Faculteit Wetenschappen

Ideals of three dimensional Artin-Schelter regular algebras

Proefschrift voorgelegd tot het behalen van de graad
Doctor in de Wetenschappen, richting Wiskunde
aan de Universiteit Hasselt
te verdedigen door

Koen De Naeghel

Promotor
Prof. dr. M. Van den Bergh

2006

Contents

Acknowledgements

Firstly, there are many people I would like to thank for a variety of professional reasons.

I would like to gratefully acknowledge the enthusiastic supervision of Michel Van den Bergh during this work. He did a perfect job in guiding me through the mathematical landscape, and I would have been lost without him.

Thank-you to the members of the jury, for managing to read the whole thing thoroughly.

I also state my gratitude to S. Paul Smith and Nicolas Marconnet for sharing mathematical ideas with me.

Thanks are due also to several researchers at other organisations for giving me all kinds of possibilities and for their invitations: Jacques Alev, Roland Berger, Bernhard Keller, S. Paul Smith, Freddy Van Oystaeyen. I am grateful for the predoc position funded by the European Research Training Network Liegrits.

Colleagues and professors at Hasselt University are thanked for their care and attention.

Secondly, as I have always tried to separate work and personal life the latter deserves special attention.

It is difficult to overstate my gratitude to friends and special friends, in past and present time, wherever they are. Their influence on me - and therefore also on my thesis - has been immeasurable. However I do not think this is the correct place to list them, moreover it is far to delicate to write them all down. They will be thanked personally.

Lastly, and most importantly, I am forever indebted to my parents. They raised me, supported me, taught me, and loved me. Above all things they had faith in me.

To them I dedicate this thesis.

Introduction

In 1927 Heisenberg discovered that uncertainties, or imprecisions, always turned up if one tried to measure the position and the momentum of a particle at the same time. His 14-page letter to Pauli evolved into a published paper [42] in which Heisenberg presented to the world what became known as the uncertainty principle. In order to study this principle, Weyl introduced the algebra $A_1 = k\langle x, y\rangle/(xy - yx - 1)$ as the k-algebra generated by position and momentum operators x, y. For simplicity k will be an algebraically closed field k of characteristic zero. The noncommutativity of the generators reflects the Heisenberg uncertainty principle.

The algebra A_1 is called the first Weyl algebra, it is the basic example of a noncommutative noetherian domain and has been studied in various papers and books. In particular A_1 is simple i.e. there are no non-trivial two-sided ideals. However there are plenty one-sided ideals of A_1 and it is a natural question to describe them. In 1994 Cannings and Holland [22] classified right A_1-ideals, by means of the adelic Grassmanian. A few years later Wilson [84] found a relation between the adelic Grassmanian and the Calogero-Moser spaces. It turned out [17] that the orbits of the natural action of the automorphism group $\operatorname{Aut}(A_1)$ on the set of right A_1-ideals are indexed by the set of natural numbers $\mathbb{N}$, and the orbit corresponding to $n \in \mathbb{N}$ is in natural bijection with the n-th Calogero-Moser space

$$C_n = \{(\mathbb{X}, \mathbb{Y}) \in M_n(k) \times M_n(k) \mid \operatorname{rank}(\mathbb{YX} - \mathbb{XY} - \mathbb{I}) \leq 1\}/\operatorname{Gl}_n(k),$$

a connected smooth affine variety of dimension $2n$ [84]. On the other hand, in 1995 Le Bruyn [51] proposed an alternative classification method based on noncommutative algebraic geometry. His idea was to consider the homogenized Weyl algebra H, the algebra obtained by adding a third variable z of degree one to A_1 which commutes with x, y, making the relation $xy - yx - 1$ homogeneous. Then A_1 is thought of as the coordinate ring of an open affine part of a noncommutative space $\mathbb{P}^2_{\mathrm{q}} = \operatorname{Proj} H$ in the sense of Artin and Zhang [10] i.e. $\operatorname{Proj} H$ is the quotient of the abelian category of finitely generated graded right H-modules by the Serre subcategory of finite dimensional modules. The problem of describing A_1-ideals then becomes equivalent to describing certain objects on the noncommutative projective plane $\mathbb{P}^2_{\mathrm{q}}$. They may be used to describe moduli spaces, just as in the ordinary commutative case. In 2002 this idea of Le Bruyn was picked up and worked out by Berest and Wilson [16] to prove directly the relation between A_1-ideals and Calogero-Moser spaces C_n.

This work starts from the observation that the algebra H is a so-called three dimensional Artin-Schelter regular algebra, see Chapter 1 for these preliminary definitions. This class of graded algebras was introduced by Artin and Schelter [5] in 1986 and classified a few years later by Artin, Tate and Van den Bergh [7, 8] and Stephenson [72, 73]. They are all noetherian domains of Gelfand-Kirillov dimension three and may be considered as noncommutative analogues of the polynomial ring $k[x, y, z]$. To each of them there is an associated noncommutative projective surface $\operatorname{Proj} A$. Let us further assume A is generated in degree one. It turns out

[5] there are two possibilities for such an algebra A. Either there are three generators x, y, z and three quadratic relations (we say A is quadratic) or two generators x, y and two cubic relations (A is cubic). If A is quadratic then A is Koszul and has Hilbert series $h_A(t) = 1 + 3t + 6t^2 + 10t^3 + \cdots = 1/(1-t)^3$, and we may think of $\mathbb{P}^2_{\mathrm{q}} = \operatorname{Proj} A$ as a noncommutative projective plane. In case A is cubic then $h_A(t) = 1+2t+4t^2+6t^3+\cdots = 1/(1-t)^2(1-t^2)$, we then write $(\mathbb{P}^1 \times \mathbb{P}^1)_q = \operatorname{Proj} A$ which we think of as a noncommutative quadric surface.

The generic class of quadratic and cubic Artin-Schelter algebras are usually called type A-algebras [5], in which case the relations are respectively given by

$$\left\{\begin{array}{l} ayz + bzy + cx^2 = 0 \\ azx + bxz + cy^2 = 0 \\ axy + byx + cz^2 = 0 \end{array}\right. \quad \text{and} \quad \left\{\begin{array}{l} ay^2x + byxy + axy^2 + cx^3 = 0 \\ ax^2y + bxyx + ayx^2 + cy^3 = 0 \end{array}\right. \tag{1}$$

where $a, b, c \in k$ are generic scalars.

It was shown in [7] that a three dimensional Artin-Schelter regular algebra A generated in degree one is completely determined by a triple (E, σ, j) where either

- $j : E \cong \mathbb{P}^2$ if A is quadratic, resp. $j : E \cong \mathbb{P}^1 \times \mathbb{P}^1$ if A is cubic; or
- $j : E \hookrightarrow \mathbb{P}^2$ is an embedding of a divisor E of degree three if A is quadratic, resp. $j : E \hookrightarrow \mathbb{P}^1 \times \mathbb{P}^1$ where E is a divisor of bidegree $(2, 2)$ if A is cubic

and $\sigma \in \operatorname{Aut}(E)$. In the first case we say A is linear, otherwise A is called elliptic. If A is of type A and the divisor E is a smooth elliptic curve (this is the generic case) then we say A is of generic type A. In that case σ is a translation on E. Quadratic three dimensional Artin-Schelter regular algebras of generic type A are also called three dimensional Sklyanin algebras.

For most of our results below we will assume A is elliptic and σ has infinite order. In that case the degree zero part $(A_g)_0$ of the localisation A_g of A at the powers of the canonical normalizing element $g \in A$ is a simple k-algebra [8], which means that the critical modules of Gelfand-Kirillov dimension one are, up to shift of grading, exactly the so-called point modules over A i.e. cyclic graded right A-modules with Hilbert series $1/(1-t) = 1 + t + t^2 + \ldots$. The point modules over A are parameterized by the closed points on E. This will, at least implicitly, be a key ingredient in most of our proofs.

In Chapter 2 of this thesis we generalize the methods used in [51, 16] to obtain

Theorem 1 (Chapter 2). *Assume k is uncountable. Let A be an elliptic quadratic Artin-Schelter algebra for which σ has infinite order. There exist smooth locally closed varieties D_n of dimension $2n$ such that the set $R(A)$ of reflexive graded right A-ideals considered up to isomorphism and shift of grading is in natural bijection with $\coprod_n D_n$.*

In particular D_0 is a point and D_1 is the complement of E under $\mathbb{P}^2$.

In fact D_n is connected, see Theorem 5 below. In the generic case we have

Theorem 2 (Chapter 2). *Let A be a three dimensional Sklyanin algebra for which σ has infinite order. Then the varieties D_n in Theorem 1 are affine.*

Theorem 3 (Chapter 2). *Assume k is uncountable. Let A be an elliptic quadratic Artin-Schelter algebra for which σ has infinite order. Let $I \in R(A)$. Then there exists an $m \in \mathbb{N}$ together with a monomorphism $I(-m) \hookrightarrow A$ such that there is a filtration of reflexive graded right A-modules of rank one*

$$A = I_0 \supset I_1 \supset \cdots \supset I_u = I(-m)$$

with the property that up to finite length modules the quotients are shifted line modules i.e. modules of the form A/aA where $a \in A$ has degree one.

In the generic case where A is a three dimensional Sklyanin algebra we do not need the assumption k is uncountable in Theorem 1 and Theorem 3.

A result similar to Theorem 1 was proved by Nevins and Stafford [60] for all quadratic three dimensional Artin-Schelter regular algebras. They showed that D_n is an open subset in a projective variety $\mathrm{Hilb}_n(\mathbb{P}^2_{\mathrm{q}})$ of dimension $2n$, parameterizing graded right A-ideals of projective dimension one up to isomorphism and shift of grading. Thus $\mathrm{Hilb}_n(\mathbb{P}^2_{\mathrm{q}})$ is the analog of the classical Hilbert scheme of points on $\mathbb{P}^2$.

Our next objective is to determine the possible Hilbert functions of reflexive graded right ideals, or more generally, graded right ideals of projective dimension one. In Chapter 3 we prove that, as in the commutative case, they are related to so-called Castelnuovo functions [26]. These are finitely supported functions $s : \mathbb{N} \to \mathbb{C}$ of the form

$$s(0) = 1, s(1) = 2, \ldots, s(\sigma - 1) = \sigma \text{ and } s(\sigma - 1) \geq s(\sigma) \geq s(\sigma + 1) \geq \cdots \geq 0$$

for some integer $\sigma \geq 0$. We identify s with its generating function $s(t) = \sum_i s(i)t^i$.

Theorem 4 (Chapter 3). *Let A be a quadratic Artin-Schelter algebra. There is a bijective correspondence between Castelnuovo polynomials $s(t)$ of weight $\sum_i s(i) = n$ and Hilbert series $h_I(t)$ of objects I in $\mathrm{Hilb}_n(\mathbb{P}^2_q)$, given by*

$$h_I(t) = \frac{1}{(1-t)^3} - \frac{s(t)}{1-t} \tag{2}$$

Moreover if A is elliptic for which σ has infinite order this correspondence restricts to Hilbert series $h_I(t)$ of reflexive objects I in $\mathrm{Hilb}_n(\mathbb{P}^2_q)$.

The appearing formal power series (2) will be called admissible Hilbert series of degree n. In particular we were able to prove intrinsically

Theorem 5 (Chapter 3). *Let A be a quadratic Artin-Schelter algebra. Then $\mathrm{Hilb}_n(\mathbb{P}^2_q)$ is connected.*

This is done by showing that the admissible Hilbert series of degree n induce a stratification of $\mathrm{Hilb}_n(\mathbb{P}^2_{\mathrm{q}})$ into connected locally closed subvarieties. In the commutative case this was shown by Gotzmann [36]. Furthermore there is a dimension formula for these strata, from which we deduce there is an unique stratum of maximal dimension $2n$, implying $\mathrm{Hilb}_n(\mathbb{P}^2_{\mathrm{q}})$ is connected. The connectedness of $\mathrm{Hilb}_n(\mathbb{P}^2_q)$ was also proved by Nevins and Stafford [60] for almost all A using deformation-theoretic methods and relying on the commutative case $A = k[x, y, z]$. For the Weyl algebra the connectedness of the Calogero-Moser spaces C_n was shown by Wilson [84].

We may take this one step further by describing all possible minimal resolutions for objects in $\mathrm{Hilb}_n(\mathbb{P}^2_{\mathrm{q}})$. We prove a more general result which implies Theorem 4.

Theorem 6 (Chapter 3). *Let A be a quadratic Artin-Schelter algebra. Let $0 \neq q(t) \in \mathbb{Z}[t^{-1}, t]$ be a Laurent polynomial such that $q_\sigma t^\sigma$ is the lowest non-zero term of q. Then there is a torsion free graded right A-module M with Hilbert series $q(t)/(1-t)^3$ and a minimal resolution of the form*

$$0 \to \oplus_i A(-i)^{b_i} \to \oplus_i A(-i)^{a_i} \to M \to 0$$

if and only if $a_l = 0$ for $l < \sigma$, $a_\sigma = q_\sigma > 0$ and $\max(q_l, 0) \leq a_l < \sum_{i \leq l} q_i$ for $l > \sigma$.

In Chapter 4 we proceed the study of the projective variety $\mathrm{Hilb}_n(\mathbb{P}^2_{\mathrm{q}})$. This was realized in collaboration with S. Paul Smith. An object $I \in \mathrm{Hilb}_n(\mathbb{P}^2_{\mathrm{q}})$ is a submodule of its bidual $I^{**} \in R(A)$, and the quotient module I^{**}/I is Cohen-Macaulay of Gelfand-Kirillov dimension one and multiplicity $\leq n$. Thus it is natural to define the subsets

$$\mathrm{Hilb}^d_n(\mathbb{P}^2_{\mathrm{q}}) = \{I \in \mathrm{Hilb}_n(\mathbb{P}^2_{\mathrm{q}}) \mid I^{**}/I \text{ has multiplicity } d\}$$

and $\mathrm{Hilb}^{\geq d}_n(\mathbb{P}^2_{\mathrm{q}}) = \bigcup_{d' \geq d} \mathrm{Hilb}^{d'}_n(\mathbb{P}^2_{\mathrm{q}})$. We prove

Theorem 7 (Chapter 4). *Let A be an elliptic quadratic Artin-Schelter algebra for which σ has infinite order. Let $n \geq 0$ and $0 \leq d \leq n$. We have*

1. $\mathrm{Hilb}^d_n(\mathbb{P}^2_{\mathrm{q}})$ *is non-empty,*
2. $\mathrm{Hilb}^{\geq d}_n(\mathbb{P}^2_{\mathrm{q}}) \subset \mathrm{Hilb}_n(\mathbb{P}^2_{\mathrm{q}})$ *is a projective variety of dimension* $2n - d$.

In particular we observe that for elliptic A for which σ has infinite order the projective variety $\mathrm{Hilb}^n_n(\mathbb{P}^2_{\mathrm{q}}) \subset \mathrm{Hilb}_n(\mathbb{P}^2_{\mathrm{q}})$ of dimension n parameterizes the cyclic Cohen-Macaulay modules of Gelfand-Kirillov dimension one, up to isomorphism and shift of grading.

The first statement of Theorem 7 is shown by proving

Theorem 8 (Chapter 4). *Let A be an elliptic quadratic Artin-Schelter algebra for which σ has infinite order. Let $n \geq 0$ and $0 \leq d \leq n$. Then there is an object $I \in \mathrm{Hilb}^d_n(\mathbb{P}^2_{\mathrm{q}})$ with Hilbert series $h_I(t) = h_A(t) - (1 + t + \cdots + t^{n-1})/(1 - t)$.*

In a next stage of this thesis we are interested in the precise inclusion relation between the closures of the strata in $\mathrm{Hilb}_n(\mathbb{P}^2_q)$. It is therefore natural to study the methods for the classical Hilbert scheme of points $\mathrm{Hilb}_n(\mathbb{P}^2)$. But even in this situation the precise inclusion relation between the closures of the strata are still unknown. And even in the special (and simplest) case where the Hilbert series are as close as possible the inclusion relations were unclear, until in 2002 Guerimand [38] found necessary and sufficient conditions for this special case, preimposing a technical condition. Guerimand used geometrical methods to obtain his results. In Chapter 5 we present a new approach based on deformation theory. We were able to reprove Guerimand's results and furthermore we show that the technical condition is not necessary.

Theorem 9 (Chapter 5). *Assume φ, ψ are admissible Hilbert series of degree n for which $\varphi > \psi$ and such that there are no admissible Hilbert series τ of degree n for which $\varphi > \tau > \psi$. Write H_φ resp. H_ψ for the stratum of $\mathrm{Hilb}_n(\mathbb{P}^2)$ associated to φ resp. ψ. Then there are (known) necessary and sufficient conditions on φ, ψ such that $H_\varphi \subset \overline{H_\psi}$.*

In Chapter 5 we will use the stratification by the series $h_A(t) - \varphi(t)$ rather than the admissible Hilbert series $\varphi(t)$. Our methods seem to extend to the noncommutative situation which we hope to describe in forthcoming work.

Chapter 6, the final chapter of this thesis, was recently accomplished in collaboration with N. Marconnet. We show how the methods used in Chapter 2 also apply for cubic three dimensional Artin-Schelter regular algebras. In particular we provide the following analogons for Theorems 1, 2, 3.

Theorem 10 (Chapter 6). *Assume k is uncountable. Let A be an elliptic cubic Artin-Schelter algebra for which σ has infinite order. Define $N = \{(n_e, n_o) \in \mathbb{N}^2 \mid n_e - (n_e - n_o)^2 \geq 0\}$. Then for any $(n_e, n_o) \in N$ there exists a smooth locally closed variety $D_{(n_e,n_o)}$ of dimension $2(n_e - (n_e - n_o)^2)$ such that the set $R(A)$ of reflexive graded right A-ideals, considered up to isomorphism and shift of grading is in natural bijection with $\coprod_{(n_e,n_o)\in N} D_{(n_e,n_o)}$.*

In particular $D_{(0,0)}$ is a point and $D_{(1,1)}$ is the complement of E under $\mathbb{P}^1 \times \mathbb{P}^1$. We also expect $D_{(n_e,n_o)}$ to be connected, see below.

In the generic case we have

Theorem 11 (Chapter 6). *Let A be a cubic Artin-Schelter algebra of generic type A for which σ has infinite order. Then the varieties $D_{(n_e,n_o)}$ in Theorem 10 are affine.*

Theorem 12 (Chapter 6). *Assume k is uncountable. Let A be an elliptic cubic Artin-Schelter algebra and assume σ has infinite order. Let $I \in R(A)$. Then there exists an $m \in \mathbb{N}$ together with a filtration of reflexive graded right A-modules of rank one*

$$I_0 \supset I_1 \supset \cdots \supset I_u = I(-m)$$

with the property that up to finite length modules the quotients are shifted conic modules i.e. modules of the form A/bA where $b \in A$ has degree two. Moreover I_0 admits a minimal resolution of the form $0 \to A(-c-1)^c \to A(-c)^{c+1} \to I_0 \to 0$ for some integer $c \geq 0$, and I_0 is up to isomorphism uniquely determined by c.

In case A is a cubic Artin-Schelter algebra of generic type A we do not need the assumption k is uncountable in Theorem 10 and Theorem 12.

One delicate step in the proof of Theorem 10 is to show that the varieties $D_{(n_e,n_o)}$ are actually nonempty. We do this by pointing out that Theorem 6 also holds for cubic Artin-Schelter algebras A. Further, to a graded right ideal of projective dimension one we may associate $(n_e, n_o) \in N$. Let $\mathrm{Hilb}_{(n_e,n_o)}((\mathbb{P}^1 \times \mathbb{P}^1)_q)$ denote the set of all such objects, considered up to shift of grading. In case A is linear then this corresponds to the usual Hilbert scheme of points on $\mathbb{P}^1 \times \mathbb{P}^1$. We have

Theorem 13 (Chapter 6). *Let A be a cubic Artin-Schelter algebra. There is a bijective correspondence between Castelnuovo polynomials $s(t)$ of even weight $\sum_i s(2i) = n_e$ and odd weight $\sum_i s(2i+1) = n_o$ and Hilbert series $h_I(t)$ of objects I in $\mathrm{Hilb}_{(n_e,n_o)}((\mathbb{P}^1 \times \mathbb{P}^1)_q)$ given by*

$$h_I(t) = \frac{1}{(1-t)^2(1-t^2)} - \frac{s(t)}{1-t^2}$$

Moreover if A is elliptic for which σ has infinite order this correspondence restricts to Hilbert series $h_I(t)$ of reflexive objects I in $\mathrm{Hilb}_{(n_e,n_o)}((\mathbb{P}^1 \times \mathbb{P}^1)_q)$.

