. Working out the above expression yields a differential equation for $\alpha$:

$$\frac{d\alpha}{dk} = \left.\frac{1}{g(p)\frac{\partial\psi_{0x}}{\partial h}(\alpha,\mu) - f(p)\frac{\partial\psi_{0y}}{\partial h}(\alpha,\mu)}\right|_{p=(\psi_0(\alpha,\mu),\mu)}.$$

The local existence of solutions of differential equations implies the result. □

In these coordinates, we can calculate $\theta$ more easily. Writing

$$X_\mu = X_0 + \mu X_1 + \mu^2 X_2 + \cdots ,$$

and

$$\theta_\mu = \theta_0 + \mu\theta_1 + \mu^2\theta_2 + \cdots ,$$

we can give recursive formulas for $\theta_i$ (always under the assumption that $\mu$ is 1-dimensional):

$$\sum_{i,j} X_i(\theta_j)\mu^{i+j} = \sum_{i,j}(-\theta_j \operatorname{div} X_i)\mu^{i+j} \tag{3.19}$$

At zero order:

$$\theta_0 = \exp(\int_{O(p,0)} \operatorname{div} X_0 dt),$$

with $O(p,0)$ the orbit from $(p,0)$ to the intersection point at $T$. For higher orders, we have to solve differential equations at each point, but an integration along orbits of $X_0$ will always lead to solutions. This knowledge combined with the fact that the initial conditions are trivial leads to a recursion only depending on the jets of $X_\mu$ along $\mu = 0$. Note that the integrands at higher order could contain $X_i(\theta_j)$, so a way of deriving $\theta_j$ needs to be available to be able to calculate the integrals. All these integrals only involve the unperturbed vector field, but still they can be quite complicated.

Using (3.19), one finds

$$\theta_n = \theta_0 \int_{O(p,0)} \sum_{k=0}^{n-1} (\theta_k/\theta_0)(\operatorname{div} X_{n-k} + X_{n-k}(\log\theta_0) + X_{n-k}(\theta_k/\theta_0))dt. \tag{3.20}$$

### 3.5.3 Heteroclinic connections on the blow up locus

Let us apply the results of section 3.5.2 to the setting of this chapter. Take a section $T$ transverse to the flow of $\overline{X}$ on the blow up locus (in the family rescaling chart), and assume $T$ intersects the heteroclinic connection $\Gamma$ which connects the points at infinity $P_-$ and $P_+$.

Write

$$\theta(\overline{x},\overline{y}) := \exp\left(\int_{\mathcal{O}(\overline{x},\overline{y})} \operatorname{div} \overline{X}\big|_{u=A=0}\, dt\right). \tag{3.21}$$

where $\mathcal{O}(\overline{x}, \overline{y})$ denotes the orbit along the unperturbed vector field from $(\overline{x}, \overline{y})$ to the intersection point in $T$. We have chosen a coordinate system on $T$ so that $\theta = 1$ for points on $T$ (see lemma 3.15).

Let $\zeta_-$ be the intersection of an invariant manifold $W_-$ from section 3.3.2 with $T$, and let $\zeta_+$ be the intersection of $W_+$ with $T$ for some choices for the manifolds $W_-$ and $W_+$. In the chosen coordinate system, let $z_-(u, A, \lambda)$ be the graph representation of $\zeta_-$ and $z_+(u, A, \lambda)$ be the graph representation of $\zeta_+$.

Following the techniques of the previous section, we can calculate the $k$-jets $j_k(z_+ - z_-)$, with respect to $\mu$, provided that we are able to calculate $X_0(H)$ for a first integral $H$. The link between $X_0(H)$ and $\zeta_\pm$ is stated in a limit version of formula (3.17):

**Lemma 3.16** *Consider the vector field $\overline{X}|_{A=0}$ with the perturbation parameter $u$. Then, in (3.17),*

$$\lim_{p \to P_-} j_k^u H(p, u) = j_k^u z_-(0, \lambda), \qquad \lim_{q \to P_+} j_k^u H(q, u) = j_k^u z_+(0, \lambda),$$

*where $j_k^u$ is the $k$-jet w.r.t. $u$. Hence, also the $k$-jet of the right hand side of (3.17) converges:*

$$\lim_{p \to P_-, q \to P_+} \int_{\gamma(p,q)} j_k^\mu X_0(H) = \int_\Gamma j_k^\mu X_0(H) dt,$$

*where $\Gamma$ is the heteroclinic connection from $P_-$ to $P_+$.*

**Proof** Let us focus on $z_-$, and get a clear idea of what we have to prove. Given a point $p$ on the blow up locus, then the line segment $\{(p, u) | u \geq 0\}$ can be saturated with respect to $\overline{X}$. We have to prove that as $p$ gets closer to the end point $P_-$, the intersection of this saturation with $T$ is $O(u^k)$-close to any choice of invariant manifold $W_-$, and this for any $k > 0$. Close enough to $P_-$ the straight line segment in $C^k$-normal form coordinates is expressed by a $C^k$-curve $(\overline{v}(u), z(u))$ with $(\overline{v}(0), z(0))$ on the center manifold of $P_-$ (the connection on the blow up locus). Choose a $C$ with $0 < \overline{v}(0) < C$ and saturate (in normal form coordinates) the $C^k$ curve until the section $T'$: $\{\overline{v} = C\}$ is met.

Take an initial point $(u_0, \overline{v}_0, z_0)$, and let $(u, v, z)$ be the coordinates of the intersection of the orbit through $(u_0, \overline{v}_0, z_0)$ with $T'$. In the light of these remarks, we have to prove that $z = O(u^k)$.

It is easy to show that there is a $\kappa > 0$ so that the transition time $\tau$ to go from a sufficiently small neighbourhood of $(0, \overline{v}(0), 0)$ to $T'$ has a lower bound given by

$$\tau \geq \frac{\kappa}{\overline{v}_0}$$

Since $z(t) = z_0 \exp(-t)$, we find $|z| \leq |z_0| \exp(-\kappa/\overline{v}_0)$. Finally, we have a first integral $u^m \overline{v}$, hence $u_0^m \overline{v}_0 = u^m C$. We conclude

$$|z| \leq |z_0| \exp(-\tilde{\kappa}/u^{1/m})$$

for some $\tilde{\kappa} > 0$. This proves the $k$-flatness to $z = 0$ in the section $T'$. □

**Lemma 3.17** *Consider the vector field* $\overline{X}|_{u=0}$ *with the perturbation parameter* $A$. *Then, in (3.17),*

$$\lim_{p\to P_-} j_k^A H(p,A) = j_k^A z_-(0,\lambda), \qquad \lim_{q\to P_+} j_k^A H(p,A) = j_k^A z_+(0,\lambda),$$

*where* $j_k^A$ *is the* $k$*-jet w.r.t.* $A$.

**Proof** The technique of proving this lemma is slightly different. Let us again focus at $z_-$. Look at line segments $\{(p,A)|A\sim 0\}$. Taking the union for all $p$ on the heteroclinic connection, we get a manifold (not necessarily invariant) $U$. As in the previous lemma, we choose a section $T'$ close to $P_-$, and look in the phase directional rescaling chart. Consider now the normal form of lemma 3.9, restricted to $u=0$ and look in the $(\overline{v},z,A)$-space:

$$\begin{cases} \dot{A} &= 0 \\ \dot{\overline{v}} &= m\overline{v}^2 h(0,\overline{v},A,\lambda) \\ \dot{z} &= -z, \end{cases}$$

In these coordinates, the manifold $U$ is a graph $z=\varphi(\overline{v},A)$, and any choice of invariant manifold $W$ is $O(A^k)$-close to $z=0$ (in fact $\{z=0\}$ is the unique center manifold for this system—all invariant manifolds $W$ have a common intersection with the blow up locus). The normal form is integrable: if an initial condition $(\overline{v}_0,z_0,A_0)$ inside $U$ is taken, and we intersect with a section $T'$: $\{\overline{v}=C\}$, then we get

$$A=A_0, \overline{v}=C, z=\varphi(\overline{v}_0,A)\exp\left(-\int_{\overline{v}_0}^{C}\frac{ds}{ms^2h(0,s,A,\lambda)}dt\right).$$

Letting $\overline{v}_0$ tend to zero, one can prove that the $k$-jet of the last expression tends to zero as well (it is in fact a consequence of case (b) of proposition 2.8, if we apply the transformation $s=-\tilde{s}$). We conclude that the saturation of line pieces are $O(A^k)$-close to any choice of invariant manifold $W$ if the line pieces tend to the line of singularities. □

Remark that the preceding two lemma's use techniques from the previous section, with $\mu=u$ or $\mu=A$ one-dimensional. If we want to examine $j_k^{u,A}$, then the previous section needs to be formulated in a more-dimensional context.

Now, let us apply the results in the previous section, first on $\overline{X}|_{A=0}$, and on $\overline{X}|_{u=0}$. An immediate consequence of the above lemmas, and (3.17) is:

$$(j_k^u z_-(0,\lambda) - j_k^u z_+(0,\lambda)) = j_k^u \int_\Gamma \theta(\overline{f}|_{u=0}\overline{g} - \overline{g}|_{u=0}\overline{f})\Big|_{A=0} dt \tag{3.22}$$

and

$$\left(j_k^A z_-(0,\lambda) - j_k^A z_+(0,\lambda)\right) = j_k^A \int_\Gamma \theta(\overline{f}|_{A=0}\overline{g} - \overline{g}|_{A=0}\overline{f})\Big|_{u=0} dt$$

where $\Gamma$ is the heteroclinic connection for $u = A = 0$.

Define the separation function

$$\Delta(u, A, \lambda) := z_-(u, A, \lambda) - z_+(u, A, \lambda). \tag{3.23}$$

Then, using the $k$-jet formulas from above, we can easily find expressions for the 1-jet:

**Proposition 3.18**

$$\Delta(u, 0, \lambda) = u \int_\Gamma (F_u + O(u^2))dt$$

*with*

$$F_u := \theta \left( \overline{f} \frac{\partial \overline{g}}{\partial u} - \overline{g} \frac{\partial \overline{f}}{\partial u} \right) \Bigg|_{u=A=0}$$

*and with $\theta$ as in (3.21).*

**Proof** Immediate from (3.22), if one calculates the 1-jet of $\theta(\overline{f}|_{u=0}\overline{g} - \overline{g}|_{u=0}\overline{f})$ with respect to $u$. $\square$

**Proposition 3.19**

$$\Delta(0, A, \lambda) = A \int_\Gamma (F_A + O(A^2))dt,$$

*with*

$$F_A := \theta \left( \overline{f} \frac{\partial \overline{g}}{\partial A} - \overline{g} \frac{\partial \overline{f}}{\partial A} \right) \Bigg|_{u=A=0}$$

*and with $\theta$ as in (3.21). The parameter $A$ has the required regular breaking property if and only if*

$$\rho(\lambda) = \int_\Gamma F_A dt \neq 0,$$

*where $\rho(\lambda)$ has been introduced in (3.3).*

Combining these two propositions yields

**Corollary 3.20** *The first order term in $\mathcal{A}(u, \lambda)$ is*

$$\frac{\partial \mathcal{A}}{\partial u}(0, \lambda) = -\frac{\int_\Gamma F_u dt}{\int_\Gamma F_a dt}.$$

*where $F_u$ and $F_a$ are defined above.*

### 3.5.4 Higher order angles

Assuming that the first order angle is zero, it may be interesting to find the first order contact that is nonzero. Remark that even though the first order angle is zero, the invariant manifolds *can* be tilted; the angle in the forward manifolds is then the same as the angle in the backward manifolds. The appearance of a tilt will have an effect on higher order terms, and that is the reason why the knowledge of $z_\pm$ up to order $u^{k-1}$ is necessary to calculate the angle of order $u^k$. The method to work is to calculate inductively the terms of order $u^k$ of the invariant manifolds, and stop at the first order where the angle has a nonzero coefficient. The method consists in expanding (3.22) in terms of $u$. This involves calculating the integrating factor $\theta$ up to order $u^{k-1}$, or at least $\theta|_\Gamma$. To that end, the recursive formule (3.20) can be used, however, as one might expect, the formulas become quite cumbersome.

## 3.6 Proof of theorem 3.7

Under the conditions of theorem 3.7, we can explicitely calculate the optimal weights for the blow up and check the conditions for theorems 3.3 and 3.4. Let us first rewrite the vector field in some kind of standard form:

**Lemma 3.21** *Under the conditions of theorem 3.7, $X_{\epsilon,a,\lambda}$ is locally $C^\infty$ conjugate to*

$$\tilde{X}_{\epsilon,a,\lambda}: \left\{ \begin{array}{rcl} \dot{x} & = & -y + \frac{1}{2}x^2 + x^3F_1(x,y,\epsilon,a,\lambda) + x\epsilon F_2(x,y,\epsilon,a,\lambda) \\ \dot{y} & = & \epsilon G(x,y,\epsilon,a,\lambda). \end{array} \right.$$

*Furthermore,*

*(i)* $G(0,0,0,0,\lambda) = -g.f_{xx}f_y$;

*(ii)* $\frac{\partial G}{\partial x}(0,0,0,0,\lambda) = -g_x f_y$;

*(iii)* $\displaystyle\frac{\partial G}{\partial a}(0,0,0,0,\lambda) = g_x \left| \begin{array}{cc} f_y & f_a \\ f_{xy} & f_{xa} \end{array} \right| - f_{xx} \left| \begin{array}{cc} f_y & f_a \\ g_y & g_a \end{array} \right|.$

**Proof** Consider a transformation of the form

$$x = \alpha(y,a,\lambda)\tilde{x} + \beta(y,a,\lambda).$$

Since $\dot{y} = O(\epsilon)$, we have $\dot{x} = \alpha\dot{\tilde{x}} + O(\epsilon)$, and hence

$$\begin{aligned} \dot{\tilde{x}} &= \frac{1}{\alpha}f(\alpha\tilde{x}+\beta,y,0,a,\lambda) + O(\epsilon) \\ &= \frac{1}{\alpha}f(\beta,y,0,a,\lambda) + \frac{\partial f}{\partial x}(\beta,y,0,a,\lambda)\tilde{x} \\ &\quad + \alpha\frac{\partial^2 f}{\partial x^2}(\beta,y,0,a,\lambda)\frac{\tilde{x}^2}{2} + O(\tilde{x}^3) + O(\epsilon). \end{aligned}$$

Consider the mapping

$$I\colon (y,a,\lambda,\alpha,\beta) \mapsto \left(\frac{\partial f}{\partial x}(\beta,y,0,a,\lambda), \alpha\frac{\partial^2 f}{\partial x^2}(\beta,y,0,a,\lambda)\right).$$

Defining $\alpha_0 = 1/f_{xx}$, then $I(0,0,\lambda,\alpha_0,0) = (0,1)$. Furthermore,

$$D_{\alpha,\beta}(I)(0,0,\lambda,\alpha_0,0) = \begin{pmatrix} 0 & f_{xx} \\ f_{xx} & * \end{pmatrix}$$

This linear operator is invertible, hence the implicit function theorem gives us the existence of $(\alpha,\beta)$ so that

$$\dot{\tilde{x}} = F(\tilde{x},y,\epsilon,a,\lambda) := \frac{1}{\alpha(y,a,\lambda)} f(\beta(y,a,\lambda),y,0,a,\lambda) + \frac{\tilde{x}^2}{2} + O(\tilde{x}^3) + O(\epsilon).$$

Because we need it for the second part, we will give the asymptotics of $\alpha$ and $\beta$ w.r.t. $(y,a)$:

$$\alpha(y,a,\lambda) = \left(\frac{1}{f_{xx}}\right) + O(\|(y,a)\|)$$

and

$$\beta(y,a,\lambda) = \left(\frac{-f_{xy}}{f_{xx}}\right) y + \left(\frac{-f_{xa}}{f_{xx}}\right) a + O(\|(y,a)\|^2)$$

Define now

$$\tilde{y} := -F(0,y,\epsilon,a,\lambda) = \frac{-1}{\alpha(y,a,\lambda)} f(\beta(y,a,\lambda),y,0,a,\lambda) + O(\epsilon). \tag{3.24}$$

We can use $\tilde{y}$ as new $y$ coordinate, locally near $(y,\epsilon,a) = (0,0,0)$. In this form, we find

$$\dot{\tilde{x}} = -\tilde{y} + \frac{1}{2}\tilde{x}^2 + O(\tilde{x}^3) + \epsilon O(x).$$

This finishes the first part of the lemma. The second part is more elaborate. Let $y = \varphi(\tilde{y},\epsilon,a,\lambda)$ be the implicit solution of (3.24), then

$$\dot{\tilde{y}} = -\epsilon\frac{\partial F}{\partial y}(0,y,\epsilon,a,\lambda) g(x,y,\epsilon,a,\lambda)\Big|_{y=\varphi(\tilde{y},\epsilon,a,\lambda)}$$

So,

$$G(\tilde{x},\tilde{y},\epsilon,a,\lambda) = -\frac{\partial F}{\partial y}(0,\varphi,0,a,\lambda) g(\alpha\tilde{x}+\beta,\varphi,0,a,\lambda) + O(\epsilon).$$

We can calculate $\frac{\partial F}{\partial y}$ (remembering that $\frac{\partial f}{\partial x}(\beta(y,a,\lambda),y,0,a,\lambda) \equiv 0$):

$$\begin{aligned}\frac{\partial F}{\partial y}(0,y,0,a,\lambda) = &\frac{-1}{\alpha(y,a,\lambda)^2} f(\beta(y,a,\lambda),y,0,a)\frac{\partial \alpha}{\partial y}(y,a,\lambda)\\ &+ \frac{1}{\alpha}\frac{\partial f}{\partial y}(\beta(y,a,\lambda),y,0,a,\lambda).\end{aligned}$$

So,

$$\frac{\partial F}{\partial y}(0,0,0,0,\lambda) = f_{xx} f_y.$$

From this last property easily follow the expressions for $G(0,0,0,0,\lambda)$ as claimed in the lemma, and also for $\frac{\partial G}{\partial x}(0,0,0,0,\lambda)$. Let us now focus on $\frac{\partial G}{\partial a}(0,0,0,0,\lambda)$:

$$\begin{aligned}\frac{\partial G}{\partial a}(0,0,0,0,\lambda) = g_{\cdot}C(\lambda) \\ & - f_{xx} f_y \left( g_x \left( \frac{\partial \beta}{\partial a}(0,0,\lambda) + \frac{\partial \beta}{\partial y}(0,0,\lambda) \frac{\partial \varphi}{\partial a}(0,0,0,\lambda) \right) \right. \\ & \qquad \left. + g_y \frac{\partial \varphi}{\partial a}(0,0,0,\lambda) + g_a \right),\end{aligned}$$

where $C(\lambda)$ is some function of which the value is irrelevant in view of proving the lemma, since under the assumptions of theorem 3.7, $g_{\cdot} = 0$). If we find the value of $\frac{\partial \varphi}{\partial a}$ we can put all pieces together. It can be readily checked from formula (3.24) that

$$\frac{\partial \varphi}{\partial a}(0,0,0,\lambda) = -\frac{\frac{\partial F}{\partial a}(0,0,0,0,\lambda)}{\frac{\partial F}{\partial y}(0,0,0,0,\lambda)} = -\frac{f_a}{f_y}.$$

We conclude:

$$\frac{\partial G}{\partial a}(0,0,0,0,\lambda) = g_{\cdot}C(\lambda) - f_{xx} f_y \left( g_x \left(-\frac{f_{xa}}{f_{xx}} + \frac{f_{xy}}{f_{xx}} \frac{f_a}{f_y}\right) - g_y \frac{f_a}{f_y} + g_a \right).$$

Elaborating this expression yields a proof of the lemma. □

**Corollary 3.22** *Under the assumptions of theorem 3.7, the family of vector fields $X_{\epsilon,a,\lambda}$ is locally $C^\infty$-equivalent to*

$$\tilde{X}_{\epsilon,a,\lambda} \colon \left\{ \begin{array}{lcl} \dot{x} & = & -y + \frac{1}{2}x^2 + x^3 F_{1,\lambda}(x,y,\epsilon,a) + x\epsilon F_{2,\lambda}(x,y,\epsilon,a) \\ \dot{y} & = & \epsilon(a + x + G_\lambda(x,y,\epsilon,a)) \end{array} \right. \tag{3.25}$$

*with $F_{1,\lambda}$, $F_{2,\lambda}$ and $G_\lambda$ $C^\infty$ functions and*

$$G(x,y,\epsilon,a,\lambda) = O(\epsilon, y, \|(x,a)\|^2),$$

*and where the $O$ notation is uniform in $\lambda$.*

**Proof** Take the function $G$ as in lemma 3.21. From the assumptions of theorem 3.7 we know that $\frac{\partial G}{\partial x}(0,0,0,0,\lambda) > 0$. By rescaling $\epsilon$ with a positive factor, we may assume that $\frac{\partial G}{\partial x} = -1$. We also know that $\frac{\partial G}{\partial a}(0,0,0,0,\lambda) \neq 0$. By rescaling the $a$-space with a nonzero factor, we may assume that this derivative is 1. This proves the corollary. □

Let us now check all assumption T1–T6 for the normal form (3.25). The first assumption—the existence of a critical curve—is guaranteed by the implicit function theorem:

$$y - \frac{1}{2}x^2 + x^3 F_{1,\lambda}(x, y, 0, 0) = 0.$$

Clearly, there is a unique solution in the neighbourhood of $(x, y) = (0, 0)$. Here, the curve is a graph $y = \varphi(x) = \frac{1}{2}x^2 + O(x^3)$. Looking at the linear part of (3.25), we find, for $\epsilon = 0$:

$$\begin{pmatrix} x + O(x^2) & -1 + O(x^3) \\ 0 & 0 \end{pmatrix}$$

The eigenspace transverse to the curve of singularities $\gamma$ has eigenvalue $x$, so for $x < 0$ we have attraction, and for $x > 0$ there is repulsion.

Assumption T2 can be checked as follows: Substituting $y = \varphi(x)$ in $\frac{1}{\epsilon}\dot{y}$ leads to the slow dynamics

$$\varphi'(x)x' = x + G_\lambda(x, \varphi(x), 0, 0).$$

Since $\varphi'(x) = x + O(x^2)$, we find

$$x' = 1 + O(x).$$

The slow dynamics ensures movement from the attracting to the repelling part of the critical curve.

To look at assumption T3, we need to blow up the family of vector fields. We use a rescaling in the parameter space:

$$a = vA, \quad \epsilon = v^2.$$

The parameter $A$ will serve as regular breaking parameter, but we come to that later. In terms of these new parameters, the vector field yields

$$\left\{ \begin{array}{rcl} \dot{x} & = & -y + \frac{1}{2}x^2 + x^3\tilde{F}_{1,\lambda}(x, y, v, A) + xv^2\tilde{F}_{2,\lambda}(x, y, v, A) \\ \dot{y} & = & v^2(vA + x + \tilde{G}_\lambda(x, y, v, A) \end{array} \right. \tag{3.26}$$

The blow up formulas for blowing up the origin are

$$x = u\overline{x}, y = u^2\overline{y}, v = u\overline{v}.$$

In the phase directional rescaling chart, we look at the directional chart $\overline{y} = 1$, and find (after dividing through $u$) a family

$$\left\{ \begin{array}{rcl} \dot{u} & = & \frac{1}{2}u\overline{v}^2\left(\overline{v}A + \overline{x} + O(u)\right) \\ \dot{\overline{v}} & = & -\frac{1}{2}\overline{v}^3\left(\overline{v}A + \overline{x} + O(u)\right) \\ \dot{\overline{x}} & = & -1 + \frac{1}{2}\overline{x}^2 - \frac{1}{2}\overline{x}\overline{v}^2(\overline{v}A + \overline{x}) + O(u) \end{array} \right.$$

The preimage of the attracting part $\gamma_-$ is represented by $\{\overline{x} = -\sqrt{2} + O(u), \overline{v} = 0\}$, which is normally hyperbolically attracting with eigenvalue $\overline{x} < 0$. Similarly, the preimage of the repelling part will be normally hyperbolic up to the end point $P_+$.

To check the connection assumption (assumption T5), we look in the family rescaling chart $\overline{v} = 1$. Here, the family of vector fields is equivalent to

$$\left\{ \begin{array}{rcl} \dot{\overline{x}} & = & -\overline{y} + \frac{1}{2}\overline{x}^2 + O(u) \\ \dot{\overline{y}} & = & A + \overline{x} + O(u). \end{array} \right.$$

The invariant line $\overline{y} = \frac{1}{2}\overline{x}^2 - 1$ is a curve without singularities, connecting $P_-$ to $P_+$. To verify that $P_\pm$ is indeed a part of this line, one needs to look at $\overline{y} = \frac{1}{2}\overline{x}^2 - 1$ in the phase directional rescaling coordinates; there, this curve is represented by $\{\overline{x}^2 = 2(\overline{v}^2 + 1), u = 0\}$. In any case, assumption T5 is verified.

Assumption T6 can be checked easily in this case. This is because the unperturbed vector field is Hamiltonian in the family rescaling chart, with integrating factor

$$\theta(\overline{x}, \overline{y}) = \exp(-\overline{y}).$$

Using proposition 3.19, the parameter $A$ is a regular breaking parameter if

$$\int_\Gamma (-\overline{y} + \frac{1}{2}\overline{x}^2)\theta(\overline{x}, \overline{y})dt \neq 0.$$

where $\Gamma$ is the heteroclinic connection $\overline{y} = \frac{1}{2}\overline{x}^2 - 1$. The integral can be explicitely evaluated.

This proves the first part of theorem 3.7. The next lemma deals with the second part of the proof of theorem 3.7:

**Lemma 3.23** *There exist formal power series*

$$\hat{a} = \sum_{n=0}^{\infty} a_n(\lambda)\epsilon^n, \qquad \hat{y} = \sum_{n=0}^{\infty} y_n(x, \lambda)\epsilon^n$$

*so that $y_n$ is smooth in a uniform neighbourhood of $x = 0$, and so that $\hat{y}$ is formally invariant under (3.25).*

**Proof** For the sake of readibility, we drop the dependence on $\lambda$ in the notation. Let us first start by making an observation. Let $w(x, z, \epsilon, a)$ be a smooth function in the neighbourhood of the origin. Then we can define

$$\hat{w}(x, z, \epsilon, a) := \sum_{|k|=0}^{\infty} w_k(x) z^{k_1} \epsilon^{k_2} a^{k_3}$$

where $k = (k_1, k_2, k_3)$ is a multi-index and the functions $w_k$ are defined in a *uniform* neighbourhood $\Omega$ of $x = 0$. If we have formal power series $z = \hat{z}$ (w.r.t. $\epsilon$) where the coefficients are also defined in $\Omega$, and if we have a formal power series $a = \hat{a}$, then we assert that $\hat{w}(x, \hat{z}, \epsilon, \hat{a})$ makes sense, and is a formal power series where the coefficients are defined in the same neighbourhood $\Omega$, i.e. the neighbourhood does not shrink. This is true, provided that $\hat{z}$ and $\hat{a}$ have no terms in $\epsilon^0$.

Holding this property in mind, we write $y = \varphi_0(x,a) + z$, where $y = \varphi_0(x,a)$ is the graph of the curve of singularities of the unperturbed vector field ($\epsilon = 0$) (the existence of $\varphi_0$ follows from the implicit function theorem). In the search of a formal expansion of $z$, its constant term will be 0. The formula for $\dot{x}$ yields

$$\dot{x} = -z + xO(\epsilon).$$

As for $\dot{z}$, we get

$$\begin{aligned}\dot{z} &= \dot{y} - \varphi_0'(x)(z + xO(\epsilon)) \\ &= \epsilon\left(a + x + O(\epsilon, z, \|(x,a)\|^2)\right) - \varphi_0'(x)(-z + xO(\epsilon))\end{aligned}$$

Knowing that $\varphi_0'(x) = x + O(x^2)$, we find

$$\begin{cases}\dot{x} &= -z + xO(\epsilon, a) \\ \dot{z} &= \varphi_0'(x)z + \epsilon(a + x + O(\epsilon, z, \|(x,a)\|^2))\end{cases}$$

Solutions satisfy the differential equation:

$$(-z + xO(\epsilon, a))\frac{dz}{dx} = \varphi_0'(x)z + \epsilon(a + x + O(\epsilon, z, \|(x,a)\|^2)) \tag{3.27}$$

Assume that $\hat{z} = \sum_{n=1}^{\infty} z_n(x)\epsilon^n$ is known up to order $\epsilon^n$ and $\hat{a}$ is known up to order $\epsilon^{n-1}$. Look now at the coefficient of order $\epsilon^{n+1}$ of the above equation. Remembering that $\hat{z} = O(\epsilon)$ and $\hat{a} = O(\epsilon)$, observe that the coefficient of order $\epsilon^{n+1}$ of terms like $\hat{z}^2$, $\hat{z}^3$, $\epsilon\hat{z}$, $\epsilon^2\hat{a}$ etc. are functions in $x, z_1, \ldots, z_n, a_1, \ldots, a_{n-1}$. Hence, the term in $\epsilon^{n+1}$ in the lefthand side of (3.27) is a function

$$F_n(x, z_1, \ldots, z_n, \frac{dz_1}{dx}, \ldots, \frac{dz_n}{dx}, a_1, \ldots, a_{n-1}) + xH_n(x, a_n)\frac{dz_1}{dx}.$$

Similarly, the right hand side of (3.27) is of the form

$$\varphi_0'(x)z_{n+1} + a_n + G_n(x, z_1, \ldots, z_n, \frac{dz_1}{dx}, \ldots, \frac{dz_n}{dx}, a_1, \ldots, a_{n-1})$$

for some function $G_n$. Assuming that $z_1, \ldots, z_n$ and $a_1, \ldots, a_{n-1}$ are already known, looking at the $\epsilon^{n+1}$ level in (3.27) yields an equation

$$\tilde{F}_n(x) + xH_n(x, a_n)\frac{dz_1}{dx} = \varphi_0'(x)z_{n+1} + a_n + \tilde{G}_n(x).$$

From the above equation, one can find a unique $a_n$, by reducing the equation to $x = 0$, and once $a_n$ is known, the equation becomes

$$\varphi_0'(x)z_{n+1} = O(x).$$

From this equation, a smooth $z_{n+1}$ can be found, since $\varphi_0'(x) = x + O(x^2)$. This process is a recursion to find unique $\hat{a}$ and $\hat{z}$ as formal power series in $\epsilon$. The final step will define $\hat{y}$:

$$\hat{y} = \varphi(x, \hat{a}) + \hat{z}.$$

which still will be a formal power series, with coefficients smooth in a uniform neighbourhood of $x = 0$. $\square$

## 3.7 Proof of theorem 3.8

Under the conditions of theorem 3.8, we can explicitely calculate the optimal weights for the blow up and check the conditions for theorem 3.3. Let us first rewrite the vector field in some normal form:

**Lemma 3.24** *Under the conditions of theorem 3.8, $X_{\epsilon,a,\lambda}$ is locally $C^\infty$ conjugate to*

$$\tilde{X}_{\epsilon,a,\lambda}: \begin{cases} \dot{x} &= -y + \frac{1}{2n}x^{2n} + xF(x,y,\epsilon,a,\lambda) \\ \dot{y} &= \epsilon G(x,y,\epsilon,a,\lambda). \end{cases}$$

*with $F(x,0,0,0,\lambda) = O(x^{2n})$. Furthermore,*

*(i)* $F(0,0,0,a,\lambda) = \left(\frac{1}{f_y}\left|\begin{smallmatrix} f_y & f_a \\ f_{xy} & f_{xa}\end{smallmatrix}\right|\right) a + O(a^2)$;

*(ii)* $G(x,0,0,0,\lambda) = \left(\frac{-1}{\alpha^{2n-2}} f_y \frac{g_{x^{2n-1}}}{(2n-1)!}\right) x^{2n-1} + O(x^{2n})$

*(iii)* $G(0,0,0,a,\lambda) = \left(-\alpha \left|\begin{smallmatrix} f_y & f_a \\ g_y & g_a\end{smallmatrix}\right|\right) a + O(a^2)$,

*with* $\alpha = \left(\frac{f_{x^{2n}}}{(2n-1)!}\right)^{1/(2n-1)}$.

**Proof** The proof is straightforward: apply the transformation

$$\tilde{x} = \alpha x, \quad \tilde{y} = -\alpha f(0,y,\epsilon,a,\lambda)$$

for some well chosen number $\alpha$. If $\alpha$ is nonzero, then we can use $(\tilde{x},\tilde{y})$ locally as new variables, since we know that $f_y \neq 0$. Let's calculate the new vector field:

$$\dot{\tilde{x}} = \alpha f(\frac{1}{\alpha}\tilde{x},y,\epsilon,a,\lambda) = \alpha f(0,y,\epsilon,a,\lambda) + O(\tilde{x}) = -\tilde{y} + \tilde{F}(\tilde{x},\tilde{y},\epsilon,a).$$

for some function $\tilde{F} = O(\tilde{x})$. The terms of $O(\tilde{x})$ can be specified a bit more: from the conditions of theorem 3.8 follow (using the notations introduced in the announcement of this theorem)

$$\tilde{F}(\tilde{x},0,0,0,\lambda) = \frac{1}{\alpha^{2n-1}} f_{x^{2n}} \frac{\tilde{x}^{2n}}{(2n)!} + O(\tilde{x}^{2n+1}).$$

Choose $\alpha = \left(\frac{f_{x^{2n}}}{(2n-1)!}\right)^{1/(2n-1)}$ so that $\tilde{F}(\tilde{x},0,0,0,\lambda) = \frac{\tilde{x}^{2n}}{2n} + O(\tilde{x}^{2n+1})$, and conclude:

$$\dot{\tilde{x}} = -\tilde{y} + \frac{x^{2n}}{2n} + \tilde{x}F(\tilde{x},\tilde{y},\epsilon,a,\lambda),$$

with $F(\tilde{x},0,0,0,\lambda) = O(x^{2n})$. This finishes the first part of the lemma. The second part is more elborate. Let $y = \varphi(\tilde{y},\epsilon,a,\lambda)$ be the implicit solution of $\tilde{y} = \alpha f(0,y,\epsilon,a,\lambda)$, then

$$\begin{aligned}\frac{\partial F}{\partial a}(0,0,0,0,\lambda) &= \frac{\partial^2 \tilde{F}}{\partial \tilde{x}\partial a}(0,0,0,0,\lambda)\\ &= \frac{\partial^2 f}{\partial x\partial a}(0,0,0,0,\lambda) + \frac{\partial^2 f}{\partial x\partial y}(0,0,0,0,\lambda)\frac{\partial \varphi}{\partial a}(0,0,0,\lambda)\\ &= f_{xa} + f_{xy}\frac{\partial \varphi}{\partial a}(0,0,0,\lambda).\end{aligned}$$

From $\tilde{y} = -\alpha f(0,\varphi(\tilde{y},\epsilon,a,\lambda),\epsilon,a,\lambda)$, we obtain $f_a + f_y\frac{\partial\varphi}{\partial a}(0,0,0,\lambda) = 0$, and hence

$$\frac{\partial F}{\partial a}(0,0,0,0,\lambda) = \frac{1}{f_y}(f_{xa}f_y - f_{xy}f_a).$$

Let us now concentrate on $\dot{\tilde{y}}$: $\dot{\tilde{y}} = \epsilon G(\tilde{x},\tilde{y},\epsilon,a,\lambda)$, with

$$G(\tilde{x},\tilde{y},\epsilon,a,\lambda) = -\alpha\frac{\partial f}{\partial y}(0,\varphi(\tilde{y},\epsilon,a,\lambda),\epsilon,a,\lambda)g(\frac{1}{\alpha}\tilde{x},\varphi(\tilde{y},\epsilon,a,\lambda),\epsilon,a,\lambda).$$

Hence,

$$G(\tilde{x},0,0,0,\lambda) = -\alpha f_y g(\frac{1}{\alpha}\tilde{x},0,0,0,\lambda) = \frac{-1}{\alpha^{2n-2}}f_y g_{x^{2n-1}}\frac{x^{2n-1}}{(2n-1)!} + O(x^{2n}).$$

Finally it is easy to calculate that

$$\frac{\partial G}{\partial a}(0,0,0,0,\lambda) = -\alpha f_y\left(g_y\frac{-f_a}{f_y} + g_a\right) = \alpha(g_y f_a - g_a f_y).$$

This finishes the proof. □

**Corollary 3.25** *Under the assumptions of theorem 3.8, the family of vector fields* $X_{\epsilon,a,\lambda}$ *is locally* $C^\infty$*-equivalent to*

$$\tilde{X}_{\epsilon,a,\lambda}\colon \left\{\begin{array}{lcl}\dot{x} &=& -y + \frac{x^{2n}}{2n} + xF_1(x,y,\epsilon,a,\lambda)\\ \dot{y} &=& \epsilon(Ca + x^{2n-1} + G_1(x,y,\epsilon,a,\lambda))\end{array}\right. \qquad (3.28)$$

*with* $F_1(x,0,0,0,\lambda) = O(x^{2n})$*, and* $G_1(x,y,\epsilon,a,\lambda) = O(x^{2n},a^2,y,\epsilon)$*. The constant* $C$ *may depend on* $\lambda$*, and the regular breaking condition in theorem 3.8 is translated to*

$$C \neq \frac{\partial F_1}{\partial a}(0,0,0,0,\lambda).$$

**Proof** Take $F$ and $G$ as in lemma 3.24. From the assumptions of theorem 3.8, we know that the coefficient $\left(\frac{-1}{\alpha^{2n-2}} f_y \frac{g_{x^{2n-1}}}{(2n-1)!}\right)$ is strictly positive hence by rescaling $\epsilon$ with this the positive factor $\left(\frac{-1}{\alpha^{2n-2}} f_y \frac{g_{x^{2n-1}}}{(2n-1)!}\right)$, we reduce the coefficient of $x^{2n-1}$ in $G$ to $+1$. The coefficient of $a$ in $G$ will change to

$$\frac{\left(-\alpha \left| \begin{smallmatrix} f_y & f_a \\ g_y & g_a \end{smallmatrix} \right|\right)}{-\left(\frac{-1}{\alpha^{2n-2}} f_y \frac{g_{x^{2n-1}}}{(2n-1)!}\right)} = \frac{\alpha^{2n-1}(2n-1)!}{f_y g_{x^{2n-1}}} \left| \begin{smallmatrix} f_y & f_a \\ g_y & g_a \end{smallmatrix} \right|$$

Since $\alpha^{2n-1} = f_{x^{2n}}/(2n-1)!$, the coefficient of $a$ in $G$ equals

$$C := \frac{f_{x^{2n}}}{f_y g_{x^{2n-1}}} \left| \begin{smallmatrix} f_y & f_a \\ g_y & g_a \end{smallmatrix} \right|.$$

The regular breaking condition in theorem 3.8 states that this coefficient $C$ cannot be equal to $\frac{1}{f_y} \left| \begin{smallmatrix} f_y & f_a \\ f_{xy} & f_{xa} \end{smallmatrix} \right|$ which is exactly the coefficient in $a$ of $F_1$. $\square$

Let us now check all assumption T1–T6 for the normal form (3.28). The first assumption—the existence of a critical curve—is guaranteed by the implicit function theorem: we search $y$ in terms of $x$ so that

$$y - \frac{1}{2n} x^{2n} + x F_1(x, y, 0, 0, \lambda) = 0.$$

Clearly, there is a unique graph solution in the neighbourhood of $(x, y) = (0, 0)$. Here, the curve is a graph $y = \varphi(x) = \frac{1}{2n} x^{2n} + O(x^{2n+1})$ (to obtain this, remember that $F_1(x, 0, 0, 0, \lambda) = O(x^{2n})$). Looking at linear part of (3.28), we find, for $\epsilon = 0$:

$$\begin{pmatrix} x^{2n-1} + O(x^{2n}) & -1 + O(x) \\ 0 & 0 \end{pmatrix}$$

The eigenspace transverse to the curve of singularities $\gamma$ has a negative eigenvalue for $x < 0$, so there we have attraction, and for $x > 0$ there is repulsion.