For cubic Artin-Schelter algebras A we expect a similar treatment as in [60] to show that $\mathrm{Hilb}_{(n_e,n_o)}((\mathbb{P}^1 \times \mathbb{P}^1)_q)$ is a smooth projective variety of dimension $2(n_e - (n_e - n_o)^2)$. Furthermore we are quite convinced that using the same methods as in the proof of Theorem 5 will lead to a proof that $\mathrm{Hilb}_{(n_e,n_o)}((\mathbb{P}^1 \times \mathbb{P}^1)_q)$ is connected, hence also $D_{(n_e,n_o)}$ (for elliptic A for which σ has infinite order). We hope to come back on this in further research.

To end with, we may apply our results to the enveloping algebra of the Heisenberg-Lie algebra H_c, the cubic Artin-Schelter algebra of type A for which $(a, b, c) = (1, -2, 0)$ in (1). By [8] the ring of invariants $A_1^{\langle\varphi\rangle}$ of the first Weyl algebra A_1 under the automorphism $\varphi(x) = -x$, $\varphi(y) = -y$ is thought of as the coorinate ring of an open affine part of $\mathrm{Proj}\, H_c$. As in [16] for the first Weyl algebra A_1 we extract

Theorem 14 (Chapter 6). *The set $R(A_1^{\langle\varphi\rangle})$ of isomorphism classes of right $A_1^{\langle\varphi\rangle}$-ideals is in natural bijection with the points of $\coprod_{(n_e,n_o)\in N} D_{(n_e,n_o)}$ where*

$$D_{(n_e,n_o)} = \{(\mathbb{X}, \mathbb{Y}, \mathbb{X}', \mathbb{Y}') \in M_{n_e \times n_o}(k)^2 \times M_{n_o \times n_e}(k)^2 \mid \mathbb{Y}'\mathbb{X} - \mathbb{X}'\mathbb{Y} = \mathbb{I} \text{ and } \mathrm{rank}(\mathbb{Y}\mathbb{X}' - \mathbb{X}\mathbb{Y}' - \mathbb{I}) \leq 1\} / \mathrm{Gl}_{n_e}(k) \times \mathrm{Gl}_{n_o}(k)$$

is a smooth affine variety $D_{(n_e,n_o)}$ of dimension $2(n_e - (n_e - n_o)^2)$.

It would be interesting to see if the orbits of $R(A_1^{\langle\varphi\rangle})$ under the automorphism group $\mathrm{Aut}(A_1^{\langle\varphi\rangle})$ are in bijection to the varieties $D_{(n_e,n_o)}$.

Inleiding

In 1927 ontdekte Heisenberg [42] dat het niet mogelijk is om de plaats en de snelheid (of impuls) van een deeltje tegelijkertijd met onbeperkte nauwkeurigheid te meten. Dit principe is sindsdien bekend als de onzekerheidsrelatie van Heisenberg. Om dit fenomeen nader te bestuderen voerde Weyl de algebra $A_1 = k\langle x, y\rangle/(xy - yx - 1)$ in als de k-algebra voortgebracht door de plaats operator x en impuls operator y. Gemakshalve nemen we aan dat k een algebraisch gesloten veld is van karakteristiek nul. Het niet-commutatief karakter van de relatie $xy - yx - 1$ reflecteert de onzekerheidsrelatie van Heisenberg.

De algebra A_1, de eerste Weyl algebra genoemd, is het standaard voorbeeld van een niet-commutatief noethers domein. In het bijzonder is A_1 een simpele ring i.e. er zijn geen niet-triviale tweezijdige idealen. Er zijn daarintegen wel vele eenzijdige idealen en het is een natuurlijke vraag om die te beschrijven. In 1994 classificeerden Cannings en Holland [22] de rechtse A_1-idealen door middel van de adelische Grassmaniaan. Enkele jaren later vond Wilson [84] een verband tussen de adelische Grassmaniaan en de Calogero-Moser ruimten. Het bleek [17] dat de banen van de natuurlijke actie van de automorfisme groep $\operatorname{Aut}(A_1)$ op de verzameling van rechtse A_1-idealen geïndexeerd zijn door de verzameling van de natuurlijke getallen $\mathbb{N}$, waarbij de baan die overeenkomt met $n \in \mathbb{N}$ in natuurlijke bijectie is met de n-de Calogero-Moser ruimte

$$C_n = \{(\mathbb{X}, \mathbb{Y}) \in M_n(k) \times M_n(k) \mid \operatorname{rang}(\mathbb{Y}\mathbb{X} - \mathbb{X}\mathbb{Y} - \mathbb{I}) \leq 1\}/\operatorname{Gl}_n(k),$$

een samenhangende gladde affiene varieteit van dimensie $2n$ [84]. In 1995 stelde Le Bruyn [51] een alternatieve classificatie methode voor gebaseerd op niet-commutatieve algebraïsche meetkunde. Zijn idee was om de gehomogeniseerde Weyl algebra H te beschouwen, de algebra die men bekomt door aan A_1 een derde voortbrenger z van graad 1 toe te voegen die commuteert met x en y en de relatie $xy - yx - 1$ homogeen maakt. De algebra A_1 is dan beschouwd als de coordinaten ring van een open affien deel van een niet-commutatieve ruimte $\mathbb{P}^2_{\mathrm{q}} = \operatorname{Proj} H$ in de zin van Artin en Zhang [10] i.e. $\operatorname{Proj} H$ is het quotient van de abelse categorie van eindig voortgebrachte gegradeerde rechtse H-modulen met de Serre deelcategorie van de eindig dimensionale modulen. De classificatie van de rechtse A_1-idealen wordt dan equivalent met het beschrijven van zekere objecten op het niet-commutatief projectief vlak $\mathbb{P}^2_{\mathrm{q}}$. Die objecten geven aanleiding tot moduli ruimten, net als in het commutatief geval. In 2002 werd dit idee van Le Bruyn verder uitgewerkt door Berest en Wilson [16] wat aanleiding gaf tot een direct bewijs van het verband tussen de A_1-idealen en Calogero-Moser ruimten C_n.

Dit werk start met de vaststelling dat de algebra H een zogenaamde drie dimensionale Artin-Schelter reguliere algebra is, zie Hoofdstuk 1 voor inleidende definities en resultaten. Deze klasse van gegradeerde algebras werd ingevoerd door Artin en Schelter [5] in 1986 en enkele jaren later geclassificeerd door Artin, Tate en Van den Bergh [7, 8] en Stephenson [72, 73]. Deze algebras zijn allen noetherse domeinen van

Gelfand-Kirillov dimensie drie en worden beschouwd als niet-commutatieve analogons van de veeltermring $k[x, y, z]$. Bij elk van hen hoort een geassocieerd projectief oppervlak. Laat ons verder onderstellen dat A voortgebracht is in graad 1. Dan blijken [5] er twee mogelijkheden te zijn voor zo'n algebra A. Ofwel heeft A drie voortbrengers x, y, z en drie kwadratische relaties (dan zeggen we dat A kwadratisch is) ofwel heeft A twee voortbrengers en twee kubische relaties (A is kubisch). Indien A kwadratisch is dan is A Koszul met als Hilbert reeks $h_A(t) = 1 + 3t + 6t^2 + 10t^3 + \cdots = 1/(1-t)^3$ en we denken aan $\mathbb{P}^2_{\mathrm{q}} = \operatorname{Proj} A$ als een niet-commutatief projectief vlak. In geval A kubisch is dan is $h_A(t) = 1 + 2t + 4t^2 + 6t^3 + \cdots = 1/(1-t)^2(1-t^2)$, we schrijven $(\mathbb{P}^1 \times \mathbb{P}^1)_q = \operatorname{Proj} A$ wat kan beschouwd worden als een niet-commutatief analogon van een kwadriek in $\mathbb{P}^3$.

De generieke klasse van kwadratische en kubische Artin-Schelter reguliere algebras worden type A-algebras genoemd [5] en in dat geval worden de relaties gegeven door respectievelijk

$$\left\{\begin{array}{l} ayz + bzy + cx^2 = 0 \\ azx + bxz + cy^2 = 0 \\ axy + byx + cz^2 = 0 \end{array}\right. \quad \text{en} \quad \left\{\begin{array}{l} ay^2x + byxy + axy^2 + cx^3 = 0 \\ ax^2y + bxyx + ayx^2 + cy^3 = 0 \end{array}\right. \tag{1}$$

waarin $a, b, c \in k$ generiek zijn.

In [7] werd aangetoond dat een drie dimensionale Artin-Schelter reguliere algebra A voortgebracht in graad 1 volledig bepaald is door een drietal (E, σ, j) waarin ofwel

- $j : E \cong \mathbb{P}^2$ als A kwadratisch is, resp. $j : E \cong \mathbb{P}^1 \times \mathbb{P}^1$ als A kubisch is; of
- $j : E \hookrightarrow \mathbb{P}^2$ is een inbedding van een divisor E van graad drie als A kwadratisch is, resp. $j : E \hookrightarrow \mathbb{P}^1 \times \mathbb{P}^1$ waar E een divisor van bigraad $(2, 2)$ als A kubisch is

en $\sigma \in \operatorname{Aut}(E)$. In het eerste geval zeggen we dat A linear is, in het tweede geval noemen we A elliptisch. Indien A van type A is en de divisor E is een gladde elliptische kromme (dit is het generiek geval) dan zeggen we dat A van generiek type A is. In dat geval is σ een translatie op E. Kwadratische drie dimensionale Artin-Schelter reguliere algebras van generiek type A worden ook drie dimensionale Sklyanin algebras genoemd.

Voor de meeste resultaten onderaan zullen we aannemen dat A elliptisch is waarbij de orde van σ oneindig is. In dat geval is het deel van graad nul $(A_g)_0$ van de localisatie A_g van A in de machten van het canoniek normaliserend element $g \in A$ een simpele k-algebra [8], wat betekend dat op een shift na de kritische modulen van Gelfand-Kirillov dimensie een precies de zogenaamde punt modulen zijn over A i.e. cyclische gegradeerde rechtse A-modulen met Hilbert reeks $1/(1-t) = 1 + t + t^2 + \ldots$. De punt modulen over A worden geparameteriseerd door de punten van E. Deze eigenschap wordt in vele bewijzen impliciet gebruikt.

In Hoofdstuk 2 van deze thesis veralgemenen we de werkwijze gebruikt in [51, 16] om het volgend resultaat te bekomen.

Stelling 1 (Hoofdstuk 2). *Onderstel dat k overaftelbaar is. Zij A een elliptische kwadratische Artin-Schelter algebra waarbij de orde van σ oneindig is. Dan bestaan er gladde lokaal gesloten varieteiten D_n van dimensie $2n$ zodat de verzameling $R(A)$ van reflexieve gegradeerde rechtse A-idealen, beschouwd op isomorfie en gegradeerde shift na, in natuurlijke bijectie is met $\coprod_n D_n$.*

In het bijzonder is D_0 een punt en D_1 het complement van E onder $\mathbb{P}^2$.

Uit Stelling 5 zal blijken dat D_n samenhangend is. In het generiek geval hebben we

Stelling 2 (Hoofdstuk 2). *Zij A een drie dimensionale Sklyanin algebra waarbij de orde van σ oneindig is. Dan zijn de varieteiten D_n in Stelling 1 affien.*

Stelling 3 (Hoofdstuk 2). *Onderstel dat k overaftelbaar is. Zij A een elliptische kwadratische Artin-Schelter reguliere algebra waarbij de orde van σ oneindig is. Zij $I \in R(A)$. Dan bestaat er een $m \in \mathbb{N}$ samen met een monomorfisme $I(-m) \hookrightarrow A$ zodat er een filtratie is van reflexieve gegradeerde rechtse A-modulen van rang een*

$$A = I_0 \supset I_1 \supset \cdots \supset I_u = I(-m)$$

met de eigenschap dat op eindig dimensionale modulen na de quotienten op een gegradeerde shift na allen lijn modulen zijn i.e. modulen van de gedaante A/aA waarbij $a \in A$ graad een heeft.

In het generiek geval waarbij A een drie dimensionale Sklyanin algebra is, is de onderstelling dat k overaftelbaar is overbodig in Stelling 1 en Stelling 3.

Een resultaat soortgelijk aan Stelling 1 werd bekomen door Nevins en Stafford [60] voor alle kwadratische drie dimensionale Artin-Schelter algebras. Zij toonden aan dat D_n een open deel is van een projectieve varieteit $\mathrm{Hilb}_n(\mathbb{P}^2_q)$ van dimensie $2n$, die de gegradeerde rechtse A-idealen van projectieve dimensie een parameteriseert, op isomorfie en gegradeerde shift na. Dus $\mathrm{Hilb}_n(\mathbb{P}^2_q)$ is te zien als het analogon van het klassieke Hilbert schema van punten op $\mathbb{P}^2$.

Onze volgende doelstelling is het bepalen van de mogelijke Hilbert functies van reflexieve gegradeerde rechtse idealen, of meer algemeen, gegradeerde rechtse idealen van projectieve dimensie een. In Hoofdstuk 3 tonen we aan dat, net als in het commutatief geval, deze Hilbert functies gerelateerd zijn aan zogenaamde Castelnuovo functies [26]. Dat zijn functies $s : \mathbb{N} \to \mathbb{C}$ met eindige support van de gedaante

$$s(0) = 1, s(1) = 2, \ldots, s(\sigma - 1) = \sigma \text{ and } s(\sigma - 1) \geq s(\sigma) \geq s(\sigma + 1) \geq \cdots \geq 0$$

voor een natuurlijk getal $\sigma \geq 0$. We identificeren s met zijn genererende functie $s(t) = \sum_i s(i)t^i$.

Stelling 4 (Hoofdstuk 3). *Zij A een kwadratische Artin-Schelter algebra. Dan is er een bijectief verband tussen Castelnuovo polynomen $s(t)$ van gewicht $\sum_i s(i) = n$ en Hilbert reeksen $h_I(t)$ van objecten I in $\mathrm{Hilb}_n(\mathbb{P}^2_q)$, gegeven door*

$$h_I(t) = \frac{1}{(1-t)^3} - \frac{s(t)}{1-t} \tag{2}$$

In geval A elliptisch is met de orde van σ oneindig dan restringeert dit bijectief verband tot Hilbert reeksen $h_I(t)$ van reflexieve objecten I in $\mathrm{Hilb}_n(\mathbb{P}^2_q)$.

Formele machtreeksen van de gedaante (2) worden toelaatbare Hilbert reeksen van graad n genoemd. In het bijzonder konden we een intrinsiek bewijs geven van

Stelling 5 (Hoofdstuk 3). *Zij A een kwadratische Artin-Schelter algebra. Dan is* $\mathrm{Hilb}_n(\mathbb{P}^2_q)$ *samenhangend.*

Dit wordt bekomen door aan te tonen dat toelaatbare Hilbert reeksen van graad n aanleiding geven tot een stratificatie van $\mathrm{Hilb}_n(\mathbb{P}^2_{\mathrm{q}})$ in samenhangende lokaal gesloten deelvarieteiten. In het commutatief geval werd dit aangetoond door Gotzmann [36]. Verder is er een dimensie formule voor deze strata, waaruit we afleiden dat er een uniek stratum is met maximale dimensie $2n$ hetgeen impliceert dat $\mathrm{Hilb}_n(\mathbb{P}^2_{\mathrm{q}})$ samenhangend is. De samenhangendheid van $\mathrm{Hilb}_n(\mathbb{P}^2_q)$ werd ook aangetoont door Nevins en Stafford [60] voor bijna alle A gebruik makend van deformatie-theoretische technieken en steundend op het commutatief geval $A = k[x, y, z]$. Voor de Weyl algebra werd de samenhangendheid van de Calogero-Moser ruimten C_n aangetoond door Wilson [84].

We kunnen nu een stap verder gaan door de mogelijke minimale resoluties te beschrijven voor objecten in $\mathrm{Hilb}_n(\mathbb{P}^2_{\mathrm{q}})$. We bewijzen een meer algemeen resultaat waaruit Stelling 4 zal voortvloeien.

Stelling 6 (Hoofdstuk 3). *Zij A een kwadratische Artin-Schelter algebra. Zij* $0 \neq q(t) \in \mathbb{Z}[t^{-1}, t]$ *een Laurent veelterm zodat* $q_\sigma t^\sigma$ *de niet-nul term van minimale graad is in q. Dan is er een torsie vrij gegradeerd rechts A-moduul M met Hilbert reeks* $q(t)/(1-t)^3$ *en een minimale resolutie van de gedaante*

$$0 \to \oplus_i A(-i)^{b_i} \to \oplus_i A(-i)^{a_i} \to M \to 0$$

als en slechts als $a_l = 0$ *voor* $l < \sigma$, $a_\sigma = q_\sigma > 0$ *en* $\max(q_l, 0) \leq a_l < \sum_{i \leq l} q_i$ *voor* $l > \sigma$.

In Hoofdstuk 6 zetten we de studie van de projectieve varieteit $\mathrm{Hilb}_n(\mathbb{P}^2_{\mathrm{q}})$ verder. Deze resultaten werden bekomen in samenwerking met S. Paul Smith. Een object $I \in \mathrm{Hilb}_n(\mathbb{P}^2_{\mathrm{q}})$ is een deelmoduul van zijn biduale $I^{**} \in R(A)$, en het quotient I^{**}/I is Cohen-Macaulay van Gelfand-Kirillov dimension een en multipliciteit $\leq n$. Dus het ligt voor de hand om de volgende deelverzamelingen te beschouwen

$$\mathrm{Hilb}^d_n(\mathbb{P}^2_{\mathrm{q}}) = \{I \in \mathrm{Hilb}_n(\mathbb{P}^2_{\mathrm{q}}) \mid I^{**}/I \text{ heeft multipliciteit } d\}$$

en $\mathrm{Hilb}^{\geq d}_n(\mathbb{P}^2_{\mathrm{q}}) = \bigcup_{d' \geq d} \mathrm{Hilb}^{d'}_n(\mathbb{P}^2_{\mathrm{q}})$. We bekomen

Stelling 7 (Hoofdstuk 4). *Zij A een elliptische kwadratische Artin-Schelter algebra waarbij de orde van* σ *oneindig is. Zij* $n \geq 0$ *en* $0 \leq d \leq n$. *Dan geldt*

1. $\mathrm{Hilb}^d_n(\mathbb{P}^2_{\mathrm{q}})$ *is niet-ledig,*

2. $\mathrm{Hilb}_n^{\geq d}(\mathbb{P}^2_q) \subset \mathrm{Hilb}_n(\mathbb{P}^2_q)$ *is een projectieve varieteit van dimensie* $2n - d$.

In het bijzonder merken we op dat voor elliptische A waarbij σ oneindige orde heeft de projectieve varieteit $\mathrm{Hilb}_n^n(\mathbb{P}^2_q) \subset \mathrm{Hilb}_n(\mathbb{P}^2_q)$ van dimensie n de cyclische Cohen-Macaulay modulen van Gelfand-Kirillov dimensie een parameteriseert, op isomorfie en gegradeerde shift na.

Het eerste deel van Stelling 7 werd aangetoond door middel van

Stelling 8 (Hoofdstuk 4). *Zij A een elliptische kwadratische Artin-Schelter algebra waarbij de orde van σ oneindig is. Zij $n \geq 0$ en $0 \leq d \leq n$. Dan is er een object $I \in \mathrm{Hilb}_n^d(\mathbb{P}^2_q)$ met Hilbert reeks $h_I(t) = h_A(t) - (1 + t + \cdots + t^{n-1})/(1 - t)$.*

In een volgend stadium van deze thesis hebben we interesse in de preciese inclusie relaties tussen de sluitingen van de strata in $\mathrm{Hilb}_n(\mathbb{P}^2_q)$. Het is daarom aangewezen om de gebruikte methoden te bestuderen voor het klassiek Hilbert schema van punten $\mathrm{Hilb}_n(\mathbb{P}^2)$. Maar zelfs in dit geval zijn de precieze inclusie relaties tussen de sluitingen van de strata onbekend. En zelfs in het speciaal (en meest simpel) geval waarbij de Hilbert reeksen zo dicht mogelijk tegen elkaar liggen waren de inclusie relaties onduidelijk, tot in 2002 Guerimand [38] nodig en voldoende voorwaarden vond voor dit speciaal geval, uitgaande van een technische onderstelling. Guerimand gebruikte hiervoor meetkundige technieken. In Hoofdstuk 5 brengen we een nieuwe benadering naar voor, gebaseerd op deformatie theorie. We waren in staat om Guerimand's resultaten te herbewijzen en bovendien toonden we aan dat zijn technische onderstelling overbodig is.

Stelling 9 (Hoofdstuk 5). *Zij φ, ψ toelaatbare Hilbert reeksen van graad n waarbij $\varphi > \psi$ en zodat er geen toelaatbare Hilbert reeksen τ van graad n zijn waarvoor $\varphi > \tau > \psi$. Schrijf H_φ resp. H_ψ voor het stratum van $\mathrm{Hilb}_n(\mathbb{P}^2)$ geassocieerd met φ resp. ψ. Dan zijn er nodige en voldoende voorwaarden bekend op φ, ψ waarvoor $H_\varphi \subset \overline{H_\psi}$.*

In Hoofdstuk 5 zullen we de stratificatie geïnduceerd door de reeksen $h_A(t) - \varphi(t)$ gebruiken in plaats van de toelaatbare Hilbert reeksen $\varphi(t)$. Onze methode lijkt uit te breiden naar het niet-commutatief geval en we hopen dit in verder onderzoek uit te kunnen werken.

Hoofdstuk 6, het laatste hoofdstuk van deze thesis, werd onlangs volbracht in samenwerking met N. Marconnet. We tonen aan dat de werkwijze gebruikt in Hoofdstuk 2 ook toepasbaar is voor kubische drie dimensionale Artin-Schelter reguliere algebras. In het bijzonder tonen we de volgende analogons aan van Stellingen 1, 2, 3.

Stelling 10 (Hoofdstuk 6). *Onderstel dat k overaftelbaar is. Zij A een elliptische kubische Artin-Schelter algebra waarbij de orde van σ oneindig is. Definieer $N = \{(n_e, n_o) \in \mathbb{N}^2 \mid n_e - (n_e - n_o)^2 \geq 0\}$. Dan bestaat er voor elke $(n_e, n_o) \in N$ een gladde lokaal gesloten varieteit $D_{(n_e,n_o)}$ van dimensie $2(n_e-(n_e-n_o)^2)$ zodat de verzameling $R(A)$ van reflexieve gegradeerde rechtse A-idealen, beschouwd op isomorfie en gegradeerde shift na, in natuurlijke bijectie is met $\coprod_{(n_e,n_o)\in N} D_{(n_e,n_o)}$.*

In het bijzonder is $D_{(0,0)}$ een punt en $D_{(1,1)}$ het complement van E onder $\mathbb{P}^1 \times \mathbb{P}^1$. We verwachten dat $D_{(n_e,n_o)}$ samenhangend is, zie verder.

In het generiek geval hebben we

Stelling 11 (Hoofdstuk 6). *Zij A een elliptische kubische Artin-Schelter algebra van generiek type A waarbij de orde van σ oneindig is. Dan zijn de varieteiten $D_{(n_e,n_o)}$ in Stelling 10 affien.*

Stelling 12 (Hoofdstuk 6). *Onderstel dat k overaftelbaar is. Zij A een elliptische kubische Artin-Schelter reguliere algebra waarbij de orde van σ oneindig is. Zij $I \in R(A)$. Dan bestaat er een $m \in \mathbb{N}$ samen met een filtratie van reflexieve gegradeerde rechtse A-modulen van rang 1*

$$A = I_0 \supset I_1 \supset \cdots \supset I_u = I(-m)$$

met de eigenschap dat op eindig dimensionale modulen na de quotienten op een gegradeerde shift na allen kwadratische modulen zijn i.e. modulen van de gedaante A/bA waarbij $b \in A$ graad twee heeft. Verder is een minimale resolutie van I_0 van de gedaante $0 \to A(-c-1)^c \to A(-c)^{c+1} \to I_0 \to 0$ voor een zeker natuurlijk getal c, en I_0 is op isomorfie na uniek bepaald door c.

In het generiek geval waarbij A een elliptische kubische Artin-Schelter algebra is van generiek type A, is de onderstelling dat k overaftelbaar is overbodig in Stelling 10 en Stelling 12.

Een delicate stap in het bewijs van Stelling 10 is aantonen dat de varieteiten $D_{(n_e,n_o)}$ niet ledig zijn. We doen dit door op te merken dat Stelling 6 ook geldt voor kubische Artin-Schelter algebras A. Verder kunnen we aan elk gegradeerd rechts ideaal van projectieve dimensie een paar $(n_e, n_o) \in N$ associeren. Schrijf $\mathrm{Hilb}_{(n_e,n_o)}((\mathbb{P}^1 \times \mathbb{P}^1)_q)$ voor de verzameling van zo'n objecten, beschouwd op isomorfisme en gegradeerde shift na. In geval A linear is komt dit overeen met het klassiek Hilbert schema van punten op $\mathbb{P}^1 \times \mathbb{P}^1$. We hebben

Stelling 13 (Hoofdstuk 6). *Zij A een kubische Artin-Schelter algebra. Dan is er een bijectief verband tussen enerzijds Castelnuovo veeltermen $s(t)$ van even gewicht $\sum_i s(2i) = n_e$ en oneven gewicht $\sum_i s(2i+1) = n_o$ en anderzijds Hilbert reeksen $h_I(t)$ van objecten in $\mathrm{Hilb}_{(n_e,n_o)}((\mathbb{P}^1 \times \mathbb{P}^1)_q)$, gegeven door*

$$h_I(t) = \frac{1}{(1-t)^2(1-t^2)} - \frac{s(t)}{1-t^2}$$

In geval A elliptisch is met de orde van σ oneindig dan restringeert dit bijectief verband tot Hilbert reeksen $h_I(t)$ van reflexieve objecten I in $\mathrm{Hilb}_{(n_e,n_o)}((\mathbb{P}^1 \times \mathbb{P}^1)_q)$.

Voor kubische Artin-Schelter algebras A verwachten we een analoge werkwijze als in [60] om aan te tonen dat $\mathrm{Hilb}_{(n_e,n_o)}((\mathbb{P}^1 \times \mathbb{P}^1)_q)$ een gladde projectieve varieteit is van dimensie $2(n_e - (n_e - n_o)^2)$. Verder zijn we overtuigd dat dezelfde methoden als in het bewijs van Stelling 5 zullen leiden tot een bewijs dat $\mathrm{Hilb}_{(n_e,n_o)}((\mathbb{P}^1 \times \mathbb{P}^1)_q)$ samenhangend is, en dus ook $D_{(n_e,n_o)}$ (voor elliptische A waarvoor σ oneindige orde heeft). We hopen hierop terug te komen in verder onderzoek.

Tenslotte kunnen we onze resultaten toepassen op de enveloping algebra van de Heisenberg-Lie algebra H_c, de kubische Artin-Schelter algebra van type A waarvoor $(a, b, c) = (1, -2, 0)$ in (1). Uit [8] volgt dat we de invarianten ring $A_1^{\langle\varphi\rangle}$ van de eerste Weyl algebra A_1 onder het automorfisme $\varphi(x) = -x$, $\varphi(y) = -y$ kunnen zien als de coordinatenring van een open affien deel van $\mathrm{Proj}\, H_c$. Zoals in [16] voor de eerste Weyl algebra bekomen we nu

Stelling 14 (Hoofdstuk 6). *De verzameling $R(A_1^{\langle\varphi\rangle})$ van rechtse idealen van $A_1^{\langle\varphi\rangle}$ is in natuurlijke bijectie met $\coprod_{(n_e,n_o)\in N} D_{(n_e,n_o)}$ waarbij*

$$D_{(n_e,n_o)} = \{(\mathbb{X}, \mathbb{Y}, \mathbb{X}', \mathbb{Y}') \in M_{n_e \times n_o}(k)^2 \times M_{n_o \times n_e}(k)^2 \mid \mathbb{Y}'\mathbb{X} - \mathbb{X}'\mathbb{Y} = \mathbb{I} \text{ en } \mathrm{rang}(\mathbb{Y}\mathbb{X}' - \mathbb{X}\mathbb{Y}' - \mathbb{I}) \leq 1\} / \mathrm{Gl}_{n_e}(k) \times \mathrm{Gl}_{n_o}(k)$$

gladde affiene varieteiten zijn van dimensie $2(n_e - (n_e - n_o)^2)$.

Het zou interessant zijn om te zien of de banen van $R(A_1^{\langle\varphi\rangle})$ onder de automorfisme groep $\mathrm{Aut}(A_1^{\langle\varphi\rangle})$ in bijectief verband staan met de varieteiten $D_{(n_e,n_o)}$.

Chapter 1

Preliminaries and basic tools

In this first chapter we gather some basic tools and results used along the way. They are collected from [5, 7, 8, 10, 56, 57, 59, 65, 66, 67, 68, 69, 72, 73, 78].

Throughout we work over an algebraically closed field k of characteristic zero.

1.1 Categories

We assume the reader is familiar with generalities on categories and derived categories. The composition of morphisms $f : A \to B$, $g : B \to C$ in any category $\mathcal{C}$ will be written as $gf : A \to C$. We will make the following convention:

Convention 1.1.1. *Whenever* XyUvw$(\cdots)$ *denotes an abelian category then* xyuvw$(\cdots)$ *denotes the full subcategory of* XyUvw$(\cdots)$ *consisting of noetherian objects.*

1.1.1 Grothendieck group and Euler form

Let $\mathcal{C}$ be an abelian category. Write $\mathcal{C}_f$ for the full subcategory of $\mathcal{C}$ consisting of the noetherian objects. We say $\mathcal{C}$ has *finite global dimension* if there exists an n such that $\operatorname{Ext}^i_{\mathcal{C}}(A, B) = 0$ for all $A, B \in \mathcal{C}$ and for all $i > n$. The minimal such n is called the *global dimension* of $\mathcal{C}$, denoted by $\operatorname{gldim}\mathcal{C}$. Two objects A, B in $\mathcal{C}$ are *perpendicular*, denoted by $A \perp B$, if $\operatorname{Hom}_{\mathcal{C}}(A, B) = \operatorname{Ext}^1_{\mathcal{C}}(A, B) = 0$. For an object $B \in \mathcal{C}_f$ we define ${}^{\perp}B$ as the full subcategory of $\mathcal{C}_f$ which objects are

$$ {}^{\perp}B = \{A \in \mathcal{C}_f \mid A \perp B\} $$

A k-linear abelian category $\mathcal{C}$ is said to be *Ext-finite* if for all objects $A, B \in \mathcal{C}$ and $i \geq 0$ the k-vector space $\operatorname{Ext}^i_{\mathcal{C}}(A, B)$ is finite dimensional.

The *Grothendieck group* $K_0(\mathcal{C})$ of an abelian category $\mathcal{C}$ is the abelian group generated by all objects of $\mathcal{C}$ (we write $[A] \in K_0(\mathcal{C})$ for $A \in \mathcal{C}$) and for which we define

$[A] - [B] + [C] = 0$ whenever there is an exact sequence $0 \to A \to B \to C \to 0$ in $\mathcal{C}$. Assume furthermore $\mathcal{C}$ is k-linear and Ext-finite with finite global dimension. It is easy to see the following map defines a bilinear form on the Grothendieck group

$$\chi : K_0(\mathcal{C}) \times K_0(\mathcal{C}) \to \mathbb{Z}$$
$$([A],[B]) \mapsto \chi(A,B) = \sum_i (-1)^i \dim_k \operatorname{Ext}^i_{\mathcal{C}}(A,B)$$

which we call the *Euler form* for $\mathcal{C}$. We write $\chi(A,B) = \chi([A],[B])$.

1.1.2 Derived categories

To simplify notations we often use implicitly the following result

Lemma 1.1.2. *Assume $\mathcal{C}$ is a locally noetherian category. Then the natural map $D^b(\mathcal{C}_f) \to D^b_{\mathcal{C}_f}(\mathcal{C})$ is an equivalence of categories.*

Proof. This follows for example from the dual of [46, 1.7.11]. □

We will also need

Lemma 1.1.3. *Let Y be a variety of finite type over k, and let $\mathcal{M} \in D^b(\operatorname{coh}(Y))$. If $\operatorname{RHom}_Y(\mathcal{M}, \mathcal{O}_p) = k$ for all $p \in Y$ then $\mathcal{M} \in \operatorname{coh}(X)$ is a line bundle on X.*

Proof. It is clear that for all $p \in Y$

$$\operatorname{RHom}_Y(\mathcal{M}, \mathcal{O}_p) = \operatorname{RHom}_{\mathcal{O}_{Y,p}}(\mathcal{M}_p, k).$$

Replacing $\mathcal{M}_p$ by a minimal resolution $P^\cdot$ we see the differentials in $\operatorname{Hom}_{\mathcal{O}_{Y,p}}(P^\cdot, k)$ are all zero. Therefore $\operatorname{Hom}_{\mathcal{O}_{Y,p}}(P^\cdot, k) = k$ implies $\mathcal{M}_p = \mathcal{O}_{Y,p}$ for all $p \in Y$. This means $\mathcal{M}$ is locally free of rank one, proving what we want. □

1.2 Algebras and modules

We will assume the reader is familiar with basic definitions and results on algebras.

Let A be a k-algebra. We write $\operatorname{Mod}(A)$ for the category of right A-modules, and we set $\operatorname{gldim} A = \operatorname{gldim} \operatorname{Mod}(A)$. For a right A-module M its dual $M^* = \operatorname{Hom}_A(M, A)$ is a left A-module, and M is called *reflexive* if $M^{**} = M$. Recall a right A-module M is *a-torsion free* for some $a \in A$ if a acts faithfully on M i.e. if no non-zero element in M is annihilated by a. We say M is *a-torsion* if $Ma = 0$. We refer to [50] and [56, 8.1] for the definition of the Gelfand-Kirillov dimension (GK-dimension) of finitely generated modules over A.

1.3 Quivers

A *quiver* $Q = (Q_0, Q_1, h, t)$ is a quadruple consisting of a set of *vertices* Q_0, a set of *arrows* Q_1 between those vertices and maps $t, h : Q_1 \to Q_0$ which assign to each arrow its starting (tail) and terminating (head) vertex

$$\underset{t(a)}{\bullet} \xrightarrow{\;a\;} \underset{h(a)}{\bullet}$$

We say Q is *finite* if both Q_0 and Q_1 are finite sets. A *path* in Q is a sequence of arrows $p = a_l \dots a_1$ where $h(a_i) = t(a_{i+1})$ for all i. We define $t(p) = t(a_1)$, $h(p) = h(a_l)$. For each $v \in Q_0$ there is a trivial path at v, denoted by e_v, with $h(e_v) = t(e_v) = v$. A path p in Q is called an (*oriented*) *cycle* if it is not a trivial path e_v and $h(p) = t(p)$. Given two paths p and q in Q their composition pq is defined if $t(p) = h(q)$ in which case it is obtained by concatenating the paths p and q. The *path algebra* kQ of Q is defined to be the k-vector space with basis consisting of all paths in Q. The product of two paths is defined to be their composition pq if it exists and 0 otherwise. It is easy to see that the algebra kQ is finite dimensional over k if and only if Q has no oriented cycles.

Let Q be a quiver. An element $r = \sum_i \lambda_i p_i \in kQ$ (where $\lambda_i \in k$ and p_i path in Q) is called *admissible* if, for all i, $h(p_i) = v$ and $t(p_i) = w$ for some $v, w \in Q_0$. A *quiver with relations* is a couple (Q, R) where Q is a quiver and R is a subset of kQ consisting of admissible elements. An *admissible* ideal of kQ is an ideal which is generated by admissible elements of kQ. By a theorem of Gabriel [32] any basic finite dimensional k-algebra A isomorphic to kQ/I where Q is a finite quiver and I is an admissible ideal of the path algebra kQ.

A *representation* F of a quiver Q (with relations R) assigns to each vertex $v \in Q_0$ a linear space F_v and to each arrow $a \in Q_1$ a linear map $F(a) : F_{t(a)} \to F_{h(a)}$, such that for all $r = \sum_i \lambda_i p_i \in R$ we have $\sum_i \lambda_i F(p_i) = 0$. Here $F(p) = F(a_l) \dots F(a_1)$ for any path $p = a_l \dots a_1$ in Q. Thus representations of Q are always assumed to satisfy the relations R of the quiver Q. If F and G are representations then a *morphism* $\tau : F \to G$ is a collection of linear maps $\tau(v) : F_v \to G_v$ for each $v \in Q_0$ such that, for all $a \in Q_1$, $\tau(h(a))F(a) = G(a)\tau(t(a))$. We write $\mathrm{Hom}_Q(F, G)$ for all morphisms from F to G and $\mathrm{Mod}(Q)$ for the category of representations, which is an abelian category. It is equivalent with $\mathrm{Mod}(kQ/(R))$. We will identify a representation with its corresponding $kQ/(R)$-module. The *dimension vector* of $F \in \mathrm{mod}(Q)$ is $\underline{\dim} F = (\dim_k F_v)_{v \in Q_0} \in \mathbb{Z}^{Q_0}$. A *dimension vector* of Q is an integer sequence $\alpha \in \mathbb{Z}^{Q_0}$.

Let Q be a quiver and α a dimension vector of Q. Define the affine space

$$\mathrm{Rep}_\alpha(Q) = \prod_{a \in Q_1} M_{\alpha_{h(a)} \times \alpha_{t(a)}}(k)$$

where $M_{m\times n}(k)$ is the linear space of $m \times n$ matrices over k. If $m = n$ we sometimes write $M_n(k)$. Also define $\mathrm{Gl}_\alpha(k) = \prod_{v\in Q_0} \mathrm{Gl}_{\alpha_v}(k)$ where $\mathrm{Gl}_n(k)$ stands for the general linear group of $n\times n$ matrices over k. A point of $\mathrm{Rep}_\alpha(Q)$ defines a representation of Q of dimension vector α in a natural way. The isomorphism class of representations of Q of dimension vector α are in one-one correspondence with the orbits of the group $\mathrm{Gl}_\alpha(k)$ acting on $\mathrm{Rep}_\alpha(Q)$ by conjugation.

Let Q be a quiver (with relations R). For $v \in Q_0$ we write S_v for the associated simple representation. Thus $\dim S_v = (\delta_{vv'})_{v'\in Q_0}$. Write $K_0(Q)$ for the Grothendieck group $K_0(\mathrm{mod}(Q))$ of $\mathrm{mod}(Q)$. Since $\underline{\dim}(-)$ is exact on short exact sequences, it extends to a group morphism

$$\varphi : K_0(Q) \to \mathbb{Z}^{Q_0}$$

The image of $\{S_v\}_{v\in Q_0}$ under φ is a $\mathbb{Z}$-module basis of $\mathbb{Z}^{Q_0}$ hence φ is an isomorphism and $\{S_v\}_{v\in Q_0}$ is a $\mathbb{Z}$-module basis for $K_0(Q)$. In what folows we will often identify $K_0(Q) = \mathbb{Z}^{Q_0}$ and view the Euler form χ for $\mathrm{mod}(Q)$ as a bilinear form on $\mathbb{Z}^{Q_0}$.

For $v \in Q_0$ we write P_v for the projective module $e_v kQ/(R)$ in $\mathrm{mod}(Q)$. For any representation F of Q we have $\mathrm{Hom}_Q(P_v, F) = F(v)$ hence $\mathrm{Hom}_Q(P_v, P_w) = e_w kQ/(R)e_v$, the vector space spanned by the paths p in Q having $h(p) = v$ and $t(p) = w$. A basic result is that the category of finitely generated projective $kQ/(R)$-modules is equivalent to the additive category generated by the $(P_v)_{v\in Q_0}$.

Let Q be a quiver without oriented cycles and let $\theta \in \mathbb{Z}^{Q_0}$ be a dimension vector. A representation F of Q is called *θ-semistable* (resp. *stable*) if $\theta \cdot \underline{\dim}F = 0$ and $\theta \cdot \underline{\dim}N \geq 0$ (resp. > 0) for every non-trivial subrepresentation N of F. Here we denote "$\cdot$" for the standard scalar product on $\mathbb{Z}^{Q_0}$: $(\alpha_v)_v \cdot (\beta_v)_v = \sum_v \alpha_v\beta_v$.

The full subcategory of θ-semistable representations of Q forms an exact abelian subcategory of $\mathrm{mod}(Q)$ in which the simple objects are precisely the stable representations. For more details we refer to [47].

It is a fundamental fact [65, Corollary 1.1] that F is semistable for some θ if and only there exists $G \in \mathrm{mod}(Q)$ for which $F \perp G$. The relation between θ and $\underline{\dim}G$ is such that the forms $-\cdot\theta$ and $\chi(-, \underline{\dim}G)$ are proportional. Associated to $G \in \mathrm{mod}(Q)$ there is a semi-invariant function ϕ_G on $\mathrm{Rep}_\alpha(Q)$ such that the set

$$\{F \in \mathrm{Rep}_\alpha(Q) \mid F \perp G\} \tag{1.1}$$

coincides with $\{\phi_G \neq 0\}$. In particular (1.1) is affine.