To look at assumption T3, we need to blow up the family of vector fields. We use a rescaling in the parameter space:

$$a = v^{2n-1} A, \quad \epsilon = v^{2n}.$$

The parameter $A$ will serve as regular breaking parameter, but we come to that later. We then blow the origin, as follows:

$$x = u\overline{x}, y = u^{2n}\overline{y}, v = u\overline{v}.$$

In the phase directional rescaling chart, we look at the directional chart $\overline{y} = 1$, and find (after dividing through $u^{2n-1}$) a family

$$\begin{cases} \dot{u} &= \frac{1}{2n} u \overline{v}^{2n} \left(C \overline{v}^{2n-1} A + \overline{x}^{2n-1} + O(u)\right) \\ \dot{\overline{v}} &= -\frac{1}{2n} \overline{v}^{2n+1} \left(C \overline{v}^{2n-1} A + \overline{x}^{2n-1} + O(u)\right) \\ \dot{\overline{x}} &= -1 + \frac{1}{2n} \overline{x}^{2n} + DA\overline{x} - \frac{1}{2n} \overline{x}\overline{v}^{2n} (C \overline{v}^{2n-1} A + \overline{x}^{2n-1}) + O(u) \end{cases}$$

with

$$D = \frac{\partial F_1}{\partial a}(0,0,0,0,\lambda).$$

The preimage of the attracting part $\gamma_-$ is represented by $\{\overline{x} = -\sqrt[2n]{2n}+O(u), \overline{v} = 0\}$, which is normally hyperbolically attracting, with eigenvalue $-\overline{x}^{2n-1}$. Similarly, the preimage of the repelling part will be normally hyperbolicly repelling up to the end point $P_+$.

To check the connection assumption (assumption T5), we look in the family rescaling chart $\overline{v} = 1$. Here, the family of vector fields is equivalent to

$$\begin{cases} \dot{\overline{x}} &= -\overline{y} + \frac{1}{2n}\overline{x}^{2n} + D\overline{x}A + O(u) \\ \dot{\overline{y}} &= CA + \overline{x}^{2n-1} + O(u). \end{cases}$$

The invariant line $\overline{y} = \frac{1}{2n}\overline{x}^{2n} - 1$ is a curve without singularities, connecting $P_-$ to $P_+$. To verify that $P_\pm$ is indeed a part of this line, one needs to look at $\overline{y} = \frac{1}{2n}\overline{x}^{2n} - 1$ in the phase directional rescaling coordinates; there, this curve is represented by $\{\overline{x}^{2n} = 2n(\overline{v}^{2n} + 1), u = 0\}$. In any case, assumption T5 is verified.

Assumption T6 can be easily checked in this case. This is because the unperturbed vector field is Hamiltonian in the family rescaling chart, with integrating factor

$$\theta(\overline{x}, \overline{y}) = \exp(-\overline{y}).$$

Using proposition 3.19, the parameter $A$ is a regular breaking parameter if

$$\int_\Gamma \left( C(-\overline{y} + \frac{1}{2n}\overline{x}^{2n}) - D\overline{x}^{2n} \right) \theta(\overline{x}, \overline{y}) dt \neq 0.$$

where $\Gamma$ is the heteroclinic connection $\overline{y} = \frac{1}{2n}\overline{x}^{2n} - 1$. The integral can be explicitely evaluated:

$$\int_\Gamma \left( C(-\overline{y} + \frac{1}{2n}\overline{x}^{2n}) - D\overline{x}^{2n} \right) \theta(\overline{x}, \overline{y}) dt = (C - D)e \int_{-\infty}^{\infty} \exp(-s^{2n}/2n) ds,$$

where $e$ is the Euler number. Hence, the regular breaking condition is satisfied provided $C \neq D$.

This proves theorem 3.8.

## 3.8 Examples

The examples in this section are not meant to describe general classes of vector fields, but are aimed at illustrating in a rather unexpected way the theorems.

### 3.8.1 $C^1$ canard solutions

Consider

$$X_{\epsilon,a}: \left\{ \begin{array}{rcl} \dot{x} & = & y - \frac{x^4}{4} + \epsilon x^2 \\ \dot{y} & = & \epsilon(a - x^3) \end{array} \right.$$

According to theorem 3.8 and theorem 3.3, there exist manifolds of canard solutions that are $C^\infty$ in the blow up space. Also the control curves, related to $C^\infty$ boundary conditions $\Sigma_-$ and $\Sigma_+$ are $C^\infty$ in $\epsilon^{1/4}$. We show here that these manifolds can be blown down in a $C^1$ way, but not in a $C^2$ way, although the control curve itself will be smooth. Indeed, first notice that

$$y = \frac{x^4}{4} - (1 + x^2)\epsilon + O(\epsilon^2), a = O(\epsilon)$$

is a graph that is formally invariant w.r.t. $X_{\epsilon,a}$, up to order $O(\epsilon)$. Let us try to extend this to an expansion

$$y = \frac{x^4}{4} - (1 + x^2)\epsilon + y_2(x)\epsilon^2 + O(\epsilon^3), \qquad a = a_1\epsilon + O(\epsilon^2).$$

Expressing the formal invariance of this new expansion quickly yields

$$y_2(x) = \frac{a_1 - 2x}{x^3}.$$

Hence, no choice of $(a_1, y_2)$ exists so that the formal invariance is true up to order $O(\epsilon^2)$ and so that $y_2$ is continuous at the origin.

But there is one more interesting observation to be made about this family of vector fields: for $a = 0$, the vector field $X_{\epsilon,a}$ has a symmetry $\{x \mapsto -x, t \mapsto -t\}$. This means that if the two curves $\Sigma_-$ and $\Sigma_+$ in theorem 3.3 are chosen symmetrically with respect to the $y$-axis, then the control curve will be situated at $\{a = 0\}$! As a consequence, all control curves will be flat to $a = 0$. Combine this with the knowledge that the control curves are smooth in $\epsilon^{1/4}$, and we can conclude that each choice of control curve $a = \mathcal{A}(\epsilon)$ will be $C^\infty$ in $\epsilon$!

Also, if the two boundary curves $\Sigma_-$ and $\Sigma_+$ are not chosen "symmetrically" ($\Sigma_-$ is not inside the backward saturation of $\Sigma_+$ with respect to $X_{\epsilon,0}$, or equivalently $\Sigma_+$ is not inside the forward saturation of $\Sigma_-$ w.r.t. $X_{\epsilon,0}$), then the control curve cannot be analytic, even if we know that the vector field and the boundary curves are analytic.

### 3.8.2 Normal crossing of lines of singularities

Consider the scalar o.d.e.

$$\epsilon\frac{dy}{dx} = a + x^{2n-1}y + \epsilon F(x, y, \epsilon, a, \lambda)$$

and the associated vector field

$$X_{\epsilon,a,\lambda}\colon \left\{ \begin{array}{rcl} \dot{x} & = & \epsilon \\ \dot{y} & = & a + x^{2n-1}y + \epsilon F(x, y, \epsilon, a, \lambda), \end{array} \right.$$

with $F$ a $C^\infty$ function in the neighbourhood of $(x, y, \epsilon, a) = (0, 0, 0, 0)$. For $\epsilon = a = 0$, we have a crossing of the lines of singularities: $x = 0$ and $y = 0$. Along $x = 0$, it would not be possible to satisfy the regular passage property, but along $y = 0$, it will be, so we will define $y = 0$ as the critical curve.

Observe that along $y = 0$, for $x < 0$ the critical curve is attracting, and for $x > 0$ the critical curve is repelling. Given a point $p$ on the attracting part of the critical curve, then the regular passage assumption (assumption T2) is trivially satisfied, since $\dot{x} = \epsilon > 0$. To check the remaining assumptions, we blow up the origin:

$$\epsilon = v^{4n}, a = v^{4n-1}A, \qquad x = u^2\overline{x}, y = u\overline{y}, v = u\overline{v}.$$

In the phase-directional rescaling chart $\overline{x} = +1$, we find (after dividing by $u^{4n-2}$)

$$\left\{ \begin{array}{rcl} \dot{u} & = & \frac{1}{2}u\overline{\varepsilon}^{4n} \\ \dot{\overline{v}} & = & -\frac{1}{2}\overline{v}^{4n+1} \\ \dot{\overline{y}} & = & \overline{v}^{4n-1}A + \overline{y} - \frac{1}{2}\overline{v}^{4n}\overline{y} + O(u). \end{array} \right.$$

Clearly, assumption T3 is satisfied, and due to the absence of singularities outside $u\overline{v} = 0$ assumption T4 also holds. Look now in the family rescaling chart $\overline{v} = 1$:

$$\left\{ \begin{array}{rcl} \dot{\overline{x}} & = & 1 \\ \dot{\overline{y}} & = & A + \overline{x}^{2n-1}\overline{y} + O(u). \end{array} \right.$$

Clearly, for $A = 0$ there is a connection $\overline{y} = 0$ connecting $P_-$ to $P_+$, and on this connection, no singularities appear. This shows that assumption T5 is satisfied. Finally, in order to check assumption T6, one needs to calculate an integrating factor. One readily checks that

$$\theta(\overline{x}, 0) = \exp\left(\int_{\overline{x}}^{0} s^{2n-1}ds\right) = \exp\left(-\overline{x}^{2n}/2n\right).$$

Using proposition 3.19, $A$ is regular breaking parameter because

$$\int_{\overline{y}=0} \theta dt \neq 0.$$

The control curve $A = \mathcal{A}(u)$ in original coordinates is a curve

$$a(\epsilon) = \epsilon^{(4n-1)/4n} \mathcal{A}(\epsilon^{1/4n}).$$

where $\mathcal{A}$ is smooth in its argument. If $n > 1$, then generally, $a(\epsilon)$ will not be $C^\infty$ in $\epsilon$. If $n = 1$, then one can prove the existence of a formally invariant expansion, and using theorem 3.4, the smoothness in $\epsilon$ can be shown.

# Chapter 4

# Study of the transition time

We aim to study the transition time of orbits inside canard manifolds. The behaviour of the transition time, more specifically the monotonicity of this transition time, can be important in proving unicity in boundary value problems. We notice that for singularly perturbed problems of the form

$$\begin{cases} \dot{x} &= \epsilon^\sigma \\ \dot{y} &= F(x, y, \epsilon, a) \end{cases}$$

that are traditionally written down as an o.d.e. problem $\epsilon^\sigma \frac{dy}{dx} = F(x, y, \epsilon, a)$, the study of the transition time is trivial since

$$dt = \frac{1}{\epsilon^\sigma} dx.$$

In general singularly perturbed vector fields, as in (3.1), the study becomes quite more complicated. Inside canard manifolds, orbits can be divided into several parts. A part of the orbit can be seen as an $\epsilon$-perturbation of a regular orbit of the reduced vector field for $\epsilon = 0$ (this regular orbit is the regular passage towards a region near the critical curve). This part is followed by a part that "follows" the critical curve. The transition time in this part is studied in section 4.1. Finally, a part of the orbit is close to the simple passage turning point, as introduced in chapter 3; the time study of this part will be done in section 4.2.

## 4.1 Normally hyperbolic passage

When studying the transition time, assumptions N1 and N2 are too weak (see the example in section 4.1.1). Our interest goes to proving monotonicity in the transition time, and in that case we will need some kind of Łojasiewicz condition:

**Assumption N3** At points $p$ of the critical curve $\gamma$, in any $C^\infty$ center manifold $W_p$ of $X_{\epsilon,\lambda} + 0\frac{\partial}{\partial \epsilon}$ in $M \times [0, \epsilon_0[$, there is a $C > 0$ and an $N \in \mathbf{N}$ so that

$$|X_{\epsilon,\lambda}(q)| \geq C\epsilon^N, \qquad \forall q \in W_p$$

Using assumption N3 we can guarantee the existence of a largest $\sigma \in \mathbf{N}$ so that

$$|X_\lambda(q)| = O(\epsilon^\sigma), \qquad \forall q \in W_p$$

We call $\sigma$ the *order of degeneracy* of $X_{\epsilon,\lambda}$. Note that the smallest $\sigma \leq N$ for any $N$ satisfying N3.

Let $p \in \gamma$ be fixed. There exists a $C^k$-center manifold $W^c$ of $X_\lambda$ at $p$. The center manifold $W^c$ is seen as a two-dimensional manifold inside $M \times [0, \epsilon_0[$. Let $X_\lambda^c$ be the restriction of $X_\lambda$ to $W^c$ (center-manifold reduction), and define the *slow vector field*

$$X_\lambda^0(p) = \lim_{\epsilon \to 0} \epsilon^{-\sigma} X_\lambda^c(p).$$

It is easily seen that this construction is independent of the choice of center manifold $W^c$ (provided the smoothness is high enough of course), and hence a vector field along $\gamma$ is constructed:

**Definition 4.1** *The* slow vector field $X_\lambda^0$ *is a vector field along the critical curve $\gamma$ defined by the construction above. This construction desingularizes the dynamics on the critical curve $\gamma$ so that it is no longer a curve of singularities (although it may still contain isolated singularities).*

In practice, the slow vector field is easily derived from the original (fast) vector field. As an example, consider

$$\left\{ \begin{array}{rcl} \dot{x} & = & y - F(x, \lambda) \\ \dot{y} & = & \epsilon G(x, \lambda). \end{array} \right. \tag{4.1}$$

The critical curve is given by $\gamma\colon y = F(x, \lambda)$, and at points $(x, F(x, \lambda))$ where $\frac{\partial F}{\partial x}(x, \lambda) < 0$ the vector field is normally attracting. Starting with a center manifold $y = F(x, \lambda) + \epsilon F_1(x, \lambda) + O(\epsilon^2)$, one easily finds that the reduced dynamics on this center manifold is given by

$$X_\lambda^c\colon \dot{x} = \epsilon G(x, \lambda) \frac{\partial F}{\partial x}(x, \lambda)^{-1} + O(\epsilon^2).$$

The slow vector field thus yields

$$\dot{x} = (G/\frac{\partial F}{\partial x})(x, \lambda).$$

This example shows however, that sometimes not all points are regular for the slow vector field. Indeed, at individual zeroes of $G$, the slow vector field remains singular.

**Definition 4.2** *With*

$$T(\epsilon, \lambda) := \int_{O_\epsilon} dt$$

*we denote the integral over a compact piece of an orbit of $X_\epsilon$. Which piece will be clear from the context where $T(\epsilon, \lambda)$ appears. The orbit $O_\epsilon$ limits to a piece of the critical curve $\gamma$, which we denote $O_0$. With*

$$\int_{O_0} ds$$

*we define the integral over this piece of the critical curve, parametrized by the time of the slow vector field defined above.*

**Theorem 4.3** *Assume assumptions N1 and N3 are verified for the family of vector fields $X_{\epsilon,\lambda}$ on $M$. Let $\Sigma$ be a* smooth *admissible entry boundary curve and let $W$ be the saturated manifold.*

*Consider the passage from points on $\Sigma$ to a section of $M \times [0, \epsilon_0[$, intersecting $\gamma$ at a normally attracting point. Assume that the integral of the slow time converges. The transition time $T(\epsilon, \lambda)$ from $\Sigma$ to this section yields*

$$T(\epsilon, \lambda) = \frac{1}{\epsilon^\sigma}\left(\int_{O_0} ds + \varphi(\epsilon, \lambda)\right)$$

*where $\varphi$ is $C^\infty$ and is $o(1)$ as $\epsilon \to 0$ (uniformly in $\lambda$). The integral $\int_{O_0} ds$ integrates the slow time over the piece of $\gamma$ between the corner point of $W$ and the intersection of $\gamma$ with the transverse section.*

**Note 1:** In theorem 4.3 an important condition is added: the integral of the slow time must converge. More specifically, this means that the slow vector field must not contain singularities on the critical curve. It is however perfectly possible to treat isolated singularities on the critical curve. We will come back to this later.

**Note 2:** We limit the theorem to the case of smooth admissible entry boundary curves, merely for the sake of simplicity. If $\Sigma$ is non-smooth at $\epsilon = 0$, one can obtain similar results.

**Corollary 4.4** *Under the conditions of theorem 4.3 the transition time $T(\epsilon, \lambda)$ is* monotonously *increasing to $+\infty$ as $\epsilon \to 0$, at least for $\epsilon > 0$ small enough.*

Let us show how this corollary follows from the theorem. One has

$$\epsilon^\sigma T(\epsilon, \lambda) = \int_{O_0} ds + \varphi(\epsilon, \lambda).$$

Differentiating this equation w.r.t. $\epsilon$ yields

$$\sigma\epsilon^{\sigma-1}T(\epsilon, \lambda) + \epsilon^\sigma \frac{\partial T}{\partial \epsilon}(\epsilon, \lambda) = \frac{\partial \varphi}{\partial \epsilon}(\epsilon, \lambda);$$

in other words:

$$\frac{\partial T}{\partial \epsilon}(\epsilon,\lambda) = \frac{1}{\epsilon^{\sigma+1}}\left(\epsilon\frac{\partial \varphi}{\partial \epsilon}(\epsilon,\lambda) - \sigma\left(\int_{O_0} ds + \varphi(\epsilon,\lambda)\right)\right).$$

We conclude that

$$\frac{\partial T}{\partial \epsilon}(\epsilon,\lambda) = \frac{1}{\epsilon^{\sigma+1}}\left(-\sigma\int_{O_0} ds + \varphi_1(\epsilon,\lambda)\right),$$

for some $C^\infty$ function $\varphi_1$ that is $o(1)$ as $\epsilon \to 0$. For $\epsilon > 0$ small enough, this expression is negative, so $T(\epsilon,\lambda)$ will be monotonously increasing as $\epsilon \to 0$.

### 4.1.1 Non-monotonous transition time

The extra Łojasiewicz condition (assumption N3) is required to obtain the monotonicity of the transition time. As a counterexample to the monotonicity if this condition is not satisfied, consider for $0 \le \epsilon < 1$ the family of vector fields

$$X_\epsilon\colon \left\{\begin{array}{rcl} \dot{x} &=& -x \\ \dot{y} &=& \big(1+\epsilon\sin^2(1/\epsilon^2)\big)\exp(-1/\epsilon). \end{array}\right.$$

Notice that $\dot{y} > 0$ for all $\epsilon \in ]0,1[$, that the vector field has a $C^\infty$ extension at $\epsilon = 0$, and that assumptions N1 and N2 are verified. The saturation of the admissible entry boundary curve $\Sigma\colon \{x=0, y=0, \epsilon \ge 0\}$ is a $C^\infty$ manifold $W_\Sigma$ (theorem 2.5) outside $(x,y,\epsilon) = (0,0,0)$. However, the transition time towards the section $\{y=1\}$ is given by

$$T(\epsilon) = \frac{\exp(1/\epsilon)}{1+\epsilon\sin^2(1/\epsilon^2)},$$

which is certainly not monotonous as $\epsilon \to 0$. Indeed, one can check that its derivative has the property

$$\frac{dT}{d\epsilon}(\epsilon) \sim \frac{1}{\epsilon^2}\exp(1/\epsilon)(-1+2\sin(2/\epsilon^2)+o(1))$$

which changes sign infinitely many times as $\epsilon \to 0$.

### 4.1.2 Proof of theorem 4.3

We split the integration path in several parts: a passage of type I and several passages of type II. To that end, we cover the compact piece $O_0$ of $\gamma$ by a finite number of neighbourhoods $\{U_i\}_{i=0,\ldots,N}$ where $C^k$-normal forms are valid: we assume that in each $U_i$ the vector field has the form of the vector field in lemma 2.6, at least for the conjugacy version of this lemma. More specifically, because of assumption N3, we can divide $g$ by $\epsilon^\sigma$ and find that the vector field is $C^k$-conjugate to

$$\left\{\begin{array}{rcl} \dot{x} &=& -xh(x,y,\epsilon,\lambda) \\ \dot{y} &=& \epsilon^\sigma g(y,\epsilon,\lambda)h(x,y,\epsilon,\lambda), \end{array}\right. \tag{4.2}$$

for some strictly positive function $h$ and some positive function $g$. Note that a priori $g$ may have individual zeros (refer to section 4.3 for the treatment of such an example). The slow vector field is easily deduced from the above normal form:

$$\frac{dy}{ds} = g(y, 0, \lambda)h(0, y, 0, \lambda).$$

The slow transition time hence yields

$$\int_{O_0} \frac{dy}{g(y, 0, \lambda)h(0, y, 0, \lambda)}.$$

The assumption in the formulation of theorem 4.3 stating that the integral of the slow time must converge will imply that $g(y, 0, \lambda)$ cannot be zero along $O_0$.

Let us describe now the passage types:

Type I. A passage between the curve $\Sigma$ and a curve $\Sigma_0$, where $\Sigma_0$ is any boundary curve inside $W$ for which the end point on $\{\epsilon = 0\}$ is inside the neighbourhood $U_0$ and outside $\gamma$. A good choice for $\Sigma_0$ could be an image of $\Sigma$ under the time-$\tau$ map of the vector field, for a well-chosen time $\tau$. In that case, $\Sigma_0$ inherits the regularity properties of $\Sigma$, and will hence be $C^\infty$.

Type II. A passage between a curve $\Sigma_i$ and $\Sigma_{i+1}$, where the end point of $\Sigma_i$ is inside $U_i$, and the end point of $\Sigma_{i+1}$ is inside $U_i \cap U_{i+1}$. A good choice for $\Sigma_{i+1}$ would be the intersection of the manifold $W$ with a $C^\infty$ transverse section in $U_i \cap U_{i+1}$; for $i \geq 1$ this yields $C^\infty$ curves. A special case is the passage between $\Sigma_0$ and $\Sigma_1$, but below it will need no different treatment.

Clearly, the transition time $T(\epsilon, \lambda)$ is the sum of $T_I(\epsilon, \lambda)$ and $T_{II}(\epsilon, \lambda)$. Notice already that

$$T_I(\epsilon, \lambda) = O(1), \qquad \text{as } \epsilon \to 0,$$

and that $T_I$ is $C^\infty$. Concerning $T_{II}$, remember that the orbits $O_\epsilon$ lie inside an invariant manifold $x = \psi(y, \epsilon, \lambda)$, so the transition time for the passage from $\Sigma_i$ to $\Sigma_{i+1}$ is given by

$$T_{II}(\epsilon, \lambda) = \frac{1}{\epsilon^\sigma} \int_{y_i(\epsilon,\lambda)}^{y_{i+1}(\epsilon,\lambda)} \frac{dy}{g(y, \epsilon, \lambda)h(\psi(y, \epsilon, \lambda), y, \epsilon, \lambda)}.$$

whereas the integral of the slow time yields

$$\int_{O_0^{II}} ds = \int_{y_i(0,\lambda)}^{y_{i+1}(0,\lambda)} \frac{dy}{g(y, 0, \lambda)h(0, y, 0, \lambda)}.$$

Since we have assumed in the formulation theorem 4.3 that the integral of the slow time converges, the segment $[y_i, y_{i+1}]$ cannot exhibit zeros of $g(y, 0, \lambda)$. We thus can safely assume that $g(y, \epsilon, \lambda)h(\psi(y, \epsilon, \lambda), y, \epsilon, \lambda)$ is nonzero in this segment. As a consequence, $\epsilon^\sigma T_{II}(\epsilon, \lambda) - \int_{O_0^{II}} ds$ is a function that is as smooth as the boundary curve, and is $o(1)$ as $\epsilon \to 0$.

## 4.2 Passing through a turning point

We position ourselves in the framework of chapter 3, and assume now that the curve of singularities $\gamma$ has a "simple passage turning point", dividing the curve $\gamma$ in a normally attracting part, $\gamma_-$, and a normally repelling part $\gamma_+$. Consider a pair of admissible entry–exit boundary curves $(\Sigma_-, \Sigma_+)$. These two curves are connected by a "canard manifold" $W$, and this manifold is invariant under the flow of the vector field, for one exceptional curve $a = \mathcal{A}(\epsilon, \lambda)$ in parameter space—see theorem 3.3. It is our goal to investigate the transition time from $\Sigma_-$ to $\Sigma_+$, restricting the parameters to this exceptional control manifold $a = \mathcal{A}(\epsilon, \lambda)$.

The results in this section will be written down in terms of $(v, A)$, which is a rescaled version of $(\epsilon, a)$:

$$(a, \epsilon) = (v^k A, v^\ell).$$

Once this blow up in parameter space has been performed, we assume that in an admissible chart near the simple passage turning point, the vector field looks like (3.5), which we repeat here for the sake of convenience:

$$X_{v,A,\lambda}\colon \left\{ \begin{array}{rcl} \dot{x} & = & f(x, y, v, A, \lambda) \\ \dot{y} & = & vg(x, y, v, A, \lambda) \end{array} \right. \tag{4.3}$$

Of course, when studying the transition time *inside canard manifolds*, we restrict parameter space to a manifold $A = \mathcal{A}(v, \lambda)$, and study one specific invariant manifold of the family

$$X_{v,\lambda}\colon \left\{ \begin{array}{rcl} \dot{x} & = & f(x, y, v, \mathcal{A}(v, \lambda), \lambda) \\ \dot{y} & = & vg(x, y, v, \mathcal{A}(v, \lambda), \lambda) \end{array} \right. \tag{4.4}$$

The vector field in (4.4) is blown up, using the blow up map

$$\Phi\colon \mathbf{R}^+ \times S^2 \to \mathbf{R}^3\colon (u, (\overline{x}, \overline{y}, \overline{v})) \mapsto (x, y, v) = (u^p\overline{x}, u^q\overline{y}, u^m\overline{v}).$$

We remind the reader that if $\ell \neq 1$ in the first parameter rescaling, then in practice $m = 1$ in the second rescaling, and conversely, if $m \neq 1$, then in practice $\ell = 1$ in the first rescaling. See section 1.1 for a clear example. The results in this section will be written down in terms of the singular parameter $v$, but can easily be translated to results in terms of $\epsilon$.

**Definition 4.5** *Let $\Phi$ be the blow up map defined above, used for blowing up a singular point of a family of vector fields $X_\lambda := X_{v,\lambda} + 0\frac{\partial}{\partial v}$. Assume that the blown up vector field yields*

$$\overline{X}_\lambda = u^{-\alpha}\Phi^*(X_\lambda)$$

*(letting $\alpha$ be the largest index $i \in \mathbf{N}$ so that $\Phi^*(X_\lambda)$ is divisible by $u^i$). We define the blow up constant $b$ as the constant*

$$b := \sigma - \frac{\alpha}{m}.$$

*The constant $\sigma$ is chosen as the order of degeneracy along $\gamma$ (the existence of such an index is guaranteed under assumption N3; see section 4.1).*

One can think of the blow up constant $b$ as being strictly positive in the sequel. An uninteresting and technical proof would indicate that in all successful blow up constructions, the constant $b$ is always positive. But there is really no need to do this since it is easy to check once the blow up weights are identified. Instead, we will impose this as a condition in all future results.

To pass through the turning point, a regularity condition as in assumption N3 is needed. A priori, the "order of degeneracy" $\sigma$ may be different for the limit point $P_\pm$. To give an example, consider the vector field

$$\left\{ \begin{array}{lcl} \dot{x} & = & v(x^2+v) \\ \dot{y} & = & xy + O(v^2), \end{array} \right.$$

which after the directional blow up $(x, y, v) = (-u, u\overline{y}, u\overline{v})$ yields

$$\left\{ \begin{array}{lcl} \dot{u} & = & -u\overline{v}(u+\overline{v}) \\ \dot{\overline{v}} & = & \overline{v}^2(u+\overline{v}) \\ \dot{\overline{y}} & = & -\overline{y} + O(u). \end{array} \right.$$

where the order of degeneracy, $\sigma$, is 1 outside $u = 0$, but 2 at $u = 0$. This indicates a problem, and extra blow ups are needed to completely desingularize the system. (In this example, the problem can be bypassed by using the weights $(2, 1, 2)$ instead of $(1, 1, 1)$ in the blow up.)

**Assumption T7 (Corner Regularity)**

*In the normal form that is obtained in lemma 4.8, the function $h$ is nonzero at $(u, \overline{v}) = (0, 0)$.*

This assumption expresses the fact that the order of degeneracy $\sigma$ does not change at the limit point $P_-$.

We remind the reader that the saturation of a smooth entry boundary curve $\Sigma_-$ under the flow of $X_\lambda$ is smooth, except at a *corner point* $c_-$, which is defined as the $\omega$-limit of the the base point $\Sigma_- \cap \{v = 0\}$ of the entry boundary curve. Because the base point of this boundary curve lies in the basin of attraction of $\gamma_-$, the corner point is a point on the critical curve.

**Theorem 4.6** *Assume that the conditions of theorem 3.3 are verified together with assumption N3 and assumption T7 for the family of vector fields $X_{v,A,\lambda}$ on $M$. Let $\Sigma_\pm$ be a pair of* smooth *admissible entry-exit boundary curves, and let $W$ be the canard manifold connecting $\Sigma_-$ to $\Sigma_+$ along the control curve $A = \mathcal{A}(v, \lambda)$ in parameter space (so that $W$ is an invariant manifold for the subfamily $X_{v,\mathcal{A}(v,\lambda),\lambda}$).*

*Assume also that the transition time of the slow vector field converges between the corner points $c_-$ and $c_+$ as defined above (see definitions 4.1 and 4.2), and denote this transition time by*

$$\int_{O_0} ds.$$

*Let $k \in \mathbf{N}_1$ be arbitrary. The transition time $T(v,\lambda)$ from $\Sigma_-$ to $\Sigma_+$ along the parameter manifold $A = \mathcal{A}(v,\lambda)$ is given by*

$$T(v,\lambda) = \frac{1}{v^\sigma}\left(\int_{O_0} ds + v^b\varphi(v,\lambda) + \tilde{\varphi}(v,\lambda)\right),$$

*where $\varphi$ is $C^k$-smooth w.r.t. $v^{1/m}$, and where $\tilde{\varphi} = O(v)$ as $v \to 0$. The smoothness of $\tilde{\varphi}$ is detailed in proposition 4.9. The index $b$ is the blow up constant as defined in definition 4.5, and is assumed to be strictly positive. The index $m$ is the weight of the $v$-variable in the family blow up.*

Because the splitting between $\varphi$ and $\tilde{\varphi}$ is nonunique (i.e. depends on $k$), one cannot immediately assume that one can take $\varphi$ to be $C^\infty$. On the other hand, due to the low regularity of $\tilde{\varphi}$ there is really no need to in obtaining a smooth contribution in $\varphi$.

**Corollary 4.7** *For $v > 0$ small enough, the transition time in theorem 4.6 tends monotonously to $+\infty$.*

The proof of theorem 4.6 and of this corollary is written down in section 4.2.2.

### 4.2.1 Example: periodic orbits

Consider the van der Pol equation

$$X_{\epsilon,a}\colon \left\{ \begin{array}{lcl} \dot{x} & = & y - \frac{1}{2}x^2 - \frac{1}{3}x^3 \\ \dot{y} & = & \epsilon(a-x) \end{array} \right.$$

and the rescaled variant

$$X_{v,A}\colon \left\{ \begin{array}{lcl} \dot{x} & = & y - \frac{1}{2}x^2 - \frac{1}{3}x^3 \\ \dot{y} & = & v^2(vA-x) \end{array} \right. .$$

Consider the boundary curve $\Sigma\colon \{x = 0, y = Y, v \geq 0\}$ with $0 < Y < \frac{1}{6}$. The base point $(x,y) = (0,Y)$ lies in the basin of attraction of $\gamma_-\colon \{y = \frac{1}{2}x^2 + \frac{1}{3}x^3, x \in ]-1,0[\}$, but it also lies in the basin of repulsion of $\gamma_+\colon \{y = \frac{1}{2}x^2 + \frac{1}{3}x^3, x > 0\}$. So we can take $\Sigma_- = \Sigma_+ = \Sigma$ and use theorem 3.3 to create a manifold $W$ consisting of periodic orbits for the family $X_{v,\mathcal{A}(v)}$. We demonstrate how the period can be characterized with theorem 4.6.

The slow vector field along $\gamma = \gamma_- \cup \gamma_+$ is defined as

$$\dot{x} = \frac{-1}{1+x},$$

(the vector field is of the form (4.1) and the slow vector field can hence be similarly deduced), so

$$\int_{O_0} ds = \int_{x_0(Y)}^{x_1(Y)} (1+x)dx,$$

where $x_0(Y)$ and $x_1(Y)$ are respectively the largest negative and the unique positive solutions of $\frac{1}{2}x^2 + \frac{1}{2}x^3 = Y$. We conclude using theorem 4.6 that the period of the periodic orbits inside $W$ is given by

$$T(v,Y) = \frac{1}{v^2}\left(\int_{x_0(Y)}^{x_1(Y)} (1+x)dx + O(v)\right),$$

and furthermore that $T(v,Y)$ tends monotonously to infinity as $v \to 0$, uniformly for $Y$ in compact subsets of $]0, 1/6[$. Replacing $v$ by $\sqrt{\epsilon}$ yields a characterization in terms of $\epsilon$ instead of $v$.

### 4.2.2 Proof of theorem 4.6

The study of the transition time from $\Sigma_-$ to $\Sigma_+$ can be divided into the study of four types of transitions. A passage from $\Sigma_-$ along any compact piece of the critical curve $\gamma_-$ is described by passages of types I and II as in section 4.1.2. Similarly, upon reversing time, the passage from $\Sigma_+$ along any compact piece of the critical curve $\gamma_+$ can be studied using the techniques in section 4.1.2. To pass near the turning point, we have to study two more types of passages:

Type III. In an admissible chart where the turning point is represented by the origin, we blow up the origin, as in chapter 3. A type III passage will cover the passage near $P_-$ in a phase-directional rescaling chart (where we can find a normal form (for $C^k$-equivalence) for the family as in lemma 3.9. Also the passage near $P_+$ is, upon reversing time, to be studied as a type III passage.

Type IV. A passage along compact pieces of the connection $\Gamma$ in the family rescaling chart of the blow up.

Before considering a passage of type III, let us first consider a type IV passage. In the family rescaling, the blown up vector field is defined as

$$\overline{X}_\lambda = \frac{1}{u^\alpha}\Phi^*(X_\lambda),$$

where $X_\lambda = X_{v,\lambda} + 0\frac{\partial}{\partial v}$ is the extended family on $M \times [0, v_0[$, and where $\Phi$ is the blow up map as defined in by the equations in (1.3). The factor $u^\alpha$ is the common divisor that is divided away when blowing up. In the family rescaling chart, the canard manifold $W$ is a perturbation of the heteroclinic connection $\Gamma$ on the blow up locus.