1.4 Graded algebras and modules

Let $A = \oplus_{i\in\mathbb{Z}} A_i$ be a $\mathbb{Z}$-graded k-algebra. We say A is *connected* if in addition $A_i = 0$ for all $i < 0$ and $A_0 = k$. Any graded connected noetherian k-algebra A is *locally finite*, i.e. $\dim_k A_i < \infty$, for all $i \in \mathbb{Z}$.

We write $\mathrm{GrMod}(A)$ for the category of graded right A-modules with morphisms the A-module homomorphisms of degree zero. Since $\mathrm{GrMod}(A)$ is an abelian category with enough injective objects we may define the functors $\mathrm{Ext}^n_A(M,-)$ on $\mathrm{GrMod}(A)$ as the right derived functors of $\mathrm{Hom}_A(M,-)$. It is convenient to write (for $n \geq 0$)

$$\underline{\mathrm{Ext}}^n_A(M,N) := \bigoplus_{d\in\mathbb{Z}} \mathrm{Ext}^n_A(M,N(d))$$

whence $\underline{\mathrm{Ext}}^n_A(M,-)$ are the right derived functors of $\underline{\mathrm{Ext}}^0_A(M,-) := \underline{\mathrm{Hom}}_A(M,-)$, for $n \geq 1$.

A graded right A-module M is a *graded right ideal* in A if $M \subset A$ i.e. $M_i \subset A_i$ for all i. Let M be a graded right A-module. We use the notation (for all $n \in \mathbb{Z}$) $M_{\geq n} = \oplus_{d\geq n} M_d$ and $M_{\leq n} = \oplus_{d\leq n} M_d$. We say M is *left* (resp. *right*) *bounded* if $M_{\leq n} = 0$ (resp. $M_{\geq n} = 0$) for some $n \in \mathbb{Z}$. For any integer n, define $M(n)$ as the graded A-module equal to M with its original A action, but which is graded by $M(n)_i = M_{n+i}$. We refer to the functor $M \mapsto M(n)$ as the n-th *shift functor*. If A is connected note that $k = A/A_{\geq 1}$ is both a graded left and a graded right A-module, concentrated in degree zero. We write ${}_Ak$ resp. k_A if we want to stress the left resp. right A-module structure of k.

The k-dual of a k-vector space V is $V' = \mathrm{Hom}_k(V,k)$. The graded dual of a graded right A-module M is $M^* = \underline{\mathrm{Hom}}_A(M,A)$ and M is said to be *reflexive* if $M^{**} = M$. We also write $(-)'$ for the functor on graded k-vector spaces which sends M to its Matlis dual

$$M' = \underline{\mathrm{Hom}}_k(M,k) = \oplus_n (M_{-n})'$$

1.5 Tails

Let A be a noetherian connected graded k-algebra. We denote by τ the functor which sends a graded right A-module to the sum of all its finite dimensional submodules. Denote by $\mathrm{Tors}(A)$ the full subcategory of $\mathrm{GrMod}(A)$ consisting of all modules M such that $\tau M = M$ and write $\mathrm{Tails}(A)$ for the quotient category $\mathrm{GrMod}(A)/\mathrm{Tors}(A)$. We write $\pi : \mathrm{GrMod}(A) \to \mathrm{Tails}(A)$ for the (exact) quotient functor. By localization theory [70] π has a right adjoint which we denote by ω. It is well-known that $\pi \circ \omega = \mathrm{id}$. The object πA in $\mathrm{Tails}(A)$ will be denoted by $\mathcal{O}$ and it is easy to see $\omega = \underline{\mathrm{Hom}}_{\mathrm{Tails}(A)}(\mathcal{O},-)$. Objects in $\mathrm{Tails}(A)$ will be denoted by script letters like $\mathcal{M}$.

The shift functor induces an automorphism $\mathrm{sh} : \mathcal{M} \mapsto \mathcal{M}(1)$ on $\mathrm{Tails}(A)$ which we also call the shift functor (in analogy with algebraic geometry it should perhaps be called the “twist” functor).

When there is no possible confusion we write Hom instead of Hom_A and $\mathrm{Hom}_{\mathrm{Tails}(A)}$. The context will make clear in which category we work.

If $\mathcal{M} \in \mathrm{Tails}(A)$ then $\mathrm{Hom}(\mathcal{M},-)$ is left exact and since $\mathrm{Tails}(A)$ has enough injectives [10] we may define its right derived functors $\mathrm{Ext}^n(\mathcal{M},-)$. We also use the notation

$$\underline{\mathrm{Ext}}^n(\mathcal{M},\mathcal{N}) := \bigoplus_{d\in\mathbb{Z}} \mathrm{Ext}^n(\mathcal{M},\mathcal{N}(d))$$

and we set $\underline{\mathrm{Hom}}(\mathcal{M},\mathcal{N}) = \underline{\mathrm{Ext}}^0(\mathcal{M},\mathcal{N})$.

Convention 1.1.1 fixes the meaning of grmod(A), tors(A) and tails(A). It is easy to see tors(A) consists of the finite dimensional graded A-modules. Furthermore $\mathrm{tails}(A) = \mathrm{grmod}(A)/\mathrm{tors}(A)$.

If M is finitely generated and N is arbitrary we have

$$\underline{\mathrm{Ext}}^n(\pi M, \pi N) \cong \varinjlim \underline{\mathrm{Ext}}^n_A(M_{\geq m}, N). \tag{1.2}$$

If M and N are both finitely generated, then (1.2) implies

$$\pi M \cong \pi N \text{ in } \mathrm{tails}(A) \quad \Leftrightarrow \quad M_{\geq n} \cong N_{\geq n} \text{ in } \mathrm{grmod}(A) \text{ for some } n \in \mathbb{Z}$$

explaining the word "tails". The right derived functors of τ are given by

$$R^i\tau = \varinjlim \underline{\mathrm{Ext}}^i_A(A/A_{\geq n}, -)$$

and for $M \in \mathrm{GrMod}(A)$ there is an exact sequence (see [10], Proposition 7.2)

$$0 \to \tau M \to M \to \omega\pi M \to R^1\tau M \to 0. \tag{1.3}$$

We say A satisfies condition χ if $\dim_k \mathrm{Ext}^j_A(k, M) < \infty$ for all j and all $M \in \mathrm{grmod}(A)$. In case A satisfies condition χ then for every $M \in \mathrm{grmod}(A)$ the cokernel of the map $M \to \omega\pi M$ in the exact sequence (1.3) is right bounded. In particular, for $M \in \mathrm{grmod}(A)$ we have $M_{\geq d} \cong (\omega\pi M)_{\geq d}$ for some d.

Every graded quotient of a polynomial ring satisfies condition χ and so do most noncommutative algebras of importance. The condition is essential to get a theory for noncommutative schemes which resembles the commutative theory.

Proposition 1.5.1. *[10] Let A be a right noetherian connected k-algebra satisfying condition χ. Then $\mathrm{Ext}^j(\mathcal{M},\mathcal{N})$ is finite dimensional for all j and all $\mathcal{M},\mathcal{N} \in \mathrm{tails}(A)$.*

1.6 Projective schemes

Let A be a noetherian graded k-algebra. As suggested by Artin and Zhang [10] we define the (polarized) projective scheme $\mathrm{Proj}\, A$ of A as the triple $(\mathrm{Tails}(A), \mathcal{O}, \mathrm{sh})$. In analogy with classical projetive schemes we shall refer to the objects of tails(A) (resp. Tails(A)) as the *coherent* (resp. *quasicoherent*) sheaves on $X = \mathrm{Proj}\, A$, even when A is not commutative, and we shall use the notation $\mathrm{coh}(X) := \mathrm{tails}(A)$, $\mathrm{Qcoh}(X) := \mathrm{Tails}(A)$. By analogy we sometimes write $\mathcal{O}_X = \mathcal{O} = \pi A$. We write $\mathrm{Ext}^i_X(\mathcal{M},\mathcal{N})$ instead of $\mathrm{Ext}^i_{\mathrm{Tails}(A)}(\mathcal{M},\mathcal{N})$.

The following definitions agree with the classical ones for projective schemes.

If $\mathcal{M}$ is be a quasicoherent sheaf on $X = \mathrm{Proj}\, A$, we define the *cohomology groups* of $\mathcal{M}$ by

$$H^n(X, \mathcal{M}) := \mathrm{Ext}^n_X(\mathcal{O}_X, \mathcal{M}).$$

We refer to the graded right A-modules

$$\underline{H}^n(X, \mathcal{M}) := \bigoplus_{d \in \mathbb{Z}} H^n(X, \mathcal{M}(d))$$

as the *full cohomology modules* of $\mathcal{M}$. Finally, we mention the *cohomological dimension* of τ

$$\operatorname{cd} \tau := \max\{n \in \mathbb{N} \mid R^n\tau(-) \neq 0\}$$

and the *cohomological dimension* of X

$$\operatorname{cd} X := \max\{n \in \mathbb{N} \mid H^n(X, -) \neq 0\}.$$

It is easy to prove

$$\operatorname{cd} X = \max(0, \operatorname{cd} \tau - 1).$$

1.7 Hilbert series

The *Hilbert series* of a graded k-vector space V having finite dimensional components is the formal series

$$h_V(t) = \sum_{i=-\infty}^{+\infty} (\dim_k V_i) t^i \in \mathbb{Z}((t)).$$

Let A be a noetherian connected graded k-algebra. Then the Hilbert series $h_M(t)$ of $M \in \operatorname{grmod}(A)$ makes sense since A is right noetherian. Note $h_k(t) = 1$, $h_{M(l)}(t) = t^{-l} h_M(t)$ and $h_{M'} t = h_M(t^{-1})$.

Assume further A has finite global dimension. We denote by $\operatorname{pd} M$ the projective dimension of M. Given a projective resolution of $M \neq 0$

$$0 \to P^r \to \ldots \to P^1 \to P^0 \to M \to 0$$

we have

$$h_M(t) = \sum_{i=0}^{r} (-1)^i h_{P^i}(t).$$

Since A is connected, left bounded graded right A-modules are projective if and only if they are free hence isomorphic to a sum of shifts of A. So if we write

$$P^i = \bigoplus_{j=0}^{r_i} A(-l_{ij})$$

we obtain

$$h_M(t) = \sum_{i=0}^{r} (-1)^i h_{\oplus_{j=0}^{r_i} A(-l_{ij})}(t) = \underbrace{\sum_{i=0}^{r} (-1)^i \sum_{j=0}^{r_i} t^{l_{ij}}}_{q_M(t)} h_A(t)$$

where $q_M(t)$ is the so-called *characteristic polynomial of* M. Thus we have the formula

$$q_M(t) = h_M(t)h_A(t)^{-1} \tag{1.4}$$

Note $q_{M(l)} = t^{-l}q_M(t)$, $q_A(t) = 1$ and $q_k(t) = h_A(t)^{-1}$.

Put $X = \operatorname{Proj} A$. We will write $K_0(X)$ for the Grothendieck group $K_0(\mathrm{coh}(X))$ of $\mathrm{coh}(X)$. The shift functor on $\mathrm{coh}(X)$ induces a group automorphism

$$\mathrm{sh} : K_0(X) \to K_0(X) : [\mathcal{M}] \mapsto [\mathcal{M}(1)]$$

We may view $K_0(X)$ as a $\mathbb{Z}[t, t^{-1}]$-module with t acting as the shift functor sh^{-1}. In [57] it was shown how $K_0(X)$ may be described in terms of Hilbert series.

Theorem 1.7.1. *[57, Theorem 2.3] Let A be a noetherian connected graded k-algebra of finite global dimension and set $X = \operatorname{Proj} A$. Then there is an isomorphism of $\mathbb{Z}[t, t^{-1}]$-modules*

$$\begin{aligned} \theta : K_0(X) &\xrightarrow{\cong} \mathbb{Z}[t, t^{-1}]\,/(q_k(t)) \\ [\mathcal{M}] &\mapsto \overline{q_M(t)} \qquad \textit{where } M \in \mathrm{grmod}(A),\ \mathcal{M} = \pi M. \end{aligned} \tag{1.5}$$

In particular, $[\mathcal{O}(n)]$ is sent to t^{-n}.

1.8 Artin-Schelter regular algebras

Now we come to the definition of regular algebras, introduced by Artin and Schelter [5] in 1986. They may be considered as noncommutative analogues of polynomial rings.

Definition 1.8.1. [5] A connected graded k-algebra A is called an *Artin-Schelter regular algebra of dimension d* if it has the following properties:

(i) A has finite global dimension d;

(ii) A has polynomial growth i.e. there are positive real numbers c, e such that $\dim_k A_n \leq cn^e$ for all positive integers n;

(iii) A is Gorenstein, meaning there is an integer l such that

$$\underline{\mathrm{Ext}}^i_A(k_A, A) \cong \begin{cases} {}_Ak(l) & \text{if } i = d, \\ 0 & \text{otherwise.} \end{cases}$$

where l is called the *Gorenstein parameter* of A.

It is easy to see the Gorenstein parameter l is equal to the degree of $q_k(t)$.

If A is commutative then the condition (i) already implies A is isomorphic to a polynomial ring $k[x_1, \ldots, x_n]$ with some positive grading, if the grading is standard then $n = l$.

The Gorenstein property determines the full cohomology modules of $\mathcal{O}$.

Theorem 1.8.2. *[10] Let A be a noetherian Artin-Schelter regular algebra of dimension $d = n+1$, and let $X = \operatorname{Proj} A$. Let l denote the Gorenstein parameter of A. Then $\operatorname{cd} X = n$, and the full cohomology modules of $\mathcal{O} = \pi A$ are given by*

$$\underline{H}^i(X, \mathcal{O}) \cong \begin{cases} A & \text{if } i = 0 \\ 0 & \text{if } i \neq 0, n \\ A'(l) & \text{if } i = n \end{cases}$$

The following questions for an Artin-Schelter regular algebra A of dimension d are still open in general.

1. Is $e+1 = d$, where e is the minimal choice in Definition 1.8.1(ii)? Or equivalently, is $\operatorname{GKdim} A = \operatorname{gldim} A$?

2. Is A a domain?

3. Is A noetherian?

The ultimate objective is of course to classify all Artin-Schelter regular algebras of dimension d. At this moment this is still unknown for $d \geq 4$, but completely solved for $d \leq 3$

- If $d = 1$ then $A \cong k[x]$.

- If $d = 2$ then [71, Lemma 2.2.5] A is either isomorphic to

  $$k\langle x, y\rangle \,/(ax^2 + byx + cxy + dy^2) \quad \text{where } a, b, c, d \in k \text{ and } ad - bc \neq 0$$

  (in this case $\deg x = \deg y > 0$) or A is isomorphic to the skew polynomial ring $k[x][y; \sigma, \delta]$ where σ is a graded algebra morphism of $k[x]$ and δ is a graded σ-derivation (then $\deg y > \deg x > 0$).
  If we restrict to the case where A is generated in degree one then A is either isomorphic to a so-called quantum plane

  $$k\langle x, y\rangle \,/(yx - \lambda xy) \quad \text{where } \lambda \in k \setminus \{0\}$$

  or to the Jordan quantum plane

  $$k\langle x, y\rangle \,/(x^2 - yx + xy)$$

  and the category $\operatorname{GrMod}(A)$ is equivalent with $\operatorname{GrMod}(k[x, y])$, see [87].

- If $d = 3$ then there also exists a complete classification for Artin-Schelter regular algebras of dimension three [5, 7, 8, 72, 73]. They are all left and right noetherian domains with Hilbert series of a weighted polynomial ring $k[x, y, z]$.

The significance of conditions (i) and (ii) in Definition 1.8.1 is shown in the following examples.

Example 1.8.3. The algebra $A = k\langle x, y\rangle/(yx)$ is *not* an Artin-Schelter regular algebra. Although it has global dimension two and polynomial growth (even $\operatorname{GKdim} A = 2$), it does *not* satisfy the Gorenstein condition since $\underline{\operatorname{Ext}}^1_A(k_A, A) \neq 0$. This algebra is also the only graded algebra of global dimension two and GK-dimension two which is not noetherian [5].

Example 1.8.4 ([74]). The algebra $A = k\langle x, y, z\rangle/(x^2 + y^2 + z^2)$ is *not* an Artin-Schelter regular algebra. It has global dimension two and satisfies the Gorenstein condition, but it is not noetherian. By [74, Theorem 1.2] this implies A does not have polynomial growth.

1.9 Three dimensional Artin-Schelter algebras

In this manuscript we will restrict ourselves to three dimensional Artin-Schelter regular algebras A which are generated in degree one. In this section A will be such an algebra. As proved in [5] there are two possibilities.

- k_A has a minimal resolution of the form

$$0 \to A(-3) \to A(-2)^3 \to A(-1)^3 \to A \to k_A \to 0$$

  Thus A has three generators and three defining homogeneous relations in degree two. Hence A is Koszul and the Gorenstein parameter is $l = 3$. The Hilbert series of A is given by $h_A(t) = (1-t)^3$, i.e.

$$\dim_k A_n = \frac{(n+1)(n+2)}{2} \qquad \text{for } n \geq 0$$

  which is the same as the Hilbert series of the commutative polynomial algebra $k[x, y, z]$ with standard grading. Such algebras A are called *quantum polynomial rings in three variables*. Since the relations have degree two we also refer to these algebras as *quadratic Artin-Schelter algebras*. The corresponding $\operatorname{Proj} A$ will be called a *quantum projective plane*, denoted by $\mathbb{P}^2_q$.

- k_A has a minimal resolution of the form

$$0 \to A(-4) \to A(-3)^2 \to A(-1)^2 \to A \to k_A \to 0 \tag{1.6}$$

  Now A has two generators and two defining homogeneous relations in degree three. The Gorenstein parameter is $l = 4$. We deduce

$$h_A(t) = \frac{1}{(1-t)^2(1-t^2)}$$

  which means

$$\dim_k A_n = \begin{cases} (n+2)^2/4 & \text{if } n \geq 0 \text{ even} \\ (n+1)(n+3)/4 & \text{if } n \geq 0 \text{ odd} \end{cases}$$

which is the same as the Hilbert series of the commutative polynomial algebra $k[x, y, z]$ with grading $\deg x = \deg y = 1$, $\deg z = 2$. We refer to these algebras as *cubic Artin-Schelter algebras*. The Hilbert series of the Veronese subalgebra $A^{(2)} = k \oplus A_2 \oplus A_4 \oplus \ldots$ of A is the same as the Hilbert series of the commutative ring

$$k[x_0, x_1, x_2, x_3] / (x_0 x_1 - x_2 x_3)$$

which is the homogeneous coordinate ring of a quadratic surface (quadric) in $\mathbb{P}^3$. Since [81] $\mathrm{Tails}(A) \cong \mathrm{Tails}(A^{(2)})$ and $\mathrm{Proj}\, A$ has cohomological dimension two we should think of $\mathrm{Proj}\, A$ as a *quantum quadric*, a noncommutative analogue of the quadric surface $\mathbb{P}^1 \times \mathbb{P}^1$. We sometimes denote $\mathrm{Proj}\, A = \left(\mathbb{P}^1 \times \mathbb{P}^1\right)_{\mathrm{q}}$.

1.9.1 Examples

Example 1.9.1. The commutative polynomial ring in three variables $k[x, y, z]$ with standard grading is a quadratic Artin-Schelter algebra, and $\mathrm{Proj}\, A = \mathbb{P}^2$. In contrast, the weighted polynomial ring $k[x, y, z]$ where $\deg x = \deg y = 1$, $\deg z = 2$ is neither a quadratic nor a cubic Artin-Schelter algebra since it is not generated in degree one.

Example 1.9.2. Other standard examples are provided from homogenizations of the first Weyl algebra

$$A_1 = k\langle x, y\rangle / (xy - yx - 1).$$

- Introduce a third variable z which commutes with x and y, and for which $yx - xy - z^2 = 0$. Thus $\deg z = 1$, and we obtain the quadratic Artin-Schelter algebra

$$H = H_q = k\langle x, y, z\rangle / (yz - zy, zx - xz, xy - yx - z^2) \tag{1.7}$$

  to which we refer as the *homogenized Weyl algebra*. It is easy to see H is the Rees algebra with respect to the standard Bernstein filtration on A_1, see Example 1.11.1 below.

- Introduce a third variable z which commutes with x and y and for which $xy - yx - z = 0$. Thus $\deg z = 2$ and we obtain the enveloping algebra of the Heisenberg-Lie algebra, which is a cubic Artin-Schelter algebra

$$\begin{aligned} H_c &= k\langle x, y, z\rangle / (yz - zy, xz - zx, xy - yx - z) \\ &= k\langle x, y\rangle / (y^2 x - 2yxy + xy^2, x^2 y - 2xyx + yx^2) \\ &= k\langle x, y\rangle / ([y, [y, x]], [x, [x, y]]) \end{aligned} \tag{1.8}$$

  We refer to H_c as the *enveloping algebra* for short.

Example 1.9.3. The generic three dimensional Artin-Schelter regular algebras generated in degree one are the so-called type A-algebras [5], they are of the form

- quadratic:
$$A = k\langle x, y, z\rangle/(f_1, f_2, f_3)$$
where f_1, f_2, f_3 are the quadratic equations
$$\begin{cases} f_1 = ayz + bzy + cx^2 \\ f_2 = azx + bxz + cy^2 \\ f_3 = axy + byx + cz^2 \end{cases} \tag{1.9}$$

- cubic:
$$k\langle x, y\rangle/(f_1, f_2)$$
where f_1, f_2 are the cubic equations
$$\begin{cases} f_1 = ay^2x + byxy + axy^2 + cx^3 \\ f_2 = ax^2y + bxyx + ayx^2 + cy^3 \end{cases} \tag{1.10}$$

where $(a, b, c) \in \mathbb{P}^2 \setminus F$ where F is some finite set. In order to describe F, we recall from [7, Theorem 1] that the regular algebras of global dimension three generated in degree one are exactly the nondegenerate standard algebras. The algebra A with above relations is nondegenerate (and hence regular since A is already standard) unless $(a, b, c) \in F$ where

$$\text{quadratic: } F = \{(a, b, c) \in \mathbb{P}^2 \mid a^3 = b^3 = c^3\} \cup \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$$
$$\text{cubic: } F = \{(a, b, c) \in \mathbb{P}^2 \mid a^2 = b^2 = c^2\} \cup \{(0, 1, 0), (0, 0, 1)\}$$

The generic subclass of three dimensional Artin-Schelter regular algebras of type A are given by the more restrictive condition $(a, b, c) \in \mathbb{P}^2 \setminus F'$ where

$$\text{quadratic: } F' = \{(a, b, c) \in \mathbb{P}^2 \mid abc = 0 \text{ or } (3abc)^3 = (a^3 + b^3 + c^3)^3\}$$
$$\text{cubic: } F' = \{(a, b, c) \in \mathbb{P}^2 \mid abc = 0 \text{ or } b^2 = c^2 \text{ or } (2bc)^2 = (4a^2 - b^2 - c^2)^2\}$$

We will refer to quadratic resp. cubic Artin-Schelter algebras A of type A for which $(a, b, c) \in \mathbb{P}^2 \setminus F'$ as *generic type* A. Quadratic algebras of generic type A are also called *three dimensional Sklyanin algebras*. The particular choice of F' will become clear in Example 1.9.15 below.

Remark 1.9.4. The homogenized Weyl algebra H is not of type A. It is also clear the enveloping algebra of the Heisenberg-Lie algebra H_c is a cubic Artin-Schelter algebra of type A, where $(a, b, c) = (1, -2, 0)$. However since $abc = 0$ we conclude H_c is not of generic type A.

Example 1.9.5. Our final example is that of a cubic Artin-Schelter algebra [8]
$$A(0, 1) = k\langle x, y\rangle/(f_1, f_2)$$
where f_1, f_2 are the cubic relations
$$\begin{cases} f_1 = xy^2 + y^2x \\ f_2 = yx^2 + x^2y + y^3 \end{cases}$$
Note $A(0, 1)$ is not of type A.

1.9.2 Dimension, multiplicity and Hilbert series

Let $0 \neq M \in \text{grmod}(A)$. As shown in [8] we may compute the Gelfand-Kirillov dimension (or GK-dimension or dimension for short) $\text{GKdim}\, M$ as the order of the pole of $h_M(t)$ at $t = 1$. The GK-dimension is the only dimension for graded modules we will use in this manuscript, and therefore there is no confusion by putting $\dim M = \text{GKdim}\, M$. From the Hilbert series of A we find $\text{GKdim}\, A = 3$. If $\text{GKdim}\, M \leq n$ then we define $e_n(M)$ as

$$e_n(M) = \lim_{t \to 1} (1-t)^n h_M(t)$$

We have $e_n(M) \geq 0$ and furthermore $e_n(M) = 0$ if and only if $\text{GKdim}\, M < n$. We define the *rank* of M as $\text{rank}\, M = e_3(M)/e(A)$. If $\text{GKdim}\, M = n$ we put $e(M) = e_n(M)$ and call this the *multiplicity* (or *Bernstein number*) of M. In other words $e(M)$ is the first nonvanishing coefficient of the expansion of $h_M(t)$ in powers of $1-t$

$$e(M) = \lim_{t \to 1} (1-t)^{\text{GKdim}\, M} h_M(t) \tag{1.11}$$

and one computes

$$e(A) = \begin{cases} 1 & \text{if } A \text{ is quadratic,} \\ 1/2 & \text{if } A \text{ is cubic.} \end{cases}$$

It is more convenient to work with $\epsilon(M) := e(M)e(A)^{-1}$ rather than $e(M)$. For $0 \neq \mathcal{M} \in \text{tails}(A)$ we put $\dim \mathcal{M} = \text{GKdim}\, M - 1$ and $e(\mathcal{M}) = e(M)$, $\epsilon(\mathcal{M}) = \epsilon(M)$ where $M \in \text{grmod}(A)$, $\pi M = \mathcal{M}$. We will often need

Lemma 1.9.6. *Consider a short exact sequence* $0 \to N' \to N \to N'' \to 0$ *in* $\text{grmod}(A)$ *or* $\text{tails}(A)$ *with dimension* $\leq n$. *Then* $\dim N = \max\{\dim N', \dim N''\}$ *and* $e_n(N) = e_n(N') + e_n(N'')$.

Proof. Taking Hilbert series is additive on short exact sequences in $\text{grmod}(A)$ i.e. if $0 \to N' \to N \to N'' \to 0$ in $\text{grmod}(A)$ then $h_M(t) = h_{M'}(t) + h_{M''}(t)$. This easily proves what we want. □

An object in $\text{grmod}(A)$ or $\text{tails}(A)$ is said to be *pure* if it contains no subobjects of strictly smaller dimension. It is *critical* if every proper quotient has lower dimension, or equivalently, if all nontrivial subobjects have the same multiplicity. Note A is critical and for a critical A-module M we have $\text{Hom}_A(M, M) = k$, see [8, Proposition 2.30]. We say $M \in \text{grmod}(A)$ is *Cohen-Macaulay* if $\text{pd}\, M = 3 - \text{GKdim}\, M$.

The following result is well-known. By lack of reference we have included a proof.

Lemma 1.9.7.

1. *If* $M \in \text{grmod}(A)$ *is pure (resp. critical) then* $\pi M \in \text{tails}(A)$ *is pure (resp. critical).*
2. *If* $\mathcal{M} \in \text{tails}(A)$ *is pure (resp. critical) then* $\mathcal{M} = \pi M$ *for some pure (resp. critical) object in* $\text{grmod}(A)$.

3. *Let* $M, N \in \operatorname{grmod}(A)$ *(resp.* $\operatorname{tails}(A)$*) are of the same dimension and assume* M *is critical and* N *is pure. Then every non-zero morphism in* $\operatorname{Hom}(M, N)$ *is injective.*

Proof. For the first statement, assume by contradicition $\pi M \in \operatorname{tails}(A)$ is not pure and let $0 \neq \mathcal{N} \in \operatorname{tails}(A)$ be a subobject of smaller dimension. Since M is pure we have in particular $\tau M = 0$ hence (1.3) gives $M \subset \omega\mathcal{M}$. Also, $W = \omega\mathcal{N} \cap M$ is a submodule of M hence $W \in \operatorname{grmod}(A)$. If W would be non-zero then $0 \neq \pi W \subset \mathcal{N}$ hence $\dim \pi W \leq \dim \mathcal{N} < \dim \mathcal{M}$. This implies $\operatorname{GKdim} W < \operatorname{GKdim} M$, which is impossible by the pureness of M. Thus $W = \omega\mathcal{N} \cap M = 0$ and we may consider $\omega\mathcal{N}$ as a subobject of the quotient $(\omega\mathcal{M})/M$. Since the cokernel of the map $M \to \omega\pi M$ is right bounded this implies $\omega\mathcal{N} \in \operatorname{tors}(A)$, which contradicts $0 \neq \mathcal{N} \in \operatorname{tails}(A)$. Analogous reasoning in case M is critical.

Second, let $\mathcal{M} \in \operatorname{tails}(A)$ be pure. Let $M \in \operatorname{grmod}(A)$ such that $\pi M = \mathcal{M}$. We may assume M contains no subobject in $\operatorname{tors}(A)$. Assuming M has a non-zero subobject of lower GK-dimension then application of π shows $\pi M = \mathcal{M}$ has a subobject of lower dimension which is impossible. Hence $\mathcal{M}$ is pure. Analogous for the critical case.

For the final part of the lemma, assume by contradiction $0 \neq f \in \operatorname{Hom}(M, N)$ is not injective. Since M is critical $\dim \ker f = \dim M$ and $e(\ker f) = e(M)$. From the short exact sequence $0 \to \ker f \to M \to \operatorname{im} f \to 0$ we find $\dim \operatorname{im} f < \dim N$ which contradicts the pureness of N. Thus f is injective, completing the proof. □

We also recall the following frequently used result on the dimensions of dual modules and the duality between left and right A-modules.

Theorem 1.9.8. *[8, Theorem 4.1 and Corollary 4.2] Let* $M \in \operatorname{grmod}(A)$*,* $M \neq 0$*. Write* $m = \operatorname{GKdim} M$ *and denote* $M^\vee = \underline{\operatorname{Ext}}_A^{3-m}(M, A)$*. Then*

1. $\underline{\operatorname{Ext}}_A^j(M, A) = 0$ *for* $j < 3 - m$
2. $\operatorname{GKdim} M^\vee = \operatorname{GKdim} M$ *and* $e(M^\vee) = e(M)$
3. $\operatorname{GKdim} \underline{\operatorname{Ext}}_A^j(M, A) \leq 3 - j$ *for all* j*, and moreover following conditions are equivalent:*
   (a) $\operatorname{GKdim} \underline{\operatorname{Ext}}_A^j(M, A) = 3 - j$*,*
   (b) $\underline{\operatorname{Ext}}_A^j(\underline{\operatorname{Ext}}_A^j(M, A), A) \neq 0$
   (c) M *contains a non-zero submodule of GK-dimension* $3 - j$
4. *There is a canonical map* $\mu : M \to M^{\vee\vee}$ *which is an isomorphism if* M *is Cohen-Macaulay*
5. *If* $m < 3$ *then* $M^\vee$ *is Cohen-Macaulay*
6. $M^\vee$ *is pure* m*-dimensional*

7. $\ker \mu$ *is the maximal submodule of* M *which has GK-dimension* $< m$*, and* $\text{GKdim}\,(\text{coker}\,\mu) \leq m-2$

Remark 1.9.9. In [1, Proposition 1.4(iv)] it is stated that the converse of Theorem 1.9.8(4) also holds, however this is faulty. For example, Proposition 2.2.14 below implies, under suitable hypotheses on A, there are modules of GK-dimension 3 and of projective dimension 1 for which μ is an isomorphism.

We will now discuss an useful expansion of the Hilbert series $h_M(t)$ of $M \in \text{grmod}(A)$, where we treat the quadratic case and the cubic case separately.

Quadratic Artin-Schelter algebras

Assume A is a quadratic Artin-Schelter algebra and let $M \in \text{grmod}(A)$. We expand the characteristic polynomial $q_M(t) \in \mathbb{Z}[t, t^{-1}]$ in powers of $(1-t)$

$$q_M(t) = r + a(1-t) + b(1-t)^2 + f(t)(1-t)^3 \tag{1.12}$$

where $r, a, b \in \mathbb{Z}$ and $f(t) \in \mathbb{Z}[t, t^{-1}]$ is a Laurent polynomial. Needless to say

$$r = q_M(1), \quad a = -\frac{q'_M(1)}{1!}, \quad b = \frac{q''_M(1)}{2!}$$

Multiplying both sides of (1.12) with $h_A(t)$, equation (1.4) implies

$$h_M(t) = \frac{r}{(1-t)^3} + \frac{a}{(1-t)^2} + \frac{b}{(1-t)} + f(t)$$

Note $r = \text{rank}\, M$. By definition of $\text{GKdim}\, M$ and $e(M)$ we find $0 \leq \text{GKdim}\, M \leq 3$ and $e(M) > 0$ for $M \neq 0$. We easily deduce

Lemma 1.9.10. *Assume* A *is a quadratic Artin-Schelter algebra. Let* $M \in \text{grmod}(A)$*. Then there exist integers* r, a, b *and* $f(t) \in \mathbb{Z}[t, t^{-1}]$ *such that the Hilbert series of* M *is of the form*

$$h_M(t) = \frac{r}{(1-t)^3} + \frac{a}{(1-t)^2} + \frac{b}{(1-t)} + f(t) \tag{1.13}$$

Furthermore, if $M \neq 0$ *then one of the following possibilities occurs*

1. $\text{GKdim}\, M = 3$ *and* $\epsilon(M) = e_3(M) = r > 0$*,*
2. $\text{GKdim}\, M = 2$ *and* $r = 0$*,* $\epsilon(M) = e_2(M) = a > 0$*,*
3. $\text{GKdim}\, M = 1$ *and* $r = a = 0$*,* $\epsilon(M) = e_1(M) = b > 0$*,*
4. $\text{GKdim}\, M = 0$ *and* $r = a = b = 0$*,* $\epsilon(M) = e_0(M) = h_M(1) = f(1) = \sum_i \dim_k M_i > 0$*.*

We would like to refer to (1.13) as the *standard form* of $h_M(t)$. It is not hard to find the standard form of $h_{M(l)}(t)$. Indeed, from $q_{M(l)}(t) = t^{-l} q_M(t)$ we find

$$q_{M(l)}(1) = r, \quad -q'_{M(l)}(1) = lr + a, \quad \frac{q''_{M(l)}(1)}{2} = \frac{1}{2} l(l+1)r + la + b$$

and therefore

$$h_{M(l)}(t) = \frac{r}{(1-t)^3} + \frac{lr+a}{(1-t)^2} + \frac{l(l+1)r/2 + la + b}{(1-t)} + t^{-l} f(t) \tag{1.14}$$

In particular we have shown that the rank, multiplicity en Gelfand-Kirillov dimension of M are invariant under shift of grading. We will see some special types of modules in §1.9.3 below.

Cubic Artin-Schelter algebras

Assume A is a cubic Artin-Schelter algebra and let $M \in \operatorname{grmod}(A)$. Expand the characteristic polynomial $q_M(t) \in \mathbb{Z}[t, t^{-1}]$ in powers of $1 - t$

$$q_M(t) = r + (1-t) f'(t)$$

where $r \in \mathbb{Z}$ and $f'(t) \in \mathbb{Z}[t, t^{-1}]$. Now expand $f'(t)$ in powers of $1 + t$

$$q_M(t) = r + (1-t)(a + (1+t) f''(t)) = r + a(1-t) + (1-t^2) f''(t)$$

where $a \in \mathbb{Z}$ and $f''(t) \in \mathbb{Z}[t, t^{-1}]$. Finally, expand $f''(t)$ in powers of $1 - t$

$$\begin{aligned} q_M(t) &= r + a(1-t) + (1-t^2)(b + c(1-t) + f(t)(1-t)^2) \\ &= r + a(1-t) + b(1-t^2) + c(1-t)(1-t^2) + f(t)(1-t)^2(1-t^2) \end{aligned} \tag{1.15}$$

where $b, c \in \mathbb{Z}$ and $f(t) \in \mathbb{Z}[t, t^{-1}]$. Substituting this expansion in (1.4) yields

$$h_M(t) = \frac{r}{(1-t)^2(1-t^2)} + \frac{a}{(1-t)(1-t^2)} + \frac{b}{(1-t)^2} + \frac{c}{1-t} + f(t)$$

The specific choice of the expansion will become clear in Chapter 6. Again we find $r = \operatorname{rank} M$ and $0 \leq \operatorname{GKdim} M \leq 3$ for $M \neq 0$. However in the cubic case the expression of $\epsilon(M)$ in terms of r, a, b, c is more subtle.

Lemma 1.9.11. *Assume A is a cubic Artin-Schelter algebra. Let $M \in \operatorname{grmod}(A)$. Then there exist integers r, a, b, c and $f(t) \in \mathbb{Z}[t, t^{-1}]$ such that the Hilbert series of M is of the form*

$$h_M(t) = \frac{r}{(1-t)^2(1-t^2)} + \frac{a}{(1-t)(1-t^2)} + \frac{b}{(1-t)^2} + \frac{c}{1-t} + f(t) \tag{1.16}$$

Furthermore, if $M \neq 0$ then one of the following possibilities occurs

1. GKdim $M = 3$ *and* $\epsilon(M) = 2e_3(M) = r > 0$,
2. GKdim $M = 2$ *and* $r = 0$, $\epsilon(M) = 2e_2(M) = a + 2b > 0$,
3. GKdim $M = 1$ *and* $r = 0$, $a + 2b = 0$, $\epsilon(M) = 2e_1(M) = -b + 2c > 0$,
4. GKdim $M = 0$ *and* $r = 0$, $a+2b = 0$, $-b+2c = 0$, $\epsilon(M) = 2e_0(M) = 2h_M(1) = -c + 2f(1) = 2\sum_i \dim_k M_i > 0$. *Thus* $c \leq 0$.

We refer to (1.16) as the *standard form* of $h_M(t)$. Analogous as in the quadratic case one may may compute the standard form of $h_{M(l)}(t)$ for any $l \in \mathbb{Z}$. This is left as an exercise for the reader. Again rank, multiplicity en GK-dimension of M are invariant under shift of grading.

1.9.3 Linear modules

A *linear module of dimension* d over A is a cyclic graded right A-module M generated in degree zero with Hilbert series $(1-t)^{-d}$. Clearly $0 \leq d \leq 3$, GKdim $M = d$, $e(M) = 1$ and a linear module of dimension zero is isomorphic to k_A. Concerning $d = 3$ we note

Proposition 1.9.12. *If A is quadratic then a linear module of dimension three is isomorphic to A. If A is cubic then there exists no linear module of dimension three.*

Proof. Assume $M \in \operatorname{grmod}(A)$ is a linear module of dimension three. Thus we have a surjective map $A \to M$. Let N be the kernel of that map. Then

$$h_N(t) = h_A(t) - h_M(t) = \begin{cases} 0 & \text{if } A \text{ is quadratic} \\ -t(1-t)^{-3}(1+t)^{-1} & \text{if } A \text{ is cubic} \end{cases}$$

Thus if A is quadratic then $N = 0$ hence $A \cong M$, while if A is cubic then $h_N(t)$ would have strictly negative coefficients, which is absurd. □

We now discuss linear modules of dimension one and two. A linear module of dimension one is called a *point module*. They were classified in [7, 8]. Although the methods used for quadratic and cubic Artin-Schelter algebras are similar, we prefer to discuss this classification separately.

Quadratic Artin-Schelter algebras

Linear modules of dimension two are called a *line modules*, they are of the form $A/uA = S$ with $u \in A_1$. Hence line modules correspond naturally to lines in $\mathbb{P}^2$. The image under π of a point module P (resp. line module S) over A will be called a *point object* on $\mathbb{P}^2_q$ (resp. *line object*). In particular, $\dim \mathcal{O} = 2$, $\dim \mathcal{S} = 1$, $\dim \mathcal{P} = 0$ where $\mathcal{S} = \pi S$ and $\mathcal{P} = \pi P$. A minimal resolution of a line module S and a point module P over A is of the form [8]

$$0 \to A(-1) \to A \to S \to 0, \quad 0 \to A(-2) \to A(-1)^2 \to A \to P \to 0 \qquad (1.17)$$

Since line objects on $\mathbb{P}^2_q$ are of the form $\pi(A/uA)$ they are naturally parametrized by points in $\mathbb{P}(A_1)$.

We now show how point modules were classified in [7, 8]. Write the relations of A as

$$\begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix} = M_A \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} \tag{1.18}$$

where $M_A = (m_{ij})$ has entries $m_{ij} \in A_1$. We introduce auxiliary (commuting) variables $x^{(l)}, y^{(l)}, z^{(l)}$ (for $l \in \mathbb{Z}$) and for a monomial $m = a_0 \cdots a_n$ where $a_i \in \{x, y, z\}$ we define the *multilinearization* of m as $\widetilde{m}$ as $a_0^{(0)} \cdots a_n^{(n)}$. We extend this operation linearly to homogeneous polynomials in the variables x, y, z.