Because this connection is a regular orbit (assumption T5), we can assume that $\overline{X}_\lambda$ does not have any singularities inside $W$. Hence, the transition time between two sections cutting $W$ transversally is a function $\varphi_{IV}(u,\lambda)$ that is $C^\infty$ smooth in $(u,\lambda)$ and that is $O(1)$ as $u \to 0$. Hence,

$$T_{IV}(v,\lambda) = \frac{1}{v^{\alpha/m}}\varphi_{IV}(v^{1/m},\lambda) = \frac{1}{v^\sigma}v^b\varphi_{IV}(v^{1/m},\lambda),$$

with $b = \sigma - \frac{\alpha}{m}$ as in definition 4.5. Assuming $b > 0$ this shows that the time spent in type IV passages is small with respect to the time spent in type II passages.

So let us now take care of type III passages.

**Lemma 4.8** *The vector field $X_{v,A,\lambda}$ restricted to the canard manifold $W$ is, restricting the parameters to the manifold $A = \mathcal{A}(v,\lambda)$, $C^k$-conjugate to the vector field*

$$\left\{\begin{array}{rcl} \dot{u} & = & -u^{\alpha+1}\overline{v}^\sigma h(u,\overline{v},\lambda) \\ \dot{\overline{v}} & = & mu^\alpha\overline{v}^{\sigma+1}h(u,\overline{v},\lambda) \end{array}\right. \tag{4.5}$$

*in a region near $P_-$ where the phase-directional rescaling coordinates are valid. The function $h$ does not have any zeros at $u = 0$, except maybe at $(u,\overline{v}) = (0,0)$. If assumption T7 is satisfied, then also at $(u,\overline{v}) = (0,0)$, $h$ cannot be a zero.*

**Proof** Starting point is the normal form for $C^{k+\sigma}$-equivalence in lemma 3.9. We can transform it to a normal form for $C^{k+\sigma}$-equivalence, by multiplying the vector field with a strictly positive $C^{k+\sigma}$ function. This way, we get a vector field that is $C^{k+\sigma}$-conjugate to the blown up vector field. By multiplying with a factor $u^\alpha$, we get a vector field that is $C^{k+\sigma}$-conjugate to the original vector field, at least in the region where the phase-directional rescaling coordinates are valid:

$$\left\{\begin{array}{rcl} \dot{u} & = & -u^{\alpha+1}\overline{v}h(u,\overline{v},A,\lambda)f(u,\overline{v},z,A,\lambda) \\ \dot{\overline{v}} & = & mu^\alpha\overline{v}^2h(u,\overline{v},A,\lambda)f(u,\overline{v},z,A,\lambda) \\ \dot{z} & = & -u^\alpha zf(u,\overline{v},z,A,\lambda), \end{array}\right.$$

where $f$ is $C^{k+\sigma}$ and strictly positive, and where $h$ is $C^{k+\sigma}$ and positive. Restricting to the canard manifold $W$, which is a $C^{k+\sigma}$ graph $z = \psi(u,\overline{v},\lambda)$, and restricting the parameter space to the manifold $A = \mathcal{A}(v,\lambda)$ we get a vector field on $W$:

$$\left\{\begin{array}{rcl} \dot{u} & = & -u^{\alpha+1}\overline{v}\tilde{h}(u,\overline{v},\lambda) \\ \dot{\overline{v}} & = & mu^\alpha\overline{v}^2\tilde{h}(u,\overline{v},\lambda) \end{array}\right.$$

where $\tilde{h}$ is $C^{k+\sigma}$ and positive. The index $\sigma$ is chosen so that $\tilde{h}(u,\overline{v},\lambda)$ is divisible by $\overline{v}^\sigma$ for all $u > 0$. By continuity, this will be automatically true for $u = 0$ as well. Upon writing $\tilde{h}(u,\overline{v},\lambda) = \overline{v}^\sigma\overline{h}(u,\overline{v},\lambda)$ (so that $\overline{h}$ is a $C^k$ function), and upon dropping the bars, we obtain the equations in the statement of the lemma. Finally, we remark that the absence of zeros at $u = 0$ follows from assumption T5, which states that at the blow up locus there is a heteroclinic connection without any singular points. □

Using the above lemma, one finds that the transition time from a section $\{u = u_0\}$ to a section $\{\overline{v} = \overline{v}_1\}$ is the same as the transition time of the vector field in (4.5). Remembering that $v = u^m\overline{v}$, an orbit at "height" $v$ is an orbit starting at $(u, \overline{v}) = (u_0, vu_0^{-m})$ and finishing at $(u, \overline{v}) = ((v/\overline{v}_1)^{1/m}, \overline{v}_1)$. Using the normal form of the above lemma, one finds

$$T_{III}(v, \lambda) = \int_{u_0}^{(v/\overline{v}_1)^{1/m}} \left. \frac{du}{-u^{\alpha+1}\overline{v}^{\sigma} h(u, \overline{v}, \lambda)} \right|_{\overline{v}=vu^{-m}}.$$

The above can now be rewritten as

$$T_{III}(v, \lambda) = \frac{1}{v^{\sigma}} \int_{(v/\overline{v}_1)^{1/m}}^{u_0} \frac{du}{u^{\alpha+1-m\sigma} h(u, vu^{-m}, \lambda)} = \frac{1}{v^{\sigma}} \int_{(v/\overline{v}_1)^{1/m}}^{u_0} \frac{u^{mb-1} du}{h(u, vu^{-m}, \lambda)},$$

where $b$ is the blow up constant as defined in definition 4.5. To guarantee the continuity of the righthand side, we need to assume $b > 0$ and $h(u, 0, \lambda) \neq 0$, which is done in assumption T7.

Now that we have assumed $b$ to be strictly positive, the continuity of the righthand side is guaranteed, and tends to

$$-\int_{u_0}^{0} \frac{u^{mb-1} du}{h(u, 0, \lambda)},$$

which is an integral of the slow time along the part $u \in [0, u_0]$ of the critical curve $\gamma_-$. This shows that

$$T_{III}(v, \lambda) = \frac{1}{v^{\sigma}} \left( \int_{O_0^{III}} ds + \varphi_{III}(v, \lambda) \right),$$

where

$$\varphi_{III}(v, \lambda) = \int_{(v/\overline{v}_1)^{1/m}}^{u_0} \frac{u^{mb-1} du}{h(u, vu^{-m}, \lambda)} - \int_0^{u_0} \frac{u^{mb-1} du}{h(u, 0, \lambda)} \tag{4.6}$$

is a function that is at least $C^0$ and $o(1)$ as $v \to 0$. We will investigate this function more precisely in the following proposition, thereby finishing the proof of theorem 4.6:

**Proposition 4.9** *The function $\tilde{\varphi}$ from theorem 4.6 has the following properties: $\tilde{\varphi}$ is $C^k$ for all $v > 0$. Furthermore:*

*(i) If $b \in ]0, 1[$, then $\tilde{\varphi}(v, \lambda) = O(v^b)$, $\frac{\partial \tilde{\varphi}}{\partial v}(v, \lambda) = O(v^{b-1})$ as $v \to 0$.*

*(ii) If $b = 1$, then $\tilde{\varphi}(v, \lambda) = O(v \log v)$, $\frac{\partial \tilde{\varphi}}{\partial v}(v, \lambda) = O(\log v)$ as $v \to 0$.*

*(iii) If $b > 1$, then $\tilde{\varphi}(v, \lambda) = O(v)$, $\frac{\partial \tilde{\varphi}}{\partial v}(v, \lambda) = O(1)$ as $v \to 0$.*

**Proof** The smoothness outside $v=0$ is trivial since it is assumed that the canard manifolds do not have singularities outside $v=0$. Let us focus on the smoothness at $v=0$, and to that end we use $C^k$-coordinates. The function $\tilde{\varphi}$ is explicitely written down in $C^k$-coordinates in formula (4.6).

Implicitely it is assumed that

$$\int_0^{u_0} \frac{u^{mb-1}du}{h(u,0,\lambda)}$$

converges (this is a part of the integral of the slow time, along the critical curve $\gamma_-$). Because of this, and because $h$ is $C^k$-smooth (with $k$ high enough), $h$ cannot have zeros in $u \in ]0,1]$, and, in extremis, $h$ can have a zero of at most order $mb-1$ at $u=0$, but the presence of a zero at $u=0$ is disproved in lemma 4.8. This means that for $\overline{v}$ small enough, also $h(u,\overline{v},\lambda)$ does not have any zeros. Using the theorem of dominated convergence, it follows that $\tilde{\varphi}(v,\lambda)=o(1)$ as $v\to 0$. Write now

$$f(u,\overline{v},\lambda)=\frac{1}{h(u,\overline{v},\lambda)}.$$

Then $f$ is $C^k$ in the region $(u,\overline{\varepsilon},\lambda)\in[0,1]^2\times\Lambda$. We find

$$\begin{aligned}\tilde{\varphi}(v,\lambda) &= \int_{(v/\overline{v}_1)^{1/m}}^{u_0} u^{mb-1}\left(f(u,vu^{-m},\lambda)-f(u,0,\lambda)\right)du\\ &\quad -\int_0^{(v/\overline{v}_1)^{1/m}} u^{mb-1}f(u,0,\lambda)du\end{aligned}$$

The second integral gives a contribution that is $C^k$ in $v^{1/m}$, and it is clearly $O(v^b)$ as $v\to 0$. To examine the first part, we write $f(u,\overline{v},\lambda)=f(u,0,\lambda)+\overline{v}f_1(u,\overline{v},\lambda)$ for some $C^{k-1}$ function $f_1$. The first part can now be rewritten as

$$\tilde{\varphi}_1(v,\lambda):=v\int_{(v/\overline{v}_1)^{1/m}}^{u_0} u^{mb-1-m}f_1(u,vu^{-m},\lambda)du,$$

Bound $|f_1|$ by a constant $M$ to find

$$|\tilde{\varphi}_1(v,\lambda)|\leq Mv\int_{(v/\overline{v}_1)^{1/m}}^{u_0} u^{mb-1-m}du.$$

If $b>1$, this is simply $O(v)$; if $b=1$ the function is clearly $O(v\log v)$, and if $b<1$, one finds an $O(v^b)$ expression.

Let us now concentrate on the derivative. A function that is $O(v^b)$ and $C^k$ in terms of $v^{1/m}$ clearly has a derivative that is $O(v^{b-1})$ as $v\to 0$. So let us concentrate on the first part $\tilde{\varphi}_1$, which has a worse-behaving derivative:

$$\begin{aligned}\frac{\partial\tilde{\varphi}_1}{\partial v}(v,\lambda) &= \frac{1}{v}\tilde{\varphi}_1(v,\lambda)-\frac{1}{m}\overline{v}_1^{1-b}v^{b-1}f_1((v/\overline{v}_1)^{1/m},\overline{v}_1,\lambda)\\ &\quad +v\int_{(\frac{v}{\overline{v}_1})^{1/m}}^{u_0} u^{mb-1-2m}\frac{\partial f_1}{\partial\overline{v}}(u,vu^{-m},\lambda)du.\end{aligned}$$

This expression has three contributions: the middle contribution is $O(v^{b-1})$; the left contribution is $O(v^{-1})O(\tilde{\varphi}_1)$. Depending on the case (i), (ii) or (iii), this is respectively $O(v^{b-1})$, $O(\log v)$ or $O(1)$, but in all cases the required statement regarding the derivative is true. Let us now focus on the contribution of the third part:

$$v \int_{(v/\overline{v}_1)^{1/m}}^{u_0} u^{mb-1-2m} \frac{\partial f_1}{\partial \overline{v}}(u, vu^{-m}, \lambda) du.$$

Bound $|\frac{\partial f_1}{\partial \overline{v}}|$ by a constant $\tilde{M}$ to find that this contribution is in absolute value bounded by

$$\tilde{M} v \int_{(v/\overline{v}_1)^{1/m}}^{u_0} u^{mb-1-2m} du.$$

As before, this is $O(v)$ if $b > 2$, $O(v \log v)$ if $b = 2$, or $O(v^{b-1})$ if $b < 2$. □

Remains to proof corollary 4.7:

**Proof** Applying the theorem 4.6, one has

$$v^\sigma T(v, \lambda) = \int_{O_0} ds + v^b \varphi(v, \lambda) + \tilde{\varphi}(v, \lambda),$$

so after derivation this yields

$$\sigma v^{\sigma-1} T + v^\sigma \frac{\partial T}{\partial v} = b v^{b-1} \varphi + v^b \frac{\partial \varphi}{\partial v} + \frac{\partial \tilde{\varphi}}{\partial v}.$$

Replacing $T$ in the above expression by the expression in theorem 4.6, and collecting the terms gives us

$$\begin{aligned} v^\sigma \frac{\partial T}{\partial v} &= b v^{b-1} \varphi + v^b \frac{\partial \varphi}{\partial v} + \frac{\partial \tilde{\varphi}}{\partial v} - \frac{\sigma}{v} \left( \int_{O_0} ds + v^b \varphi + \tilde{\varphi} \right) \\ &= \frac{1}{v} \left( -\sigma \int_{O_0} ds + b v^b \varphi + v^b (v \frac{\partial \varphi}{\partial v}) + (v \frac{\partial \tilde{\varphi}}{\partial v}) - \sigma v^b \varphi - \sigma \tilde{\varphi} \right). \end{aligned}$$

Examining all terms of this last expression (using proposition 4.9 and the fact that $\varphi$ is smooth w.r.t. $v^{1/m}$, so $v \frac{\partial \varphi}{\partial v} = O(v^{1/m})$) gives us

$$\frac{\partial T}{\partial v}(v, \lambda) = \frac{1}{v^{\sigma+1}} \left( -\sigma \int_{O_0} ds + \rho(v, \lambda) \right),$$

where $\rho(v, \lambda) = O(v^b)$ as $v \to 0$ if $b \in ]0, 1[$ or $\rho(v, \lambda) = O(v \log v)$ as $v \to 0$ if $b = 1$ or where $\rho(v, \lambda) = O(v)$ as $v \to 0$ if $b > 1$. In any case, it is $o(1)$ as $v \to 0$, meaning that the $\frac{\partial T}{\partial v}$ becomes negative for $v > 0$ small enough, making the transition time tend monotonously to $+\infty$ as $v \to 0$. □

## 4.3 Desingularizing the slow dynamics

In theorem 4.3 an important condition is added: the integral of the slow time must converge. More specifically, this means that the slow vector field must not contain singularities on the critical curve. Here it is shown how to tackle a situation where an isolated singularity appears on the critical manifold. The main tool is a family blow up at this point.

Consider as an example the smooth family of vector fields

$$X_\epsilon\colon \left\{ \begin{array}{rcl} \dot{x} & = & \epsilon(x^2+\epsilon^2+\epsilon^3 f(x,y,\epsilon,\lambda)) \\ \dot{y} & = & -y+\epsilon g(x,y,\epsilon,\lambda) \end{array} \right. \tag{4.7}$$

and we want to study the transition time of orbits starting from an entry boundary curve $\Sigma\colon \{x=x_0, y=0, \epsilon\geq 0\}$ until the orbits reach the plane $\{x=x_1\}$.

Apparently, the slow dynamics is given (after division by $\epsilon^\sigma := \epsilon$) by

$$X^0\colon \dot{x}=x^2.$$

The slow vector field has a singularity at $x=0$, and integrating the time over the critical curve $\{y=0\}$ yields

$$\int_{x_0}^{x_1} \frac{ds}{s^2}.$$

If $0\notin[x_0,x_1]$, then we can apply the results of this chapter and show that the transition time is $O(1/\epsilon)$, and that the leading term in the asymptotic expansion is given by the above integral. Some more analysis is needed if $0\in(x_0,x_1)$, because the above integral is then divergent.

We therefore blow up the vector field. In this specific situation we perform a cylindrical blow up

$$x=u\overline{x}, \epsilon=u\overline{\varepsilon}$$

The resulting object is no longer a family of vector fields, but instead a foliated vector field: at the chart $\{\overline{x}=\pm 1\}$ one finds:

$$\overline{X}\colon \left\{ \begin{array}{rcl} \dot{u} & = & \pm u^3\overline{\varepsilon}(1+\overline{\varepsilon}^2+u\overline{\varepsilon}^3 f(\pm u,y,u\overline{\varepsilon},\lambda)) \\ \dot{\overline{\varepsilon}} & = & \mp u^2\overline{\varepsilon}^2(1+\overline{\varepsilon}^2+u\overline{\varepsilon}^3 f(\pm u,y,u\overline{\varepsilon},\lambda)) \\ \dot{y} & = & -y+u\overline{\varepsilon}g(\pm u,y,u\overline{\varepsilon},\lambda) \end{array} \right.$$

When saturating the boundary curve $\Sigma$, an invariant manifold $W$ is formed that in our case can be written as a graph $y=\psi(x,\epsilon,\lambda)$, for $x\in[x_0,x_1]$ and $\epsilon>0$. This graph has a $C^\infty$ extension to $\epsilon=0$ (theorem 2.5) for all $x\neq x_0$. Hence, after blow up and for $u$ small, the blown up manifold $y=\psi(\pm u,u\overline{\varepsilon},\lambda)$ is smooth in $(u,\overline{\varepsilon},\lambda)$. The restriction of $\overline{X}$ to this manifold yields

$$\left\{ \begin{array}{rcl} \dot{u} & = & \pm u^3\overline{\varepsilon}(1+\overline{\varepsilon}^2+u\overline{\varepsilon}^3 h(u,\overline{\varepsilon},\lambda)) \\ \dot{\overline{\varepsilon}} & = & \mp u^2\overline{\varepsilon}^2(1+\overline{\varepsilon}^2+u\overline{\varepsilon}^3 h(u,\overline{\varepsilon},\lambda)) \end{array} \right.$$

with $h(u,\overline{\varepsilon},\lambda) := f(\pm u, \psi(\pm u, u\overline{\varepsilon},\lambda), u\overline{\varepsilon},\lambda)$ a $C^\infty$ function. Using $\dot{\overline{\varepsilon}} = d\overline{\varepsilon}/dt$, one calculates the transition time in the chart $\{\overline{x} = -1\}$ from a section $\{u = \delta\}$ to a section $\{\overline{\varepsilon} = \delta\}$ (corresponding to sections $\{x = -\delta\}$ and $\{x = -\epsilon/\delta\}$):

$$T_1(\epsilon,\lambda) = \int_{\epsilon/\delta}^{\delta} \frac{d\overline{\varepsilon}}{u^2\overline{\varepsilon}^2(1+\overline{\varepsilon}^2+u\overline{\varepsilon}^3 h(u,\overline{\varepsilon},\lambda))}\Bigg|_{u=\epsilon/\overline{\varepsilon}}.$$

In other words,

$$\epsilon^2 T_1(\epsilon,\lambda) = \int_{\epsilon/\delta}^{\delta} \frac{d\overline{\varepsilon}}{1+\overline{\varepsilon}^2+\epsilon\overline{\varepsilon}^2 h(\epsilon/\overline{\varepsilon},\overline{\varepsilon},\lambda)}$$

Using the theorem on dominated convergence of Lebesgue, the above expression has a well-defined limit as $\epsilon \to 0$:

$$\epsilon^2 T_1(\epsilon,\lambda) = \int_0^{\delta} \frac{d\overline{\varepsilon}}{1+\overline{\varepsilon}^2} + o(1), \qquad \text{as } \epsilon \to 0.$$

Similarly, in the chart $\{\overline{x} = 1\}$ a transition from $\{\overline{\varepsilon} = \delta\}$ to $\{u = \delta\}$ (corresponding to sections $\{x = \epsilon/\delta\}$ and $\{x = \delta\}$) is given by

$$\epsilon^2 T_3(\epsilon,\lambda) = \int_0^{\delta} \frac{d\overline{\varepsilon}}{1+\overline{\varepsilon}^2} + o(1), \qquad \text{as } \epsilon \to 0.$$

Remains to study the passage from $\{x = -\epsilon/\delta\}$ to $\{x = +\epsilon/\delta\}$, which can be seen in the family rescaling chart $\{\overline{\varepsilon} = 1\}$. There, the blow up of the vector field yields

$$\overline{X}\colon \begin{cases} \dot{\overline{x}} &=& u^2(\overline{x}^2+1+uf(u\overline{x},y,u,\lambda)) \\ \dot{y} &=& -y+ug(u\overline{x},y,u,\lambda) \\ \dot{u} &=& 0. \end{cases}$$

The passage in these coordinates is a passage from $\{\overline{x} = -1/\delta\}$ to $\{\overline{x} = +1/\delta\}$. One easily shows that

$$T_2(\epsilon,\lambda) = \frac{1}{\epsilon^2}\left(\int_{-1/\delta}^{+1/\delta} \frac{d\overline{x}}{\overline{x}^2+1} + o(1)\right), \qquad \text{as } \epsilon \to 0.$$

We conclude that the passage from $\{x = -\delta\}$ to $\{x = +\delta\}$ has a transition time given by

$$T_\delta(\epsilon,\lambda) = \sum_{i=1}^{3} T_i(\epsilon,\lambda) = \frac{1}{\epsilon^2}\left(2\int_0^{\delta}\frac{ds}{1+s^2} + \int_{-1/\delta}^{1/\delta}\frac{ds}{1+s^2} + o(1)\right)$$

as $\epsilon \to 0$. The leading term yields $2(\arctan(\delta) + \arctan(1/\delta))$. Because transitions from $\{x = \delta_1\}$ to $\{x = \delta_2\}$ are $O(\epsilon^{-1})$, one expects that the leading term cannot depend on $\delta$, and indeed it does not:

$$2(\arctan(\delta) + \arctan(1/\delta)) = \pi \qquad \forall \delta > 0.$$

(One can prove this geometrically.) Note also that as $\delta \to 0$, the leading term is given by an integral over a single orbit in the family rescaling chart, from $-\infty$ to $+\infty$! We conclude:

**Proposition 4.10** *The transition time of orbits of (4.7) from a boundary curve $\Sigma$ in the plane $\{x = x_0\}$ to a section $\{x = x_1\}$ (with $x_0 < x_1$) is given by*

$$T(\epsilon, \lambda) = \epsilon^{-2}(\pi + o(1))$$

*if $x_0 < 0 < x_1$, or*

$$T(\epsilon, \lambda) = \epsilon^{-1}\left(\int_{x_0}^{x_1} \frac{ds}{s^2} + o(1)\right)$$

*if $0 \notin [x_0, x_1]$.*

Using techniques similar to the ones in previous sections, one could calculate next to the leading term also additional terms, and prove more precise results (such as handling the case $x_0 = 0$ or $x_1 = 0$), but this would lead us to far.

# Chapter 5

# Distance between canard manifolds

Our aim is to perform a study of the transition maps near the critical curve. A key element in the study is the integral of the divergence of the vector field. For a background on this divergence and the relation to transition maps, we refer to the digression in section 1.3. In section 5.1, we define the notion “integral of the slow divergence”, where a measure of the attraction along a part of the critical curve is given. In the literature, one can find this notion as well; in particular in [BFSW], its real part is called “le relief”.

In section 5.2, we determine how saturated manifolds along a normally hyperbolic branch of the critical curve depend on the choice of the initial condition. This section does not treat the passage through turning points.

In section 5.3 our aim is to study how canard manifolds depend on the choice of the initial conditions. We will not operate directly on manifolds of canard solutions; instead we will study these properties on center manifolds (as in definition 3.5). We have a pair of such invariant manifolds: the saturation in positive time of an initial entry boundary curve $\Sigma_-$ (this is the “attracting center manifold”) and a saturation in negative time (“repelling center manifold”) of an initial exit boundary curve $\Sigma_+$. These two manifolds meet in a chart near the turning point, where the breaking parameter $a$ is resolved in terms of $\epsilon$. It is important to realize that these center manifolds depend regularly on $(a, \lambda)$, whereas the manifold of canard solutions depends regularly on $\lambda$; the parameter $a$ is in the latter resolved in terms of the singular parameter $\epsilon$ (and $\lambda$).

In section 5.4, we show how the estimates on center manifolds can be translated to estimates on canard manifolds. This leads to a result regarding the exponential closeness of control curves. From the results in this last section, one can deduce an entry–exit relationship for the vector field $X_{v,A,\lambda}$ along individual curves $A = \mathcal{A}(v, \lambda)$ in parameter space.

In the last section, we give some idea how the "maximum bifurcation delay" can be estimated using complex techniques, and introduce the next chapter.

## 5.1 Definition of slow divergence

Let $X_{\epsilon,\lambda}$ be a family of vector fields on a Riemannian manifold $M$, and let $\Omega$ be a volume form on $M$. We assume that $X_{\epsilon,\lambda}$ satisfies assumptions N1 and N3, putting ourselves in the framework of chapter 4.

**Definition 5.1** *The* slow divergence *of the vector field* $X_\lambda$ *is the divergence of* $X_{0,\lambda}$*, and will be denoted* $\operatorname{div} X^0_\lambda$.

This notion is intrinsically associated to a volume form, in the sense that it can be calculated in any chart (see section 1.3).

To give a quick example, look at the vector field

$$X_\epsilon \colon \left\{ \begin{array}{rcl} \dot{x} & = & \alpha(x,y,\epsilon)x + \epsilon f(x,y,\epsilon) \\ \dot{y} & = & \epsilon^3 g(x,y,\epsilon) \end{array} \right.$$

The critical curve is the curve $\{x = 0\}$. The slow divergence is given by $\alpha(x, y, 0) + x\frac{\partial\alpha}{\partial x}(x, y, 0)$, and restricting to the critical curve it yields $\alpha(0, y, 0)$. Associated to the critical curve a "slow vector field" can be defined (reducing to a center manifold and dividing away the factor $\epsilon^\sigma := \epsilon^3$):

$$\frac{dy}{ds} = g(0, y, 0)$$

We define the integral of the slow divergence as the integral along a part of the critical curve $\gamma$ of the slow divergence

$$\int \operatorname{div} X^0 \, ds = \int_{y_0}^{y_1} \frac{\alpha(0, y, 0)}{g(0, y, 0)} dy.$$

Apparently, the divergence along orbits of $X_\epsilon$ are related to the above expression.

Our aim is to relate the divergence integral (integral of the divergence of a vector field along an orbit of this vector field) to the integral of the slow divergence along an orbit of the slow vector field (for the definition of the slow vector field, we refer to chapter 4), and first we show that such a relation can be searched using any coordinate system, and any equivalent volume form:

**Lemma 5.2** *Let $J$ be a regular orbit of $X_{\epsilon,\lambda}$, and let $I$ be a regular orbit of $X^0_\lambda$, where $X^0_\lambda$ is the slow vector field along $\gamma$. Define*

$$H_\Omega(I,J) := \int_J \operatorname{div}_\Omega X_\lambda dt - \frac{1}{\epsilon^\sigma}\int_I \operatorname{div}_\Omega X^0_\lambda ds$$

*where $I$ is parametrized by the slow time $s$. Then, if $\Omega' = f\Omega$ is an equivalent volume form:*

$$H_{\Omega'}(I,J) = H_\Omega(I,J) + \log\frac{f(q_J)f(p_I)}{f(q_I)f(p_J)},$$

*where $p_I$, $p_J$, $q_I$ and $q_J$ are the begin points resp. end point of $I$ resp. $J$. If in particular $J = J_\epsilon$ is an orbit that tends in Hausdorff sense to $I$, then $H_{\Omega'}(I,J_\epsilon) = H_\Omega(I,J_\epsilon) + o(1)$ as $\epsilon \to 0$.*

**Proof** Using lemma 1.18, one has $\operatorname{div}_{\Omega'} X^0_\lambda = \operatorname{div}_\Omega X^0_\lambda + X_{0,\lambda}(f)/f$. At points of $I$, $X_{0,\lambda} = X^c_\lambda = \epsilon^\sigma X^0_\lambda$:

$$\operatorname{div}_{\Omega'} X^0_\lambda = \operatorname{div}_\Omega X^0_\lambda + \epsilon^\sigma \frac{X^0_\lambda(f)}{f}.$$

One continues by saying that

$$\int_I \operatorname{div}_{\Omega'} X^0_\lambda ds = \int_I \operatorname{div}_\Omega X^0_\lambda ds + \epsilon^\sigma \log(f(q_I)/f(p_I)).$$

Combine this with lemma 1.18 applied to the vector field $X_\lambda$ and the orbit $J$, we get the result. □

As such, if a normal form coordinate change is used, one need not worry about transforming the volume form—simply take an easy volume form in the normal form coordinates. The difference with respect to calculating divergence integrals is $o(1)$. Similarly, one can reparametrize time without any consequences:

**Lemma 5.3** *Let $J$ be a regular orbit of $X_\lambda$, and let $I$ be a regular orbit of $X^0_\lambda$. Define*

$$H(I,J) := \int_J \operatorname{div} X_\lambda dt - \frac{1}{\epsilon^\sigma}\int_I \operatorname{div} X^0_\lambda ds$$

*Let $\tilde{X} = hX$ be an equivalent vector field, and let $\tilde{H}$, $\tilde{X}^0$ be analogously defined. Then,*

$$\tilde{H}(I,J) = H(I,J) + \log\frac{h(q_J)h(p_I)}{h(q_I)h(p_J)},$$

*where $p_I$, $p_J$, $q_I$ and $q_J$ are the begin points resp. end point of $I$ resp. $J$. If in particular $J = J_\epsilon$ is an orbit that tends in Hausdorff sense to $I$, then $\tilde{H}(I,J_\epsilon) = H(I,J_\epsilon) + o(1)$, as $\epsilon \to 0$.*

## 5.2 Normally hyperbolic passage

Our aim is to find out how individual invariant manifolds are related to each other. The main result is theorem 5.5, from which follows that the saturation of two admissible entry boundary curves are manifolds that are exponentially close to each other. To give a quick example, consider the family

$$\begin{cases} \dot{x} &=& f(x,y,\epsilon)x + \epsilon g(x,y,\epsilon) \\ \dot{y} &=& \epsilon^\sigma, \end{cases}$$

where $f$ and $g$ are $C^\infty$ in appropiate domains, and $f(0,y,0) < 0$ for all $y$ (normal attraction). We will consider sections

$$S_1 = \{x = a\}, \quad S_2 = \{y = y_0\}.$$

Provided $S_1 \cap \{\epsilon = 0\}$ lies in the basin of attraction of the critical curve $\gamma\colon \{x = \epsilon = 0\}$, the transition map $S_1 \to S_2$ is well-defined. Let $\Sigma^{(1)}$ and $\Sigma^{(2)}$ be two smooth entry boundary curves in $S_1$ with base points $(a, y^{(1)}, 0)$ and $(a, y^{(2)}, 0)$ (both assumed to be smaller than $y_0$). The saturation of $\Sigma^{(i)}$ intersects $S_2$ in a curve $x = \theta^{(i)}(\epsilon, y_0)$. From the next sections it will become clear that

$$(\theta^{(2)} - \theta^{(1)})(\epsilon, y_0) = \exp\left(\frac{1}{\epsilon^\sigma}\left(\int_{\max(y^{(1)},y^{(2)})}^{y_0} f(0,s,0)ds + \varphi(\epsilon, y_0)\right)\right)$$

for some smooth function $\varphi$ that is $O(\epsilon)$.

### 5.2.1 Study of the divergence integral

In this section, we calculate the integral of the divergence of a singularly perturbed vector field along a normally hyperbolic curve of singularities, and relate this to the "slow divergence". We position ourselves in the framework of chapter 2, and consider a 2-dimensional Riemannian manifold $M$ with volume form $\Omega$. We fix a family of vector fields $X_{\epsilon,\lambda}$ on $M$, and denote $X_\lambda := X_{\epsilon,\lambda} + 0\frac{\partial}{\partial\epsilon}$ the lift on $M \times [0, \epsilon_0[$.

**Proposition 5.4** *Assume $\gamma$ is a normally attracting critical curve for $X_\lambda$ satisfying assumptions T1 and N3. Let $\Sigma$ be a smooth boundary curve, and let $W$ be the saturation of $\Sigma$. Let $\Sigma'$ be another smooth boundary curve inside $W$ (that could be obtained as the intersection of $W$ with a transverse plane). Let $p$ be the $\omega$-limit of the end point $s(0,\lambda)$ of $\Sigma$, and assume the end point $p'$ of $\Sigma'$ lies on $\gamma$. Assuming the piece $[p,p'] \subset \gamma$ is a regular part of an orbit of the slow vector field $X^0_\lambda$, we have*

$$\int_{O_\epsilon} \operatorname{div} X_\lambda dt = \frac{1}{\epsilon^\sigma}\left(\int_{[p,p']} \operatorname{div} X^0_\lambda\, ds + \varphi(\epsilon,\lambda)\right) \tag{5.1}$$

*where $O_\epsilon$ is the orbit along $X_{\epsilon,\lambda}$ from $s(\epsilon,\lambda)$ to $s'(\epsilon,\lambda)$, and where $\varphi$ is $C^\infty$, with $\varphi(0,\lambda) = 0$.*

**Note 1:** We have taken smooth boundary curves, although it is not necessary; if $\Sigma$ and $\Sigma'$ are nonsmooth at $\epsilon = 0$ (but are still admissible), then the function $\varphi$ will also not be smooth, but its regularity is in a sense given by the regularity of $\Sigma$.

**Note 2:** Instead of a pair of boundary curves $\epsilon \to s(\epsilon, \lambda)$ one can consider families $\epsilon \to s(\epsilon, \lambda, \alpha)$, with $\alpha \in A$, $A$ being a compact index family. In that case, the limit property in (5.1) is uniform in the parameter $\alpha$. The proof is easy: one can replace $\Lambda$ by $\Lambda \times A$ in order to obtain uniformity in $\alpha$.

**Proof** Define $W$ as the saturation of the boundary curve $\Sigma$; its smoothness is already described in earlier chapters. Keeping in mind lemma 5.2 and lemma 5.3, we can show relation (5.1) using $C^k$ normal form theory (normal forms for equivalence). As in the transition time analysis, we cover the part $[p, p']$ of the critical curve $\gamma$ by a finite number of neighbourhoods $(U_i)$ where $C^k$ normal forms are valid (see also section 4.1.2). The integration path can now be divided in a finite number of segments, being of type I or type II; it clearly suffices to consider one segment of each type:

Type I. A passage between the curve $\Sigma$ and a curve $\Sigma_0$, where $\Sigma_0$ is any boundary curve inside $W$ for which the end point on $\{\epsilon = 0\}$ is inside the neighbourhood $U_0$ and outside $\gamma$. A good choice for $\Sigma_0$ could be an image of $\Sigma$ under the time-$\tau$ map of the vector field, for a well-chosen time $\tau$. In that case, $\Sigma_0$ inherits the regularity properties of $\Sigma$, i.e. $\Sigma_0$ is smooth.

Type II. A passage between a curve $\Sigma_i$ and $\Sigma_{i+1}$, where the end point of $\Sigma_i$ is inside $U_i$, and the end point of $\Sigma_{i+1}$ is inside $U_i \cap U_{i+1}$. A good choice for $\Sigma_{i+1}$ would be the intersection of the manifold $W$ with a $C^\infty$ transverse section in $U_i \cap U_{i+1}$; for $i \geq 1$ this yields $C^\infty$ curves. A special case is the passage between $\Sigma_0$ and $\Sigma_1$, but below it will need no different treatment.

Restricting to a segment of type I one can uniformly bound the transition time and the divergence, meaning that there is a $C^\infty$ function $\varphi_I$ so that

$$\int_{O_\epsilon^I} \operatorname{div} X_\lambda dt = \varphi_I(\epsilon, \lambda).$$

(To see this, notice that in this part, the vector field is equivalent to a divergence-free flow box, hence the original divergence only consists of extra terms that appear as in lemmas 5.2 and 5.3.)