Let $\Gamma \subset \mathbb{P}^2 \times \mathbb{P}^2$ denote the locus of common zeroes of the $\widetilde{f}_i$. It turns out Γ is the graph of an automorphism σ of $E = \mathrm{pr}_1(\Gamma)$, the locus of zeroes of the multihomogenized polynomial $\det(\widetilde{M_A})$ where $\widetilde{M_A}$ is the matrix $(\widetilde{m}_{ij})$. If $\det(\widetilde{M_A})$ is not identically zero then E is a divisor of degree 3 in $\mathbb{P}^2$. We then say A is *elliptic*. Otherwise, E is all of $\mathbb{P}^2$ and we call A *linear* in this case.

The connection between E and point modules is as follows. Let P be a point module over A. Since $\dim_k P_i = 1$ for $i \geq 0$ we may choose a basis e_i for each k-vector space P_i. Thus $P = \sum k e_i$. Multiplication by the generators $x, y, z \in A_1$ of A induce linear maps $P_i \to P_{i+1}$. Thus

$$\begin{cases} e_i x = \alpha_i e_{i+1} \\ e_i y = \beta_i e_{i+1} \\ e_i z = \gamma_i e_{i+1} \end{cases} \quad \text{for some } \alpha_i, \beta_i, \gamma_i \in k$$

Now since P is generated in degree one $(\alpha_i, \beta_i, \gamma_i)$ determines a point on $\mathbb{P}^2$, which is independent of the choice of our basis e_i. From the relations $f_i = 0$ we have $e_0 f_i = 0$ thus $((\alpha_0, \beta_0, \gamma_0), (\alpha_1, \beta_1, \gamma_1)) \in \Gamma$ thus $(\alpha_0, \beta_0, \gamma_0) \in E$. This construction is reversible and defines a bijection between the closed points of E and the point modules over A. If $p \in E$ corresponds to the point module P then $(P_{\geq 1})(1)$ is the point module associated to σp.

Example 1.9.13. Consider the commutative polynomial ring $A = k[x, y, z]$. Then it is easy to see $E = \mathbb{P}^2$ and $\sigma = \mathrm{id}$. Thus $k[x, y, z]$ is a linear quadratic Artin-Schelter algebra.

Example 1.9.14. Consider the homogenized Weyl algebra H from Example 1.9.2. Then

$$\widetilde{M_H} = \begin{pmatrix} 0 & -z_0 & y_0 \\ z_0 & 0 & -x_0 \\ -y_0 & x_0 & -z_0 \end{pmatrix} \tag{1.19}$$

hence $\det(\widetilde{M_H}) = -z_0^3$, thus E is the "triple" line $z = 0$ in $\mathbb{P}^2$: The points (x, y, ϵ) such that $\epsilon^3 = 0$. Since $\det(\widetilde{M_H})$ is not identically zero, H is an elliptic quadratic

Artin-Schelter algebra. Using the affine coordinates $u = y/x$, $v = z/x$ in $\mathbb{P}^2$ it is easy to check that the automorphism σ is given by $\sigma(1, u, \epsilon) = (1, u + \epsilon^2, \epsilon)$. Hence σ is an infinitesimal translation. Note that in particular σ has infinite order.

Example 1.9.15. Consider a quadratic Artin-Schelter algebra of type A. Then E is given by the equation

$$(a^3 + b^3 + c^3)xyz = abc(x^3 + y^3 + z^3).$$

A is elliptic and one checks A is of generic type A (i.e. A is a three dimensional Sklyanin algebra) if and only if E is smooth curve. In that case E is an elliptic curve in $\mathbb{P}^2$ and σ is given by translation by some point $\xi \in E$ under the group law. Choosing the rational point $(1, -1, 0)$ on E as the origin we have $\xi = (a, b, c)$.

Cubic Artin-Schelter algebras

We refer to a linear modules of dimension two as a *conic* modules. Conic modules are are of the form A/vA with $v \in A_2$. Hence they correspond naturally to conics in $\mathbb{P}^1 \times \mathbb{P}^1$, the zero sets of quadratic forms. Further it is also natural to consider *line* modules which are of the form $A/uA = S$ with $u \in A_1$. Line modules correspond naturally to lines in $\mathbb{P}^1 \times \mathbb{P}^1$, where we use the convention that a line in $\mathbb{P}^1 \times \mathbb{P}^1$ will mean a set of the form $p \times \mathbb{P}^1$ where $p : \{u = 0\}$ is a point of $\mathbb{P}^1$.

From the minimal resolutions for a line module S and a conic module Q

$$0 \to A(-1) \to A \to S \to 0, \quad 0 \to A(-2) \to A \to Q \to 0 \tag{1.20}$$

one computes the Hilbert series

$$h_S(t) = \frac{1}{(1-t)(1-t^2)}, \quad h_Q(t) = \frac{1}{(1-t)^2}$$

thus

$$\dim_k S_n = \begin{cases} n/2 + 1 & \text{if } n \text{ is even} \\ (n+1)/2 & \text{if } n \text{ is odd} \end{cases}, \quad \dim_k Q_n = n + 1.$$

The image under π of a point module P (resp. line module S or a conic module Q) over A will be called a *point object* on $\left(\mathbb{P}^1 \times \mathbb{P}^1\right)_{\mathrm{q}}$ (resp. *line object* or *conic object*). In particular, $\dim \mathcal{O} = 2$, $\dim \mathcal{S} = \dim \mathcal{Q} = 1$, $\dim \mathcal{P} = 0$ where $\mathcal{S} = \pi S$, $\mathcal{Q} = \pi Q$ and $\mathcal{P} = \pi P$. A minimal resolution for a point module P over A is of the form [8]

$$0 \to A(-3) \to A(-2) \oplus A(-1) \to A \to P \to 0$$

Let us show how point modules were classified [7, 8]. We write the relations of A as

$$\begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = M_A \cdot \begin{pmatrix} x \\ y \end{pmatrix} \tag{1.21}$$

where $M_A = (m_{ij})$ has entries $m_{ij} \in A_2$. Again we introduce auxiliary (commuting) variables $x^{(l)}, y^{(l)}$ (for $l \in \mathbb{Z}$) and for a monomial $m = a_0 \cdots a_n$ in A where $a_i \in \{x, y\}$ we define the *multilinearization* of m as $\widetilde{m}$ as $a_0^{(0)} \cdots a_n^{(n)}$. We extend this operation linearly to homogeneous polynomials in the variables x, y.

Let $\Gamma \subset \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ denote the locus of common zeroes of the $\tilde{f}_i$. Define the projections

$$\begin{aligned} \mathrm{pr}_{12} : \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 &\to \mathbb{P}^1 \times \mathbb{P}^1 \\ (q_1, q_2, q_3) &\mapsto (q_1, q_2) \quad \text{drop the last component} \\ \mathrm{pr}_{23} : \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 &\to \mathbb{P}^1 \times \mathbb{P}^1 \\ (q_1, q_2, q_3) &\mapsto (q_2, q_3) \quad \text{drop the first component} \end{aligned}$$

As in the quadratic case it turns out the images of Γ under these two projections are the same (denoted by E), given by the zeroes of the multihomogenized polynomial $\det(\widetilde{M_A})$. Thus Γ is the graph of an automorphism $\sigma : E \to E$. There are two distinguished cases

- $\det(\widetilde{M_A})$ is identically zero. Then $E = \mathbb{P}^1 \times \mathbb{P}^1$ and in this case we call A *linear*. It follows that $\sigma \in \mathrm{Aut}(\mathbb{P}^1 \times \mathbb{P}^1)$ is of the form $\sigma(q_1, q_2) = (q_2, \tau(q_1))$ where $\tau \in \mathrm{Aut}(\mathbb{P}^1)$.

- $\det(\widetilde{M_A})$ is not identically zero. Then E is a divisor of bidegree $(2, 2)$ in $\mathbb{P}^1 \times \mathbb{P}^1$. We then say A is *elliptic*. We now have that $\sigma \in \mathrm{Aut}(E)$ is of the form $\sigma(q_1, q_2) = (q_2, f(q_1, q_2))$ for some map $f : E \to \mathbb{P}^1$.

The connection between E and point modules is as follows. Let P be a point module over A. Since $\dim_k P_i = 1$ for $i \geq 0$ we may choose a basis e_i for each k-vector space P_i. Thus $P = \sum ke_i$. Multiplication by the generators $x, y \in A_1$ of A induce linear maps $P_i \to P_{i+1}$. Thus

$$\begin{cases} e_i x = \alpha_i e_{i+1} \\ e_i y = \beta_i e_{i+1} \end{cases} \quad \text{for some } \alpha_i, \beta_i \in k$$

Now since P is generated in degree one it is not hard to see $q_i = (\alpha_i, \beta_i) \in \mathbb{P}^1$. Further, $e_0 f_i = 0$ hence $(q_1, q_2, q_3) \in \Gamma$ and therefore $(q_1, q_2) \in E$. This construction is reversible and defines a bijection between the closed points of E and the point modules over A. If $p \in E$ corresponds to the point module P then $(P_{\geq 1})(1)$ is the point module associated to σp.

For other properties of point modules and line modules over three dimensional Artin-Schelter regular algebras we refer to [1, 7, 8].

Example 1.9.16. Consider the enveloping algebra H_c from Example 1.9.2. Then

$$\widetilde{M_{H_c}} = \begin{pmatrix} y_1 y_2 & x_0 y_1 - 2y_0 x_1 \\ y_0 x_1 - 2x_0 y_1 & x_0 x_1 \end{pmatrix}$$

hence $\det(\widetilde{M}_{H_c}) = -2(x_0y_1 - x_1y_0)^2$, thus E is the double diagonal on $\mathbb{P}^1 \times \mathbb{P}^1$ i.e. the points $((x,y),(x+\epsilon,y+\epsilon)) \in \mathbb{P}^1 \times \mathbb{P}^1$ such that $\epsilon^2 = 0$. As a consequence the enveloping algebra H_c is elliptic. From the computation

$$\begin{pmatrix} y(y+\epsilon) & x(y+\epsilon) - 2y(x+\epsilon) \\ y(x+\epsilon) - 2x(y+\epsilon) & x(x+\epsilon) \end{pmatrix} \cdot \begin{pmatrix} x+2\epsilon \\ y+2\epsilon \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

It follows that $\sigma((x,y),(x+\epsilon,y+\epsilon)) = ((x+\epsilon,y+\epsilon),(x+2\epsilon,y+2\epsilon))$. In other words, σ is an infinitesimal translation by the point $((\epsilon,\epsilon),(\epsilon,\epsilon)) \in E$. In particular σ has infinite order on E.

Example 1.9.17. Consider a cubic Artin-Schelter algebra of type A. Then E is the divisor of bidegree $(2,2)$ on $\mathbb{P}^1 \times \mathbb{P}^1$ given by all $((x_0,y_0),(x_1,y_1)) \in \mathbb{P}^1 \times \mathbb{P}^1$ for which

$$(c^2 - b^2)x_0y_0x_1y_1 + ax_0^2(cx_1^2 - by_1^2) + ay_0^2(cy_1^2 - bx_1^2) = 0$$

and we deduce A is elliptic. Now E is smooth unless $abc = 0$ or $b^2 = c^2$ or $(2bc)^2 = (4a^2 - b^2 - c^2)^2$, i.e. E is smooth if and only if A is of generic type A. In this case σ is given by translation under the group law of E.

1.9.4 Geometric data

Let A be a three dimensional Artin-Schelter regular algebra and put $X = \operatorname{Proj} A$. As previously we denote by E the locus of zeroes of $\det(\widetilde{M}_A)$. Let j be the inclusion $j : E \hookrightarrow \mathbb{P}^2$ (resp. $j : E \hookrightarrow \mathbb{P}^1 \times \mathbb{P}^1$) if A is quadratic (resp. cubic).

Assume A is elliptic. Then [7, 41] the canonical sheaf ω_E is isomorphic to $\mathcal{O}_E$ and E has arithmetic genus 1. We will use the notations $\det \mathcal{E} := \wedge^{\operatorname{rank} \mathcal{E}} \mathcal{E}$ and $\deg \mathcal{E} := \deg(\det \mathcal{E})$ for vector bundles $\mathcal{E} \in \operatorname{coh}(E)$, and the Riemann-Roch theorem and Serre duality are given by

$$\chi(\mathcal{O}_E, \mathcal{E}) = \dim_k \operatorname{Hom}_E(\mathcal{O}_E, \mathcal{E}) - \dim_k \operatorname{Ext}^1_E(\mathcal{O}_E, \mathcal{E}) = \deg \mathcal{E}$$
$$\operatorname{Ext}^1_E(\mathcal{O}_E, \mathcal{E}) \cong \operatorname{Hom}_E(\mathcal{E}, \mathcal{O}_E)'$$

Assume furthermore A is of generic type A (see Example 1.9.3). Thus E is a smooth elliptic curve and σ is given by a translation on E. In particular E is a reduced and irreducible scheme. According to [41, Ex II. 6.11] we have a group isomorphism

$$\operatorname{Pic}(E) \oplus \mathbb{Z} \to K_0(E) : (\mathcal{O}(D), r) \mapsto r[\mathcal{O}_E] + \psi(D)$$

where ψ is the group homomorphism

$$\psi : \operatorname{Cl}(E) \to K_0(E) : \sum_i n_i p_i \mapsto \sum_i n_i [\mathcal{O}_{p_i}]$$

The projection $K_0(E) \to \mathbb{Z}$ is given by the rank and the projection $K_0(E) \to \operatorname{Pic}(E)$ is given the first Chern class. If $\mathcal{E}$ is a vector bundle on E then $c_1(\mathcal{E}) = \det \mathcal{E} = \wedge^{\operatorname{rank} \mathcal{E}} \mathcal{E}$.

We also have for $q \in E$: $c_1(\mathcal{O}_q) = \mathcal{O}_E(q)$. There is a homomorphism $\deg : \operatorname{Pic}(E) \to \mathbb{Z}$ which assigns to a line bundle its degree. For simplicity we will denote the composition $\deg \circ c_1$ also by deg. If $\mathcal{U}$ is a line bundle then $\deg[\mathcal{U}] := \deg \mathcal{U}$. If $F \in \operatorname{coh}(E)$ has finite length then $\deg[F] = \operatorname{length} F$ [41, Ex. 6.12].

We now return to the general situation i.e. let A be a three dimensional Artin-Schelter regular algebra. Put $\mathcal{O}_E(1) = j^*\mathcal{O}_{\mathbb{P}^2}(1)$ (resp. $j^* \operatorname{pr}_1^* \mathcal{O}_{\mathbb{P}^1}$). Associated to the geometric data $(E, \sigma, \mathcal{O}_E(1))$ is a so-called "twisted" homogeneous coordinate ring $B = B(E, \sigma, \mathcal{O}_E(1))$. This is a special case of a general construction in [10]. See also [9], or the construction below. If A is linear then $A \cong B$. If A is elliptic there exists, up to a scalar in k, a canonical normal element $g \in A_3$ (resp. $g \in A_4$) if A is quadratic (resp. cubic). The factor ring A/gA is isomorphic to the twisted homogeneous coordinate ring $B = B(E, \sigma, \mathcal{O}_E(1))$, see [8, 9, 10]. All point modules are B-modules. In other words g annihilates all point modules P i.e. $P\,g = 0$. If in addition A is elliptic and the automorphism σ has infinite order then g turns out to be central.

The fact that A may be linear or elliptic presents a notational problems and the fact that E may be non-reduced also presents some challenges. We side step these problems by defining $C = E_{\text{red}}$ if A is elliptic and letting C be a σ invariant line in $\mathbb{P}^2$ (resp. $\mathbb{P}^1 \times \mathbb{P}^1$). The geometric data $(E, \sigma, \mathcal{O}_E(1))$ then restricts to geometric data $(C, \sigma_C, \mathcal{O}_C(1))$. Note that in the elliptic case, writing $E = \sum_i n_i C_i$ where C_i are the irreducible components of the support of E we have $C = E_{\text{red}} = \sum_i C_i$ and the irreducible components C_i of C form a single σ-orbit.

As the examples in §1.9.3 indicate it may occur that the order of σ is different from the order of σ_C, being the restriction of σ to C. For example when A is the homogenized Weyl algebra then σ has infinite order, C is the line in $\mathbb{P}^2$ given by $z = 0$ and it follows that σ_C is the identity. Similar for the enveloping algebra of the Heisenberg-Lie algebra.

Warning. *To simplify further expressions we write $(C, \sigma, \mathcal{O}_C(1))$ for the triple $(C, \sigma_C, \mathcal{O}_C(1))$. Below we will often assume σ has infinite order. By this we will always mean the automorphism σ in the geometric data $(E, \sigma, \mathcal{O}_E(1))$ has infinite order and* not *the restriction of σ to C.*

We will now recall the construction of the homogeneous coordinate ring $B(C, \sigma, \mathcal{O}_C(1))$. To simplify notations we will write $\mathcal{L} = \mathcal{O}_C(1)$ and we denote the auto-equivalence $\sigma_*(- \otimes_C \mathcal{L})$ by $- \otimes \mathcal{L}_\sigma$. It is easy to check [69, (3.1)] for $n \geq 0$ one has

$$\begin{aligned} \mathcal{M} \otimes (\mathcal{L}_\sigma)^{\otimes n} &= \sigma_*^n(\mathcal{M} \otimes_C \mathcal{L} \otimes_C \sigma^*\mathcal{L} \otimes_C \cdots \otimes_C (\sigma^*)^{n-1}\mathcal{L}) \\ &= \sigma_*^n \mathcal{M} \otimes_C \sigma_*^n \mathcal{L} \otimes_C \sigma_*^{n-1}\mathcal{L} \otimes_C \cdots \otimes_C \sigma_*\mathcal{L} \end{aligned} \tag{1.22}$$

and since $(- \otimes \mathcal{L}_\sigma)^{-1} = \sigma^*(-) \otimes_C \mathcal{L}^{-1}$ we find for $n \geq 0$

$$\begin{aligned} \mathcal{M} \otimes (\mathcal{L}_\sigma)^{\otimes -n} &= (\sigma^*)^n(\mathcal{M} \otimes_C \sigma_*\mathcal{L}^{-1} \otimes_C \sigma_*^2\mathcal{L}^{-1} \otimes_C \cdots \otimes_C \sigma_*^n\mathcal{L}^{-1}) \\ &= (\sigma^*)^n\mathcal{M} \otimes_C (\sigma^*)^{n-1}\mathcal{L}^{-1} \otimes_C (\sigma^*)^{n-2}\mathcal{L}^{-1} \otimes_C \cdots \otimes_C \mathcal{L}^{-1} \end{aligned} \tag{1.23}$$

For $\mathcal{M} \in \operatorname{Qcoh}(X)$ put $\Gamma_*(\mathcal{M}) = \oplus_{n\geq 0}\Gamma(C, \mathcal{M} \otimes (\mathcal{L}_\sigma)^{\otimes n})$ and $D = B(C, \sigma, \mathcal{L}) \stackrel{\text{def}}{=} \Gamma_*(\mathcal{O}_C)$. Now D has a natural ring structure and $\Gamma_*(\mathcal{M})$ is a right D-module.

Notation. It will be convenient below to let the shift functors $-(n)$ on $\operatorname{coh}(C)$ be the ones obtained from the equivalence $\operatorname{coh}(C) \cong \operatorname{tails}(D)$ and *not* the ones coming from the embedding j. Thus for all $\mathcal{M} \in \operatorname{coh}(C)$ we write $\mathcal{M}(n) = \mathcal{M} \otimes (\mathcal{L}_\sigma)^{\otimes n} = \mathcal{M} \otimes (\mathcal{O}_C(1)_\sigma)^{\otimes n}$.

In [8, §5] it is shown there is a surjective morphism $A \to D = B(C, \sigma, \mathcal{O}_C(1))$ of graded k-algebras whose kernel is generated by a normalizing element h. In the elliptic case h divides g and D is a prime ring. However D may not be a domain since C may have multiple components C_i.

For a homogeneous element $a \in A$ we denote by $\overline{a}$ its image in $D = A/hA$. For homogeneous elements $d_1, d_2 \in D$ of degrees m and n respectively the multiplication $d_1 d_2$ in D is by definition

$$d_1 d_2 = d_1 \otimes_k \sigma^m d_2 \in H^0(C, \mathcal{O}_C(m+n))$$

where we have used the notation $\sigma^m d_2 = d_2 \circ \sigma^m$.

For $d \in D_n = H^0(C, \mathcal{O}_C(n))$ we denote $d(p)$ for the evaluation of the global section d in a point $p \in C$ and $\operatorname{div}(d)$ for the divisor of d consisting of all points p of C vanishing at d i.e. $d(p) = 0$. Thus for $p \in C$ we obtain $(d_1 d_2)(p) = d_1(p) d_2(\sigma^m p) \in k$ and

$$\operatorname{div}(d_1 d_2) = \operatorname{div}(d_1 \otimes_k d_2^{\sigma^m}) = \operatorname{div}(d_1) + \sigma^{-m} \operatorname{div}(d_2)$$

Example 1.9.18. Let A be quadratic and $a = (\lambda x + \mu y + \nu z)(\lambda' x + \mu' y + \nu' z) \in A_2$. Let $p \in E$. Writing $\sigma^i p = (\alpha_i, \beta_i, \gamma_i)$ we have $\overline{a}(p) = (\lambda\alpha_0 + \mu\beta_0 + \nu\gamma_0)(\lambda'\alpha_1 + \mu'\beta_1 + \nu'\gamma_1) \in k$.

Let A be cubic and $a = (\lambda x + \mu y)(\lambda' x + \mu' y) \in A_2$. Let $p \in E$. We may write $\sigma^i p = ((\alpha_i, \beta_i), (\alpha_{i+1}, \beta_{i+1})$ and it follows that $\overline{a}(p) = (\lambda\alpha_0 + \mu\beta_0)(\lambda'\alpha_1 + \mu'\beta_1) \in k$.

By analogy with the commutative case we may say $\operatorname{Proj} A$ contains $\operatorname{Proj} D$ as a "closed" subscheme. Though the structure of $\operatorname{Proj} A$ is somewhat obscure, that of $\operatorname{Proj} D$ is well understood.

Indeed it follows from [10, 9] that the functor $\Gamma_* : \operatorname{Qcoh}(C) \to \operatorname{GrMod}(D)$ defines an equivalence $\operatorname{Qcoh}(C) \cong \operatorname{Tails}(D)$. The inverse of this equivalence and its composition with $\pi : \operatorname{GrMod}(D) \to \operatorname{Tails}(D)$ are both denoted by $\widetilde{(-)}$.

In case A is elliptic the map $p \mapsto \Gamma_*(\mathcal{O}_p)$ defines the bijection from §1.9.3 between the points of C (hence the closed points of E) and the point modules over A. We will denote $N_p = \Gamma_*(\mathcal{O}_p)$ and $\mathcal{N}_p = \pi N_p$. In particular all point modules over A are D-modules i.e. $N_p h = 0$. We will frequently use

$$(N_p)_{\geq m} = N_{\sigma^m p}(-m), \quad \mathcal{N}_p = \mathcal{N}_{\sigma^m p}(-m) \tag{1.24}$$

where $p \in C$ and $m \in \mathbb{Z}$. We will use the following observation.

Lemma 1.9.19. *Assume A is elliptic. Let $M \in \operatorname{grmod}(A)$ be such that $M/Mh \in \operatorname{tors}(A)$. Then $\operatorname{GKdim} M = 1$. If σ has infinite order then $M \in \operatorname{tors}(A)$.*

Proof. Let $c = \deg h$. Multiplication by h induces an isomorphism $M_n \cong M_{n+c}$ for large n. Hence $\operatorname{GKdim} M \leq 1$.

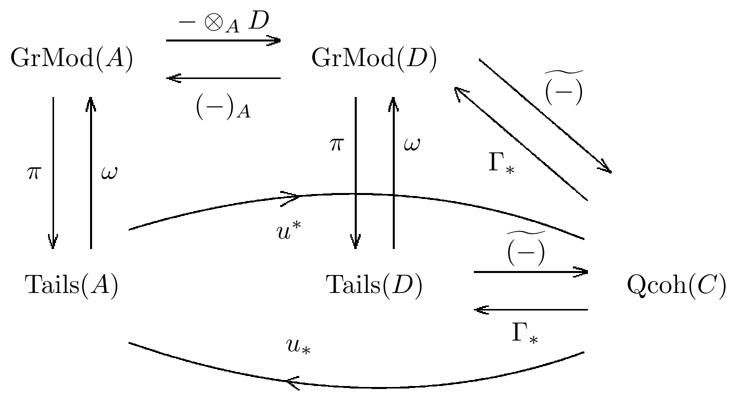
For the second part it suffices to prove $Mh = 0$. Assume by contradiction $Mh \neq 0$. Write $T \subset M$ for the submodule of h-torsion elements of M. Then $N = M/T$ is a non-zero h-torsion free module of GK-dimension ≤ 1. Write $(N_h)_0$ for the degree zero part of the localization of N at the powers of h. We find $(N_h)_0$ is a finite dimensional representation of $(A_h)_0$. By [8, Proposition 5.18] it is not difficult to see $(A_h)_0 = (A_g)_0$. If σ has infinite order then $(A_g)_0$ is a simple ring [8]. In particular it has no finite dimensional representations. Thus $(N_h)_0 = 0$ i.e. there is a positive integer i for which $Nh^i = 0$. For such a minimal i this implies Nh^{i-1} is a non-zero submodule of N which satisfies $Nh^{i-1}h = 0$, a contradiction to the fact that $N = M/T$ is h-torsion free. This proves what we want. □

In the sequel it will be useful to cast the relationship between the noncommutative graded ring A and the commutative scheme C into the language of noncommutative algebraic geometry exhibited in [67, 78] although we will use this language only in an intuitive way.

We define a map of noncommutative schemes $u : C \to X$ by

$$\begin{aligned} u^*\pi M &= (M \otimes_A D)^{\sim} && \text{for } M \in \operatorname{GrMod}(A), \\ u_*\mathcal{M} &= \pi(\Gamma_*(\mathcal{M})_A) && \text{for } \mathcal{M} \in \operatorname{Qcoh}(C) \end{aligned}$$

We will call $u^*(\pi M)$ the *restriction* of πM to C. The above functors are indicated in the following commutative diagram



Note u_* is an exact functor. For the left derived functor of u^* we have

Lemma 1.9.20. *If* $M \in D^-(\mathrm{GrMod}(A))$ *then* $Lu^*(\pi M) = (M \overset{\mathbf{L}}{\otimes}_A D)\tilde{}$

Proof. One shows first that the objects πF where F is a finitely generated graded free A-module are acyclic for u^* in the sense of [40]. Then the lemma follows by replacing M by a resolution of finitely generated free A-modules. ☐

We easily obtain the following consequence.

Lemma 1.9.21. *Assume A is elliptic and let $\mathcal{M} \in D^-(\mathrm{Tails}(A))$. Then there are short exact sequences*

$$0 \to u^* H^j(\mathcal{M}) \to H^j(Lu^*\mathcal{M}) \to L_1 u^* H^{j+1}(\mathcal{M}) \to 0$$

Proof. Take $M \in D^-(\mathrm{GrMod}(A))$ for which $\mathcal{M} = \pi M$. We may assume M is given by a right bounded complex of graded projective A-modules. The lemma now follows by applying π to the long exact homology sequence associate to the short exact sequence of complexes

$$0 \to Mh \to M \to M/Mh \to 0 \quad \square$$

1.10 Serre duality

It was shown in [86] that under reasonable hypotheses on a graded connected algebra A the category tails(A) satisfies a classical form of Serre duality. However we will need a stronger form of Serre duality introduced by Bondal and Kapranov in [19].

Let $\mathcal{A}$ be a k-linear Ext-finite triangulated category. By this we mean that for all $\mathcal{M}, \mathcal{N} \in \mathcal{A}$ we have $\sum_n \dim_k \mathrm{Hom}_{\mathcal{A}}(\mathcal{M}, \mathcal{N}[n]) < \infty$. The category $\mathcal{A}$ is said to satisfy Bondal-Kapranov-Serre (BKS) duality if there is an auto-equivalence $F : \mathcal{A} \to \mathcal{A}$ together with for all $\mathcal{M}, \mathcal{N} \in \mathcal{A}$ natural isomorphisms

$$\mathrm{Hom}_{\mathcal{A}}(\mathcal{M}, \mathcal{N}) \to \mathrm{Hom}_{\mathcal{A}}(\mathcal{N}, F\mathcal{M})'$$

We now assume A is a noetherian connected graded k-algebra. If we use notations which refer to the left structure of A then we adorn them with a superscript "$\circ$".

We make the following additional assumptions on A

1. A satisfies χ and the functor τ has finite cohomological dimension.
2. A satisfies χ° and the functor τ° has finite cohomological dimension.
3. tails(A) has finite global dimension.

Note that if A has finite global dimension then so does tails(A) by (1.2).

Put $R = R\tau(A)'$. According to [77] R is a complex of bimodules with finitely generated cohomology on the left and on the right, which in addition has finite injective dimension, also on the left and on the right. We now have the following result

Theorem 1.10.1. *The category* $D^b(\mathrm{tails}(A))$ *satisfies BKS-duality with Serre functor defined by*

$$F(\pi M) = \pi(M \overset{\mathbf{L}}{\otimes} R)[-1]$$

We refer to Appendix A for a more general version of Theorem 1.10.1 and its proof.

Now let A be an Artin-Schelter regular algebra of dimension $d = n+1$. Let l denote the Gorenstein parameter of A. It is easy to see A satisfies the hypotheses for Theoreom 1.10.1. In this case the Serre functor has a particularly simple form. Indeed in [10] it is shown that $R = (R^{n+1}\tau A)' \cong A[n+1](-l)$ as left A-modules and in [77] it is proved that $R\tau A \cong R\tau^\circ A$ as complexes of bimodules. Thus we also have $R = A[n+1](-l)$ as right A-modules. In other words $R = A_\phi[n+1](-l)$ where ϕ is some graded automorphism of A. The automorphism $M \mapsto M_\phi$ of $\mathrm{GrMod}(A)$ passes to an automorphism $\mathrm{Tails}(A)$ for which we also use the notation $(-)_\phi$.

We find the the following formula for the Serre functor on $\mathrm{tails}(A)$.

$$F\mathcal{M} = \mathcal{M}_\phi(-l)[n]$$

From this we easily obtain:

Proposition 1.10.2. *One has* $\operatorname{gldim}\mathrm{tails}(A) = \operatorname{gldim} A - 1$.

Proof. As above put $\operatorname{gldim} A = n+1$. The inequality $\operatorname{gldim}\mathrm{tails}(A) \le n$ follows directly from BKS-duality and the above discussion. Indeed, for all $\mathcal{M}, \mathcal{N} \in \mathrm{tails}(A)$ and $i > n$ we have

$$\begin{aligned}\mathrm{Ext}^i_{\mathrm{tails}(A)}(\mathcal{M},\mathcal{N}) &\cong \mathrm{Hom}_{D^b(\mathrm{tails}(A))}(\mathcal{M},\mathcal{N}[i]) \\ &\cong \mathrm{Hom}_{D^b(\mathrm{tails}(A))}(\mathcal{N}[i],\mathcal{M}_\phi(-l)[n])' \\ &\cong \mathrm{Ext}^{n-i}_{\mathrm{tails}(A)}(\mathcal{N},\mathcal{M}_\phi(-l))'\end{aligned}$$

which is zero. The other inequality follows from Theorem 1.8.2. □

We now assume A is an Artin-Schelter regular algebra of dimension 3. We will look for an algebra $\widehat{A}$ for which $\mathrm{GrMod}(A) \cong \mathrm{GrMod}(\widehat{A})$ and for which Serre duality for $\mathrm{tails}(\widehat{A})$ takes a particulary simple form.

In [87] Zhang found an elegant answer to the question when $\mathrm{GrMod}(A) \cong \mathrm{GrMod}(\widehat{A})$ for two $\mathbb{Z}$-graded connected k-algebras $A, \widehat{A}$. A *Zhang-system* of A is a set of graded isomorphisms $\tau = (\tau_i)_{i\in\mathbb{Z}}$ for which $\tau_n(a\tau_m(b)) = \tau_n(a)\tau_{m+n}(b)$ for all $n, m \in \mathbb{Z}$ and all homogeneous elements a, b in A with $a \in A_m$. The *Zhang-twist* of A by τ, denoted by A^τ, is the graded k-algebra A with a new multiplication defined by $a \cdot b = a\tau_n b$ for $a \in A_n$, $b \in A$. It was shown in [87, Theorem 1.2] that for two $\mathbb{Z}$-graded connected k-algebras $A, \widehat{A}$ generated in degree one we have $\mathrm{GrMod}(A) \cong \mathrm{GrMod}(\widehat{A})$ if and only if A is isomorphic to a Zhang-twist of $\widehat{A}$. Furthermore Gelfand-Kirillov dimension, global dimension, noetherian, domain and Artin-Schelter are Zhang-twisting invariant properties.

Let A be an Artin-Schelter regular algebra of dimension 3 with Gorenstein parameter l. As in (1.18), (1.21) we write the relations f of A as $f = M_A x$. With a suitable choice of the relations f we have $x^t M_A = (Q_A f)^t$ for some invertible matrix Q_A with scalar entries, see [5, Theorem 1.5]. It now turns out there exists a Zhang-twist A^τ of A such that Q_{A^τ} is the identity matrix. This was pointed out by M. Van den Bergh, see also [79]. Note again $\mathrm{GrMod}(A) \cong \mathrm{GrMod}(A^\tau)$ and by [87, Theorem 1.4] also $\mathrm{Tails}(A) \cong \mathrm{Tails}(A^\tau)$ where $(\pi A)(n)$ is sent to $(\pi A^\tau)(n)$.

If A is of type A then writing the relations as in (1.9), (1.10) yields $Q_A = \mathrm{id}$ whence we may put $A = A^\tau$. By (1.7) it is easy to check this is also true for the homogenized Weyl algebra $A = H$.

Convention 1.10.3. *From now on we will replace any quadratic or cubic Artin-Schelter algebra A with a Zhang-twist A^τ for which Q_{A^τ} is the identity matrix.*

Remark 1.10.4. We are allowed to use Convention 1.10.3 in this thesis since

- we will only specify to elliptic algebras for which σ has infinite order (but these are invariant properties under Zhang-twisting), the homogenized Weyl algebra and algebras of (generic) type A,
- we will not rely on the relations of A except for specific relations (1.9), (1.10) for algebras of type A, (1.7) for the homogenized Weyl algebra and (1.8) for the enveloping algebra of the Heisenberg-Lie algebra.

Using this convention we see Serre duality for $\mathrm{tails}(A)$ takes a particulary simple form.

Theorem 1.10.5. *(Serre duality) Let A be a quadratic or cubic Artin-Schelter algebra. Let l denote the Gorenstein parameter A. Let $\mathcal{M}, \mathcal{N} \in D^b(\mathrm{tails}(A))$. Then there are natural isomorphisms*

$$\mathrm{Ext}^i_{D^b(\mathrm{tails}(A))}(\mathcal{M}, \mathcal{N}) \cong \mathrm{Ext}^{n-i}_{D^b(\mathrm{tails}(A))}(\mathcal{N}, \mathcal{M}(-l))' \quad \textit{for all } i \in \mathbb{Z}.$$

Proof. As pointed out above the balanced dualizing complex R of A is given by $R = A_\varphi[3](-l)$ for some graded automorphism φ of A. By [77, Corollary 9.3] and an extended version in the cubic case (also communicated by M. Van den Bergh) this automorphism is given by $x \mapsto (Q^{-1})^t x$. By Convention 1.10.3 we have $Q_A = \mathrm{id}$ whence $R = A[3](-l)$. We conclude by Theorem 1.10.1. □

Finally we will often need the special case where $\mathcal{N}$ is a point object on $\mathrm{Proj}\, A$. From the previous theorem we deduce

Corollary 1.10.6. *Let A be a quadratic or cubic Artin-Schelter algebra. Let l denote the Gorenstein parameter A. Let $\mathcal{M} \in D^b(\mathrm{tails}(A))$ and $p \in E$ a closed point. Then there are natural isomorphisms*

$$\mathrm{Ext}^i_{D^b(\mathrm{tails}(A))}(\mathcal{M}, \mathcal{N}_p) \cong \mathrm{Ext}^{2-i}_{D^b(\mathrm{tails}(A))}(\mathcal{N}_{\sigma^l p}, \mathcal{M})'$$

Proof. By Theorem 1.10.5 we have

$$\mathrm{Ext}^i_{D^b(\mathrm{tails}(A))}(\mathcal{M}, \mathcal{N}_p) \cong \mathrm{Ext}^{2-i}_{D^b(\mathrm{tails}(A))}(\mathcal{N}_p(l), \mathcal{M})'$$

Invoking (1.24) ends the proof. ☐

1.11 Filtered algebras and modules

Let A be a three dimensional Artin-Schelter algebra generated in degree zero. Let h be the corresponding normalizing element as defined in §1.9.4. We denote by A_h the localisation of A at the multiplicative set $\{1, h, h^2, \dots, h^n, \dots\}$. It follows that A_h is a strongly $\mathbb{Z}$-graded k-algebra. Write $(A_h)_0$ for the degree zero part of A_h. Recall from [8] and the proof of Lemma 1.9.19 that in case σ has infinite order, $(A_h)_0$ is a simple hereditary ring of GK-dimension two. As a frequently used consequence all critical graded right A-modules of GK-dimension one are shifted point modules, and any A-module of GK-dimension one maps surjectively to a shifted point module. In order to describe the correspondence between graded right A-modules and representations of $(A_h)_0$ we recall some facts about filtered algebras and modules [14, Appendix A.3].

Let A be any k-algebra. A *filtration* of A is an ascending chain of linear subspaces

$$\dots \subset V_{i-1} \subset V_i \subset V_{i+1} \subset \dots$$

such that $1 \in V_0$, $\bigcup_{i\in\mathbb{Z}} V_i = A$ and $V_i V_j \subset V_{i+j}$ for all $i, j \in \mathbb{Z}$. A filtration is *positive* if in addition $V_i = 0$ for $i < 0$. The *Rees algebra* of such a filtered algebra A is

$$\mathrm{Rees}(A) = \bigoplus_{i\in\mathbb{Z}} V_i$$

which is identified with the subring $\bigoplus_{i\in\mathbb{Z}} V_i t^i$ of the ring of Laurent polynomials $A[t, t^{-1}]$. The Rees algebra becomes a $\mathbb{Z}$-graded k-algebra by setting $\deg t = 1$ and $\deg a = 0$ for all $a \in A$. We denote $\mathrm{gr}(A)$ for the *associated graded algebra* of A

$$\mathrm{gr}(A) = \bigoplus_{i\in\mathbb{Z}} V_i/V_{i-1}$$

We have $\mathrm{Rees}(A)/t\,\mathrm{Rees}(A) \cong \mathrm{gr}(A)$ and $\mathrm{Rees}(A)/(t-1)\,\mathrm{Rees}(A) \cong A$. Note that $\mathrm{Rees}(A)_t$, the localisation of $\mathrm{Rees}(A)$ at the multiplicative set $\{1, t, \dots, t^n, \dots\}$, is isomorphic to $A[t, t^{-1}]$. Thus $(\mathrm{Rees}(A)_t)_0$, the degree zero part of $\mathrm{Rees}(A)_t$, is isomorphic to A.

A right A-module of a filtered k-algebra A is called a *filtered* A-module if there is an ascending chain of linear subspaces

$$\dots \subset M_{i-1} \subset M_i \subset M_{i+1} \subset \dots$$

such that $\bigcup_{i\in\mathbb{Z}} M_i = M$ and $M_iA_j \subset M_{i+j}$ for all $i, j \in \mathbb{Z}$. We shall assume such a filtration is *separated*, meaning $\bigcap_{i\in\mathbb{Z}} M_i = 0$. For such a module M we have the *Rees module*

$$\mathrm{Rees}(M) = \bigoplus_{i\in\mathbb{Z}} M_i \in \mathrm{GrMod}(\mathrm{Rees}(A))$$

where we identify $\mathrm{Rees}(M) = \bigoplus_{i\in\mathbb{Z}} M_it^i \subset M[t,t^{-1}] = M \otimes_A A[t,t^{-1}]$, and the *associated graded module*

$$\mathrm{gr}(M) = \bigoplus_{i\in\mathbb{Z}} M_i/M_{i-1} \in \mathrm{GrMod}(\mathrm{gr}(A)).$$

We have $\mathrm{gr}(M) \cong \mathrm{Rees}(M)/t\,\mathrm{Rees}(M) \cong \mathrm{Rees}(M) \otimes_{\mathrm{Rees}(A)} \mathrm{gr}(A)$. When A is commutative, $X = \mathrm{Proj}(\mathrm{Rees}(A))$ is a projective scheme containing the affine scheme $\mathrm{Spec}(A)$ as an open subset, and the sheaf $\mathcal{M} = \pi\,\mathrm{Rees}(M)$ is an extension of the sheaf $\widetilde{M}$ on $\mathrm{Spec}(A)$ corresponding to M. Furthermore $\mathrm{Proj}(\mathrm{gr}(A))$ is the hypersurface at infinity in X, and $\pi\,\mathrm{gr}(M)$ is the restriction of $\mathcal{M}$ to this hypersurface. This justifies the similar language we used in the noncommutative case §1.9.4.