Consider now a segment of type II inside $U_i$ where a normal form for conjugacy is given in (4.2). Unlike in chapter 4, we are not limited to normal forms for conjugacy, and continue with the $C^k$-normal form for equivalence

$$\begin{cases} \dot{x} &= -x \\ \dot{y} &= \epsilon^\sigma g(y, \epsilon, \lambda), \end{cases}$$

where $g$ is a $C^k$ function. Because the slow vector field in these coordinates is given by $\dot{y} = g(y, 0, \lambda)$, and because it is assumed that the segment $[p, p']$ does not have singularities, we know that we can assume that $g(y,\epsilon,\lambda)$ is nonzero in the neighbourhood $U_i$. It is well-known that under these circumstances, one can use $C^k$ normal form theory to take $g = 1$:

$$Y_{\epsilon,\lambda}: \left\{ \begin{array}{lcl} \dot{x} & = & -x \\ \dot{y} & = & \epsilon^\sigma. \end{array} \right.$$

Let $\Sigma_i$ be given by the graph of $(\epsilon,\lambda) \to (x_i(\epsilon,\lambda), y_i(\epsilon,\lambda))$ in local coordinates, and similarly for $\Sigma_{i+1}$. One finds

$$\int_{O_\epsilon^{II}} \operatorname{div} X_\lambda dt = \int_{y_i(\epsilon,\lambda)}^{y_{i+1}(\epsilon,\lambda)} \frac{dx}{\epsilon^\sigma} = \frac{1}{\epsilon^\sigma}(y_{i+1}(\epsilon,\lambda) - y_i(\epsilon,\lambda)).$$

On the other hand, the slow vector field is given by $\dot{y} = 1$; if $p_i$ is the $\omega$-limit of $(x_i(0,\lambda), y_i(0,\lambda))$ w.r.t. $Y_{0,\lambda}$, then the $y$-coordinate of $p_i$ is given by $y_i(0,\lambda)$. We conclude

$$\int_{[p_i,p_{i+1}]} \operatorname{div} Y_\lambda^0 ds = (y_{i+1}(0,\lambda) - y_i(0,\lambda)).$$

Because the coordinate functions can be chosen $C^k$-smooth (in fact $C^k$ for any $k$), we find

$$\int_{O_\epsilon^{II}} \operatorname{div} Y_\lambda dt - \frac{1}{\epsilon^\sigma} \int_{[p_i,p_{i+1}]} \operatorname{div} Y_\lambda^0 ds = \frac{1}{\epsilon^\sigma} \varphi_{II}(\epsilon,\lambda)$$

with $\varphi_{II}$ is a $C^k$ function that is $O(\epsilon)$. A similar statement is now also true for the equivalent vector field $X_{\epsilon,\lambda}$. We can repeat this proof to obtain $C^k$-smoothness for any $k$. Because $\varphi$ does not depend on $k$ and because outside $\epsilon = 0$ the resulting objects are automatically $C^\infty$ (proving that the domain of $C^k$-smoothness does not shrink as $k$ increases), we obtain $C^\infty$ smoothness in the final result. $\square$

### 5.2.2 Transition map

Let us now link the study of the divergence integral to transition maps. Define the integral of the slow divergence along a compact piece $[p,q] \subset \gamma$ as

$$I(p,q,\lambda) := \int_{[p,q]\subset\gamma} \operatorname{div} X_\lambda^0 ds.$$

Notice that if the orbit of the slow vector field goes from $p$ to $q$ along $\gamma_-$, then $I(p,q,\lambda)$ is strictly negative, whereas if the orbit goes from $p$ to $q$ along $\gamma_+$, then $I(p,q,\lambda)$ is strictly positive.

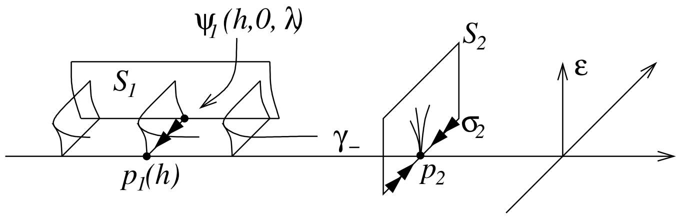


Figure 5.1: Transition map in the normally hyperbolic case

**Theorem 5.5** *Let $S_1$ be a smooth section of $M \times [0, \epsilon_0[\times\Lambda$ that is a graph*

$$\psi_1 \colon (h, \epsilon, \lambda) \mapsto (s_1(h, \epsilon, \lambda), \epsilon, \lambda).$$

*Assume that $S_1$ is transverse to the flow of $X_{\epsilon,\lambda}$. Let $\sigma_2$ be a section of $M$ transverse to $\gamma_-$ that consists of an orbit and its $\omega$-limit on $\gamma_-$ (one orbit on both sides of $\gamma_-$), and define $S_2 = \sigma_2 \times [0, \epsilon_0[$. Let $S_2$ be parametrized by*

$$\psi_2 \colon (z, \epsilon, \lambda) \mapsto (s_2(z, \lambda), \epsilon, \lambda)$$

*(where $s_2(z, \lambda)$ is a parametrization for $\sigma_2$) and let the transition map*

$$P \colon S_1 \to S_2$$

*be well-defined (see the text below the theorem). The composition $\psi_2^{-1} \circ P \circ \psi_1$ is then of the form*

$$(\psi_2^{-1} \circ P \circ \psi_1)(h, \epsilon, \lambda) = (\theta(h, \epsilon, \lambda), \epsilon, \lambda),$$

*where $\theta$ a $C^\infty$ function satisfying*

$$\epsilon^\sigma \frac{\partial \theta}{\partial h} = \pm \exp\left(\frac{1}{\epsilon^\sigma}\left(I(p_1(h), p_2, \lambda) + \varphi(h, \epsilon, \lambda)\right)\right) \tag{5.2}$$

*for some $C^\infty$ function $\varphi$ that is $O(\epsilon)$, and where $p_1(h) = p_1(h, \lambda)$ is the $\omega$-limit of $\psi_1(h, 0, \lambda)$, and where $p_2 = p_2(\lambda)$ is the $\omega$-limit of the orbit $\sigma_2$. The functions $I$ and $\varphi$ depend on the chosen volume form $\Omega$ on $M$.*

Note that the transition map $P$ is well-defined provided that the section $S_2$ is chosen so that the $\omega$-limits of $S_1 \cap \{\epsilon = 0\}$ are points of $\gamma$ that "come before $S_2$", in the sense that the orbit w.r.t. the slow vector fields through these $\omega$-limitpoints reach $\sigma_2$ in finite positive time. In that case, it follows from the results in chapter 2 that $P$ is a $C^\infty$ map.

**Proof** The smoothness of $\theta$ follows from the smoothness of $P$. On the other hand, it is a direct application of proposition 1.21 that

$$\frac{\partial \theta}{\partial h}(h, \epsilon, \lambda) = \frac{\langle \Omega(p), D\psi_1(h, \epsilon, \lambda) \times X(p) \rangle}{\langle \Omega(q), D\psi_2(\theta(h, \epsilon, \lambda), \epsilon, \lambda) \times X(q) \rangle} \exp\left(\int_{O(p,q)} \operatorname{div}_\Omega X_{\epsilon,\lambda} dt\right),$$

for any volume form $\Omega$ on $M$, and where the integration takes place along an orbit starting at $p = \psi_1(h, \epsilon, \lambda)$ until it meets $S_2$ in a point $q = \psi_2(\theta(h, \epsilon, \lambda), \epsilon, \lambda)$.

The transversality of $S_1$ w.r.t. the flow of the vector field shows that the numerator in the above expression is a smooth nonzero function.

The denominator will be calculated in $C^k$-normal form coordinates, and an equivalent $C^k$-volume form. Near $p_2$ the family has a $C^k$-normal form for equivalence given by

$$\begin{cases} \dot{x} &= -x \\ \dot{y} &= \epsilon^\sigma, \end{cases}$$

and in these coordinates $S_2$ is a section for which the intersection with $\{\epsilon = 0\}$ is a straight line $\{y = y_2\}$. The coordinate function $\psi_2$ has the expression

$$\psi_2\colon (z, \epsilon, \lambda) \mapsto (x, y, \epsilon, \lambda) := (x_2(z, \epsilon, \lambda), y_2(z, \epsilon, \lambda), \epsilon, \lambda).$$

with $y_2(z, 0, \lambda) = y_2$. Using $\Omega_2 = dx \wedge dy$ on $M$, one can calculate that

$$\langle \Omega(q), D\psi_2(\theta(h, \epsilon, \lambda), \epsilon, \lambda) \times X(q) \rangle = \epsilon^\sigma \frac{\partial x_2}{\partial z} - x_2 \frac{\partial y_2}{\partial z}\bigg|_{z=\theta(h,\epsilon,\lambda)}.$$

Now, notice first that $\frac{\partial x_2}{\partial z}(z, 0, \lambda) \neq 0$; this is because $\frac{\partial y_2}{\partial z}(z, 0, \lambda) = 0$, and because of the fact that $\psi_2$ is a regular coordinate system for $S_2$. Second, notice that $x_2(\theta(h, 0, \lambda), 0, \lambda) = 0$; even more so: one has

$$x_2(\theta(h, \epsilon, \lambda), \epsilon, \lambda) = O(\epsilon^N), \qquad \forall N \leq k$$

This is because the saturation of orbits of $S_2$ form an invariant manifold that is $C^k$-flat to $\{x = 0\}$, which is a $C^k$-center manifold in the chosen coordinate system. For $k$ high enough, this shows that

$$\epsilon^\sigma \frac{\partial \theta}{\partial h}(h, \epsilon, \lambda) = f(h, \epsilon, \lambda) \exp\left( \int_{O(p,q)} \operatorname{div}_\Omega X_{\epsilon,\lambda} dt \right),$$

for some nonzero $C^k$ function $f$. Application of proposition 5.4 yields the existence of a $C^k$-function $\varphi$ as claimed in the formulation of the theorem. Because we can increase $k$ arbitrarily, and keeping in mind that $\varphi$ is unique, we can conclude that $\varphi$ is $C^\infty$. The domain of smoothness does not shrink, since outside $\epsilon = 0$ the function $\varphi$ is clearly $C^\infty$ because it is given by

$$\varphi(h, \epsilon, \lambda) = \epsilon^\sigma \log\left( \pm \epsilon^\sigma \frac{\partial \theta}{\partial h}(h, \epsilon, \lambda) \right) - I(p_1(h), p_2, \lambda).$$

$\square$

Using this theorem, one can measure the difference between two invariant manifolds:

$$\theta(h^{(2)}, \epsilon, \lambda) - \theta(h^{(1)}, \epsilon, \lambda) = \int_{h^{(2)}}^{h^{(1)}} \frac{\partial \theta}{\partial h} dh.$$

It is now easily shown that (using the lemma below), provided $\frac{d}{dh}(p_1(h)) \neq 0$ one has

$$\theta(h^{(2)}) - \theta(h^{(1)}) = \pm \exp\left(\frac{1}{\epsilon^\sigma}(I(p_1(h), p_2, \lambda) + \varphi_{12}(h, \epsilon, \lambda))\right),$$

for some $C^0$ function $\varphi_{12}$, and where $p_1(h)$ is the $\omega$-limit of either $\psi_1(h^{(2)}, 0, \lambda)$ or $\psi_1(h^{(1)}, 0, \lambda)$, whichever lies closer to $p_2$ w.r.t. the slow motion on the critical curve $\gamma_-$.

**Lemma 5.6** *Let $\alpha(h)$ be a $C^1$ function, and let $R(h,v)$ be continuous in $(h,v)$ and continuously differentiable w.r.t. $h$, so that $R(h,0) = 0$. Assume $\frac{d\alpha}{dh}$ is strictly negative along $[a,b]$ (with $a < b$) and $\alpha(a) < 0$. Then,*

$$\int_a^b \frac{1}{v^\sigma} \exp\left(\frac{\alpha(h) + R(h,v)}{v^\sigma}\right) dh = \exp\left(\frac{\alpha(a) + \tilde{R}(v)}{v^\sigma}\right),$$

*for some continuous function $\tilde{R}(v)$ with $\tilde{R}(0) = 0$.*

**Proof** Define

$$\beta(h,v) = \alpha(h) + R(h,v) - \alpha(a) - R(a,v)$$

Then, $\beta(a,v) = 0$ and $\frac{\partial \beta}{\partial h}(h,v)$ is strictly negative along $h \in [a,b]$, for $v$ small enough. Consider

$$F(v) := \int_a^b \frac{1}{v^\sigma} \exp\left(\frac{\beta(h,v)}{v^\sigma}\right) dh.$$

It suffices now to show that this forms a strictly positive continuous function. We write $\beta(h,v) = \eta(h,v)(h-a)$ for some $C^0$-function $\eta$. One can rewrite the above integral as

$$F(v) := \int_0^{(b-a)/v^\sigma} \exp\left(\eta(a + uv^\sigma, v)u\right) du.$$

For $v > 0$ small enough, the above integral is bounded (use $\eta < -r$ for some $r > 0$ to find that the integral is less than $1/r$. By the theorem of dominated convergence of Lebesgue, the above expression tends continuously to

$$\int_0^\infty \exp((\eta(a,0)u)\, du = \frac{-1}{\eta(a,0)} = \frac{-1}{\frac{\partial \beta}{\partial h}(a,0)} = \frac{-1}{\alpha'(a)}.$$

So, $F(v)$ tends to a value that is strictly positive as $v \to 0$, meaning that we can write

$$F(v) = \exp\left(\frac{v^\sigma \log F(v)}{v^\sigma}\right) = \exp((R_1(v)/v^\sigma),$$

with $R_1$ a continuous function. One concludes:

$$\int_a^b \frac{1}{v^\sigma} \exp\left(\frac{\alpha(h) + R(h,v)}{v^\sigma}\right) dh = \exp\left(\frac{\alpha(a) + \tilde{R}(v)}{v^\sigma}\right),$$

with

$$\tilde{R}(v) = R(a,v) + R_1(v) = R(a,v) + v^{\sigma} \log F(v).$$

This proves the lemma. □

Similar techniques would allow one to prove additional regularity properties on $\tilde{R}$. In fact one can see immediately that $\tilde{R}$ retains the smoothness that $R$ has w.r.t. additional (non-essential) parameters $\lambda$.

## 5.3 Passage through a turning point

At a point $p_*$ where the normal hyperbolicity of the reduced vector field $X_{0,\lambda}$ is lost, one can try to desingularize the vector field by means of a weighted family blow up. As we position ourselves in the framework of chapter 3, we assume that the family of vector fields on $M$ is already in the form (3.5), i.e. we assume that a rescaling

$$(a, \epsilon) = (v^k A, v^{\ell})$$

has already taken place, so that the "breaking parameter" $A$ is brought in a form that is required to prove the existence of canard manifolds. From this moment, we assume that $v$ is the singular parameter.

Consider hence a family of vector fields $X_{v,A,\lambda}$ on a 2-manifold $M$ with a volume form $\Omega$. We will assume all conditions imposed on $X_{v,A,\lambda}$ in chapter 3.

Our aim is to treat the behaviour of a transition map from a section $S_1$, just as in section 5.2, to a section $T$. This time, the section $T$ is a section positioned on top of the blow up locus, i.e. it is a section cutting the critical curve $\gamma$ transversally at the turning point.

The treatment will be based on a study of the divergence integral. Since the divergence is dependent on the volume form, we first ask ourselves how the volume form undergoes a blow up transformation:

**Lemma 5.7** *Consider a blow up map*

$$\Phi\colon \mathbf{R}^+ \times S^2 \to \mathbf{R}^3 \colon (u, (\overline{x}, \overline{y}, \overline{v})) \to (x, y, v) = (u^p \overline{x}, u^q \overline{y}, u^m \overline{v}), \tag{5.3}$$

*where $(\overline{x}, \overline{y}, \overline{v}) \in S^2$, i.e. $\overline{x}^2 + \overline{y}^2 + \overline{v}^2 = 1$. In the family rescaling chart $\{\overline{v} = 1\}$, this map is diffeomorphic to the map*

$$\Phi_1\colon \mathbf{R}^+ \times U_1 \to \mathbf{R}^3 \colon (u, (\overline{x}, \overline{y})) \to (u^p \overline{x}, u^q \overline{y}, u^m),$$

*The standard volume form $\Omega = dx \wedge dy \wedge dv$ in $\mathbf{R}^3$ is equivalent to $u^{p+q+m-1} d\overline{x} \wedge d\overline{y} \wedge du$ in the family rescaling chart. Similarly, in any phase-directional rescaling chart, $\Omega$ is equivalent to $u^{p+q+m-1} du \wedge d\overline{v} \wedge dz$, where $z$ is an angular coordinate on $S^1$ (representing $(\overline{x}, \overline{y})$).*

**Proof** Using $\Phi_1$, one simply has to calculate the jacobian determinant. For the phase-directional rescaling charts, they are locally diffeomorphic to any of the charts $\overline{x} = 1$, $\overline{x} = -1$, $\overline{y} = 1$, or $\overline{y} = -1$. For symmetry reasons, it suffices to check the chart $\overline{x} = 1$, where the blow up map is given by

$$\Phi_2 \colon \mathbf{R}^+ \times U_2 \to \mathbf{R}^3 \colon (u, (\overline{y}, \overline{v})) \to (u^p, u^q\overline{y}, u^m\overline{v}),$$

The jacobian determinant is equivalent to $u^{p+q+m-1}$. □

The treatment of the divergence integral along orbits of $X_\lambda$ will be done in several parts; the parts may be divided into four types, the already-existing types I and II (see the proof in the normally hyperbolic passage), and two additional types:

Type III A type III passage will cover the passage in a phase-directional rescaling chart in the neighbourhood of the intersection point of the critical curve $\gamma$ with the blow up locus.

Type IV A type IV passage will cover the passage in a compact piece of the family rescaling chart.

We assume that there exist $C^k$-coordinates and a phase-directional rescaling chart so that $X_\lambda$ has the following form (in fact this is a version of the result in lemma 3.9):

$$X_{A,\lambda} \colon \left\{ \begin{array}{rcl} \dot{u} & = & -u^{\alpha+1}\overline{v}^{\sigma} h(u, \overline{v}, A, \lambda) \\ \dot{\overline{v}} & = & m u^{\alpha}\overline{v}^{\sigma+1} h(u, \overline{v}, A, \lambda) \\ \dot{z} & = & -u^{\alpha} z \end{array} \right. \tag{5.4}$$

where $h$ is a strictly positive $C^k$ function. The critical curve $\gamma$ appears in these coordinates as the line $z = \overline{v} = 0$, whereas the intersection point $P_-$ of $\gamma$ with the blow up locus $\{u = 0\}$ is the origin. The factor $u^\alpha$ appears as a common factor which is divided out in the blown up vector field, i.e. the blow up of $X_\lambda$ is defined as

$$\overline{X}_{A,\lambda} = u^{-\alpha} X_{A,\lambda} \tag{5.5}$$

for a well-chosen $\alpha \in \mathbf{N}$. To study the passage of type IV in the family rescaling chart, we assume that there exist $C^k$-coordinates so that $X_\lambda$ has the following form:

$$X_{A,\lambda} \colon \left\{ \begin{array}{rcl} \dot{\overline{x}} & = & u^{\alpha} f(\overline{x}, \overline{y}, u, A, \lambda) \\ \dot{\overline{y}} & = & 0 \\ \dot{u} & = & 0 \end{array} \right. \tag{5.6}$$

The function $f$ is strictly positive; the connection $\Gamma$ connecting $P_-$ to $P_+$ is in these coordinates given by $\{\overline{y} = u = 0\}$. (These coordinates could be referred to as degenerate flow-box coordinates.)

### 5.3.1 Study of the divergence integral

The next proposition is a continuation of proposition 5.4; here the divergence inside invariant manifolds is calculated up to the turning point.

**Proposition 5.8** *Let $p_*$ be the limit of $\gamma_-$, where the normal hyperbolicity is lost, and assume that $p_*$ is a simple passage turning point (so that assumptions T1–T6 are satisfied for the vector field $X_{v,A,\lambda}$). Assume that the vector field $X_{v,A,\lambda}$ satisfies assumptions N1 and T7, and let $\sigma$ be the "order of degeneracy" as introduced in chapter 4.*

*Let $\Sigma$ be a smooth boundary curve, and let $W$ be the saturation of $\Sigma$. Let $\Sigma'$ be another smooth boundary curve inside $W$, so that the end point of $\Sigma'$ coincides with $p_*$ and so that $\Sigma'$ has a smooth blow up ($\Sigma'$ could be obtained as the intersection of $W$ with a transverse plane $T$ in the family rescaling chart). Let $p = p(A, \lambda)$ be the $\omega$-limit of the end point of $\Sigma$. Assume the piece $[p, p_*[ \subset \gamma$ is a regular orbit of the slow vector field $X^0_{A,\lambda}$ so that*

$$I(p, p_*, A, \lambda) := \int_{[p,p_*]} \operatorname{div} X^0_{A,\lambda}\, ds$$

*is convergent. Let $k \in \mathbf{N}_1$ be fixed. Then,*

$$\int_{O_v} \operatorname{div} X_{A,\lambda} dt = C \log v + \frac{I(p, p_*, A, \lambda) + \tilde{\varphi}(v, A, \lambda) + v^\sigma \varphi(v, A, \lambda)}{v^\sigma} \tag{5.7}$$

*where $O_v$ is the orbit from $\Sigma$ to $\Sigma'$ at height $v$ (i.e. $O_{v_1} = W \cap \{v = v_1\}$), where $\tilde{\varphi}$ is a continuous $o(1)$ function (detailed regularity properties of $\tilde{\varphi}$ are explained in proposition 5.9), where $\varphi$ is a function that is $C^k$ in terms of $v^{1/m}$. The constant $C$ is given by*

$$C = \frac{p+q}{m} - b; \qquad b := \frac{m\sigma - \alpha}{m}.$$

*The "blow up constant" $b$ is assumed to be strictly positive.*

Note: the function $\tilde{\varphi} + v^\sigma \varphi$ is unique (i.e. does not depend on $k$), and can therefore be proved to be $C^\infty$ outside $v = 0$; this is not necessarily true for the individual functions $\tilde{\varphi}$ and $\varphi$, since the splitting does depend on $k$.

**Proof** We divide the passage inside the manifold $W$ into several parts; parts of type I and type II are studied in the proof of proposition 5.4. During passages of type I and II, the divergence has no contribution in $\log v$; it only contributes to a part of $\varphi$ in the statement of the proposition. Remains to study the passages of types III and IV. Consider $C^k$-coordinates so that the vector field is given by (5.4) near the end point $P_-$ of the critical curve on the blow up locus $\{u = 0\}$. Let $\Sigma_0 = \{u = u_0\} \cap W$ and let $\Sigma_1 = \{\overline{v} = \overline{v}_1\} \cap W$. Both curves are $C^k$, at all points up to and including their end points. The study of the type $III$ passage from $\Sigma_0$ to $\Sigma_1$ will be performed using an equivalent vector field

$$Y_{A,\lambda}: \begin{cases} \dot{u} &= -u^{\alpha+1}\overline{v}^{\sigma} \\ \dot{\overline{v}} &= m u^{\alpha}\overline{v}^{\sigma+1} \\ \dot{z} &= -u^{\alpha} z/h(u,\overline{v},A,\lambda) \end{cases} \tag{5.8}$$

Define $\overline{Y}_{A,\lambda}$ to be the blow up of the vector field $Y_{A,\lambda}$, i.e. $\overline{Y}_{A,\lambda} = u^{-\alpha}Y_{A,\lambda}$. Let $\Omega = u^{p+q+m-1}\Omega_0$, with $\Omega_0 = du \wedge d\overline{v} \wedge dz$, in accordance to lemma 5.7. We are interested in the divergence of $X_{A,\lambda}$ with respect to $\Omega$, and notice that this is related to $\operatorname{div}_\Omega Y_{A,\lambda}$ (see lemma 1.19). So concentrate first on $\operatorname{div}_\Omega Y_{A,\lambda}$:

$$\begin{aligned} \operatorname{div}_\Omega Y_{A,\lambda} &= \operatorname{div}_{\Omega_0} Y_{A,\lambda} + Y_{A,\lambda}(u^{p+q+m-1})/u^{p+q+m-1} \\ &= \operatorname{div}_{\Omega_0} Y_{A,\lambda} - (p+q+m-1)u^{\alpha}\overline{v}^{\sigma} \\ &= \operatorname{div}_{\Omega_0}(u^{\alpha}\overline{Y}_{A,\lambda}) - (p+q+m-1)u^{\alpha}\overline{v}^{\sigma} \\ &= u^{\alpha}\operatorname{div}_{\Omega_0}\overline{Y}_{A,\lambda} + \overline{Y}_{A,\lambda}(u^{\alpha}) - (p+q+m-1)u^{\alpha}\overline{v}^{\sigma}. \end{aligned}$$

In other words, letting $C_1 = (p+q+m-1+\alpha)$:

$$\operatorname{div}_\Omega Y_{A,\lambda} = u^{\alpha}\left(\operatorname{div}_{\Omega_0}\overline{Y}_{A,\lambda} - C_1\overline{v}^{\sigma}\right).$$

We find

$$\begin{aligned} \int_{O_v^{III}} \operatorname{div}_\Omega Y_{A,\lambda} dt &= \int_{O_v^{III}} \left(\operatorname{div}_{\Omega_0}\overline{Y}_{A,\lambda} - C_1\overline{v}^{\sigma}\right)(u^{\alpha}dt) \\ &= -\int_{O_v^{III}} \frac{\operatorname{div}_{\Omega_0}\overline{Y}_{A,\lambda} - C_1\overline{v}^{\sigma}}{u\overline{v}^{\sigma}} du \end{aligned}$$

The orbit $O_v^{III}$ goes through the point $(u,\overline{v},z) = (u_0,\overline{v},\zeta_0(\overline{v},A,\lambda))$ for some $C^k$ function $\zeta_0$ determined by $\Sigma_0 = W \cap \{u = u_0\}$. We stop integrating the orbit $O_v^{III}$ at the point $(u,\overline{v},z) = (u_1,\overline{v}_1,\zeta_1(u_1))$ for some $u_1 > 0$. Since $v = u^m\overline{v}$ is a constant of the flow, one has $u_1^m = \frac{v}{\overline{v}_1}$:

$$\int_{O_v^{III}} \operatorname{div}_\Omega Y_{A,\lambda} dt = -\int_{u_0}^{(v/\overline{v}_1)^{1/m}} \left.\frac{\operatorname{div}_{\Omega_0}\overline{Y}_{A,\lambda}}{u\overline{v}^{\sigma}}\right|_{\overline{v}=vu^{-m}} du + C_1[\log u]_{u=u_0}^{u=(v/\overline{v}_1)^{1/m}} \tag{5.9}$$

One can calculate $\operatorname{div}_{\Omega_0}\overline{Y}_{A,\lambda}$ easily:

$$\operatorname{div}_{\Omega_0}\overline{Y}_{A,\lambda} = (m\sigma + m - 1)\overline{v}^{\sigma} - 1/h(u,\overline{v},A,\lambda).$$

Replacing this expression in the integral formula yields

$$\int_{O_v^{III}} \operatorname{div}_\Omega Y_{A,\lambda} dt = \int_{u_0}^{(v/\overline{v}_1)^{1/m}} \left.\frac{1}{u\overline{v}^{\sigma}h(u,\overline{v},A,\lambda)}\right|_{\overline{v}=vu^{-m}} du + C\log\frac{v}{u_0^m\overline{v}_1},$$

with $C = (\alpha - m\sigma + p + q)/m$. Define now

$$\tilde{\varphi}_{III}(v, A, \lambda) = v^\sigma \int_{u_0}^{(v/\overline{v}_1)^{1/m}} \frac{1}{u\overline{v}^\sigma h(u, \overline{v}, A, \lambda)}\bigg|_{\overline{v}=vu^{-m}} du - \int_{u_0}^{0} \frac{du}{u^{1-m\sigma} h(u, 0, A, \lambda)}$$

Then,

$$\begin{aligned} v^\sigma \int_{O_v^{III}} \operatorname{div} X_{A,\lambda} dt &= \int_{O_0^{III}} \operatorname{div} X_{A,\lambda}^0 ds + C v^\sigma \log v \\ &\quad + \tilde{\varphi}_{III}(v, A, \lambda) + v^\sigma \log \varphi_{III}(v^{1/m}, A, \lambda), \end{aligned}$$

for some strictly positive $C^k$ function $\varphi_{III}$ (defined by the equivalence $h$ between $X_{A,\lambda}$ and $Y_{A,\lambda}$), where $O_0^{III}$ is the piece of the critical curve between $u = u_0$ and $u = 0$, parametrized by the slow time of the vector field $X_{A,\lambda}^0$. Remains to prove regularity properties of $\tilde{\varphi}_{III}$. One has

$$\tilde{\varphi}_{III}(v, A, \lambda) = \int_{u_0}^{(v/\overline{v}_1)^{1/m}} \frac{u^{m\sigma-1} du}{h(u, vu^{-m}, A, \lambda)} - \int_{u_0}^{0} \frac{u^{m\sigma-1} du}{h(u, 0, A, \lambda)}. \tag{5.10}$$

The regularity properties are shown in proposition 5.9. Let us now consider a passage of type IV, using normal form (5.6). First, since the volume forms $\Omega$ and $\Omega_0$ only differ by a power of $u$, and since $\dot{u} = 0$ along orbits of $X_{A,\lambda}$, we can—keeping in mind lemma 5.2—immediately proceed with the volume form $\Omega_0$ instead of $\Omega$. Similarly, one can continue with the blown up vector field $\{\dot{\overline{x}} = f(\overline{x}, \overline{y}, u, A, \lambda), \dot{\overline{y}} = 0, \dot{u} = 0\}$ for the same reason. For this regular vector field however, integrals of the divergence along compact pieces of orbits are $C^k$, so

$$v^\sigma \int_{O_v^{IV}} \operatorname{div} X_{A,\lambda} dt = v^\sigma \varphi_{IV}(v, A, \lambda)$$

for some function $\varphi_{IV}$ that is $C^k$ in terms of $u = v^{1/m}$. □

**Proposition 5.9** *The function $\tilde{\varphi}$ from proposition 5.8 has the following properties: $\tilde{\varphi}$ is $C^k$ for all $v > 0$. Furthermore:*

*(i) If $\sigma = 1$, then $\tilde{\varphi}(v, \lambda) = O(v \log v)$, $\frac{\partial \tilde{\varphi}}{\partial v}(v, \lambda) = O(\log v)$ as $v \to 0$.*

*(ii) If $\sigma \geq 2$, then $\tilde{\varphi}(v, \lambda) = O(v)$, $\frac{\partial \tilde{\varphi}}{\partial v}(v, \lambda) = O(1)$ as $v \to 0$.*

**Proof** See the proof of proposition 4.9—we only have to replace $b$ by $\sigma$. □

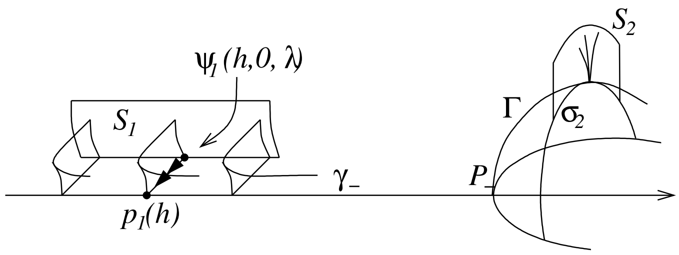


Figure 5.2: Transition map towards sections in the blow up locus

### 5.3.2 The transition map

As before, the previous subsection will be used to study the dependence of attracting and repelling center manifolds on the initial boundary curves. Define the integral of the slow divergence along a compact piece $[p, p_*] \subset \gamma_-$ as

$$I(p, p_*, \lambda) := \int_{[p,p_*[\subset\gamma_-} \operatorname{div} X^0_{A,\lambda} ds.$$

Note that this integral does not depend on $A$ (remember that $A$ is a rescaled version of $a$, making the critical curve independent of $A$). For $p \in \gamma_+$, we define

$$I(p, p_*, \lambda) := -\int_{]p_*,p]\subset\gamma_+} \operatorname{div} X^0_{A,\lambda} ds.$$

In all circumstances, $I(p, p_*, \lambda)$ will be strictly negative! For example, for the family of vector fields

$$\begin{cases} \dot{x} &= \epsilon^\sigma \\ \dot{y} &= f(x, y, \epsilon)y + \epsilon g(x, y, \epsilon) \end{cases}$$

The integral of the slow divergence is given by

$$I(x, 0) := \int_x^0 f(x, 0, 0)dx = -\int_0^x f(x, 0, 0)dx$$

For a generic turning point one can take $f(x, 0, 0) = x$ and hence

$$I(x, 0) = -\frac{x^2}{2} < 0, \quad \forall (x, 0) \in \gamma.$$

**Theorem 5.10** *Let $S_1$ be a smooth section of $M \times [0, v_0[\times] - A_0, A_0[\times\Lambda$ that is a graph*

$$\psi_1 \colon (h, v, A, \lambda) \mapsto (s_1(h, v, \lambda), v, A, \lambda).$$

*Assume that $S_1$ is transverse to the flow of $X_{v,A,\lambda}$. Let $\sigma_2$ be a section of the blow up locus in the family rescaling chart, transverse to the heteroclinic connection $\Gamma$ and define $S_2 = \sigma_2 \times [0, v_0^{1/m}[$. Let $S_2$ be parametrized by*

$$\psi_2 \colon (z, v, A, \lambda) \mapsto ((\overline{x}, \overline{y}), u, A, \lambda) = (s_2(z, A, \lambda), v^{1/m}, A, \lambda)$$

*and let the transition map*

$$P\colon S_1 \to S_2$$

*be well-defined (see the text below the theorem). The composition* $\psi_2^{-1} \circ P \circ \psi_1$ *is then of the form*

$$(\psi_2^{-1} \circ P \circ \psi_1)(h, v, A, \lambda) = (\theta(h, v, A, \lambda), v^{1/m}, A, \lambda),$$

*where* $\theta$ *a* $C^\infty$ *function w.r.t.* $v^{1/m}$ *satisfying*

$$v^\sigma \frac{\partial \theta}{\partial h} = \pm \exp\left( \frac{1}{v^\sigma} \left( I(p_1(h), p_*, \lambda) + \tilde{\varphi}(h, v, A, \lambda) \right) \right) \tag{5.11}$$

*for some function* $\tilde{\varphi}$ *with the properties explained in proposition 5.9, where* $p_1(h) = p_1(h, A, \lambda)$ *is the* $\omega$*-limit of* $\psi_1(h, 0, A, \lambda)$*. The functions* $I$ *and* $\varphi$ *depend on the chosen volume form* $\Omega$ *on* $M$*.*

Note that the transition map $P$ is well-defined provided that the section $S_2$ is chosen so that the $\omega$-limits of $S_1 \cap \{\epsilon = 0\}$ are in the basin of attraction of $\gamma_-$. In that case, it follows from the results in chapter 3 that $P$ is a $C^\infty$ map w.r.t. $u = v^{1/m}$.

Note also that the front factor $v^\sigma$ in formula (5.11) can be removed, upon changing $\varphi$; this does not decrease the continuity properties of $\varphi$ any further.

**Proof** The smoothness of $\theta$ follows from the smoothness of $P$. As in the normally hyperbolic situation, we directly apply proposition 1.21 to show that

$$\frac{\partial \theta}{\partial h} = \frac{\langle \Omega(p), D\psi_1(h, v, A, \lambda) \times X(p) \rangle}{\langle \Omega(q), D\psi_2(\theta(h, v, A, \lambda), v, A, \lambda) \times X(q) \rangle} \exp\left( \int_{O(p,q)} \operatorname{div}_\Omega X_{v,A,\lambda} dt \right),$$

for any volume form $\Omega$ on $M$, and where the integration takes place along an orbit starting at $p = \psi_1(h, v, A, \lambda)$ until it meets $S_2$ in a point $q = \psi_2(\theta(h, v, A, \lambda), v, A, \lambda)$.

The numerator in the above formula is nonzero (see the proof in the normally hyperbolic case); the denominator needs to be studied more in detail. To that end we use $C^k$-coordinates in the family rescaling chart, where the vector field is given by (5.6). The volume form in these coordinates is equivalent to

$$u^{p+q+m-1} d\overline{x} \wedge d\overline{y} \wedge du,$$

and the section $S_2$ is in these coordinates a graph

$$\psi_2\colon (z, v, A, \lambda) \mapsto (\overline{x}, \overline{y}, v, A, \lambda) = (\overline{x}_2(z, v, A, \lambda), \overline{y}_2(z, v, A, \lambda), v^{1/m}, A, \lambda)$$

Using these components, one shows that

$$\langle \Omega(q), D\psi_2(\theta(h, v, A, \lambda), v, A, \lambda) \times X(q) \rangle =$$

$$v^{(p+q+m-1)/m} g(h, v, A, \lambda) \left| \begin{matrix} \frac{\partial \overline{x}_2}{\partial z} & \frac{\partial \overline{x}_2}{\partial v} & v^{\alpha/m} f \circ \psi_2 \\ \frac{\partial \overline{y}_2}{\partial z} & \frac{\partial \overline{y}_2}{\partial v} & 0 \\ 0 & m v^{1/m-1} & 0 \end{matrix} \right|_{z=\theta(h,v,A,\lambda)}$$

where $g$ expresses the equivalence of two volume forms (and is hence a nonzero $C^k$ function). Noticing that $\frac{\partial \overline{y}_2}{\partial z}$ is nonzero (transversality argument) we conclude that

$$\frac{\partial \theta}{\partial h}(h, v, A, \lambda) = \frac{F(h, v, A, \lambda)}{v^{(p+q+\alpha)/m}} \exp\left(\int_{O(p,q)} \operatorname{div}_{\Omega} X_{v,A,\lambda} dt\right),$$

for some nonzero function $F$ that is $C^k$ w.r.t. $v^{1/m}$. In combination with proposition 5.8, this yields the result. □

Using this theorem and lemma 5.6, one can measure the difference between two center manifolds:

$$\theta(h^{(2)}, v, A, \lambda) - \theta(h^{(1)}, v, A, \lambda) = \int_{h^{(2)}}^{h^{(1)}} \frac{\partial \theta}{\partial h} dh.$$

It is now easily shown that (provided $\frac{d}{dh}(p(h)) \neq 0$):

$$\theta(h^{(2)}) - \theta(h^{(1)}) = \exp\left(\frac{1}{v^\sigma}(I(p(h), p_*, \lambda) + \varphi_{12}(h, v, A, \lambda))\right), \tag{5.12}$$

for some continuous function $\varphi_{12}$ ($o(1)$ as $v \to 0$), and where $p(h)$ is either $p(h^{(1)}, A, \lambda)$ or $p(h^{(2)}, A, \lambda)$, whichever is closer to the turning point $p_*$ (closer according to the slow dynamics on $\gamma$).

**Corollary 5.11** *Let $\sigma_1$ be a compact curve transversally intersecting $\gamma_-$ at a point $p_-$, and assume $\sigma_1$ is the union of two orbits towards $\gamma_-$ of $X_{0,A,\lambda}$. Define $S_1 = \sigma_1 \times [0, v_1]$, and consider the transition map towards a section $S_2$ as in theorem 5.10 on top of the blow up locus. The image of $S_1$ under the transition map $P$ is a wedge. In coordinates, this yields*

$$P(S_1) = \{(z, v) \colon v \in [0, v_1], \varphi_1(v, A, \lambda) \leq z \leq \varphi_2(v, A, \lambda)\},$$

*and the wedge is exponentially small, i.e.*

$$(\varphi_1 - \varphi_2)(v, A, \lambda) = \exp(\frac{1}{v^\sigma}(Ip_-, p_*, \lambda) + o(1))).$$

**Proof** It is easily seen that the border of $P(K)$ is formed by the image of the border of $K$. (In fact this follows from the topological equivalences to what could be called degenerate flow boxes in assumptions N2 and T4.) This last border is a union of three pieces: two admissible entry boundary curves, and a level curve $\{v = v_0\}$. □

In vague terms, this corollary states that the saturation of a curve with entry point farther from $p_*$ than $p_-$ is a curve inside an exponentially small wedge.

## 5.4 Comparing manifolds of canard solutions

One can use theorems 5.5 and 5.10 to compare different manifolds of canard solutions, simply because the estimates in this theorems depend regularly on $A$. Just replacing $A$ by the control curve $\mathcal{A}$ is enough to show that both theorems are applicable to the manifolds of canard solutions as well. Let us work it out in more detail.

Consider two sections $S_\pm$ as in theorems 5.5 and 5.10: the section $S_-$ is chosen so that $S_- \cap \{v = 0\}$ lies in the basin of attraction of $\gamma_-$, whereas $S_+$ is chosen so that $S_+ \cap \{v = 0\}$ lies in the basin of repulsion of $\gamma_+$. Choose a common section $T$ on the blow up locus, transverse to the heteroclinic connection $\Gamma$. We can apply theorem 5.10 for the sections $S_-$ and $S_+$ separately. In the notations of this theorem, we can define

$$\Delta(h_-, h_+, v, A, \lambda) := \theta_-(h_-, v, A, \lambda) - \theta_+(h_+, v, A, \lambda),$$

where $h_-$ is a parameter for $S_-$, $h_+$ is a parameter for $S_+$, where $z$ is a regular coordinate for the section $T$. Under the assumption T1–T6, we know that

$$\Delta|_{v=A=0} = 0, \qquad \left.\frac{\partial \Delta}{\partial A}\right|_{v=A=0} \neq 0.$$

Hence, we can use the implicit function theorem to find $A = \mathcal{A}(h_-, h_+, v, \lambda)$. We see that $\mathcal{A}$ depends regularly on the parameters $h_\pm$. Before stating the main result of this section, we give a definition:

**Definition 5.12** *Let $p, q \in \gamma$. We say that $p$ is closer to $p_*$ than $q$ if*

$$|I(p, p_*, \lambda)| \leq |I(q, p_*, \lambda)|.$$

*which is equivalent to saying that*

$$I(p, p_*, \lambda) \geq I(q, p_*, \lambda).$$

*(We remind the reader that $I(p, p_*, \lambda)$ is negative for all $p$ on the attracting branch as well as on the repelling branch of the critical curve.)*

**Theorem 5.13** *Let the sections $S_\pm$ of $M \times [0, v_0[$ be chosen as above, and let these sections be parametrized by $(h_\pm, v)$. We define $p(h_\pm, \lambda) \in \gamma_\pm$ to be the $\omega$-limit (resp. $\alpha$-limit) of a point in $S_\pm$ with parameter value $(h_\pm, 0)$, and assume*

$$\frac{d}{dh_\pm} p(h_\pm) \neq 0.$$

*Given a pair of entry–exit points $(h_-^{(1)}, h_+^{(1)})$ corresponding to two boundary curves in $S_\pm$, there is a control curve*

$$A = \mathcal{A}^{(1)}(v, \lambda)$$

*along which the two boundary curves are connected by a manifold of canard solutions* $W^{(1)}$*. Given a second pair of entry–exit points* $(h_-^{(2)}, h_+^{(2)})$ *and associated control curve* $\mathcal{A}^{(2)}$*, and manifold of canard solutions* $W^{(2)}$*. Assume finally that the closest of the points* $p(h_-^{(1)}, \lambda)$*,* $p(h_+^{(1)}, \lambda)$*,* $p(h_-^{(2)}, \lambda)$*,* $p(h_+^{(2)}, \lambda)$ *to* $p_*$ *(according to definition 5.12), which we will denote* $p$*, is strictly closer to* $p_*$ *than the three others. Then one has*

$$(\mathcal{A}^{(2)} - \mathcal{A}^{(1)})(v, \lambda) = \exp\left(\frac{1}{v^\sigma}\left(I(p, p_*, \lambda) + o(1)\right)\right) \quad \textit{as } v \to 0.$$

*Define also* $p_\pm \in \gamma_\pm$ *to be equal to* $p(h_\pm^{(1)}, \lambda)$ *or* $p(h_\pm^{(2)}, \lambda)$*, whichever is closest to* $p_*$*.*

*Let* $S_2$ *be a section of* $M \times [0, v_0[$ *intersecting* $\gamma$ *transversally at a point* $q \neq p_*$*, and let* $q \in \gamma_-$ *resp.* $\gamma_+$*. Then, the intersection of* $W^{(i)}$ *with* $S_2$ *are two graphs* $z = \varphi^{(i)}(v, \lambda)$ *(given a coordinate system* $(z, v)$ *on* $S_2$*) and writing* $\Delta_{12} = \varphi^{(1)} - \varphi^{(2)}$ *one has*

$$\begin{aligned}\Delta_{12}(v, \lambda) = \exp&\left(\frac{1}{v^\sigma}\left(I(p_\pm, p_*, \lambda) - I(q, p_*, \lambda) + o(1)\right)\right)\\ &+ f(v, \lambda)\exp\left(\frac{1}{v^\sigma}(I(p, p_*, \lambda) + o(1))\right),\end{aligned}$$

*as* $v \to 0$*, for some smooth function* $f$ *(depending on* $q$*) and this for all* $q$ *strictly closer to* $p_*$ *than* $p_\pm$ *(as in definition 5.12).*

**Proof** Continuing with the notations above one has

$$\Delta(h_-, h_+, v, \mathcal{A}(h_-, h_+, v, \lambda), \lambda) = 0,$$

so after derivation, one finds

$$\begin{aligned}0 =& \frac{\partial \Delta}{\partial h_\pm}(h_-, h_+, v, \mathcal{A}(h_-, h_+, v, \lambda), \lambda)\\ &+ \frac{\partial \Delta}{\partial A}(h_-, h_+, \mathcal{A}(h_-, h_+, v, \lambda), \lambda)\frac{\partial \mathcal{A}}{\partial h_\pm}(h_-, h_+, v, \lambda).\end{aligned}$$

Write $-1/f(h_-, h_+, v, \lambda) = \frac{\partial \Delta}{\partial A}(h_-, h_+, \mathcal{A}(h_-, h_+, v, \lambda), \lambda)$. Shortening the notations this yields

$$\frac{\partial \mathcal{A}}{\partial h_\pm} = f \cdot \frac{\partial \Delta}{\partial h_\pm}.$$

Using theorem 5.10, we get a differential equation for $\mathcal{A}$:

$$v^\sigma \frac{\partial \mathcal{A}}{\partial h_\pm} = \exp\left(\frac{1}{v^\sigma}(I(p(h_\pm), p_*, \lambda) + o(1))\right)$$

Now,

$$\begin{aligned}\mathcal{A}(h_-^{(1)},h_+^{(1)},v,\lambda)-\mathcal{A}(h_-^{(2)},h_+^{(2)},v,\lambda)=&(\mathcal{A}(h_-^{(1)},h_+^{(1)},v,\lambda)-\mathcal{A}(h_-^{(1)},h_+^{(2)},v,\lambda))\\&+(\mathcal{A}(h_-^{(1)},h_+^{(2)},v,\lambda)-\mathcal{A}(h_-^{(2)},h_+^{(2)},v,\lambda))\\=&\int_{h_+^{(1)}}^{h_+^{(2)}}\frac{\partial\mathcal{A}}{\partial h_+}(h_-^{(1)},h_+,v,\lambda)dh_+\\&-\int_{h_-^{(1)}}^{h_-^{(2)}}\frac{\partial\mathcal{A}}{\partial h_-}(h_-,h_+^{(2)},v,\lambda)dh_-\end{aligned}$$

Let us concentrate on the first integral; the second is treated in an analogous manner:

$$\int_{h_+^{(1)}}^{h_+^{(2)}}\frac{\partial\mathcal{A}}{\partial h_+}(h_-^{(1)},h_+,v,\lambda)dh_+=\pm\int_{h_+^{(1)}}^{h_+^{(2)}}\frac{1}{v^\sigma}\exp\left(\frac{1}{v^\sigma}(I(h_+)+o(1))\right)dh_+,$$

with $I(h_+):=I(p(h_+),p_*,\lambda)$. Both integrals can be rewritten as exponentials in one of the end points of these integrals (see lemma 5.6), and a sum of two exponentials, can be reduced to one exponential: if $a_1=\max(a_1,a_2)$ (with $a_i<0$), then

$$\sum_{i=1}^{2}\exp\left(\frac{1}{v^\sigma}(a_i+o(1))\right)=\exp\left(\frac{1}{v^\sigma}(a_1+o(1))\right).$$

This proves the statement regarding the exponential closeness of control curves.

For the second part of the theorem, notice

$$\Delta_{12}:=\theta(h^{(2)},v,\mathcal{A}^{(2)}(v,\lambda),\lambda)-\theta(h^{(1)},v,\mathcal{A}^{(1)}(v,\lambda),\lambda),$$

where $\theta$ is obtained from theorem 5.5. Apparently, one can decompose this difference in two parts: in a part

$$\Delta_{12}^a:=\theta(h^{(2)},v,\mathcal{A}^{(2)}(v,\lambda),\lambda)-\theta(h^{(1)},v,\mathcal{A}^{(2)}(v,\lambda),\lambda),$$

and in a part

$$\Delta_{12}^b:=\theta(h^{(1)},v,\mathcal{A}^{(2)}(v,\lambda),\lambda)-\theta(h^{(1)},v,\mathcal{A}^{(1)}(v,\lambda),\lambda).$$

The part $\Delta_{12}^a$ has been investigated at the end of the previous section—one simply has to substitute $A=\mathcal{A}^{(2)}$ in (5.12). Define

$$H(v,A,\lambda)=\int_0^1\frac{\partial\theta}{\partial A}(h^{(1)},v,sA+\mathcal{A}^{(1)}(v,\lambda),\lambda)ds,$$

then one can see that

$$\Delta_{12}^b=(A^{(2)}-A^{(1)})H(v,A^{(2)},\lambda).$$

Using the first part of the theorem suffices to conclude the proof. □

Notice the strong resemblence between the above theorem and theorem F in [BFSW], where the same result is obtained in an analytic framework (and—for our specific situation—under stronger transversality hypothesis).

This theorem gives the basis for an entry–exit relation for a vector field $X_{v,\mathcal{A}(v,\lambda),\lambda}$. In particular, it gives an expression for the distance of two control curves associated to two different entry–exit relations, meaning that if this distance is "too large", then both entry–exit relations cannot co-exist. The specification "too large" can be made explicit: if $(h_-, h_+)$ defines a pair of entry–exit points, with associated control curve $\mathcal{A}$, then this exit point is preserved along parameter curves $A(v, \lambda)$ for which

$$|A(v,\lambda) - \mathcal{A}(v,\lambda)| \leq \exp\left(\frac{1}{v^\sigma}(I + o(1))\right),$$

where

$$I = I(p(h_+,\lambda), p_*, \lambda).$$

Let us combine this fact with the results of the previous theorem. Given a fixed pair $(h_-, h_+)$ and associated to this pair a control curve $\mathcal{A}(v, \lambda)$. Consider the reduced family

$$X_{v,\lambda} := X_{v,\mathcal{A}(v,\lambda),\lambda},$$

and we ask ourselves the following question: given a fixed entry point $\tilde{h}_-$ what is the corresponding exit point, i.e. what is the unique point $\tilde{h}_+$ before which the saturation of the entry boundary curve at $\tilde{h}_-$ stays $o(1)$-close to the critical curve, and beyond which point the orbits in the saturated manifold are repelled away from this critical curve in an exponentially fast way? We assume that

$$p(h_+,\lambda) \text{ is closer to } p_* \text{ than } p(h_-,\lambda).$$

(The other case can be treated similarly, upon reversing time.) We define the following set:

$$F = \{\tilde{h}_- : p(h+,\lambda) \text{ is closer to } p_* \text{ than } p(\tilde{h}_-,\lambda)\}.$$

Clearly, $h_- \in F$ and for any $\tilde{h}_- \in F$, assume that there is an exit point specified by $\tilde{h}_+$. There are *a priori* three possibilities:

1. The exit point $p(\tilde{h}_+,\lambda)$ lies closer to $p_*$ than $p(h_+,\lambda)$.

2. The exit point coincides with $p(h_+,\lambda)$.

3. The exit point lies further away from $p_*$ than $p(h_+,\lambda)$.

One can exclude the first case as follows: any control curve connecting $\tilde{h}_-$ to $\tilde{h}_+$ in that configuration must lie on a distance with the given control curve (due to the preceding theorem), and this distance is given by

$$\exp\left(\frac{1}{v^\sigma}(I(p(\tilde{h}_+,\lambda),p_*,\lambda)+o(1))\right)$$

(the point $p(\tilde{h}_+,\lambda)$ is the closest to $p_*$ of the four relevant entry–exit points). Hence, the regions in parameter space where an entry–exit pair $(h_-,h_+)$ and $(\tilde{h}_-,\tilde{h}_+)$ are possible do not intersect, and lie relatively far apart from each other. Similarly one excludes the third case. In the second case, we cannot apply the preceding theorem, and the two entry–exit pairs can co-exist. We conclude that the saturation of all entry boundary curves specified with parameter values in $F$ leave the critical curve at the same point $p(h_+,\lambda)$. This phenomenom is called a *funneling effect* (and $F$ is called a funnel).

A natural question appears now: what happens outside the funnel? Consider the set

$$T=\{\tilde{h}_-: p(\tilde{h}-,\lambda) \text{ is closer to } p_* \text{ than } p(h_+,\lambda)\}.$$

It should be observed that

$$\gamma_-=F\cup\{p_i\}\cup T,$$

for a unique point $p_i$ for which

$$I(p_i,p_*,\lambda)=I(p(h_+,\lambda),p_*,\lambda).$$

With similar arguments, it can be seen that the corresponding exit point of an entry point $\tilde{h}_-$ in $T$ is given by the symmetric point $\tilde{h}_+$ for which

$$I(p(\tilde{h}_-,\lambda),p_*,\lambda)=I(p(\tilde{h}_+,\lambda),p_*,\lambda),$$

in other words: the saturation of an entry boundary curve specified with a parameter value in $T$ leaves the critical curve at an exit point that is given by the above integral relation. This is a one-to-one correspondence between entry and exit point. This phenomenom is called *bifurcation delay.*

Remains to find out what happens to initial boundary curves that have an entry point specified by $p_i$. It is clear that there is no unique exit point attached to this entry point, i.e. initial conditions are in a sense much more sensitive around this point. The reason is that when considering an exit point $p(\tilde{h}_+,\lambda)$ for which $p(h_+,\lambda)$ lies closer to $p_*$, then upon reversing time one can show that the corresponding entry point must coincide with $p_i$. We conclude: the saturation of an entry boundary curve with entry point $p_i$ can leave the critical curve at an exit point that is either equal to $p(h_+,\lambda)$ or it can be any point farther away from $p_*$. We summarize these results in the following

**Theorem 5.14** *(**Entry–exit relation**) Let $(h_-, h_+)$ be a pair of entry–exit points so that*

$$I(p(h_-, \lambda), p_*, \lambda) < I(p(h_+, \lambda), p_*, \lambda)$$

*(the other case can be treated similarly, upon reversing time). Define $p_i$ to be point on $\gamma_-$ symmetric to $p(h_+, \lambda)$, i.e.*

$$I(p_i, p_*, \lambda) = I(p(h_+, \lambda), p_*, \lambda).$$

*Let*

$$A = \mathcal{A}(v, \lambda)$$

*be the unique control curve connecting the curve defined by the coordinate equation $\{h = h_-\}$ in $S_-$ to the curve defined by the coordinate equation $\{h = h_+\}$ in $S_+$.*

*Then, restricting to this control curve in parameter space, the saturation of any curve in $S_-$ specified by the equation $\{h = \tilde{h}_-\}$ leaves the critical curve at a point $p_o$, that is defined by:*

1. *If $p(\tilde{h}_-, \lambda)$ lies (strictly) closer to $p_*$ than $p_i$, then $p_o$ is the unique point on $\gamma_+$ for which $I(p_o, p_*, \lambda) = I(p(\tilde{h}_-, \lambda), p_*, \lambda)$.*

2. *If $p_i$ lies (strictly) closer to $p_*$ than $p(\tilde{h}_-, \lambda)$, then $p_o = p(h_+, \lambda)$ (funneling: the exit point does not depend on the entry point inside the funneling region).*

3. *If $p(\tilde{h}_-, \lambda)$ is equal to $p_i$, then the exit point is unknown, but at least is equal to or lies farther way from $p_*$ than $p(\tilde{h}_+, \lambda)$.*

*In these circumstances, one says that $p(\tilde{h}_+, \lambda)$ is a point of maximum bifurcation delay (it is a* buffer point*).*

## 5.5 Some notes on buffer points

The above proposition makes it clear that the calculation of the "buffer point" is very sensitive to perturbations. Indeed, one can put the buffer point anywhere upon changing the control curve $A = \mathcal{A}(v, \lambda)$. The inverse question, i.e. finding the buffer point for a given family $X_{v,\lambda}$ without control parameter is hence not obvious.

Throughout the literature, one can find several results on calculating the buffer point. Most of them however are concerned about 3-dimensional slow-fast systems, with one fast dimension and a 2-dimensional critical manifold. A famous example is described by the complex vector field

$$\left\{ \begin{array}{rcl} \dot{x} & = & \epsilon \\ \dot{y} & = & (x + i)y + \epsilon. \end{array} \right.$$

By writing $y = u + iv$, and keeping $(x, \epsilon)$ real, one reduces to the 3-dimensional real situation. In any case, what is typical about this family is that the eigenvalue along the critical manifold crosses the imaginary axis *outside the origin.* This kind of vector fields always possess a formal expansion, and always exhibit a canard phenomenom.

The determination of the buffer point (or equivalently, the maximum size of a canard) is done using the fact that the points on the attracting branch of the critical manifold are in a sense connected to the points on the repelling branch of the critical manifold (see [N1], [N2]). To roughly illustrate the idea using the above example, allow us to be a bit vague; there is really no need to be very precise since it will serve only as an introduction to our case. The divergence in this example is given by $x+i$, whereas the slow dynamics is simply $\dot{x} = 1$. The integral of the divergence can be complexified and it equals

$$I(x) = \int_x^0 (s+i)ds = -\frac{1}{2}x^2 - ix.$$

Whereas in real dynamics, the attraction is governed by this integral, in complex dynamics one should only consider its real part $R(x) = \Re(I(x))$. Notice now that

$$R(a+ib) := \Re(I(a+ib)) = -\frac{1}{2}(a^2 - b^2 + 2b)$$

The real point $(-x, 0)$ on the attracting side of the critical curve is connected to the real point $(x, 0)$ on the repelling side of this curve, by means of the complex path

$$S_1 := \{(a+ib) \in \mathbf{C}\colon b \geq 0, a^2 - b^2 + 2b = x^2\} = \{(a+ib) \in \mathbf{C}\colon b = 1 - \sqrt{1 - x^2 + a^2}\},$$

which is connected provided $x < 1$. An extension of the results in this chapter would allow us to parametrize $S_1$ by $a$, and consider initial entry boundary curves in $S_1$, just as in theorem 5.10. The transition map towards a section in a family rescaling chart on top of the blow up locus of the turning point would be parametrized regularly by $a$, and its derivative w.r.t. $a$ is an expression of the form

$$\exp((R(a+ib) + o(1))/\epsilon).$$

The exponent however is constant due to the definition of $S_1$, allowing us to prove that the angle between the attracting center manifold through $(-x, 0)$ and the repelling center manifold through $(+x, 0)$ is of the order $\exp(R(x)/\epsilon)$, and this suffices to show that the attracting center manifold will stay $o(1)$-close to the critical curve along the path $[0, x[$, i.e. one proves the existence of canard manifolds of length 1.

This technique does not directly apply to our 2-dimensional real situation, where the eigenvalue passes the origin. In that case, there is no way to connect the points of the attracting side to the points on the repelling side by means of paths having a constant divergence integral. In the language of [BFSW], one would say that the complex plane is divided into valleys and mountains, and $R(x)$ would be the relief function giving at each point the "height". Let us illustrate this again with a basic example:

$$\begin{cases} \dot{x} &= \epsilon \\ \dot{y} &= xy + \epsilon^2 f(x, y, \epsilon). \end{cases}$$

In this example, the integral of the divergence is simply a function $I(x) = -\frac{1}{2}x^2$, with real part $R(a+ib) := \Re(I(a+ib)) = -\frac{1}{2}(a^2-b^2)$. The locus

$$\{a^2 - b^2 = x^2\} = \{b = \sqrt{a^2 - x^2}\}$$

is not a connected path from $(-x, 0)$ to $(x, 0)$.

In the next chapter, this problem is bypassed, using the theory of Gevrey functions in combination with the geometric tool ‘family blow up’. Instead of considering $x \in \mathbf{C}$ and $\epsilon \in \mathbf{R}^+$, we will consider $x \in \mathbf{C}$, but keeping $\epsilon/x^2$ real. This is best viewed in charts of the blow up.

# Chapter 6

# Gevrey-analysis

Part of my work relates to the study of Gevrey properties of center manifolds in dynamical systems. A great amount of work has already been done for one-dimensional center manifolds in the plane. We recall a theorem that is most relevant to us. In the context of slow-fast vector fields with one slow variable, the study of two-dimensional center manifolds becomes important. In that case, we study the divergence properties with respect to the small parameter $\epsilon$. Also in this area, a lot is known by now. The work that is presented below is devoted to *real* analytic vector fields, and although the techniques may require some complex notions, the results will still be formulated in a real language.

We will limit ourselves to an open set $M \subset \mathbf{R}^2$, and we will refer to it as an analytic manifold, meaning that we impose analyticity on local charts.

## 6.1 Gevrey properties of 1-dimensional center manifolds

Consider a real analytic vector field $X$ on $\mathbf{R}^2$, with a singularity in $(x_0, y_0)$:

$$X\colon \left\{ \begin{array}{rcl} \dot{x} & = & f(x,y) \\ \dot{y} & = & g(x,y) \end{array} \right. \tag{6.1}$$

It is well known that if a singularity $(x_0, y_0)$ is an hyperbolic saddle, then the stable and unstable manifolds are unique and real analytic in the origin. In that case, both eigenvalues of $DX_{(x_0,y_0)}$ are nonzero. It becomes interesting if one eigenvalue is zero and there locally exist one-dimensional $C^k$ center manifolds. It follows already from the results in [WA] (and many others) that in the analytic setting those center manifolds can be taken to be $C^\infty$. Still, more can be said about this center manifold. It was shown by Braaksma and by Ramis and Sibuya that center manifolds are $\sigma$-

summable for some $\sigma \in \mathbf{N} \setminus \{0\}$. We remind the reader that one possible definition of summability can be given in the following terms:

**Definition 6.1** *A formal power series $\hat{f}(x)$ is called $\sigma$-summable in the complex direction $d \in [0, 2\pi]$ in $x$ if there exists a good sectorial covering $(S_j)_{j=1,\ldots,n}$ and a sequence of functions $(f_j)_j$ with $f_j \colon S_j \to \mathbf{C}$ analytic and bounded such that all $f_j$ are Gevrey-$1/\sigma$ asymptotic to $\hat{f}$ in the sector $S_j$ and such that $S_1 = S_{r_0,d,\theta_0}$ for some $r_0 > 0$ and $\theta_0 > \pi/2\sigma$ (i.e. for the sector containing the direction $d$, we can choose an opening angle $> \pi/\sigma$). A formal power series $\hat{f}(x)$ is called $\sigma$-summable if it is $\sigma$-summable in all but at most a finite number of complex directions.*

I have found [Ba] to be a good reference for a (Braaksma-style) proof of the summability of center manifolds. There, ordinary differential equations of the form

$$x^{\sigma+1}\frac{dy}{dx} = F(x, y) \tag{6.2}$$

(and higher-dimensional variants) are considered, with $F$ analytic near $(0, 0) \in \mathbf{C}^2$, $F(0, 0) = 0$ and $\lambda := \frac{\partial F}{\partial y}(0, 0) \neq 0$. The index $\sigma$ is called the *Poincaré index* of the differential equation. The conclusion is the following:

**Proposition 6.2** *There is a unique formal power series $y = \hat{\varphi}(x)$ that solves (6.2). This series is Gevrey-$1/\sigma$ in $x$ of type $1/|\lambda|$. The series is $\sigma$-summable in all directions, except in the directions $d \in S^1 \subset \mathbf{C}$ for which $\operatorname{Arg}(\lambda d^\sigma) = 0$ (i.e. $\lambda d^\sigma$ lies on the positive real axis).*

Our interest goes to the cases $\lambda \in \mathbf{R}$. We distinguish four cases, and relate to the corresponding topological pictures of the corresponding vector field

$$X \colon \left\{ \begin{array}{rcl} \dot{x} & = & x^{\sigma+1} \\ \dot{y} & = & F(x, y) \end{array} \right.$$

(a) $\lambda < 0$ and $\sigma$ is even. The series $\hat{y}$ is $\sigma$-summable in the direction $x < 0$ and $x > 0$. The origin is a topological saddle, and the center manifolds for both $x < 0$ as $x > 0$ separate two saddle sectors and are therefore unique.

(b) $\lambda < 0$ and $\sigma$ is odd. The series $\hat{y}$ is $\sigma$-summable in the positive real direction, but not in the negative real direction. The origin is a saddle-node in this case; in the direction $x > 0$ it divides two saddle sectors and is unique while in the direction $x < 0$ the center manifold is not unique (the center manifold is an orbit in a parabolic sector of the origin).

(c) $\lambda > 0$ and $\sigma$ is even. The series $\hat{y}$ is not $\sigma$-summable in the two real directions. The origin is a topological source. In this case, the center manifold is not unique, not for $x < 0$ and not for $x > 0$.

(d) $\lambda > 0$ and $\sigma$ is odd. The series $\hat{y}$ is $\sigma$-summable in the negative real direction, but not in the positive real direction. The origin is again a saddle-node, and in the half-plane $x < 0$ the center manifold is unique since it divides two saddle sectors. The half-plane $x > 0$ is parabolic, and there the center manifold is nonunique.

It is an easy exercise to reduce the study of the partially hyperbolic fixed point of (6.1) to a study of an ordinary differential equation like (6.2) (except if the center manifold is a curve of singularities, but then this curve is analytic):

**Proposition 6.3** *Let* $0$ *be an isolated singularity of a vector field* $X$ *on* $(\mathbf{R}^2, 0)$*, and assume* $X$ *is analytic near the origin. Assume also that* $DX_0$ *has one zero and one nonzero eigenvalue. Then* $X$ *is locally near* $0$ *analytically conjugated (a precise definition of analytic conjugation is given in definition 6.8) to*

$$\left\{ \begin{array}{rcl} \dot{x} & = & x^{\sigma+1}\alpha_1(x,y) + xy\alpha_2(x,y) \\ \dot{y} & = & \beta_1(x,y)y + x^{\sigma+1}\beta_2(x,y) \end{array} \right.$$

*for some* $\sigma \in \mathbf{N} \setminus \{0,1\}$*, and where* $\alpha_1$*,* $\alpha_2$*,* $\beta_1$*,* $\beta_2$ *are analytic near the origin. Futhermore both* $\alpha_1(0,0)$ *and* $\beta_1(0,0)$ *are nonzero. The number* $\sigma$ *is unique and is called the "Poincaré index of* $X$ *at* $0$*". After performing the singular change of coordinates* $y = x^\sigma Y$*, the vector field is given by*

$$\left\{ \begin{array}{rcl} \dot{x} & = & x^{\sigma+1}\alpha(x,Y) \\ \dot{Y} & = & \beta(x,Y)Y + xg(x,Y) \end{array} \right.$$

*with* $\alpha$*,* $\beta$ *and* $g$ *analytic near* $0$*, and where* $\alpha(0,0) \neq 0$ *and* $\beta(0,0) \neq 0$*.*

## 6.2 Singular perturbations

The theory of Gevrey asymptotics has proved to be very successful in applying it to singular perturbation problems. As our attention goes to planar singular perturbation problems, the results of Canalis-Durand, Ramis, Schäfke, Sibuya, Benoït, Fruchard, Wallet ([CRSS], [BFSW], [FS], ...) and many others are important to us. More specifically, both local as more global results have been obtained in describing the normally hyperbolic passage as well as passages through nongeneric turning points.

Here, we present an addition to this set of results in the following way. By restricting ourselves to results inside the set of reals, we are able to prove the existence of "canard manifolds" that are not visible through asymptotic theory. Indeed, in general these canard manifolds do not have an asymptotic power series in terms of the traditional variables (see also [FS2]).

As a motivating example, consider the equation

$$\epsilon \frac{dy}{dx} = a + x^3 y + \epsilon^N x + \epsilon^{N+1} F(x, y, \epsilon, a),$$

where $F$ is real analytic and all variables are one-dimensional. Associated to this o.d.e. is the vector field

$$X_{\epsilon,a}: \left\{ \begin{array}{lcl} \dot{x} & = & \epsilon \\ \dot{y} & = & a + x^3 y + \epsilon^N x + \epsilon^{N+1} F(x,y,\epsilon,a) \end{array} \right. \tag{6.3}$$

One can check that this vector field has a curve of singularities with one turning point, and that it satisfies the conditions of theorem 3.3, proving the presence of smooth canard manifolds (such equations were also examined in [PA]). The smoothness of such canard manifolds is everywhere except at the turning point, where in general the smoothness is at most $C^0$. The lack of smoothness at the turning point is reflected in the absence of a formal power series satisfying the above o.d.e. For a formal canard manifold $\hat{y}(x,\epsilon)$ and canard curve $\hat{a}(\epsilon)$ to satisfy the above o.d.e., it is easily shown that $\hat{a} = O(\epsilon^N)$ and $\hat{y} = O(\epsilon^N)$. For the coefficients $(a_N, y_N(x))$ one needs to solve

$$0 = a_N + x^3 y_N(x),$$

which is impossible without introducing a pole at the origin.

In this chapter, we will show that the canard manifolds satisfy some Gevrey estimates with respect to blown up variables, and, which may be more important, that the control curve $a = \mathcal{A}(\epsilon)$ also is Gevrey in terms of $\epsilon^{1/m}$ for some $m$. Indeed, although the canard manifolds are in general only $C^0$ at the turning point, the control curve is smooth and has an asymptotic expansion.

Although we focus on obtaining results in the reals, we are automatically lead to use features in the rich domain of complex analysis. Our result will strongly depend on a theorem of Ramis-Sibuya, relating Gevrey expansions to chains of complex functions. Also the technique of majorating series will come in handy in proving Gevrey estimates. The main obstacle in transposing all these techniques to our situation is the fact that the blow up transformation replaces the trivial foliation $d\epsilon = 0$ by the foliation $d(u^p \bar{\varepsilon}^q) = 0$, with $p, q \in \mathbf{N}_1$. As a consequence, both $u$ and $v$ play the role of a singular parameter and expansions with respect to $u$ will have to be investigated independently of the expansions with respect to $\bar{\varepsilon}$.

Beside on turning point problems, some work will be done on normally hyperbolic singular perturbation problems. As a motivating example, consider the o.d.e.

$$\epsilon(\epsilon + x)\frac{dx}{dy} = -x + \epsilon F(x,y,\epsilon), \qquad x(y_0) = x_0(\epsilon)$$

with $F$ analytic. We will attach to this kind of equation the order of degeneracy $\sigma = 2$, and will prove that the solution to this boundary value problem $\varphi(y,\epsilon)$ is Gevrey-1/2 w.r.t. $\epsilon$ uniformly for $y$ in compact subsets inside $]y_0, y_1[$.

### 6.2.1 General setting and results

We again position ourselves in the framework of chapter 3, this time assuming that the manifold $M$ is a subset of $\mathbf{R}^2$, that the family $X_{\epsilon,a,\lambda}$ is real analytic (although the analyticity w.r.t. the "nonessential" parameters $\lambda$ is not necessary and can be replaced by $C^k$-smoothness). Further assumptions on the vector field are briefly recalled below.

As in preceding chapters, we consider families like in (3.5), where the parameters $(\epsilon, a)$ have already been rescaled to make $a$ a "regular breaking parameter". In the new family $X_{v,A,\lambda}$ the singular parameter is $v$ and $(A, \lambda)$ are regular parameters. For the sake of convenience, the family $X_{v,A,\lambda}$ is repeated here:

$$\left\{ \begin{array}{rcl} \dot{x} & = & f(x,y,v,A,\lambda) \\ \dot{y} & = & vg(x,y,v,A,\lambda). \end{array} \right. \tag{6.4}$$

Let us recall the definition of "order of degeneracy" (from section 4.1). Assume $\gamma$ is a curve of singularities of $X_{0,A,\lambda}$, and let $p \in \gamma$ be normally hyperbolic for $X_{0,A,\lambda}$. Then there exists $C^k$-center manifolds $W_p$ for $X_{v,A,\lambda}$ at $p$. The reduction of $X_{v,A,\lambda}$ to the center manifold $W_p$ is a vector field that is $O(v^\sigma)$. We define $\sigma$ to be the largest number $i$ for which this vector field is divisible by $v^i$. For vector fields associated to ordinary differential equations like $v^\sigma \frac{dy}{dx} = F(x,y,v,A,\lambda)$, the vector field reduced to any center manifold is equivalent to $\dot{x} = v^\sigma$, and one trivially finds $\sigma$ to be the order of degeneracy. In fact, the class of vector fields

$$\left\{ \begin{array}{rcl} \dot{x} & = & \epsilon \\ \dot{y} & = & F(x,y,\epsilon,a) \end{array} \right. \tag{6.5}$$

is treated entirely in [PA]: Panazzolo carefully examined under what circumstances such vector fields can be blown up and have a breaking parameter.

We refer to section 4.1 for the definition of the slow vector field, and also recall the definition of slow divergence: the slow divergence is defined as the divergence of the reduced vector field $X|_{v=0}$, and reduced to the critical manifold $\gamma$:

$$\operatorname{div} X^0_{v,A,\lambda} := \operatorname{div} X_{0,A,\lambda}.$$

Letting $\varphi(s; s_0)$ be the flow under the slow vector field $X^0$, then one can integrate the slow divergence along compact pieces of the critical curve. This construction is demonstrated in chapter 5. Such integrals will be important in the specification of the Gevrey type of the control curve. The integral is well-known in the literature; in [BFSW] its real part is called "le relief". To clarify, let us assume $y = \varphi_0(x)$ is a curve of singularities for $\epsilon = a = 0$ in (6.5), then the integral of the slow divergence yields

$$\int_{[x_0,x_1]} \operatorname{div} X^0 \, ds := \int_{x_0}^{x_1} \frac{\partial F}{\partial y}(x, \varphi_0(x), 0, 0) dx.$$

Finally, we briefly recall assumptions T1–T6 (as in chapter 3) and assumption T7 (as in chapter 4).

**Assumption A1** (Analytic version of assumption T1) There exists a real analytic chart of the manifold $M$ where the family of vector fields takes the form of (6.4), and where the critical curve $\gamma$ is defined as a solution of $f(x, y, 0, A, \lambda) = 0$. The curve $\gamma$ is a union of two curves $\gamma_- \cup \gamma_+$ divided by a point $p_*$. We assume that the point $p_*$ has coordinates $(0, 0)$ in this chart. Some extra properties on $\gamma$ are required to ensure that we can write $\gamma$ as a graph (see chapter 3 for details).

Important remark: because we work with the parameters $(v, A)$ instead of $(\epsilon, a)$, the critical curve will not depend on $A$; indeed $X_{0,A} = X_{0,A'}$ for all $A, A'$ if the family is obtained by a parametric blow up $(a, \epsilon) = (v^k A, v^\ell)$.

**Assumption A2** We assume that N1 and N2 are verified (chapter 2), hereby choosing an orientation on $\gamma$ so that $\gamma_+$ comes after $\gamma_-$. The analyticity of the vector field allows one to assume N3 (chapter 4) without loss of generality. We assume that the slow vector field (as introduced in chapter 4) has no singularities on $\gamma$. Roughly this states that along the critical curve $\gamma_-$ we have hyperbolic attraction, along $\gamma_+$ we have hyperbolic repulsion, and the slow dynamics is oriented from $\gamma_-$ to $\gamma_+$ (leading to a situation that is known as a dynamic bifurcation).

These two assumptions will already suffice to prove theorem 6.12, roughly stating that near the real branch of the normally attracting critical curve $\gamma_-$ the saturation of admissible entry boundary curves forms an invariant manifold that satisfies some Gevrey estimates. We refer to theorem 6.12 for more information.

To pass through a turning point, we need additionally

**Assumption A3** We blow up (in the reals) the family of vector fields $X_{A,\lambda} := X_{v,A,\lambda} + 0\frac{\partial}{\partial v}$ in $\mathbf{R}^3$ by a weighted family blow up $(x, y, v) = (u^p\overline{x}, u^q\overline{y}, u^m\overline{v})$, with $u \geq 0$ and $(\overline{x}, \overline{y}, \overline{v}) \in S^2$. After blowing up at $(0, 0, 0)$ the $(x, y, v)$-variables — in an admissible real analytic chart — of $X_{v,A,\lambda}$, we get the following:

The preimages of $\gamma_-$ and $\gamma_+$ in the blow up space (including the endpoints of $\gamma_\pm$ on the blow up locus) are normally hyperbolic. Define $P_\pm = \gamma_\pm \cap \Sigma$, where $\Sigma$ is the blow up locus, i.e. the preimage of $(x, y, v) = (0, 0, 0)$ under the blow up map.

**Assumption A4** Let $p = P_-$ be the end point of $\gamma_-$ in a blown up admissible (real analytic) chart, then there exists a sufficiently small neighbourhood $V$ of $p$ so that in $V$ there are no singularities for $\overline{v} > 0$. A similar requirement is made for the end point $P_+$ of $\gamma_+$. Up to this point, this coincides with assumption T4. Here, we also assume that in this blown up chart the vector field can be complexified in a neighbourhood of $P_- \in \mathbf{C}^3$. We assume that there are no singularities for $\overline{v} \neq 0$, and not just on the real axis.

**Assumption A5** Under the conditions of assumption A3, there is a heteroclinic connection $\Gamma$ (for $A = 0$) on the blow up locus $\Sigma$ connecting $P_-$ to $P_+$. We assume that this connection consists of one orbit going from $P_-$ to $P_+$. This is the same as assumption T5 from chapter 3.

**Assumption A6** Using the family rescaling chart expressed in (1.3) $A$ is a regular breaking parameter. Let us explain briefly: if, in extremis, $X_{v,A,\lambda}$ would not depend on $A$, we cannot expect to find a control curve $A = \mathcal{A}(v, \lambda)$ to match center manifolds. This assumption guarantees that the family of vector fields does depend on $A$, and in a way that is regular enough for the matching to take place. In other words, the connection $\Gamma$ breaks as $A \neq 0$ with a nonzero breaking speed w.r.t. $A$. This condition is explained in full detail as assumption T6 in chapter 3.

We need one more extra condition in order to be able to proceed. The heteroclinic connection near $P_-$ (as in condition A5) is locally a center manifold at $P_-$ for the vector field reduced to the blow up locus. This analytic vector field is of the form (6.1), and we assume that the Poincaré index of the equation is equal to $\sigma$, the order of degeneracy of our vector field. In other words, the order of degeneracy at points of the normally hyperbolic branch $\gamma_-$ does not change when passing to the limit point $P_-$ on the blow up locus. Essentially, we want to avoid situations like

$$\left\{ \begin{array}{lcl} \dot{x} & = & yx + O(\epsilon) \\ \dot{y} & = & \epsilon^\sigma(\epsilon^2 + y^2), \end{array} \right.$$

where the order of degeneracy changes from $\sigma$ to $\sigma + 2$ at the turning point $(x, y) = (0, 0)$. It is in fact a restatement of assumption T7:

**Assumption A7** The Poincaré index of the blown up vector field at $P_\pm$, reduced to the blow up locus $\{u = 0\}$ is equal to $\sigma$.

**Theorem 6.4** *Let $X_{v,A,\lambda}$ be a real analytic family of vector fields on a (real) analytic 2-manifold $M \subset \mathbf{R}^2$, and let assumptions A1–A7 be verified. Let $\Sigma_\pm$ be real analytic admissible entry/exit boundary curves. Denote by $c_\pm \in \gamma_\pm$ the two corner points ($c_-$ resp. $c_+$ is the $\omega$-limit resp. $\alpha$-limit of the base point of $\Sigma_-$ resp. $\Sigma_+$ on $\{v = 0\}$). (These corner points could also be called entry–exit points.) Then*

1. *For some $m \in \mathbf{N}_1$ and for $v \in [0, v_0[$ with $v_0 > 0$ sufficiently small, there exists a unique smooth curve $A = \mathcal{A}(v^{1/m}, \lambda)$ so that $\mathcal{A}(0, \lambda) = 0$ and so that the saturation of $\Sigma_-$ along $X_{v,\mathcal{A}(v,\lambda),\lambda}$ forms a manifold with boundary $W$ of canard solutions containing $\Sigma_+$ as well. The manifold with boundary is smooth in the blow up space, everywhere except at the two corner points $c_\pm$ defined above.*

2. *There exists a $T > 0$ such that the Taylor series of $\mathcal{A}$ w.r.t. $u := v^{1/m}$ is unique, and is Gevrey-$1/m\sigma$ of type $T'$, for all $T' > T$. The function $\mathcal{A}$ is real analytic in $]0, u_0[$ and can be continued analytically to a sector $S_{u_0,0,\theta}$ (for any opening angle*

*$2\theta < \pi/m\sigma$), where it is Gevrey-$1/m\sigma$ of type $T'$ asymptotic to its Taylor series at the origin (w.r.t. the variable $\overline{v}$). An upperbound for the type $T$ is given in proposition 6.28, and is given in terms of the opening angle.*

**Remark:** The manifold $W$ itself satisfies Gevrey estimates as well. This is made more precise in the theorems 6.12, 6.22 and 6.25.

**Corollary 6.5** *If the Taylor series of $\mathcal{A}$ only contains powers of $u^m$, then $\mathcal{A}$ is Gevrey-$1/\sigma$ of type $T'$ in $v = u^m$, for all $T' > T$. (The Gevrey type is the same as the type in the theorem.)*

**Theorem 6.6** *Under the assumptions of theorem 3.4 (i.e. if a unique asymptotic expansion for the canard solutions exist in original coordinates), then the unique formal expansion for the canard solutions satisfies Gevrey-$1/\sigma$ estimates.*

The proofs of these results are spread over the next sections. A sketch of the contents:

1. In section 6.2.2, we saturate entry boundary curves along normally hyperbolic parts of the critical curve; in particular we show in extension to the results of chapter 2 that the saturation is analytically continuable to complex sectors. A cylindrical blow up along $\gamma$ is needed to find good analytic normal forms.

2. In section 6.2.3, we find analytic normal forms at the limit points $P_\pm$ of the critical curves $\gamma_\pm$ on the blow up locus.

3. In the same section, we show that the formal expansions that exist over the normally hyperbolic part can be extended to the limit point $P_-$.

4. In section 6.2.4, we show that this expansion too is Gevrey-$1/\sigma$; this result will only serve to prove theorem 6.6 in section 6.2.9.

5. We show in section 6.2.5 that there exist invariant manifolds near $P_\pm$ in complex sectors.

6. We make a covering of such sectors in section 6.2.6 and prove the exponential closeness between the manifolds over these sectors in the intersection of their domains, and apply the theorem 1.15 to obtain the Gevrey property of the invariant manifolds.

7. We proceed and saturate the manifolds along the heteroclinic connection $\Gamma$ in section 6.2.7, and apply the implicit function theorem to find the Gevrey property of the control curve $\mathcal{A}$, thereby finishing the proof of theorem 6.4.

8. The proof of theorem 6.6 is outlined in section 6.2.9.

### 6.2.2 The normally hyperbolic part

The $C^k$-normal form theory cannot be applied to derive the required results. We need some kind of analytic normal forms, near points $p$ of the normally hyperbolic parts of the curve, but also near the limit points $P_-$ and $P_+$ on the blow up locus and near points $\overline{p}$ of the connection $\Gamma$ in the family rescaling chart. Let us focus in this section on points of $\gamma_-$.

#### Analytic normal forms

**Definition 6.7** *A Gevrey-chart of $M$ is a $(v, A, \lambda)$-dependent analytic chart of $M$ for which the coordinate functions are uniformly Gevrey-$1/\sigma$ w.r.t. $v$ in a complex sector $S_{v_0,0,\theta}$.*

**Definition 6.8** *Two local vector fields $X_1$ and $X_2$ on $(\mathbf{C}^2, 0)$ are called $C^\omega$-*conjugate *if there exists an analytic transformation $\varphi\colon (\mathbf{C}^2, 0) \to \mathbf{C}$ with $\varphi(0) = 0$ and $\varphi^* X_1 = X_2$. This definition can be trivially extended to families of vector fields.*

**Definition 6.9** *Two local vector fields $X_1$ and $X_2$ on $(\mathbf{C}^2, 0)$ are called $C^\omega$-*equivalent *if there exists a strictly nonzero analytic function $h\colon (\mathbf{C}^2, 0) \to \mathbf{C}$ so that $X_1$ is $C^\omega$-conjugate to $hX_2$. This definition can be trivially extended to families of vector fields.*

In (6.6), a normal form is given near normally hyperbolic points of the critical curve. Compare this to the $C^k$-normal forms under equivalent conditions in (4.2), essentially stating that in $C^k$-normal forms one can drop the $h_1$-contribution in (6.6). One can do the same in analytic terms, using a singular coordinate change; this yields the normal forms in (6.7) and (6.8).

Note that the singular change of coordinates is sometimes not required, but in general it is; consider for example the vector field

$$\begin{cases} \dot{x} &= -x + \epsilon^3 \\ \dot{y} &= \epsilon(\epsilon^2 + x^2), \end{cases}$$

which does not have singularities outside $\{x = \epsilon = 0\}$ (in a sufficiently small neighbourhood), and for which the order of degeneracy is 3, but still without rescaling $x$ one cannot factor out $\epsilon^3$ in $\dot{y}$.

**Proposition 6.10** *(Analytic normal forms) Let assumptions A1 and A2 be verified. Given a point $p \in \gamma_-$, and an angle $0 < \theta < \pi/2\sigma$, then there exists an Gevrey-chart of $M$ near $p$ having the following properties:*

*i) In $(x, y)$-coordinates, the plane $\{x = 0\}$ inside $M \times [0, v_0[$ is locally invariant under the flow of the vector field, and the point $p$ is located at $(x, y) = (0, 0)$. Locally, the vector field (6.4) can be written in these coordinates:*

$$X_{v,A,\lambda}\colon \begin{cases} \dot{x} &= \alpha_1(x, y, v, A, \lambda)x \\ \dot{y} &= v^\sigma g_1(y, v, A, \lambda) + x h_1(x, y, v, A, \lambda), \end{cases} \tag{6.6}$$

*where $\alpha_1$, $g_1$ and $h_1$ are functions that are real analytic in all its variables except in $v$; in $v$ these functions are uniformly Gevrey-$1/\sigma$. Furthermore, we have $\alpha_1(0,y,0,A,\lambda)<0$ and $g_1(y,0,A,\lambda)\neq 0$. The function $h_1$ vanishes at $v=0$.*

*ii) Upon performing the singular change of coordinates $\{x=v^\sigma X\}$ in this chart, the vector field yields*

$$\tilde{X}_{v,A,\lambda}\colon \left\{ \begin{array}{rcl} \dot{X} & = & \alpha(X,y,v,A,\lambda)X \\ \dot{y} & = & v^\sigma g(X,y,v,A,\lambda) \end{array} \right. \tag{6.7}$$

*where $\alpha(0,y,0,A,\lambda)<0$ and $g(0,y,0,A,\lambda)$ is nonzero. The functions $\alpha$ and $g$ are real analytic in all variables except $v$, and uniformly Gevrey-$1/\sigma$ w.r.t. $v$ in the sector $S_{v_0,0,\theta}$ (and real-analytic for $v\in]0,v_0[$).*

*iii) After this singular change of coordinates the vector field $\tilde{X}_{v,A,\lambda}$ is $C^\omega$-equivalent (as in definition 6.9) to a vector field*

$$\left\{ \begin{array}{rcl} \dot{X} & = & \beta(X,y,v,A,\lambda)X \\ \dot{y} & = & v^\sigma \end{array} \right. \tag{6.8}$$

*where $\beta$ is real analytic in all its variables, except w.r.t. $v$, and it is Gevrey-$1/\sigma$ w.r.t. $v$. Furthermore, $\beta(0,y,0,A,\lambda)<0$.*

**Proof** Our starting point is a chart where the vector field is given by (6.4), where $p=(0,0)$, and where the critical curve is given by $\{x=0\}$. It is well-known that upon an analytic change of coordinates, one can ensure that $C^\infty$ center manifolds are $O(v^N)$-close to $\{x=0\}$ (because the center manifolds $x=\psi(y,v,A,\lambda)$ have a unique asymptotic expansion w.r.t. $v$). This immediately yields that the vector field takes the form

$$\left\{ \begin{array}{rcl} \dot{x} & = & \alpha_0(x,y,v,A,\lambda)x+O(v^N) \\ \dot{y} & = & vg_0(y,v,A,\lambda)+xh_0(x,y,v,A,\lambda). \end{array} \right.$$

In this form, the order of degeneracy $\sigma$ is the highest index $i$ for which $vg_0(y,v,A,\lambda)+xh_0(y,v,A,\lambda)$, restricted to the center manifold $x=O(v^N)$ is divisible by $v^i$. Taking $N$ high enough, this means that $g_0$ is divisible by $v^{\sigma-1}$, i.e.

$$\left\{ \begin{array}{rcl} \dot{x} & = & \alpha_0(x,y,v,A,\lambda)x+O(v^N) \\ \dot{y} & = & v^\sigma\tilde{g}_0(y,v,A,\lambda)+xh_0(x,y,v,A,\lambda). \end{array} \right.$$

Let us now show that this analytic vector field has a local center manifold that Gevrey-$1/\sigma$ w.r.t. $v$. In fact, this follows from a result of Sibuya: the center manifold is given by $x=v^{N-1}\psi(y,v,A,\lambda)$, where $\{X=\psi(y,v,A,\lambda)\}$ is a solution to

$$\begin{array}{rl} v^\sigma\left(\tilde{g}_0(y,v,A,\lambda)+v^{N-\sigma-1}Xh_0(v^{N-1}X,y,v,A,\lambda)\right)\dfrac{dX}{dy} & \\ = & \alpha_0(v^{N-1}X,y,v,A,\lambda)X+O(v), \end{array}$$

and this latter o.d.e. is treated in:

**Theorem 6.11** *(Sibuya) Let $y_0$ be fixed, and choose an opening angle $2\theta < \pi/\sigma$. There exists an $r > 0$, $\delta > 0$ and a function $X = \psi(y, v, A, \lambda)$ that solves the above o.d.e. and that is analytic for $y \in B(y_0, \delta)$ and $v \in S_{r,0,\theta}$. This function is Gevrey-$1/\sigma$ w.r.t. $v$ in this chosen sector.*

In any case, by means of a Gevrey-translation $x \mapsto x - v^{N-1}\psi(y, v, A, \lambda)$ one obtains the normal form (6.6). The two remaining statements of the proposition follow easily now. □

### Saturating boundary curves

**Theorem 6.12** *Let $\Sigma_-$ be an analytic admissible entry boundary curve for an analytic family of vector fields satisfying assumptions A1, A2 and assume that the $\omega$-limit of the base point of $\Sigma_-$ is a point $c$ on the critical curve $\gamma_-$ (this is known as the "corner point", or equivalently the "entry-point"). Let $]c, d[\subset \gamma$ be a bounded part of $\gamma_-$. Choose an opening angle $0 < 2\theta < \pi/\sigma$. There exists a $v_0 > 0$ so that*

1. *The saturation of $\Sigma_-$ is an invariant manifold $W$ inside $M \times [0, r[$, that is smooth for all $v \in ]0, v_0[$ and also smooth for $v = 0$ along $]c, d[$.*

2. *In any analytic chart of $M$ near a point $p$ on $]c, d[$ where the critical curve is written as a graph $x = \varphi_0(y, A, \lambda)$, the invariant manifold can be locally written as a graph $x = \varphi(y, v, A, \lambda)$ (with $\varphi_0 = \varphi|_{v=0}$); the function $\varphi$ is real analytic w.r.t. $v$ in $]0, v_0[$, and can be analytically continued w.r.t. $y$ to a complex neighbourhood $U_p \subset \mathbf{C}$, and w.r.t. $v$ to a complex sector $S_{v_0,0,\theta}$.*

3. *The complexified invariant manifold satisfies Gevrey-$1/\sigma$ estimates w.r.t. $v$ in the complex sector $S_{v_0,0,\theta}$, uniformly for $y$ in compact sets inside $]y_c, y_d]$, where $y_c$ resp. $y_d$ is the local $y$-coordinate of the point $c$ resp. $d$.*

The uniformity of the Gevrey estimates is lost as one approaches the corner point $c$. To see this, consider the trivial example

$$X_v : \begin{cases} \dot{x} & = & xy \\ \dot{y} & = & v \end{cases}$$

with an admissible entry boundary curve $\{x = 1, y = -1\}$. The saturated manifold is given by

$$x = \exp((y^2 - 1)/2v).$$

The above function is Gevrey-1 asymptotic to 0 in complex sectors $S_{\infty,0,\theta}$, with type $T_y := 2/(y^2-1)\cos\theta$. Apparently, the type diverges to $+\infty$ as one approaches $y = -1$. In compact sets inside $] - 1, 0[$ one can give an upper bound on the type though, as stated in the theorem.

**Proof of theorem 6.12**

The first statement is shown in theorem 2.5. This proof is devoted to the complexification. By allowing $v$ to take complex values in a sector $S_{r,0,\theta}$, we are automatically led to consider complex phase variables as well. However, we will restrict ourselves to regions near the real component of $\gamma_-$.

Let us cover the segment $[c,d]$ by a finite number of neighbourhoods where normal forms as in (6.6) are valid, and where after the singular change of coordinates $\{x = v^\sigma X\}$ the normal forms as in (6.7) and (6.8) are valid. After complexifying (6.8) we will also assume that in each neighbourhood the normal form (6.8) has the additional property

$$\Re\left(\frac{y-y_c}{v^\sigma}\beta(X,y,v,A,\lambda)\right) \leq -\frac{\nu|y-y_c|}{|v|^\sigma} < 0 \tag{6.9}$$

for $X$ in a complex neighbourhood of 0 and $v$ in the sector $S_{v_0,0,\theta}$ (with $\nu$ small but fixed) and for $y$ in a complex neighbourhood of the segment $]y_c, y_d]$. This can be demanded upon restricting $X$ and $v$ to sufficiently small neighbourhoods (since $\beta(0,y,0,A,\lambda) < 0$), and upon restricting the neighbourhood for $y$ to a complex sector with vertex $y_c$ (so that $\mathrm{Arg}(y-y_c)$ is small).

We claim that we may assume that the admissible boundary curve is bounded and analytic for $v$ in the sector $S_{v_0,0,\theta}$ after application of the singular change of coordinates $x = v^\sigma X$. This is clearly non-trivial (as can be seen in the example below this proof), for the entry boundary curve $\{x = 1\}$ is mapped to the singular boundary curve $\{x = 1/v^\sigma\}$. Let us first assume this to be true, i.e. we continue with the normal form (6.8), and a boundary curve

$$\{y = \varphi_{0y}(v,A,\lambda), X = \varphi_{0x}(v,A,\lambda)\},$$

so that both component functions are bounded analytic for $v$ in the sector $S_{v_0,0,\theta}$. Let

$$(\tilde{X}(t), \tilde{y}(t)) := (\tilde{X}(t; X_0, y_0), \tilde{y}(t; X_0, y_0))$$

be the flow w.r.t. (6.8) through $(X_0, y_0)$ at $t = 0$ (we have made the dependence on the initial conditions silent for the sake of readibility). Given a fixed point $(X, y)$ on the saturated manifold, we want to express the fact that this point lies on this manifold through the chosen initial boundary curve.

One has

$$\tilde{y}(t) = \varphi_{0y}(v,A,\lambda) + v^\sigma t,$$

so the transition time to go from $\varphi_{0y}(v,A,\lambda)$ to some fixed $y$ yields

$$T := T(y,v,A,\lambda) = \frac{y - \varphi_{0y}(v,A,\lambda)}{v^\sigma}.$$

The saturation of the initial boundary curve is hence a graph of the form

$$\psi\colon (y,v,A,\lambda) \mapsto \tilde{X}(T(y,v,A,\lambda); \varphi_{0x}(v,A,\lambda), \varphi_{0y}(v,A,\lambda)) \tag{6.10}$$

Below, we prove that $\tilde{X}(T)$ remains bounded as $v \to 0$ so that the above graph expression is bounded analytic for $v$ in a sector $S_{v_0,0,\theta}$.

To that end, consider complex times $t$ on the segment $[0, T]$, by writing $t = sT$:

$$\frac{d}{ds}\tilde{X}(sT) = T\beta(\tilde{X}(sT), \tilde{y}(sT), v, A, \lambda)\tilde{X}(sT).$$

Keeping $s \in [0, 1]$, and letting $\tilde{X}$ take complex values, notice that

$$\tilde{X}(sT)|_{s=1} = \tilde{X}(0) \exp\left(\int_0^1 T\beta(\tilde{X}(sT), \tilde{y}(sT), v, A, \lambda)ds\right).$$

We hence have a bound

$$|\tilde{X}(T)| \leq |\varphi_{0x}(v, A, \lambda)| \exp\left(\int_0^1 \Re(T\beta(\tilde{X}(sT), \tilde{y}(sT), v, A, \lambda))ds\right).$$

Using (6.9), it follows that

$$\begin{aligned}|\tilde{X}(T)| &\leq |\varphi_{0x}(v, A, \lambda)| \exp\left(-\frac{\nu}{|v|^\sigma}\int_0^1 |\tilde{y}(sT) - y_c|ds\right)\\ &= |\varphi_{0x}(v, A, \lambda)| \exp\left(-\frac{\nu}{|v|^\sigma}\int_0^1 |y - y_c|sds\right)\end{aligned}$$

The expression $\tilde{X}(T)$ depends on the point $T = T(y, v, A, \lambda)$ of course, and hence defines a graph $X = \psi(y, v, A, \lambda)$ as in (6.10) and this graph is invariant under the flow of the vector field (because it is a union of orbits through $\Sigma_-$). Not only does it follow now that $\psi$ is a bounded analytic function, but one can even apply proposition 1.12 to conclude that $\psi$ is Gevrey-$1/\sigma$-asymptotic to 0 in the chosen sector $S_{v_0,0,\theta}$, uniformly for $y$ in compact subsets in the chosen sectorial neighbourhood of $[y_c, y_d]$.

We remark that any analytic change of coordinates preserves this Gevrey property; this is a consequence of the Gevrey substitution theorem and Gevrey implicit function theorem (to write the manifold as a graph in a different set of coordinates).

Let us now return to our initial claim regarding the admissible boundary curve. To that end, we consider the singular change of coordinates $x = v^\sigma X$ as part of a cylindrical blow up (along the curve of singularities)

$$(x, v) = (u^\sigma \overline{x}, u\overline{v}).$$

Indeed, in the chart $\{\overline{v} = 1\}$, one gets back the singular rescaling $x = v^\sigma X$. Our initial claim regarding $\Sigma$ comes down to proving that the saturation of $\Sigma$ can be done until we meet a section that is visible in the chart $\{\overline{v} = 1\}$. In fact, we only have to consider the other chart $\{\overline{x} = 1\}$:

$$(x, v) = (u^\sigma, u\overline{v}),$$

and show that we can saturate $\Sigma$ until the saturation intersects the plane $\{\overline{v}=1\}$. The blow up of the normal form (6.6) using this transformation yields

$$\begin{cases} \dot{u} &=& \frac{1}{\sigma}\alpha_1(u^\sigma, y, u\overline{v}, A, \lambda)u \\ \dot{\overline{v}} &=& -\frac{1}{\sigma}\alpha_1(u^\sigma, y, u\overline{v}, A, \lambda)\overline{v} \\ \dot{y} &=& u^\sigma(g_1(u\overline{v}, y, A, \lambda)\overline{v}^\sigma + h_1(u^\sigma, u\overline{v}, y, A, \lambda)). \end{cases}$$

After division by the positive function $-\frac{1}{\sigma}\alpha_1$, we get an analytic normal form for equivalence:

$$\begin{cases} \dot{u} &=& -u \\ \dot{\overline{v}} &=& \overline{v} \\ \dot{y} &=& u^\sigma F(u, \overline{v}, y, A, \lambda). \end{cases}$$

The passage from the entry boundary curve $\Sigma$ to the section $u = u_0 > 0$ is a regular passage, and can be treated using proposition 1.11, i.e. we assume now that $\Sigma$ is a boundary curve in this section:

$$\Sigma\colon \{u = u_0, y = s(\overline{v}, A, \lambda)\}.$$

Without loss of generality, we impose $s(\overline{v}, A, \lambda) = 0$ (after all, this can be forced, by performing the translation $y \mapsto y - s(\overline{v}, A, \lambda)$). Consider the orbit through $(u_0, \overline{v}_0, 0)$ (with $u_0 \in \mathbf{R}^+$ fixed, and $\overline{v}_0$ in a complex sector $S_{\infty,0,\theta}$. We have

$$u(t) = u_0 e^{-t}, \quad \overline{v}(t) = \overline{v}_0 e^t.$$

The section $\{\overline{v} = 1\}$ is reached after a complex time

$$T = -\log \overline{v}_0 = -(\log |\overline{v}_0| + i \operatorname{Arg} \overline{v}_0)$$

Hence, we have to prove that

$$\frac{dy}{dt} = u(t)^\sigma F(u(t), \overline{v}(t), y, A, \lambda), \quad y(0) = 0$$

has a bounded solution on the ray $t \in [0, T]$. It is easily shown that if $|F| \leq C$ for all $y \in B(0, \delta)$, then for $s \in [0,1] \subset \mathbf{R}^+$:

$$|y(sT)| \leq \frac{C}{\sigma} u_0^\sigma \frac{|T|}{-\log |v_0|}, \quad \forall s \in [0,1],$$

which is still smaller than $\delta$, provided one takes $u_0$ small enough. This means that the saturation of the entry boundary curve $\Sigma$ has an intersection $(u_0\overline{v}_0, 1, y(\overline{v}_0))$ with the plane $\{\overline{v} = 1\}$. In other words, it is a curve $\{\overline{v} = 1, y = y(u/u_0)\}$. After blow down, it forms a curve $\{x = v^\sigma, y = y(v/u_0)\}$. Finally, the application of the singular rescaling $x = v^\sigma X$ to this curve yields a boundary curve $\{X = 1, y = y(v/u_0)\}$. This proves our claim, and finishes the proof of the theorem. $\square$

**Example**

This example shows that when saturating a Gevrey boundary curve $\Sigma$, the Gevrey character is lost near the corner point (eg. entry point) on the critical curve, but is regained after this point. It also shows that, changing to the normal form after rescaling (6.7), one cannot limit to Gevrey boundary curves, but one needs to consider more general bounded curves.

Consider the simple equation

$$\left\{\begin{array}{rcl}\dot{x} & = & -x \\ \dot{y} & = & \epsilon(\epsilon + x^2)\end{array}\right.$$

with the entry boundary curve $\{x = 1, y = 0\}$. Clearly, one has

$$x(t) = e^{-t}, \qquad y(t) = \epsilon^2 t + \frac{1}{2}\epsilon(1 - e^{-2t}).$$

The order of degeneracy is 2, and upon performing the singular change of coordinates $x = \epsilon^2 X$, we determine the intersecting curve with the plane $\{X = 1\}$:

$$\epsilon^2 = e^{-t}, \qquad y(t) = \epsilon^2 t + \frac{1}{2}\epsilon(1 - e^{-2t})$$

In other words, for the rescaled equation

$$\left\{\begin{array}{rcl}\dot{X} & = & -X \\ \dot{y} & = & \epsilon^2(1 + \epsilon X^2),\end{array}\right.$$

one has to consider the entry boundary curve

$$\{X = 1, Y = -2\epsilon^2 \log \epsilon + \frac{1}{2}\epsilon(1 - \epsilon^4)\},$$

which is certainly not a Gevrey curve but is still bounded in the sector $\epsilon \in S_{1,0,\pi/4}$. Nevertheless, its saturation towards sections $\{y = y_0\}$ with $y_0 > 0$ form Gevrey-1/2 curves in $\epsilon$!

### 6.2.3 Analytic normal forms at $P_\pm$

In the previous section, the passage along compact pieces of the critical curve $\gamma$ is studied, provided the compact piece does not contain the turning point. Here, we will study a passage along a compact piece of the critical curve containing the limit point $P_\pm$. This is the first step in the proof of theorem 6.4.

We remind the reader that we have blown up (6.4), using blow up formulas

$$(x, y, v) = (u^p \overline{x}, u^q \overline{y}, u^m \overline{v}).$$

Near $\overline{v} = 0$, we can assume that $(\overline{x}, \overline{y}) \in S^1$, and we define a local chart near $P_-$ by considering an angular coordinate $z$ on $S^1$ (for example, near $(\overline{x}, \overline{y}) = (1, 0)$, we can set $z = \overline{y}$, near $(\overline{x}, \overline{y}) = (0, 1)$ we can set $z = \overline{x}$). The blow up space near $P_-$ is then parametrized by $(u, \overline{v}, z)$.

**Lemma 6.13** *Assume that in an (analytic) admissible chart near the turning point $p_*$, this point is positioned at the origin, and after blow up of (6.4) in a phase-directional rescaling chart, the point $P_-$ is situated at the origin and near $P_-$ the blow up space is parametrized by $(u, \overline{v}, z)$ (see the reminder above this lemma). Then, for all $N \in \mathbf{N}$ there is a $C^\omega$ normal form (for conjugacy) at $P_-$ for the family (in blow up coordinates, after division by a common factor $u^\alpha$ for some $\alpha \in \mathbf{N}$):*

$$\left\{ \begin{array}{lcl} \dot{u} & = & -u\left(\overline{v}^\sigma h_1(u, \overline{v}, A, \lambda) + z f_1(u, \overline{v}, z, A, \lambda)\right) \\ \dot{\overline{v}} & = & m\overline{v}\left(\overline{v}^\sigma h_1(u, \overline{v}, A, \lambda) + z f_1(u, \overline{v}, z, A, \lambda)\right) \\ \dot{z} & = & \beta_1(u, \overline{v}, z, A, \lambda) z + \overline{v}^N g_1(u, \overline{v}, A, \lambda), \end{array} \right. \tag{6.11}$$

*with $\beta_1$, $h_1$, $f_1$ and $g_1$ real analytic, $\beta_1(u, 0, 0, A, \lambda) < 0$. Furthermore, $h_1(u, 0, A, \lambda)$ is strictly positive. If, after blow up, one performs the singular change of coordinates*

$$z = v^\sigma Z,$$

*then w.r.t. the new variables the vector field yields, after division by a nonzero function*

$$\left\{ \begin{array}{lcl} \dot{u} & = & -u\overline{v}^\sigma \\ \dot{\overline{v}} & = & m\overline{v}^{\sigma+1} \\ \dot{Z} & = & \beta(u, \overline{v}, Z, A, \lambda) Z + \overline{v}^{N-\sigma} g(u, \overline{v}, A, \lambda), \end{array} \right. \tag{6.12}$$

*with $\beta$ and $g$ real analytic. Furthermore, $\beta(u, 0, 0, A, \lambda) < 0$.*

**Proof** The blown up vector field is, after division by $u^\alpha$, still analytic. The existence of $P_-$ as limit point on $u = 0$ of the blown up critical curve $\overline{\gamma}$ is assumed in A3. Furthermore, we know that there is an invariant foliation $dv = d(u^m \overline{v}) = 0$. Hence, we can write the blown up vector field yields

$$\left\{ \begin{array}{lcl} \dot{u} & = & -u h_1(u, \overline{v}, z, A, \lambda) \\ \dot{\overline{v}} & = & m\overline{v} h_1(u, \overline{v}, z, A, \lambda) \\ \dot{z} & = & F_1(u, \overline{v}, z, A, \lambda) \end{array} \right.$$

The blow up of the critical curve is an analytic curve in the plane $\{\overline{v} = 0\}$; assume that we can write this critical curve as

$$z = \varphi(u, \lambda)$$

(remember that $A$ is a rescaled version of $a$ making the critical curve independent of $A$). By a simple translation, we may assume $\varphi(u, \lambda) = 0$. This means that

$$F_1(u, 0, 0, A, \lambda) = 0, \quad h_1(u, 0, 0, A, \lambda) = 0.$$

Furthermore, the hyperbolicity along the critical curve states that

$$\frac{\partial F_1}{\partial z}(u, 0, 0, A, \lambda) < 0.$$

We can now rewrite the blown up vector field as

$$\left\{\begin{array}{lcl} \dot{u} & = & -u(\overline{v}h_{11}(u,\overline{v},A,\lambda)+zh_{12}(u,\overline{v},z,A,\lambda)) \\ \dot{\overline{v}} & = & m\overline{v}(\overline{v}h_{11}(u,\overline{v},A,\lambda)+zh_{12}(u,\overline{v},z,A,\lambda)) \\ \dot{z} & = & F_{11}(u,\overline{v},z,A,\lambda)z+\overline{v}F_{12}(u,\overline{v},A,\lambda) \end{array}\right. \tag{6.13}$$

Assume now there exists a formal expansion

$$z=\hat{\varphi}(u,v):=\sum_{n=0}^{\infty}\varphi_n(u,A,\lambda)\overline{v}^n$$

that is formally invariant under the above vector field (this is shown in lemma 6.14). Then, upon an analytic translation we may assume that this series is $O(\overline{v}^N)$. In other words,

$$\dot{z}=F_{11}(u,\overline{v},z,A,\lambda)z+\overline{v}^N\tilde{F}_{12}(u,\overline{v},A,\lambda).$$

In this form, we want to introduce $\sigma$ as the order of degeneracy. To that end, notice that near $u=u_0\neq 0$ the original vector field is $C^\omega$-conjugate (as in definition 6.8) to

$$\left\{\begin{array}{lcl} \dot{u} & = & -u((v/u^m)h_{11}(u,v/u^m,A,\lambda)+zh_{12}(u,v/u^m,z,A,\lambda)) \\ \dot{z} & = & F_{11}(u,v/u^m,z,A,\lambda)z+\overline{v}^N\tilde{F}_{12}(u,v/u^m,A,\lambda) \\ \dot{v} & = & 0 \end{array}\right.$$

The restriction to a center manifold $z=O(v^N)$ yields

$$\left\{\begin{array}{lcl} \dot{u} & = & -u((v/u^m)h_{11}(u,v/u^m,A,\lambda)+O(v^N)) \\ \dot{v} & = & 0 \end{array}\right.$$

This restriction must be $O(v^\sigma)$ and not $O(v^{\sigma+1})$ as $v\to 0$ ($\sigma$ is defined as the highest index $i$ so that the reduction of the vector field to any center manifold is $O(v^i)$). This means that for $N$ high enough:

$$-u((v/u^m)h_{11}(u,v/u^m,A,\lambda)=O(v^\sigma).$$

Since $\sigma$ is the largest number with this property, we can write

$$vh_{11}(u,v/u^m,A,\lambda)=v^\sigma\tilde{h}(u,v,A,\lambda),$$

for some nonzero function $\tilde{h}$, locally near $u=u_0$. By means of analytic continuation, it follows

$$\overline{v}h_{11}(u,\overline{v},A,\lambda)=\overline{v}^\sigma h(u,\overline{v},A,\lambda)$$

for some analytic function $h$. This finally leads to the vector field

$$\left\{\begin{array}{lcl} \dot{u} & = & -u(\overline{v}^\sigma h(u,\overline{v},A,\lambda)+zf(u,\overline{v},z,A,\lambda)) \\ \dot{\overline{v}} & = & m\overline{v}(\overline{v}^\sigma h(u,\overline{v},A,\lambda)+zf(u,\overline{v},z,A,\lambda)) \\ \dot{z} & = & \alpha(u,\overline{v},z,A,\lambda)z+\overline{v}^N g(u,\overline{v},A,\lambda) \end{array}\right.$$

with $\alpha(u, 0, 0, A, \lambda) < 0$. The fact that $h$ is nonzero for $u > 0$ follows from assumption A2; for $u = 0$ it follows from assumption A7. $\square$

We intend to study formal power series

$$z = \hat{\varphi}(u, \overline{v}, A, \lambda) := \sum_{n=0}^{\infty} \varphi_n(u, A, \lambda)\overline{v}^n$$

that define formally invariant manifolds of (6.11) near $u = 0$:

**Lemma 6.14** *Let the blown up vector field in $(u, \overline{v}, z)$-coordinates be analytic for $u \in \Omega_1 \supset B(0, \delta_1)$, $\overline{v} \in \Omega_2 \supset B(0, \delta_2)$, $z \in \Omega_3 \supset B(0, \delta_3)$, $A \in ]-A_0, A_0[$, $\lambda \in \Lambda$. Let $\Omega$ be the subset of $\Omega_1$ for which the curve of singularities is given by*

$$\{z = \varphi_0(u, A, \lambda), \overline{v} = 0\}, \quad (u, A, \lambda) \in \Omega_1 \times ]-A_0, A_0[\times \Lambda$$

*and so that along this curve inside $\Omega$ the divergence of the vector field (w.r.t. the volume form $du \wedge d\overline{v} \wedge dz$) has a strictly negative real part.*
*Then there exists unique functions*

$$\varphi_n \colon \Omega \times ]-A_0, A_0[\times \Lambda \to \mathbf{C}$$

*that are real analytic, and so that the formal power series*

$$z = \sum_{n=0}^{\infty} \varphi_n(u, A, \lambda)\overline{v}^n \tag{6.14}$$

*is formally invariant under the vector field (6.11).*

**Proof** As required in the proof of the proposition above, we have to show the existence of a formal power series of the slightly more general vector field (6.13), where the curve of singularities $z = \varphi_0$ has already been transformed to $z = 0$ (i.e. $\varphi_0 = 0$ in these coordinates).

Expressing the formal invariance amounts to solving

$$(-u\frac{\partial\hat{\varphi}}{\partial u} + m\overline{v}\frac{\partial\hat{\varphi}}{\partial\overline{v}})(\overline{v}h_{11} + \hat{\varphi}h_{12}|_{z=\hat{\varphi}}) = F_{11}|_{z=\hat{\varphi}}\hat{\varphi} + \overline{v}F_{12},$$

which we rewrite as

$$\hat{\varphi} = G|_{z=\hat{\varphi}}\left( (-u\frac{\partial\hat{\varphi}}{\partial u} + m\overline{v}\frac{\partial\hat{\varphi}}{\partial\overline{v}})(\overline{v}h_{11} + \hat{\varphi}h_{12}|_{z=\hat{\varphi}}) - \overline{v}F_{12} \right)$$

where we have defined $G = 1/F_{11}$. Notice now that the righthand side is a formal power series in $\overline{v}$, and the term of order $\overline{v}^n$ is determined completely by the coefficient functions $\varphi_1, \ldots, \varphi_{n-1}$. (The first coefficient, $\varphi_0$, is 0.) Hence, the above equation is a recurrence relation making it possible to solve for all $\varphi_n$. It should be clear that the domain of analyticity is only bounded by the zero set of $F_{11}$ (and the domain of analyticity of the vector field). $\square$

### 6.2.4 Uniform Gevrey estimates along $\gamma$

**Proposition 6.15** *The series (6.14) is Gevrey-$1/\sigma$ w.r.t. $\overline{v}$, uniformly for $(u, A, \lambda)$ in compact subsets of $\Omega\times]-A_0, A_0[\times\Lambda$.*

The proof is based on the majorant method. It is well-known that this majorant method is an excellent tool in proving the Gevrey property, but fails to provide good estimates for the Gevrey type. Also in the proof below, an upper bound for the Gevrey type will be absent. Nevertheless, in later sections we will find a relevant upper bound for the Gevrey type, when considering formal expansions w.r.t. $u$ instead of w.r.t. $\overline{v}$. We also mention that although the coefficient functions in this series are analytic w.r.t. $u$, this does not form a guarantee that this analyticity has any immediate relevance; indeed consider the simple blown up vector field

$$\begin{cases} \dot{u} &= -u\overline{v} \\ \dot{\overline{v}} &= \overline{v}^2 \\ \dot{z} &= -z + e^{-1/\overline{v}} \end{cases}$$

The saturation of the boundary curve $\{u = 1, z = 0\}$ is a manifold

$$z = \frac{1}{2}e^{-1/\overline{v}} - \frac{1}{2}e^{(u-2)/u\overline{v}};$$

and its intersection with $\{\overline{v} = 1\}$ is a curve $z = \frac{1}{2}e^{-1} - \frac{e}{2}e^{-1/u}$, which is certainly not analytic in w.r.t. $u$ in the origin!

#### Proof of proposition 6.15

The proof is based on an idea of R. Schäfke in the proof of Sibuya's theorem (the majorant method), but is different in the sense that we are developing w.r.t. $\overline{v}$ and $\overline{v}$ is a variable, not a parameter.

Clearly, the proof of the Gevrey estimates may be performed after the singular rescaling $z = v^\sigma Z$, making it possible to use the normal form (6.12). As before, we express the formal invariance w.r.t. this vector field and rewrite it as a recurrence relation:

$$\hat{\varphi} = f(u, \overline{v}, \hat{\varphi}, A, \lambda)\left(-u\overline{v}^\sigma\frac{\partial\hat{\varphi}}{\partial u} + m\overline{v}^{\sigma+1}\frac{\partial\hat{\varphi}}{\partial\overline{v}} - \overline{v}^N g(u, \overline{v}, A, \lambda)\right), \tag{6.15}$$

where $f = 1/\beta$.

As in the proof of the above lemma, this equation induces a recurrence relation on the coefficients of $\hat{\varphi}$. If for the determination of $\varphi_n$, the righthand side of this equation, which in its turn is polynomial in $\varphi_1, \ldots, \varphi_{n-1}$, is replaced by a majorant equation, and where all coefficients $\varphi_1, \ldots, \varphi_{n-1}$ are replaced by majorants, then the outcome is a majorant for $\varphi_n$. This method is called the *majorant method.* Remains to find a good way of majorating the coefficients. The basis for this is the use of Nagumo norms.

Let $K \subset \Omega$ be a compact set, and let $K_1 = B(K, \nu)$ be the set of points with distance $< \nu$ to $K$, for some small $\nu > 0$ so that $\overline{K_1} \subset \Omega$. We define

$$d(u) = \operatorname{dist}(u, \partial K_1) \qquad \forall u \in K_1$$

the distance to the border of $K_1$. Because $\overline{K_1}$ is compact inside $\Omega$, all coefficient functions are bounded and analytic on $K_1$, so that it makes sense to define

$$\|f\|_k := \sup_{u \in K_1} \left| f(u) d(u)^{k/\sigma} \right|$$

Such norms are called "Nagumo norms". see e.g. [CRSS]. Immediate properties:

**Lemma 6.16** *1.* $\|f + g\|_k \leq \|f\|_k + \|g\|_k$

*2.* $\|f.g\|_{k+\ell} \leq \|f\|_k \|g\|_\ell$

*3.* $\|\frac{\partial f}{\partial u}\|_{k+\sigma} \leq e^{1/\sigma}(k+1)\|f\|_k$ *(Nagumo's lemma) (e is the Euler number.)*

**Proof** The first two properties are elementary. The third property is known as Nagumo's lemma, and the key element in the proof of this property is Cauchy's estimate for analytic functions:

$$|f'(u)| \leq \frac{\sup_{|t-u| \leq \delta} |f(t)|}{\delta},$$

(for the sake of convenience, we drop the dependence on $\lambda$) for all small $\delta > 0$. By varying $\delta$ with respect to $u$, we can obtain the result: use $\delta = \frac{1}{k+1} d(u)$. Then,

$$\begin{aligned} |f'(u)| &\leq \frac{k+1}{d(u)} \sup_{|t-u|=\delta} |f(t)| \\ &\leq \frac{k+1}{d(u)} \|f\|_k \sup_{|t-u|=\delta} \frac{1}{d(t)^{k/\sigma}} \end{aligned}$$

Now, $d(t) \overset{(*)}{\geq} d(u) - |t-u| = d(u) - \delta = d(u)(1 - \frac{1}{k+1})$, so

$$|f'(u)| \leq \frac{k+1}{d(u)} \|f\|_k \frac{1}{d(u)^{k/\sigma}} \left( \frac{k+1}{k} \right)^{k/\sigma}$$

We find that $\|f'\|_{k+\sigma} \leq (k+1)e^{1/\sigma}\|f\|_k$. The relation $(*)$ follows from the triangle inequality

$$d(u) \leq d(t) + |t-u|$$

(which is easily shown, by noticing that $|u-q| \leq |t-q| + |t-u|$ for all $q \in \partial K_1$ and taking infimum on both sides of the equation). $\square$

**Definition 6.17** *We say that a formal series*

$$\hat{f}(u, \overline{v}, A, \lambda) := \sum_{n=0}^{\infty} f_n(u, A, \lambda)\overline{v}^n$$

*is* <u>*majorated*</u> *by the series*

$$F(V, A, \lambda) := \sum_{n=0}^{\infty} F_n(A, \lambda)V^n$$

*if*

$$\|f_n\|_n \leq (n!)^{1/\sigma} F_n, \qquad \forall (A, \lambda).$$

*($V$ is just a formal variable.) We denote this property by $\hat{f}(u, \overline{v}, \lambda) \ll F(V)$.*

**Lemma 6.18** *We define $r$ as a number larger than $\sup_{u \in K_1} d(u)$ and larger than $\sup_{u \in K_1} |u|$. If $\hat{f}(u, \overline{v}, A, \lambda) \ll F(V, A, \lambda)$ and $\hat{g}(u, \overline{v}, A, \lambda) \ll G(V, A, \lambda)$, then*

1. *$\hat{f}(u, \overline{v}, A, \lambda)\hat{g}(u, \overline{v}, A, \lambda) \ll F(V, A, \lambda)G(V, A, \lambda)$ (according to the Cauchy product of series)*

2. *$\hat{f}(u, \overline{v}, A, \lambda) + \hat{g}(u, \overline{v}, A, \lambda) \ll F(V, A, \lambda) + G(V, A, \lambda)$*

3. *$u \ll r$, $\overline{v} \ll r^{1/\sigma}V$.*

4. *Associated to every analytic function $\Phi(u, \overline{v}, \overline{y}, A, \lambda)$ there is an analytic function $\Phi^+(V, \overline{y}, A, \lambda)$ (not depending on $\hat{f}$) such that*

$$\Phi(u, \overline{v}, \hat{f}(u, \overline{v}, A, \lambda), A, \lambda) \ll \Phi^+(V, F(V, A, \lambda), A, \lambda).$$

5. *$\overline{v}^\sigma \frac{\partial \hat{f}}{\partial u}(u, \overline{v}, \lambda) \ll e^{1/\sigma} V^\sigma F(V)$.*

6. *$\overline{v}^{\sigma+1} \frac{\partial \hat{f}}{\partial \overline{v}}(u, \overline{v}, \lambda) \ll r V^\sigma F(V)$.*

**Proof** Write $[\hat{f}]_n$ for the $n$-th coefficient of the power series $\hat{f}$. For the sake of convenience, we drop the dependence on $(A, \lambda)$.

1. Using the Cauchy product, we have

$$\begin{aligned} \left\| [\hat{f}.\hat{g}]_n \right\|_n &= \left\| \sum_{i=0}^{n} f_i g_{n-i} \right\|_n \\ &\leq \sum_{i=0}^{n} \|f_i\|_i \, \|g_{n-i}\|_{n-i} \end{aligned}$$

Using the bounds on $f_i$ and $g_{n-i}$ we continue:

$$\begin{aligned}\left\|[\hat{f}.\hat{g}]_n\right\|_n &\leq \sum_{i=0}^{n} i!^{1/\sigma}(n-i)!^{1/\sigma} F_i G_{n-i} \\ &\leq n!^{1/\sigma} \sum_{i=0}^{n} \binom{n}{i}^{-1/\sigma} F_i G_{n-i} \\ &\leq n!^{1/\sigma} \sum_{i=0}^{n} F_i G_{n-i} = n!^{1/\sigma} \left[F.G\right]_n .\end{aligned}$$

2. This property is easier to prove.

3. To prove that $u \ll r$, one looks at the coefficients with index 0, and observe that $\|u\|_0 = r$. For the second inequality, look at the coefficients with index 1, and observe that $\|1\|_1 = r^{1/\sigma}$.

4. Let $\Phi(u, \overline{v}, \overline{y}) = \sum_{k=0,\ell=0}^{\infty} \Phi_{k\ell}(u)\overline{v}^k\overline{y}^\ell$, then define

$$\Phi^+(V, \overline{y}) := \sum_{k=0,\ell=0} \sup |\Phi_{k\ell}(u)| r^{k/\sigma} V^k \overline{y}^\ell.$$

   If one substitutes $\overline{y}$ by $\hat{f}$ in the expression of $\Phi$, then one may substitute $\overline{y}$ by $F$ in $\Phi^+$ and the result is still a majorant (taking into account the first three properties of this lemma). Furthermore, $\Phi^+$ defines an analytic function, due to the absolute convergence of the Taylor series of $\Phi$.

5. Take a look at the power series $\overline{v}^\sigma \frac{\partial \hat{f}}{\partial u}$. Since the coefficients start at order $\sigma$, we majorate the $(n+\sigma)$-th coefficient, for $n \geq 0$:

$$\begin{aligned}\left\|\left[\overline{v}^\sigma \frac{\partial \hat{f}}{\partial u}\right]_{n+\sigma}\right\|_{n+\sigma} &= \left\|\frac{\partial f_n}{\partial u}\right\|_{n+\sigma} \\ &\leq (n+1)e^{1/\sigma}\|f_n\|_n \\ &\leq (n+1)e^{1/\sigma} n!^{1/\sigma} F_n \\ &= e^{1/\sigma}(n+\sigma)!^{1/\sigma} F_n \left(\frac{(n+1)^\sigma n!}{(n+\sigma)!}\right)^{1/\sigma} \\ &\leq (n+\sigma)!^{1/\sigma}\left[e^{1/\sigma}F(V)V^\sigma\right]_{n+\sigma}.\end{aligned}$$

   Hence, $\overline{v}^\sigma \frac{\partial \hat{f}}{\partial u} \ll e^{1/\sigma} V^\sigma F(V)$.

6. The proof is analogous:

$$\begin{aligned}
\left\| \left[ \overline{v}^{\sigma+1} \frac{\partial \hat{f}}{\partial \overline{v}} \right]_{n+\sigma} \right\|_{n+\sigma} &= \left\| [\overline{v} \frac{\partial \hat{f}}{\partial \overline{v}}]_n \right\|_{n+\sigma} \\
&= \| n f_n \|_{n+\sigma} \\
&= n \sup_{|u| \leq r} |f_n(u)| (r - |u|)^{(n+\sigma)/\sigma} \\
&\leq n r \| f_n \|_n \leq n r (n!)^{1/\sigma} F_n \\
&\leq (n+\sigma)!^{1/\sigma} r F_n \\
&= (n+\sigma)!^{1/\sigma} [r v^\sigma F(V)]_{n+\sigma}.
\end{aligned}$$

□

With this technique, we go back to our recurrence relation (6.15). Let $f \ll f^+$, and $g \ll g^+$ as in the previous lemma. Then, $\varphi \ll \Phi$, where $\Phi$ is a solution of the equation

$$\Phi = f^+(V, \Phi, A, \lambda) \left( r e^{1/\sigma} V^\sigma \Phi + m r V^\sigma \Phi + r^{N/\sigma} V^N g^+(V, A, \lambda) \right), \quad \Phi(0) = 0. \tag{6.16}$$

Clearly the above expression has an analytic solution $\Phi(V, A, \lambda)$, and therefore the coefficients of $\Phi$ are majorated by a geometric sequence, implying that there exist constants $C_0$, $C_1$ so that

$$\| \varphi_n(u, A, \lambda) \|_n \leq C_0 C_1^n (n!)^{1/\sigma}.$$

One can replace the Nagumo norm by a traditional norm, upon restricting $u$ to $K \subset K_1$, and noticing that for all functions $f$ one has

$$\sup_{u \in K} |f(u)| \leq \| f \|_n \left( \frac{1}{\nu} \right)^{n/\sigma}$$

where $\nu$ was introduced as the distance between $K$ and $K_1$. One obtains

$$\sup_{u \in K} |\varphi_n(u, A, \lambda)| \leq C_0 (C_1^\sigma / \nu)^{n/\sigma} (n!)^{1/\sigma}.$$

Hence, $\hat{\varphi}$ is uniformly Gevrey-$1/\sigma$, of type at most $C_1^\sigma/\nu$, but obviously this is not the optimal Gevrey type. In any case, it finishes the proof of proposition 6.15.

**Corollary 6.19** *There is a coordinate transformation*

$$Z = \Delta + \varphi(u, \overline{v}, A, \lambda),$$

*transforming (6.12) into*

$$\left\{ \begin{array}{rcl} \dot{u} & = & -u\overline{v}^{\sigma} \\ \dot{\overline{v}} & = & m\overline{v}^{\sigma+1} \\ \dot{\Delta} & = & \tilde{\beta}(u, \overline{v}, \Delta, A, \lambda)\Delta + \tilde{g}(u, \overline{v}, A, \lambda) \end{array} \right. \tag{6.17}$$

*with $\varphi$, $\tilde{\beta}$ and $\tilde{g}$ real analytic w.r.t. all variables, except w.r.t. $\overline{v}$. The functions are uniformly Gevrey-$1/\sigma$ w.r.t. $\overline{v}$ in a sector $S_{\overline{v}_0,0,\theta}$, with opening angle $0 < \theta < 2\pi/\sigma$. The function $\varphi$ is Gevrey-$1/\sigma$ asymptotic to the series (6.14). The function $\tilde{\beta}(u, 0, 0, A, \lambda)$ has strictly negative real part for $u \in \Omega$, and for all $(A, \lambda)$ (where $\Omega$ is defined in lemma 6.14). The function $\tilde{g}$ is Gevrey-$1/\sigma$ asymptotic to $0$:*

$$\tilde{g}(u, \overline{v}, \Delta, A, \lambda) = O(\exp(-1/T|\overline{v}|^{\sigma})),$$

*as $\overline{v} \to 0$, for some $T > 0$ (uniformly in the other variables).*

**Proof** Define $\varphi$ using theorem 1.5. By making the translation $z = \Delta + \varphi$, it is clear that the new vector field has the analyticity properties as stated in the formulation of the corollary, and it is clear that the formal expansion (6.14) is identically 0 in the new coordinates. Expressing this property using (6.17) yields that $\tilde{g}$ is formally 0, in other words exponentially small (proposition 1.12). □

### 6.2.5 Analytic invariant manifolds near $P_{\pm}$

Here, we want to prove that we can saturate entry boundary curves beyond the point $P_{\pm}$ on the blow up locus. The saturation up to a section close to this turning point is treated by theorem 6.12. Close to $P_{\pm}$, we can use normal form (6.11), and consider an entry boundary curve

$$\{u = u_0, z = s_-(\overline{v}, A, \lambda)\},$$

with $s_-$ Gevrey-$1/\sigma$ w.r.t. $\overline{v}$ in a sector $S_{v_0,0,\theta}$, $0 < \theta < \pi/2\sigma$. Since the center manifold in (6.11) is given by $z = O(v^N)$, we may assume that also $s_- = O(v^N)$. The reason is that the above curve is the saturation of $\Sigma_-$ inside the plane $\{u = u_0\}$, and the saturation of any boundary curve is infinitely flat to any center manifold. In any case, it is possible to define

$$\{u = u_0, Z = \frac{1}{v^{\sigma}} s_-(\overline{v}, A, \lambda)\}$$

as Gevrey-$1/\sigma$ entry boundary curve after the rescaling

$$z = v^{\sigma} Z.$$

We can hence work with the simpler normal form (6.12), which we have repeated here for the sake of convenience:

$$\left\{\begin{array}{rcl}\dot{u} & = & -u\overline{v}^{\sigma} \\ \dot{\overline{v}} & = & m\overline{v}^{\sigma+1} \\ \dot{Z} & = & \beta(u,\overline{v},Z,A,\lambda)Z+\overline{v}^{N}g(u,\overline{v},A,\lambda).\end{array}\right. \tag{6.18}$$

**Proposition 6.20** *Let $u_0 \in \Omega \cap \mathbf{R}^+$ so that the segment $[0,u_0] \subset \mathbf{C}$ does not leave $\Omega$. Let $\Sigma_-$ be an admissible entry boundary curve in the plane $\{u = u_0\}$, with $u_0 \in \Omega$ (in a chart where the vector field is as in (6.11)) that is Gevrey-$1/\sigma$ in $\overline{v}$ in a sector $S_{r,0,\theta}$ with $0 < 2\theta < \pi/\sigma$.*

*Choose $\theta_1, \theta_2 > 0$ so that*

$$m\theta_1 + \theta_2 \leq \theta$$

*There exists a $\theta_3 > 0$ with*

$$m\sigma\theta_1 + \sigma\theta_2 + \theta_3 < \frac{\pi}{2}$$

*and for which there exists an $r > 0$ and an analytic and bounded graph*

$$z = \varphi(u,\overline{v},A,\lambda),$$

*with $u \in \Omega(u_0;\theta_1,\theta_3)$, $\overline{v} \in S_{r,0,\theta_2}$, $A \in ]-A_0, A_0[$, $\lambda \in \Lambda$ that is invariant under the flow of (6.11) and so that*

$$\{z = \varphi(u_0,\overline{v},A,\lambda)\} \subset \Sigma_-$$

*The domain $\Omega(u_0;\theta_1,\theta_3)$ is defined as*

$$\Omega(u_0;\theta_1,\theta_3) := \{u \in \mathbf{C}\colon\ \mathrm{Arg}(u) < \theta_1, \mathrm{Arg}(u_0^{m\sigma} - u^{m\sigma}) < \theta_3\}.$$

*The angle $\theta_3$ should be chosen small enough so that $\Omega(u_0;\theta_1,\theta_3) \subset \Omega$ (note that $\Omega(u_0;\theta_1;\theta_3) \to ]0,u_0[$ as $\theta_3 \to 0$ in a Hausdorff sense).*

**Proof** The proof is based on the normal form in (6.12), and we will rely on the lemma below. Fixing $u_0$, the saturation of a point $(u_0,\overline{v}_0,s_-(\overline{v}_0,A,\lambda))$ on the boundary curve $\Sigma_-$ reaches a point with coordinates $(u,\overline{v},Z)$ if we choose $\overline{v}_0 = (u/u_0)^m\overline{v}$. If $u \in \Omega(u_0;\theta_1,\theta_3)$ and $\overline{v} \in S_{r,0,\theta_2}$, then

$$\overline{v}_0 := (u/u_0)^m\overline{v} \in S_{r,0,m\theta_1+\theta_2} \subset S_{r,0,\theta}.$$

This shows that $s_-(\overline{v}_0,A,\lambda)$ is defined. Furthermore, from the results of the lemma below follows that the orbit $O$ through $(u_0,\overline{v}_0)$ at $t = 0$ reaches $(u,\overline{v})$ at a time

$$T = \frac{1}{m\sigma}\frac{u_0^{m\sigma} - u^{m\sigma}}{(u^m\overline{v})^{\sigma}} \in S_{\infty,0,m\sigma\theta_1+\sigma\theta_2+\theta_3};$$

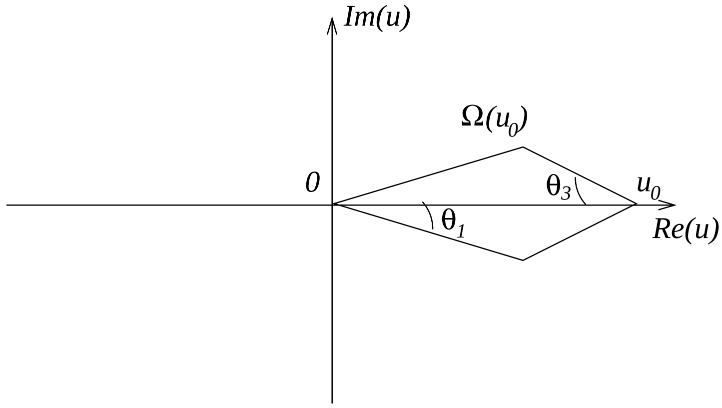


Figure 6.1: Specification of $\Omega(u_0) := \Omega(u_0; \theta_1, \theta_3)$ (in case $\sigma = 1$)

in other words $T \in S_{\infty,0,\theta_T}$ with $0 < \theta_T < \pi/2$. Finally, it is also clear that, by taking $r$ small enough, the orbit $O$ stays inside sufficiently small sectorial neighbourhoods for $t \in [0, T]$, ensuring that (6.12) is defined along this time interval. Let now

$$(\tilde{u}(t), \tilde{\overline{v}}(t), \tilde{Z}(t, A, \lambda))$$

be the flow through $(u_0, \overline{v}_0, s_-(\overline{v}_0, A, \lambda))$, and define

$$S = \sup\{s \in [0, 1] \colon |\tilde{z}(sT)| \leq R\},$$

where $R$ is yet to be defined. We show that $S = 1$, hereby proving the boundedness of the saturation of the initial boundary curve to a point $(u, \overline{v}, \varphi(u, \overline{v}, A, \lambda))$. The analyticity follows from the analytic dependence on initial conditions of regular flows. Now, let us prove that $S = 1$. Define

$$\zeta(s, A, \lambda) := \tilde{Z}(sT, A, \lambda), \qquad s \in \mathbf{R}^+.$$

One has

$$\begin{aligned} \frac{d\zeta}{ds} &= T \cdot \left(\beta(\tilde{u}(sT), \tilde{\overline{v}}(sT), \zeta, A, \lambda)\zeta + \tilde{\overline{v}}(sT)^N g(\tilde{u}(sT), \tilde{\overline{v}}(sT), A, \lambda)\right) \\ &= T\tilde{\beta}(s, \zeta, A, \lambda)\zeta + T\tilde{G}(s, A, \lambda). \end{aligned}$$

We prove that $|\zeta|$ is bounded by some $M < R$ by proving

$$|\zeta| \geq M \implies \frac{d}{ds}|\zeta(s)| < 0.$$

It is easily seen that $\frac{d}{ds}|\zeta| < 0$ provided $\Re(\overline{\zeta}\frac{d\zeta}{ds}) < 0$:

$$\Re(\overline{\zeta}\frac{d\zeta}{ds}) = \Re(T\tilde{\beta})|\zeta|^2 + \Re(\overline{\zeta}T\tilde{G}) < |T| \left(\Re\left(\frac{T}{|T|}\tilde{\beta}\right)|\zeta|^2 + R|\tilde{G}|\right).$$

Assume for a moment that we know that $\Re(\tilde{\beta}T/|T|) < -\mu < 0$, then the above expression is negative provided

$$|\zeta| \geq \left(\frac{R}{\mu}|\tilde{G}(s,A,\lambda)|\right)^{1/2}$$

Since $\tilde{G} = O(\tilde{\overline{v}}^N)$ it is easily bounded by some $M < R$, provided one chooses the sectorial neighbourhood for $\overline{v}$ sufficiently small (i.e. one chooses $r$ sufficiently small).

We still have to show that $\Re(\tilde{\beta}T/|T|) < -\mu < 0$ for some $\mu > 0$. For $u \in [0, u_0]$ one has $\beta(u, 0, 0, A, \lambda)$ is strictly negative and lies hence on a segment $]-\infty, -K] \subset \mathbf{C}$, for some $K > 0$. Multiplication with $T/|T|$ of this value rotates the segment over an angle that is given by the argument of $T$, which is less than $\theta_T < \pi/2$. This means that $\Re(\tilde{\beta}T/|T|)$ can be bounded by $-K\cos\theta_T$. For $u$ not on the reals, and for nonzero $\overline{v}$, $Z$ one shows by continuity that for sufficiently small neighbourhoods, $\Re(\tilde{\beta}T/|T|)$ is still less than $-\frac{1}{2}K\cos\theta_T =: -\mu$. □

**Lemma 6.21** *Consider the planar vector field on* $\mathbf{C}^2$

$$R\colon \left\{ \begin{array}{rcl} \dot{u} & = & -u\overline{v}^{\sigma} \\ \dot{\overline{v}} & = & m\overline{v}^{\sigma+1} \end{array} \right.$$

*Let* $(u_0, \overline{v}_0)$ *and* $(u_1, \overline{v}_1)$ *be two points in* $\mathbf{C}^2$ *with* $u_0^m\overline{v}_0 = u_1^m\overline{v}_1$. *The orbit* $O := \{(\tilde{u}(t), \tilde{\overline{v}}(t))\colon t \in \mathbf{C}\}$ *through* $(u_0, \overline{v}_0)$ *at* $t = 0$ *reaches* $(u_1, \overline{v}_1)$ *at a complex time*

$$T = \frac{1}{m\sigma}\frac{u_0^{m\sigma} - u_1^{m\sigma}}{(u_1^m\overline{v}_1)^{\sigma}}.$$

*Furthermore, let* $r_1 > 0$, $r_2 > 0$ *and choose* $\theta_1, \theta_2$ *be positive angles, so that*

$$0 < m\theta_1 + \theta_2 < \frac{\pi}{2\sigma}.$$

*Consider the domain*

$$V(r_2) := S_{r_1,0,\theta_1} \times S_{r_2,0,\theta_2} \subset \mathbf{C}^2$$

*Then, the orbit* $O$*, restricted to complex times on the segment* $[0,T]$*, has the property that* $O \subset V(r_2/\cos^{1/m\sigma}(m\sigma\theta_1))$ *provided both* $(u_0, \overline{v}_0)$ *and* $(u_1, \overline{v}_1)$ *lie in* $V(r_2)$.

**Proof** For the sake of readability, let us restrict to $m = \sigma = 1$; after all one can always reduce to this case by writing $u_1 = u^{m\sigma}$, $v_1 = v^{\sigma}$. Let

$$\tilde{u}(sT) = u_0(1-s) + u_1 s, \qquad \tilde{\overline{v}}(sT) = \frac{u_0\overline{v}_0}{u_0(1-s) + u_1 s},$$

and restrict $s$ to the real interval $[0,1]$. It is readily verified that $(\tilde{u}(t), \tilde{\overline{v}}(t))$ defines a regular orbit of $R$ going through $(u_0, \overline{v}_0)$ at $t = 0$ ($s = 0$) and through $(u_1, \overline{v}_1)$ at

$t = T$ $(s = 1)$. If $(u_0, u_1) \in S_{r_1,0,\theta_1}$, then $(u_0(1-s) + u_1 s)$ defines in $\mathbf{C}$ a straight segment from $u_0$ to $u_1$, and hence stays inside the sector $S_{r_1,0,\theta_1}$. On the other hand, one has

$$\frac{1}{\tilde{\overline{v}}(sT)} = \frac{1}{\overline{v}_0}(1-s) + \frac{1}{\overline{v}_1}s.$$

Because $1/\overline{v}_0$ and $1/\overline{v}_1$ lie in a complex sector $S_{\infty,0,\theta_2}$, also $\frac{1}{\tilde{\overline{v}}(sT)}$ lies in this complex sector (because it is a straight segment connecting $1/\overline{v}_0$ to $1/\overline{v}_1$). Remains to prove that $|\tilde{\overline{v}}(t)|$ remains bounded. This is done by bounding $|\tilde{u}(sT)|$ away from 0. It is an elementary trigoniometric exercise to show that

$$|u_0(1-s) + u_1 s| \geq \min\{|u_0|, |u_1|\} \cos\theta_1, \qquad \forall s \in [0,1].$$

Suppose that the minimum is reached at $|u_0|$, then

$$|\tilde{\overline{v}}(sT)| \leq \frac{|u_0||\overline{v}_0|}{|u_0|\cos\theta_1} < \frac{|\overline{v}_0|}{\cos\theta_1};$$

if on the other hand the minimum is reached at $|u_1|$, then

$$|\tilde{\overline{v}}(sT)| \leq \frac{|u_0||\overline{v}_0|}{|u_1|\cos\theta_1} = \frac{|u_1||\overline{v}_1|}{|u_1|\cos\theta_1} < \frac{|\overline{v}_1|}{\cos\theta_1}.$$

In both cases, we find that $\tilde{\overline{v}}(sT) \in S_{r/\cos\theta_1,0,\theta}$ provided $\overline{v}_0, \overline{v}_1 \in S_{r,0,\theta}$. □

The next theorem is a reformulation of proposition 6.20, adding the Gevrey property of the constructed invariant manifolds:

**Theorem 6.22** *Let $u_0 \in \Omega \cap \mathbf{R}^+$. Let $\Sigma_-$ be an admissible entry boundary curve in the plane $\{u = u_0\}$ (in a chart where the vector field is as in (6.11)) that is Gevrey-$1/\sigma$ in $\overline{v}$ in a sector $S_{r,0,\theta}$ with $0 < 2\theta < \pi/\sigma$. Choose $\theta_1, \theta_2 > 0$ so that*

$$m\sigma\theta_1 + \sigma\theta_2 < \frac{\pi}{2}, \qquad m\theta_1 + \theta_2 \leq \theta$$

*Assume also that*

$$\Omega(u_0; \theta_1, \theta_3) := \{u \in \mathbf{C}\colon\ \mathrm{Arg}(u) < \theta_1, \mathrm{Arg}(u_0^{m\sigma} - u^{m\sigma}) < \theta_3\} \subset \Omega$$

*There exists a $\theta_3 > 0$, an $r > 0$ and an analytic and bounded graph*

$$z = \varphi(u, \overline{v}, A, \lambda),$$

*with $u \in \Omega(u_0; \theta_1, \theta_3)$, $\overline{v} \in S_{r,0,\theta_2}$, $A \in ]-A_0, A_0[$, $\lambda \in \Lambda$ that is invariant under the flow of (6.11) and so that*

$$\{z = \varphi(u_0, \overline{v}, A, \lambda)\} \subset \Sigma_-$$

*The function is Gevrey-$1/\sigma$ asymptotic w.r.t. $\overline{v}$ to the series (6.14), uniformly for $u$ in compact subsets of $\Omega(u_0; \theta_1, \theta_3)$.*

**Proof** As said above the formulation of the theorem, everything except for the Gevrey statement is already shown in proposition 6.20. To continue, we use the normal form (6.17) obtained in corollary 6.19. We first show how we can reduce to the case where the entry boundary curve is exponentially close to $\{\Delta = 0\}$. Near $u = u_0 \neq 0$, the blow up map is a diffeomorphism, so there we can write $\overline{v} = v/u^m$, and write

$$\left\{\begin{array}{rcl} \dot{u} & = & -u^{1-m\sigma}v^{\sigma} \\ \dot{\Delta} & = & \tilde{\beta}(u, v/u^m, \Delta, A, \lambda)\Delta + \tilde{g}(u, v/u^m, A, \lambda) \\ \dot{v} & = & 0 \end{array}\right.$$

Because $\tilde{g}$ is exponentially small w.r.t. $v$, the unique formal expansion near $u = u_0$ of the invariant manifolds along $\gamma$ is given by $\hat{\Delta} = 0$; in other words, saturating $\Sigma_-$ gives in these coordinates, locally near $u = u_0$ a manifold that is exponentially close to $\Delta = 0$ (this follows from theorem 6.12). Hence, by taking a plane $u = u_1$, the intersection gives a new entry boundary curve that is exponentially close to $\Delta = 0$.

Let now $\nu > 0$ be small, consider again the vector field (6.17) and consider the coordinate transformation

$$\Delta = Z\exp(-\nu/\overline{v}^{\sigma}).$$

In these new coordinates, the entry boundary curve is still admissible (taking $\nu$ small enough), and the vector field yields

$$\left\{\begin{array}{rcl} \dot{u} & = & -u\overline{v}^{\sigma} \\ \dot{\overline{v}} & = & m\overline{v}^{\sigma+1} \\ \dot{Z} & = & (m\nu + \tilde{\beta}(u, \overline{v}, Z\exp(-\nu/\overline{v}^{\sigma}), A, \lambda))Z + \exp(\nu/\overline{v}^{\sigma})\tilde{g}(u, \overline{v}, A, \lambda). \end{array}\right.$$

Taking $\nu$ small enough (and keeping in mind that $\tilde{g}$ is exponentially small), this vector field is again of the form specified in corollary 6.19. By the previous proposition, saturations are bounded analytic, and hence going back to $(u, \overline{v}, \Delta)$ coordinates, we have a saturated manifold that is exponentially close to $\{\Delta = 0\}$. This proves the theorem. □

### 6.2.6 Manifolds over a covering of sectors

In the previous theorem, there is absolutely no reason to restrict to $u_0 \in \mathbf{R}^+$. Indeed, one can introduce the change of coordinates $u \mapsto e^{i\alpha}u$ and see that (6.11) and (6.12) will be of exactly the same form.

**Proposition 6.23** *Let $u_0 \in \Omega$ so that the segment $[0, u_0] \subset \mathbf{C}$ does not leave $\Omega$. Let*

$$\theta(u_0) := \sup_{u\in[0,u_0]} |\operatorname{Arg}(-\beta(u, 0, 0, A, \lambda))| \quad \in ]0, \frac{\pi}{2}[,$$

*where $\beta$ is the function in (6.12). Ensuring that*

$$m\sigma\theta_1 + \sigma\theta_2 + \theta_3 + \theta(u_0) < \frac{\pi}{2}, \qquad m\theta_1 + \theta_2 \leq \theta$$

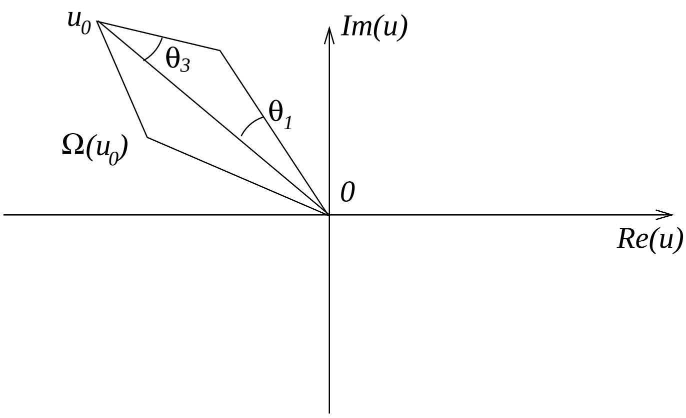


Figure 6.2: Specification of $\Omega(u_0) := \Omega(u_0; \theta_1, \theta_3)$ (in case $\sigma = 1$)

*then the same conclusions can be drawn as in theorem 6.22; in that case the domain* $\Omega(u_0; \theta_1, \theta_3)$ *is given by*

$$\Omega(u_0; \theta_1, \theta_3) := \{u \in \mathbf{C}\colon \operatorname{Arg}(u/u_0) < \theta_1, \operatorname{Arg}(1 - (u/u_0)^\sigma) < \theta_3\} \subset \Omega.$$

Note: the domain $\Omega$ is defined in lemma 6.14 as the set of points $u$ for which $\Re(\beta(u, 0, 0, A, \lambda)) < 0$. This ensures that $\theta(u_0) < \frac{\pi}{2}$.

**Proof** Only the last part of the proof of theorem 6.22 needs to be altered, where we show that $\Re(\tilde{\beta}T/|T|) < -\mu < 0$ for some $\mu > 0$. For $u \in [0, u_0]$ one has $|\operatorname{Arg}(-\beta(u, 0, 0, A, \lambda))| < \theta(u_0)$. Multiplication with $T/|T|$ of this value gives a complex argument of at most $\theta(u_0) + \theta_T$, which is still less than $\pi/2$. We find

$$\Re\left(\frac{T}{|T|}\beta(u, 0, 0, A, \lambda)\right) < -|\beta(u, 0, 0, A, \lambda)| \cos(\theta(u_0) + \theta_T) < -K \cos(\theta(u_0) + \theta_T),$$

for some $K > 0$. For $u$ not on $[0, u_0]$, and for nonzero $\overline{v}$, $Z$, one shows by continuity that for sufficiently small neighbourhoods,

$$\Re(\tilde{\beta}T/|T|) < -\frac{1}{2}K \cos(\theta(u_0) + \theta_T) =: -\mu.$$

This finishes the proof. □

For the vector field $\overline{X}_{A,\lambda}$ in (6.11), we define

$$R(u, \lambda) := \Re\left(\frac{-1}{u^{m\sigma}} \int_u^0 s^{m\sigma - 1} \frac{\beta_1(s, 0, 0, 0, \lambda)}{h_1(s, 0, 0, 0, \lambda)} ds\right)$$

For $u, \overline{v} \in \mathbf{R}^+$, this can be related to the integral of the slow divergence along a piece of the critical curve (see chapter 5). More precisely, let $p \in \gamma_-$ be a point on the critical curve that corresponds to $u = u_p$ in the coordinate system where (6.11) is valid. Then,

$$\frac{1}{v^\sigma}\left(\int_{[p,p_*]} \operatorname{div} X^0_{A,\lambda} ds\right)\Bigg|_{v=u^m\overline{v}} = \frac{1}{u^{m\sigma}\overline{v}^\sigma}(u_p^{m\sigma}R(u_p,\lambda)+o(1))$$

as $\overline{v} \to 0$.

Define now for $T > 0$

$$\mathcal{C}_T = \{u_0 \in \Omega \colon \Omega(u_0;\theta_1,\theta_3) \subset \Omega, |u_0|^{m\sigma}R(u_0,\lambda) = -T\}.$$

Assume that this is a connected subset of $\mathbf{C}$, so that along each ray from $0 \in \mathbf{C}$ there is exactly one intersection point (we will say that $\mathcal{C}_T$ is angle-parametrizable). We will use points of $\mathcal{C}_T$ as locations where the entry boundary curve is chosen. In view of theorem 1.15, we will choose a finite number of them and create a sectorial covering of invariant manifolds. The exponential closeness between them will be the subject in this section, inducing the Gevrey property of these individual manifolds. It will follow that $T$ becomes an upperbound for the Gevrey type of these manifolds (and will later form an upperbound for the Gevrey type of the control curve $\mathcal{A}$).

**Proposition 6.24** *Let $u_1, u_2 \in \mathcal{C}_T$, and let for $i = 1, 2$*

$$\varphi_i(u,\overline{v},A,\lambda)$$

*be two bounded analytic functions that are invariant under the flow of (6.11), on the domains*

$$\Omega(u_i;\theta_1,\theta_3)\times S_{r,0,\theta_2}\times]-A_0,A_0[\times\Lambda.$$

*Define*

$$\Delta(u,v,A,\lambda) = (\varphi_2-\varphi_1)(u,v,A,\lambda)$$

*on the intersecting domain $(\Omega(u_1)\cap\Omega(u_2))\times S_{r,0,\theta_2}\times]-A_0,A_0[\times\Lambda$. Then, for any $u_0 \in \Omega(u_1)\cap\Omega(u_2)$, and for any $\theta_0$ so that*

$$\{u \in \mathbf{C}\colon \operatorname{Arg}(u/u_0) < \theta_0, |u| < |u_0|\} \subset \Omega(u_1)\cap\Omega(u_2)$$

*one has*

$$\Delta(u,v,A,\lambda) = O\left(\exp\left(\frac{R(u_0,\lambda)+\nu+o(1)}{|u^m\overline{v}|^\sigma}\right)\right),$$

*as $u \to 0$, with $\nu$ an expression that is $O(\varphi_0)$, where $\varphi_0 = m\sigma\theta_0+\sigma\theta_2$. (In particular, as the intersecting domain shrinks in opening angle, $\varphi_0 \to \sigma\theta_2$.)*

**Proof** We can safely use (6.12) instead of (6.11). Then, we have

$$-u\overline{v}^\sigma\frac{\partial\Delta}{\partial u}+m\overline{v}^{\sigma+1}\frac{\partial\Delta}{\partial\overline{v}} = \beta(u,\overline{v},\varphi_2,A,\lambda)\varphi_2-\beta(u,\overline{v},\varphi_1,A,\lambda)\varphi_1.$$

This means that $\{w = \Delta\}$ is an invariant manifold for the vector field

$$\begin{cases} \dot{u} &= -u\overline{v}^{\sigma} \\ \dot{\overline{v}} &= m\overline{v}^{\sigma+1} \\ \dot{w} &= b(u,\overline{v},A,\lambda)w \end{cases}$$

with

$$b(u,\overline{v},A,\lambda) := \int_0^1 \frac{\partial(z\beta(u,\overline{v},z,A,\lambda))}{\partial z}|_{z=\varphi_1+s(\varphi_2-\varphi_1)}ds$$

In particular, observe that $b(u,0,A,\lambda) = \beta(u,0,0,A,\lambda)$. Since $u_0$ is in the intersecting domain, we can say that the invariant manifold is a saturated manifold of a boundary curve located in the plane $u = u_0$. One easily sees that one has

$$w = w_0(u^m\overline{v}/u_0^m, A, \lambda)\exp\left(\frac{1}{u^{m\sigma}\overline{v}^{\sigma}}\int_u^{u_0} s^{m\sigma-1}b(s, u^m\overline{v}/s^m, A, \lambda)ds\right)$$

First notice that

$$\int_u^{u_0} s^{m\sigma-1}b(s,u^m\overline{v}/s^m,A,\lambda)ds = \int_0^{u_0} s^{m\sigma-1}b(s,0,A,\lambda)ds + o(1), \tag{6.19}$$

as $u \to 0$. (We postpone a proof until the end of the proof of this proposition.) We hence write

$$w = w_0(u^m\overline{v}/u_0^m, A, \lambda)\exp\left(\frac{\int_0^{u_0} s^{m\sigma-1}b(s,0,A,\lambda)ds + o(1)}{u^{m\sigma}\overline{v}^{\sigma}}\right);$$

in other words

$$w = O\left(\exp\left(\Re\left(\frac{\int_0^{u_0} s^{m\sigma-1}b(s,0,A,\lambda)ds + o(1)}{u^{m\sigma}\overline{v}^{\sigma}}\right)\right)\right)$$

Remains to study the expression

$$\Re\left(\frac{u_0^{m\sigma}F(u_0)}{u^{m\sigma}\overline{v}^{\sigma}}\right),$$

with

$$F(u_0) := \frac{1}{u_0^{m\sigma}}\int_0^{u_0} s^{m\sigma-1}b(s,0,A,\lambda)ds.$$

We have

$$\Re\left(\frac{u_0^{m\sigma}F(u_0)}{u^{m\sigma}\overline{v}^{\sigma}}\right) = \frac{|u_0|^{m\sigma}}{|u|^{m\sigma}|\overline{v}|^{\sigma}}|F(u_0)|\cos\operatorname{Arg}\left(\frac{u_0^{m\sigma}}{u^{m\sigma}\overline{v}^{\sigma}}F(u_0)\right).$$

The complex argument of $u_0^{m\sigma}/u^{m\sigma}\overline{v}^\sigma$ is bounded by $\varphi_0 := m\sigma\theta_0 + \sigma\theta_2$, for $u \in \Omega(u_0;\theta_1,\theta_3)$ and $\overline{v} \in S_{r,0,\theta_2}$. We hence find

$$\begin{aligned}\Re\left(\frac{u_0^{m\sigma}F(u_0)}{u^{m\sigma}\overline{v}^\sigma}\right) &= \frac{|u_0|^{m\sigma}}{|u|^{m\sigma}|\overline{v}|^\sigma}|F(u_0)|\left(\cos(\operatorname{Arg} F(u_0)) + O(\varphi_0)\right)\\ &= \frac{|u_0|^{m\sigma}\Re(F(u_0))}{|u|^{m\sigma}|\overline{v}|^\sigma}(1+O(\varphi_0))\end{aligned}$$

We conclude that

$$w = O\left(\exp\left(\frac{|u_0|^{m\sigma}R(u_0,\lambda)+o(1)}{|u|^{m\sigma}|\overline{v}|^\sigma}(1+O(\varphi_0))\right)\right),$$

as $u \to 0$. Let us now show that (6.19) is true. The difference between lefthand side and righthand side is decomposed in two parts:

$$\int_u^{u_0} s^{m\sigma-1}(b(s,u^m\overline{v}/s^m,A,\lambda) - b(s,0,A,\lambda))ds - \int_0^u s^{m\sigma-1}b(s,0,A,\lambda)ds.$$

The second term is clearly $O(u)$ as $u \to 0$. The first term can be rewritten as

$$\int_u^{u_0} s^{m\sigma-1}H(s,u^m\overline{v},A,\lambda)u^m\overline{v}/s^m ds, \tag{6.20}$$

with

$$H(s,u^m\overline{v},A,\lambda) := \int_0^1 \frac{\partial b}{\partial \overline{v}}(s,ru^m\overline{v}/s^m,A,\lambda)dr.$$

The function $H$ is bounded along the integration path; this implies that in the worst case (remember that $m \geq 1$, $\sigma \geq 1$), expression (6.20) is $O(u\log u)$ as $u \to 0$. In any case it is $o(1)$ as $u \to 0$. $\square$

We say that a set inside $\mathbf{C}$ is *angle-parametrizable* if it is a graph $\{r(\theta)e^{i\theta}\}$ for some strictly positive continuous $2\pi$-periodic function $r$.

**Theorem 6.25** *Let $u_0 \in \mathcal{C}_T$, and assume that $\mathcal{C}_T$ is a connected set that is angle-paramatrizable. (It has a point along each ray through the origin in $\mathbf{C}$.) Then, the saturation (as in theorem 6.22) of an analytic entry boundary curve from the plane $\{u = u_0\}$ w.r.t. to flow of (6.11) is Gevrey-$1/m\sigma$ w.r.t. $u$ of type $T'|\overline{v}|^\sigma$ for any $T' > T$, and*

$$T = \frac{-1}{|u_0|^{m\sigma}R(u_0,\lambda)+O(\theta_2)},$$

*uniformly for $\overline{v} \in S_{\overline{v}_0,0,\theta_2}$. In particular, as $\theta_2 \to 0$, $T \to -1/|u_0|^{m\sigma}R(u_0,\lambda)$. The opening angle of the sector for $u$ in which the Gevrey asymptotics is guaranteed depends on $T'$ (as $T' \to T$, the opening angle tends to 0).*

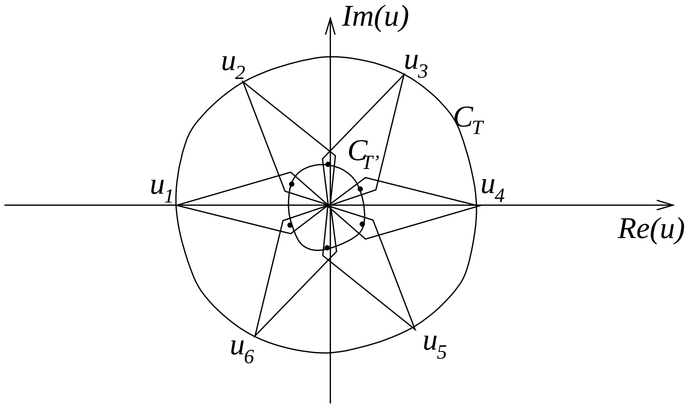


Figure 6.3: Points $u_i$ on $\mathcal{C}_T$ lead to domains $\Omega(u_i;\theta_1,\theta_3)$ that have intersection points on $\mathcal{C}_{T'}$, for well chosen angles $\theta_1$, $\theta_3$

**Proof** The proof is quite delicate. Let us first take a closer look on the shape of $\Omega(u_0;\theta_1,\theta_3)$. Notice that as $\theta_1 \to 0$ and $\theta_3 \to 0$ the domain $\Omega(u_0;\theta_1,\theta_3)$ tends to the line piece $[u_0, 0]$. Choose now $T'' > T' > T$ arbitrarily, then the interior of $\mathcal{C}_{T'}$ is a relatively compact, connected domain of $\mathbf{C}$ containing the origin. Since this interior is covered by rays $\{[u_0, 0]\}$ with $u_0 \in \mathcal{C}_{T'}$, it will also be covered a finite number of sectors $S_i := S_{|u_{0i}|,\mathrm{Arg}(u_{0i}),\theta_1}$ for any nonzero opening angle $\theta_1$.

We claim now that, for $\theta_1$ small enough, $S_i$ can be chosen inside a domain $\Omega(u_i;\theta_1,\theta_3)$, with $u_i \in \mathcal{C}_T$, and so that on $\Omega(u_i;\theta_1,\theta_3) \times S_{\overline{v}_0,0,\theta_2}$ there is defined an invariant manifold $W_i$ for the vector field. Let us first assume that this is true. Then, $\{\Omega(u_i;\theta_1,\theta_3)\}_i$ is a sectorial covering of the origin, and for two adjacent domains, the intersection $\Omega(u_i) \cap \Omega(u_j)$ contains a point of $\mathcal{C}_{T'}$; indeed, the intersection is larger than $S_i \cap S_j$ and because $\{S_i\}$ covers the interior of $\mathcal{C}_{T'}$, there must be a point of $\mathcal{C}_{T'}$ inside this intersection. The difference between the two manifold $W_i$ and $W_j$ can hence be measured using the previous proposition, and the difference is $O(\exp((-1/T' + \nu + o(1))/|u^m\overline{v}|^\sigma))$, which is $O(\exp((-1/T'' + O(\theta_2))/|u^m\overline{v}|^\sigma))$, by choosing the opening angle $\theta_1$ small enough. Upon applying theorem 1.15, one obtains the required Gevrey property.

Remains now to prove the claim that we have posed. The ray through 0 and $u_{0i}$ intersects $\mathcal{C}_T$ in a point $u_i$. Furthermore, as $\theta_1 \to 0$, one has $S_i \cap \mathcal{C}_{T'} \to \{u_0\}$ in Hausdorf manner. Hence, for $\theta_1$ small enough, we can find a $\theta_3$ so that $\Omega(u_i;\theta_1,\theta_3) \supset S_i$. By furthermore diminishing $\theta_1$ we can ensure that $\Omega(u_i;\theta_1,\theta_3)$ satisfies the conditions of proposition 6.24 in order to ensure the existence of an invariant manifold over this domain. $\square$

### 6.2.7 Passage along the connection $\Gamma$

To proceed along the heteroclinic connection between $P_-$ and $P_+$, we need to pass to the family rescaling chart. The next lemma shows that changing charts preserves the Gevrey property of saturated manifolds.

**Lemma 6.26** *Let*

$$\{\overline{y} = \varphi(u, A, \lambda), \overline{v} = \overline{v}_0\}$$

*be a curve in the phase-directional rescaling chart (with $\overline{v}_0 > 0$). Then this curve is also visible in the family-rescaling chart as a curve*

$$\{\overline{x} = -\overline{v}_0^{-p/m}, \overline{y} = \psi(u, A, \lambda) := \varphi(u\overline{v}_0^{-1/m}, A, \lambda)\}$$

*If the original function $\varphi$ is Gevrey-$1/m\sigma$ of type $T\overline{v}_0^{\sigma}$ w.r.t. $u$, then the function $\psi$ is Gevrey-$1/m\sigma$ of type $T$.*

**Proof** Elementary. □

It is assumed that along $\Gamma$ no singularities appear. Hence, one can apply corollary 1.11 to show that the Gevrey-property is preserved upon following the connection $\Gamma$ (over compact pieces).

In particular, this shows that the intersection with a section $T$ (such that $\sigma := T \cap \{u = 0\}$ is transverse to the flow of the vector field on the blow up locus) is a Gevrey-$1/m\sigma$ curve.

If the connection $\Gamma$ is a graph $\overline{y} = \overline{\varphi}(\overline{x}, \lambda)$, then this shows that the saturation of a Gevrey-$1/\sigma$ curve of type $T'$ for all $T' > T$ is a manifold

$$\overline{y} = \varphi(\overline{x}, u, A, \lambda)$$

that is uniformly Gevrey-$1/\sigma$ of type $T'$ for all $T' > T$.

### 6.2.8 The Gevrey property of the control curve

The intersection with the plane $T$ (in the family rescaling chart) of forward and backward manifolds are thus Gevrey-$1/m\sigma$ curves w.r.t. $u$. As in chapter 3, one obtains the control curve as an implicit solution of the equation describing the splitting of forward and backward center manifolds. A Gevrey implicit function theorem shows that the control curve inherits the Gevrey estimates of the intersecting curves. We refer to section 3.3.2 for a discussion on how the control curve $\mathcal{A}$ is found in a $C^\infty$-context. This text can however be repeated in the analytic setting, keeping in mind theorem 1.16.

**Definition 6.27** *We define $T_A$ as the supremum of numbers $T > 0$ for which $\mathcal{C}_T$ is a connected subset of* $\mathbf{C}$*, angle-parametrizable. The set $\mathcal{C}_T$ is defined as the set*

$$\mathcal{C}_T = \{u_0 \in \Omega \colon \Omega(u_0; \theta_1, \theta_3) \subset \Omega, |u_0|^{m\sigma} R(u_0, \lambda) = -T\};$$

*the function $R$ is defined as*

$$R(u_0, \lambda) = \Re\left(\frac{1}{u_0^{m\sigma}} \int_0^{u_0} s^{m\sigma-1}\beta_1(s, 0, 0, 0, \lambda)ds\right),$$

*where $\beta_1$ is the divergence along the critical curve of the family of vector fields after blow up (see the normal form in (6.11)). Finally, the set $\Omega$ is defined as*

$$\Omega = \{u \in \mathbf{C} \colon \Re(\beta_1(u, 0, 0, 0, \lambda)) < 0\},$$

*and so that for $u$ inside $\Omega$ the curve of singularities of (6.11) is an analytic graph in $(u, A, \lambda)$.*

**Proposition 6.28** *Let $T' > T$ with*

$$T = \max(T_{c_-}, T_{c_+}, T_A);$$

*$-1/T_{c_\pm}$ being the value of the slow divergence from the corner point $c_\pm$ on the critical curve $\gamma_\pm$ up to the turning point $p_*$:*

$$\frac{-1}{T_{c_\pm}} = |u_\pm|^{m\sigma} R(u_\pm, \lambda),$$

*where $R$ and $T_A$ are defined in the above definition.*

*There exists an opening angle $\theta_1 \in ]0, \pi/m\sigma[$ so that the control curve $A = \mathcal{A}(u, \lambda)$ is Gevrey-$1/m\sigma$ w.r.t. $u$ in a sector $S_{u_0,0,\theta_1}$ and of type $T'$.*

**Proof** The attracting center manifold through $\Sigma_-$ intersects any transverse section (in the family rescaling chart) in a curve that is Gevrey-$1/m\sigma$ of type $T_1'$ in $u$, for all $T_1' > T_1$, and where

$$T_1 = \max(T_{c_-}, T_A).$$

Similarly, the repelling center manifold through $\Sigma_+$ intersects this section in a Gevrey-$1/m\sigma$ curve of type $T_2'$ in $u$, for all $T_2' > T_2$, where

$$T_2 = \max(T_{c_+}, T_A).$$

Now apply the Gevrey implicit function theorem, to show that we can annihilate the distance between attracting and repelling center manifolds upon choosing a Gevrey control curve $A = \mathcal{A}(u, \lambda)$. □

### 6.2.9 Proof of theorem 6.6

Consider a chart where the curve of singularities is given by $y = 0$, and consider a directional rescaling $\{\overline{x} = 1\}$, i.e.

$$x = u^p, \qquad y = u^q\overline{y}, \qquad v = u^m\overline{v}.$$

Near $\overline{x} = 1$, we define $z := \overline{y}$ to be the angular coordinate, making $(u, \overline{v}, \overline{y})$ a local coordinate system for the blow up space near $P_-$. By proposition 6.15 there exists a unique formal power series

$$\overline{y} = \sum_{n=0}^{\infty} \varphi_n(u, A, \lambda)\overline{v}^n \tag{6.21}$$

that is formally invariant w.r.t. the flow of the blown up vector field, and this series is Gevrey-$1/\sigma$ w.r.t. $\overline{v}$. On the other hand, under the assumption of theorem 6.6, there is a unique formal power series

$$y = \sum_{n=0}^{\infty} \psi_n(x, \lambda)v^n, A = \sum_{n=0}^{\infty} a_n v^n. \tag{6.22}$$

We have already shown in theorem 3.4 that this formal power series coincides with the series in corollary 6.5, so $A$ is Gevrey-$1/\sigma$. By restricting (6.21) to this formal series for $A$, we find a power series

$$\overline{y} = \sum_{n=0}^{\infty} \tilde{\varphi}_n(u, \lambda)\overline{v}^n, \tag{6.23}$$

which is still Gevrey-$1/\sigma$ (Gevrey substitution theorem). After blow up, the series for $y$ in (6.22 should coincide with the series for $\overline{y}$ in (6.23, i.e.

$$u^{-q} \sum_{n=0}^{\infty} \psi_n(u^p, \lambda)u^{mn}\overline{v}^n = \sum_{n=0}^{\infty} \tilde{\varphi}_n(u, \lambda)\overline{v}^n.$$

In other words,

$$\psi_n(u^p, \lambda)u^{mn-q} = \tilde{\varphi}_n(u, \lambda)$$

One easily finds, using the maximum principle that also the sequence $(\psi_n)_n$ is of Gevrey-$1/\sigma$ growth because $(\tilde{\varphi}_n)_n$ it is.

## 6.3 Examples

### 6.3.1 Van der Pol

Consider the traditional van der pol

$$\left\{\begin{array}{rcl} \dot{x} &=& y - x^2/2 - x^3/3 \\ \dot{y} &=& \epsilon(a-x). \end{array}\right.$$

(In the literature, the equivalent vector field

$$\left\{\begin{array}{rcl} \dot{x_1} &=& y_1 - \left(\frac{x_1^3}{3} - x_1\right) \\ \dot{y_1} &=& \epsilon_1(a_1 - x_1) \end{array}\right.$$

is often encountered; one has $(x_1, y_1, \epsilon_1, a_1) = (2x+1, 8y - \frac{2}{3}, 16\epsilon, 2a+1)$).

We first perform a blow up of the parameter space: $(\epsilon, a) = (v^2, vA)$. In this form, $v$ is the singular parameter, and $A$ is the parameter having the required rotational property. The kind of blow up formulas used in [DR] are $\{x = u\overline{x}, y = u^2\overline{y}, v = u\overline{v}\}$. In the chart $\{\overline{x} = 1\}$, one has

$$\overline{X}_A\colon \left\{\begin{array}{rcl} \dot{u} &=& u(\overline{y} - \frac{1}{2} - \frac{1}{3}u) \\ \dot{\overline{v}} &=& -\overline{v}(\overline{y} - \frac{1}{2} - \frac{1}{3}u) \\ \dot{\overline{y}} &=& -2\overline{y}(\overline{y} - \frac{1}{2} - \frac{1}{3}u) + \overline{v}^2(\overline{v}A - 1) \end{array}\right.$$

Write now

$$\overline{y} = \frac{1}{2} + \frac{1}{3}u - \frac{\overline{v}^2}{1+u} + z$$

Replacing $(u, \overline{v}, \overline{y})$ by the coordinate system $(u, \overline{v}, z)$ (so that in the new coordinates the curve of singularities is given by $\{z = O(\overline{v}^3)\}$) the vector field yields

$$\overline{X}_A\colon \left\{\begin{array}{rcl} \dot{u} &=& -u\left(\frac{\overline{v}^2}{1+u} - z\right) \\ \dot{\overline{v}} &=& \overline{v}\left(\frac{\overline{v}^2}{1+u} - z\right) \\ \dot{z} &=& -2\left(\frac{1}{2} + \frac{1}{3}u - \frac{\overline{v}^2}{1+u} + z\right)\left(-\frac{\overline{v}^2}{1+u} + z\right) \\ && \quad +\overline{v}^2(\overline{v}A - 1) - \frac{1}{3}\dot{u} + \frac{2\overline{v}\dot{\overline{v}}}{1+u} - \frac{\overline{v}}{(1+u)^2}\dot{u} \end{array}\right.$$

One easily checks that this gives

$$\overline{X}_A\colon \left\{\begin{array}{rcl} \dot{u} &=& -u\left(\frac{\overline{v}^2}{1+u} - z\right) \\ \dot{\overline{v}} &=& \overline{v}\left(\frac{\overline{v}^2}{1+u} - z\right) \\ \dot{z} &=& \left(-1 - 2z - u + \frac{\overline{v}^2(2+u)}{(1+u)^2}\right) z + A\overline{v}^3 + u\frac{\overline{v}^4}{(1+u)^3} \end{array}\right.$$

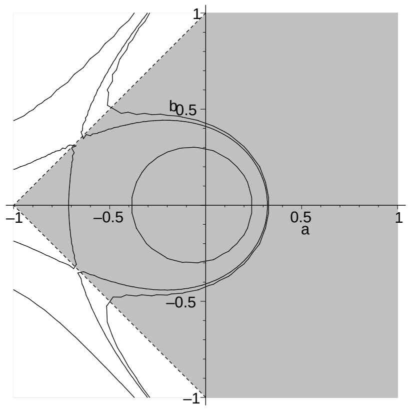


Figure 6.4: The domain $\Omega$ for the Van der Pol equation in grey, and $\mathcal{C}_T$ for $T = 24$, $T = 12.96$, $T = 12$. Observe that $\mathcal{C}_T$ is not angle-parametrizable for $T < 12.96$. The locus $\mathcal{C}_{12}$ in the above picture is the curve that is diffeomorphic to a circle; $\mathcal{C}_{12.96}$ is no longer angle-parametrizable, because it hits the boundary of $\Omega$ at two angles; the curve defining $\mathcal{C}_{12}$ crosses the boundary of $\Omega$.

This brings the blown up vector field in the form (6.11), and after applying the singular change of coordinates $z = \overline{v}^2 Z$, one finds back the normal form (6.12):

$$\begin{cases} \dot{u} &= -u\overline{v}^2 \\ \dot{\overline{v}} &= \overline{v}^3 \\ \dot{Z} &= \dfrac{\left(-(1+u)^2 - \frac{u\overline{v}^2}{(1+u)}\right) Z + A\overline{v}(1+u) + \frac{u\overline{v}^2}{(1+u)^2}}{1 - Z(1+u)} \end{cases}$$

In this form, we have

$$\beta(u, \overline{v}, Z, A) = \frac{-(1+u)^2 - \frac{u\overline{v}^2}{(1+u)}}{1 - Z(1+u)}$$

and

$$\beta(u, 0, 0, A) = -(1+u)^2.$$

The set $\Omega$, for which $\Re(\beta) < 0$ is defined as

$$\Omega = \{a + ib \in \mathbf{C} \colon |b| \leq |1 + a|\}.$$

In this form, $m = 1$ and $\sigma = 2$, so we have to consider

$$R(u_0) := \Re\left(\frac{-1}{u_0^{m\sigma}} \int_{u_0}^{0} u^{m\sigma - 1} \beta(u, 0, 0, 0) du\right) = \Re\left(\frac{1}{u_0^2} \int_{u_0}^{0} u(1+u)^2 du\right).$$

One calculates that $R(u_0) = -\frac{1}{2} - \frac{2}{3}\Re(u_0) - \frac{1}{4}\Re(u_0^2)$. Consider

$$\mathcal{C}_T := \{u_0 \in \Omega \colon |u_0|^2 R(u_0) = -1/T\}$$

One proves (it is a somewhat lengthy, straightforward calculation) that this is a connected set, angle-parametrizable, provided $T > 324/25 = 12.96$. This is the estimate for the type of the control curve that can be obtained through this work. However, numerical evidence indicates that the optimal type is 12, and it is also shown in [FS] that 12 is the optimal type (in other words for the equivalent result in $(x_1, y_1, \epsilon_1, a_1)$-variables, the type is $3/4$ w.r.t. $\epsilon_1$).

### 6.3.2 Initial example

Let us return to (6.3), which we repeat here for the sake of convenience:

$$X_{\epsilon,a} \colon \left\{ \begin{array}{rcl} \dot{x} & = & \epsilon \\ \dot{y} & = & a + x^3 y + \epsilon^N x + \epsilon^{N+1} F(x, y, \epsilon, a) \end{array} \right.$$

First, according to the technique explained in the chapter, we need a preliminary rescaling of the parameters:

$$(a, \epsilon) = (vA, v).$$

This will allow us to treat $A$ as a parameter during the family rescaling: write

$$(x, y, \tilde{v}, A) = (u\overline{x}, u\overline{y}, u^4\overline{v}, A).$$

In the chart $\overline{x} = \pm 1$, we find (after dividing through a positive time factor) the rescaled vector field:

$$\left\{ \begin{array}{rcl} \dot{u} & = & \pm u\overline{v} \\ \dot{\overline{v}} & = & \mp 4\overline{v}^2 \\ \dot{\overline{y}} & = & \pm\overline{y} + \overline{v}\overline{F}(u, v, \overline{y}, A), \end{array} \right.$$

where $\overline{F}(u, v, \overline{y}, A) = A \mp \overline{y} \mp u^{4N-3}\overline{v}^{N-1} + u^{4N}\overline{v}^N F(\pm u, u\overline{y}, u^4\overline{v}, u^4\overline{v}A)$. On the blow up locus $\{u = 0\}$, we have a connection for $A = 0$, namely the orbit $\overline{y} = 0$. It is not hard to prove that this connection is broken for $A \neq 0$ with a nonzero speed. As a consequence of the results in this chapter, we find a smooth control curve $A = \mathcal{A}(u)$, and this curve is Gevrey-$1/4$ ($m = 4$, $\sigma = 1$). Note that from the formulas above, one can see that $\mathcal{A}(u) = o(u^{4N-4})$. Write $\mathcal{A}_1(u) := \mathcal{A}(u)/u^{4N-4}$. This formula is expressed in coordinates of the family rescaling chart, so when going back to the initial coordinates, we find

$$a(\epsilon) = \epsilon^N \mathcal{A}_1(\epsilon^{1/4}).$$

Similarly, associated to this control curve, there is an overstable solution in the family rescaling chart

$$\overline{y} = \varphi(\overline{x}, u),$$

also Gevrey-$1/4$ in the second argument. We can blow down this manifold to form a continuous manifold in the original phase space, and glue this together with the blow down of the manifold in other charts. But the blown down manifold can never be $C^\infty$ in the origin (because of the remarks made in the introduction of this chapter).

# Bibliography


[Ba] W. Balser, From Divergent Power Series to Analytic Functions, Lecture Notes in Math., 1582, 1994.

[BE] E. Benoît, Applications of nonstandard analysis in ordinary differential equations, NATO adv. sci. inst. Ser. C Math. Phys. Sci, vol. 493, 1997, 153–182.

[BCDD] E. Benoît, J-L. Callot, F. Diener, M. Diener, Chasse aux canards, Collect. Math., vol. 32, no. 1–2 (1981).

[BFSW] E. Benoît, A. Fruchard, R. Schäfke, G. Wallet, solutions surstables des équations différentielles complexes lentes-rapides à point tournant, Ann. Fac. Sci. Toulouse Math. 6, vol. 7, no. 4, 1998, 627–658.

[Bon] P. Bonckaert, Partially hyperbolic fixed points with constraints, Trans. Amer. Math. Soc., vol. 348, no. 3, 1996, 997–1011.

[CD] M. Canalis-Durand, Caractère Gevrey du développement formel des solutions canard de l'équation de Van der Pol, 1990.

[CRSS] M. Canalis-Durand, J.P. Ramis, R. Schäfke, Y. Sibuya, Gevrey solutions of singularly perturbed differential equations, J. reine angew. Math., no. 588, 2000, 95–129.

[DE] A. Delcroix, A tool for the local study of slow-fast vector fields: the zoom, Lecture Notes in Mathematics, no. 1493 (1991).

[D] F. Dumortier, Techniques in the theory of local bifurcations: blow up, normal forms, nilpotent bifurcations, singular perturbations (notes written with B. Smits), in "Bifurcations and periodic orbits and vector fields" (ed. D. Schmoliuk), NATO ASI series, series C: Math. and Phys. Sciences, vol. 408, 1993, 19–73.

[DR] F. Dumortier and R. Roussarie, Canard cycles and Center manifolds, Memoirs of the Amer. Math. Soc., vol. 121, no. 577, 1996.

[DR2] F. Dumortier and R. Roussarie, Multiple canard cycles in generalized Liénard equations, Journal of differential equations, vol. 174, no. 1, 2001, 1–29.

[DR3] F. Dumortier and R. Roussarie, Geometric singular perturbation theory beyond normal hyperbolicity, in “Multiple-time-scale dynamical systems”, ed. C.K.R.T. Jones and A. Khibnik, IMA Volumes in Mathematics and its Applications, vol. 122, Springer-Verlag, 2000, 29–64 (ISBN 0-387-95126-1).

[ECK] W. Eckhaus, Asymptotic Analysis of singular perturbations, Studies in Mathematics and its applications, vol. 9, 1979, ISBN 0-444-85306-5.

[ECK2] W. Eckhaus, Fundamental concepts of matching, SIAM rev., vol. 36, no. 3, 1994, 431–439.

[FE] N. Fenichel, Geometric Singular Perturbation Theory for ordinary differential equations, J. Differential Equations, vol. 31, no. 1, 1979, 53-98.

[FS] A. Fruchard, R. Schäfke, Exceptional complex solutions of the forced van der Pol equation, Funkcial. Ekvac., vol. 12, no. 2, 1999, 201–223.

[FS2] A. Fruchard, R. Schäfke, Overstability and resonance, preprint IRMA, 2000.

[HPS] M. Hirsch, C. Pugh, M. Shub, Invariant Manifolds, Lecture Notes in Mathematics, 583, Springer-Verlag, 1977.

[JKK] C.K.R.T. Jones, T.J. Kaper, N. Kopell, Tracking invariant manifolds up to exponentially small errors, SIAM. J. Math. Anal, vol. 27, no. 2, 1996, 558-577.

[K] A. Kelley, The stable, center-stable, center, center-unstable and unstable manifolds, Appendix C in R. Abraham, J. Robbin: Transversal mappings and flows, Benjamin, New York, 1967.

[N1] A.I. Neishtadt, Persistence of stability loss for dynamical bifurcations I, Differential Equations, 12, 1988, 1385–1390.

[N2] A.I. Neishtadt, Persistence of stability loss for dynamical bifurcations II, Differential Equations, 2, 1988, 171–176.

[PA] D. Panazzolo, On the existence of canard solutions, Publ. Mat., vol. 44, no. 2, 2000, 503–592.

[PO] L.S. Pontryagin, Asymptotic behaviour of the solutions of systems of differential equations with a small parameter in the higher derivatives, Amer. Math. Soc. Transl. (2), vol. 18, 1961, 295–319.

[RA] J.P. Ramis, Dévissage Gevrey, Astérisque (Soc. Math. France), 59–70, 1978, 173–204.

[SI1] Y. Sibuya, A theorem concerning uniform simplicifation at a transition point and the problem of resonance, SIAM J. Math. Anal. , vol. 2, 1981, 653–668.

[SI2] Y. Sibuya, Linear differential equations in the complex domain, problems of analytic simplification, Trans. Math. Monographs, vol. 82 (Providence, RI: Ann. Math. Soc.), 1990.

[VDP] B. Van Der Pol, On relaxation oscillations, Phil. Mag., vol. 2, 1926, 978–992.

[WA] W. Wasow, Asymptotic expansions for ordinary differential equations, reprint, Dover publications inc, New York, 1987, ISBN 0-486-65456-7.