Example 1.11.1. Consider the homogenized Weyl algebra $A = H$ from Example 1.9.2. Then $h = z$ and $(A_h)_0$ is the first Weyl algebra $A_1 = k\langle x, y\rangle/(xy - yx - 1)$. For any positive integer l, let V_l be the k-linear space spanned by the set $\{x^jy^l \mid i+j \leq l\}$. Then $k = V_0 \subset V_1 \subset \ldots$ is a positive filtration of A_1, called the *standard Bernstein filtration*. It is then clear that $\mathrm{Rees}(A_1)$ is isomorphic to the homogenized Weyl algebra, identifying t with h, and $\mathrm{gr}(A_1) \cong k[x, y]$, the homogeneous coordinate ring of the line C in $\mathbb{P}^2$ given by the equation $h = 0$.

Example 1.11.2. Consider the enveloping algebra $A = H_c$ from Example 1.9.2. Then $h = z = xy - yx$ and it is shown in [8, Theorem 8.20] that $(A_h)_0$ is the ring of invariants $A_1^{\langle\varphi\rangle}$ of the first Weyl algebra $A_1 = k\langle u, v\rangle/(uv - vu - 1)$ under the automorphism $\varphi(u) = -u$, $\varphi(v) = -v$. For any positive integer l, let V_l be the k-linear space spanned by the set $\{x^jy^l \mid i + j \text{ even and } (i + j)/2 \leq l\}$. Then $k = V_0 \subset V_1 \subset \ldots$ is a positive filtration of $A_1^{\langle\varphi\rangle}$ and $\mathrm{Rees}(A_1^{\langle\varphi\rangle}) \cong A^{(2)}$, the 2-Veronese of A. Furthermore $\mathrm{gr}(A_1^{\langle\varphi\rangle}) \cong k[x, y]^{(2)}$.

Let A be a positively filtered k-algebra, and assume furthermore $V_0 = k$ and A is generated by V_1. Write $\mathrm{Filt}(A)$ for the category with as objects the filtered right A-modules and as morphisms the A-module morphisms $f : M \to N$ which are strict, i.e. $N_n \cap \mathrm{im}(f) = f(M_n)$ for all n. Write $\mathrm{GrMod}(\mathrm{Rees}(A))_t$ for the full subcategory of $\mathrm{GrMod}(\mathrm{Rees}(A))$ consisting of the t-torsion free modules. The exact functor

$$\mathrm{Rees}(-) : \mathrm{Filt}(A) \to \mathrm{GrMod}(\mathrm{Rees}(A))_t$$

is an equivalence, and $(\mathrm{Rees}(M)_t)_0 \cong M$. Also, for $M \in \mathrm{Filt}(A)$ we have $\mathrm{GKdim}\,\mathrm{Rees}(M) = \mathrm{GKdim}\,M + 1$. This shows the study of irreducible A-modules of GK-dimension $n \geq 0$ is equivalent to the study of the critical t-torsion free modules of GK-dimension $n + 1$.

Chapter 2

Ideals of quadratic Artin-Schelter algebras

In this chapter we classify reflexive graded right ideals of generic quadratic Artin-Schelter algebras, up to isomorphism and shift of grading. This leads to a classification of graded right ideals, up to modules of GK-dimension ≤ 1. It is similar to the classification of right ideals of the first Weyl algebra, a problem that was completely settled recently. The situation we consider is substantially more complicated however.

Most of the material presented in this chapter has been published in [27, 28]. Some results were found independently by Nevins and Stafford [60], using different methods.

2.1 Motivation, main results and analogy

We begin with a motivation of studying such modules by considering a "classical" situation, namely the Hilbert scheme of points $\mathrm{Hilb}(\mathbb{A}^2)$ on the affine plane $\mathbb{A}^2$. This survey is collected from [52, 58]. We then see what happens if we replace $k[x, y]$, the coordinate ring of $\mathbb{A}^2$, by the first Weyl algebra A_1 which we consider as the coordinate ring of an open affine part of a noncommutative space $\mathbb{P}^2_{\mathrm{q}}$. It turns out [51, 17, 16] there is an analoge for the Hilbert scheme of points. The right ideals of A_1 are now related to reflexive graded right ideals I in the homogenized Weyl algebra H, a quadratic Artin-Schelter algebra. In this chapter our goal is to generalize these results for (generic) quadratic Artin-Schelter algebras.

2.1.1 Hilbert schemes on affine planes

The commutative polynomial algebra $k[x,y]$

Let $A_0 = k[x,y]$ denote the commutative polynomial algebra in two variables, which we view as the coordinate ring of the (ordinary) affine plane $\mathbb{A}^2$. The Hilbert scheme of points on $\mathbb{A}^2$ parametizes the cyclic finite dimensional A_0-modules

$$\mathrm{Hilb}_n(\mathbb{A}^2) = \{V \in \mathrm{mod}(A_0) \mid V \text{ cyclic and } \dim_k V = n\}/\text{ iso} \tag{2.1}$$

For $V \in \mathrm{Hilb}_n(\mathbb{A}^2)$ its annihilator $\mathrm{Ann}_{A_0}(V) = \{a \in A_0 \mid a \cdot V = 0\}$ is an ideal of A_0 of finite codimension and this correspondence is reversible

$$\mathrm{Hilb}_n(\mathbb{A}^2) = \{I \subset A_0 \text{ ideal } \mid \dim_k A_0/I = n\}$$

For any ideal J of A_0 there is a unique ideal of finite codimension I such that $I \cong J$. Since the ideals of A_0 are exactly the finitely generated torsion free rank one A_0-modules we also have

$$\mathrm{Hilb}(\mathbb{A}^2) = \{M \in \mathrm{mod}(A_0) \mid M \text{ torsion free of rank one }\}/\text{ iso} \tag{2.2}$$

where $\mathrm{Hilb}(\mathbb{A}^2) = \coprod_n \mathrm{Hilb}_n(\mathbb{A}^2)$.

We rephrase this into the language of quiver representations. Let $V \in \mathrm{Hilb}_n(\mathbb{A}^2)$ be a cyclic A_0-module of dimension n. Multiplication by x and y induces linear maps on V represented by $n \times n$ matrices $\mathbb{X}, \mathbb{Y}$ for which $[\mathbb{X}, \mathbb{Y}] = 0$. We also have a vector $v \in V$ for which $v \cdot A_0 = V$. Thus

$$V \in \mathrm{Hilb}_n(\mathbb{A}^2) \mapsto \text{ data } \mathbb{X}, \mathbb{Y} \in M_n(k), v \in k^n : \begin{cases} [\mathbb{X}, \mathbb{Y}] = 0 \\ k\langle \mathbb{X}, \mathbb{Y}\rangle \cdot v = k^n \end{cases} \tag{2.3}$$

Note $k\langle \mathbb{X}, \mathbb{Y}\rangle = k[\mathbb{X}, \mathbb{Y}]$ since $[\mathbb{X}, \mathbb{Y}] = 0$. Conversely such data on the right of (2.3) determines an A_0-module structure on k^n which is cyclic, hence an object in $\mathrm{Hilb}_n(\mathbb{A}^2)$. Furthermore, isomorphism classes on the left are in one-to-one correspondence with the orbits of the group $\mathrm{Gl}_n(k)$ acting on the data on the right by (simultaneous) conjugation.

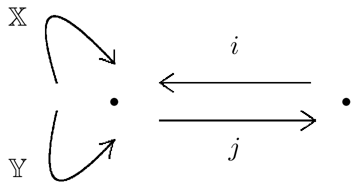
Apparently, the conditions on the right of (2.3) may be replaced by - at first sight weaker - conditions

$$V \in \mathrm{Hilb}_n(\mathbb{A}^2) \mapsto \text{ data } \mathbb{X}, \mathbb{Y} \in M_n(k), v \in k^n : \begin{cases} \mathrm{im}([\mathbb{X}, \mathbb{Y}]) \subset k \cdot v \\ k\langle \mathbb{X}, \mathbb{Y}\rangle \cdot v = k^n \end{cases} \tag{2.4}$$

Indeed, by standard arguments in linear algebra one shows that such data on the right of (2.4) imply $[\mathbb{X}, \mathbb{Y}] = 0$. See for example [58, §2.2]. Associated are the linear maps

$$\begin{aligned} &i : k \to k^n : 1 \mapsto v \\ &j : k^n \to k : u \mapsto j(u) \text{ such that } [\mathbb{X}, \mathbb{Y}] \cdot u = j(u) \cdot v \end{aligned}$$

The quadruple $(\mathbb{X}, \mathbb{Y}, i, j)$ may be visualized as



which determines a representation of the underlying quiver Q with dimension vector $(n, 1)$. We find

$$\mathrm{Hilb}_n(\mathbb{A}^2) = \{(\mathbb{X}, \mathbb{Y}, i, j) \in \mathrm{Rep}_{(n,1)}(Q) \mid [\mathbb{X}, \mathbb{Y}] = ij \\ \text{and } k\langle \mathbb{X}, \mathbb{Y}\rangle \cdot i(1) = k^n\} / \mathrm{Gl}_n(k) \quad (2.5)$$

where the group $\mathrm{Gl}_n(k)$ acts by conjugation

$$\forall g \in \mathrm{Gl}_n(k) : (\mathbb{X}, \mathbb{Y}, i, j) \mapsto (g\mathbb{X}g^{-1}, g\mathbb{Y}g^{-1}, gi, jg^{-1})$$

Note again that in fact $j = 0$ in (2.5). Also, $\mathrm{Hilb}_0(\mathbb{A}^2)$ is a point and $\mathrm{Hilb}_1(\mathbb{A}^2) = \mathbb{A}^2$.

The first Weyl algebra

Let $A_1 = k\langle x, y\rangle/(xy - yx - 1)$ be the first Weyl algebra. It is well-known A_1 is a noetherian domain of global dimension one. Thinking of A_1 as a noncommutative deformation of $A_0 = k[x, y]$, we would like to have an analogue for the Hilbert scheme of points on $\mathbb{A}^2$.

A first (naive) attempt based on (2.1) would be to consider cyclic finite dimensional right A_1-modules

$$\{V \in \mathrm{mod}(A_1) \mid V \text{ cyclic and } \dim_k V = n\}$$

But in contrast with A_0 this is the empty set for $n > 0$. Indeed, if there were such a module V then multiplication by x and y induce linear maps on $V = k^n$ represented by $n \times n$ matrices $\mathbb{X}, \mathbb{Y}$. The relation $xy - yx - 1 = 0$ in A_1 implies $[\mathbb{Y}, \mathbb{X}] - \mathbb{I} = 0$. Taking the trace of both sides we get $n = 0$. Similarly,

$$\{I \subset A_1 \text{ right ideal } \mid \dim_k A_1/I = n\} = \emptyset \text{ for } n > 0.$$

Thus there seems no reason to expect results for A_1 similar to the ones for A_0 as indicated above. But amazingly enough there are such results. The idea is to consider the alternative description (2.2) of $\mathrm{Hilb}_n(\mathbb{A}^2)$. Define the set of isomorphism classes

$$R(A_1) = \{\text{right ideals of } A_1\}/\text{ iso} \\ = \{M \in \mathrm{mod}(A_1) \mid M \text{ torsion free of rank one}\}/\text{ iso}$$

Since A_1 has global dimension one [56], the quotient A/I of a right ideal I of A_1 has projective dimension at most one hence I is projective. Note that this implies I is reflexive. We recall the basic result on this, as it was formulated by Berest and Wilson.

Theorem 2.1.1. *[17] The orbits of the natural* $\operatorname{Aut}(A_1)$*-action on the set* $R(A_1)$ *are indexed by* $\mathbb{N}$ *and the orbit corresponding to* $n \in \mathbb{N}$ *is in natural bijection with the* n*-th Calogero-Moser space*

$$C_n = \{(\mathbb{X}, \mathbb{Y}, i, j) \in \operatorname{Rep}_{(n,1)}(Q) \mid [\mathbb{X}, \mathbb{Y}] + \mathbb{I} = ij\} / \operatorname{Gl}_n(k) \tag{2.6}$$

where $\operatorname{Gl}_n(k)$ *acts by simultaneous conjugation* (gXg^{-1}, gYg^{-1})*. In particular* C_n *is a smooth connected affine variety dimension* $2n$*.*

Remark 2.1.2.

1. The first proof of Theorem 2.1.1 used the fact that there is a description of $R(A_1)$ in terms of the (infinite dimensional) adelic Grassmanian, due to Cannings and Holland [22]. Using methods from integrable systems Wilson [84] established a relation between the adelic Grassmanian and the Calogero-Moser spaces. That $R(A_1)/\operatorname{Aut}(A_1) \cong \mathbb{N}$ has also been proved by Kouakou [48, 49]. In [16] Berest and Wilson gave a new proof of Theorem 2.1.1 this using noncommutative algebraic geometry. We will come back on this in §2.1.2.

2. At first sight the description of the varieties C_n is not quite analogous as the commutative situation (2.5) since the stability condition $k\langle X, Y\rangle \cdot i(1) = k^n$ is missing. But one may prove (see for example [52]) that the representations in C_n *automatically* satisfy this condition. The fundamental reason for this is $\operatorname{gldim} A_1 = 1$ while $\operatorname{gldim} A_0 = 2$, which implies all right ideals of A_1 are reflexive (which is a stability condition) while in the case of A_0 they are not.

3. We may simplify the description of the n-th Caloger-Moser space C_n as

$$C_n = \{(\mathbb{X}, \mathbb{Y}) \in M_n^2(k) \mid \operatorname{rank}([\mathbb{Y}, \mathbb{X}] - \mathbb{I}) \le 1\} / \operatorname{Gl}_n(k) \tag{2.7}$$

where $\operatorname{Gl}_n(k)$ acts by simultaneous conjugation. Note C_0 is a point and $C_1 = \mathbb{A}^2$.

2.1.2 Hilbert schemes on projective planes

The commutative polynomial algebra $k[x, y, z]$

Let $A = k[x, y, z] \cong \operatorname{Rees}(A_0)$ denote the commutative polynomial algebra in three variables, which we view as the homogeneous coordinate ring of the projective plane $\mathbb{P}^2$. We now consider the affine plane as the open affine part $\mathbb{A}^2 = \mathbb{P}^2 \setminus l_\infty$ of $\mathbb{P}^2$ where the line l_∞ given by the equation $z = 0$. The restriction functor $u^* : \operatorname{coh}(\mathbb{P}^2) \to \operatorname{coh}(\mathbb{P}^1)$ from §1.9.4 associates with each sheaf its restriction to the line at infinity.

Let $V \in \operatorname{Hilb}_n(\mathbb{A}^2)$ be a cyclic n-dimensional A_0-module. Then V extends to a subscheme X of $\mathbb{P}^2$ of dimension zero and degree n, denoted by $X \in \operatorname{Hilb}_n(\mathbb{P}^2)$ with the property that multiplication by z induces an isomorphisms $H^0(\mathbb{P}^2, \mathcal{O}_X(l-1)) \cong H^0(\mathbb{P}^2, \mathcal{O}_X(l))$ for $l \gg 0$. This means $u^*\mathcal{O}_X = 0$. Writing $\mathcal{I}_X$ for the ideal sheaf of $\mathcal{O}_X$ we have $u^*\mathcal{I}_X = \mathcal{O}_{\mathbb{P}^1}$. These correspondences are reversible

$$\begin{aligned}\mathrm{Hilb}_n(\mathbb{A}^2) &= \{X \in \mathrm{Hilb}_n(\mathbb{P}^2) \mid u^*\mathcal{O}_X = 0\} \\ &= \{\mathcal{I} \in \mathrm{coh}(\mathbb{P}^2) \mid \mathcal{I} \text{ torsion free, rank one and } c_2(\mathcal{I}) = n, u^*\mathcal{I} = \mathcal{O}_{\mathbb{P}^1}\}/\text{ iso}\end{aligned}$$

We will now recall how these objects may be described by their homology. We have an equivalence of derived categories, known as Beilinson equivalence [15]

$$\mathrm{D}^b(\mathrm{coh}(\mathbb{P}^2)) \underset{-\overset{\mathbf{L}}{\otimes}_\Delta \mathcal{E}}{\overset{\mathrm{RHom}_{\mathbb{P}^2}(\mathcal{E},-)}{\rightleftarrows}} \mathrm{D}^b(\mathrm{mod}(\Delta)) \tag{2.8}$$

where $\mathcal{E} = \mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(-1) \oplus \mathcal{O}_{\mathbb{P}^2}(-2)$ and Δ is the quiver

$$\bullet \;\overset{u}{\underset{w}{\overset{\xrightarrow{v}}{\rightrightarrows}}}\; \bullet \;\overset{u'}{\underset{w'}{\overset{\xrightarrow{v'}}{\rightrightarrows}}}\; \bullet$$

with relations reflecting the relations in $A = k[x, y, z]$

$$\begin{cases} v'u = u'v \\ w'v = v'w \\ u'w = w'u \end{cases}$$

Under the derived equivalence (2.8) an object $X \in \mathrm{Hilb}_n(\mathbb{P}^2)$ is determined by a representation N of Δ

$$H^0(\mathbb{P}^2, \mathcal{O}_X) \overset{X}{\underset{Z}{\overset{\xrightarrow{Y}}{\rightrightarrows}}} H^0(\mathbb{P}^2, \mathcal{O}_X(1)) \overset{X'}{\underset{Z'}{\overset{\xrightarrow{Y'}}{\rightrightarrows}}} H^0(\mathbb{P}^2, \mathcal{O}_X(2))$$

where the linear map X is induced by multiplication by x, etc. and $Y'X = X'Y$ etc. (matrices will always be acting on the left). Shifting $\mathcal{I}_X$ if necessary, this representation N has dimension vector (n, n, n). As pointed out above the linear maps Z and Z' are isomorphisms. By an argument of Baer [12], N is actually determined by the linear maps X, Y, Z on the left, which is a representation of the Kronecker quiver Δ^0

$$\bullet \;\overset{u}{\underset{w}{\overset{\xrightarrow{v}}{\rightrightarrows}}}\; \bullet$$

Furthermore, consideration of matrix multiplications

$$\begin{pmatrix} X' & Y' & Z' \end{pmatrix} \cdot \underbrace{\begin{pmatrix} 0 & Z & -Y \\ -Z & 0 & X \\ Y & -X & 0 \end{pmatrix}}_{M_A(X,Y,Z)^t} = 0$$

and Z' being an isomorphism yields $\operatorname{rank} M_A(X,Y,Z) \leq 2n$. This leads to a description of $\operatorname{Hilb}_n(\mathbb{A}^2)$ in terms of Kronecker quiver representations

$$\begin{aligned}\operatorname{Hilb}_n(\mathbb{A}^2) = \{(X,Y,Z) \in \operatorname{Rep}_{(n,n)}(\Delta^0) \mid Z \text{ isomorphism }, \operatorname{rank} M_A(X,Y,Z) \leq 2n,\\ k\langle Z^{-1}X, Z^{-1}Y\rangle \cdot v = k^n \text{ for some } v \in k^n\}/\operatorname{Gl}_n(k)\end{aligned}$$

And indeed, putting $\mathbb{X} = Z^{-1}X$, $\mathbb{Y} = Z^{-1}Y$ we recover (2.3)

$$\begin{aligned}\operatorname{Hilb}_n(\mathbb{A}^2) &= \{(\mathbb{X},\mathbb{Y},\mathbb{I}) \in \operatorname{Rep}_{(n,n)}(\Delta^0) \mid \operatorname{rank} M_A(\mathbb{X},\mathbb{Y},\mathbb{I}) \leq 2n,\\ &\qquad k\langle \mathbb{X},\mathbb{Y}\rangle \cdot v = k^n \text{ for some } v \in k^n\}/\operatorname{Gl}_n(k)\\ &= \{(\mathbb{X},\mathbb{Y},\mathbb{I}) \in \operatorname{Rep}_{(n,n)}(\Delta^0) \mid [\mathbb{X},\mathbb{Y}] = 0,\\ &\qquad k\langle \mathbb{X},\mathbb{Y}\rangle \cdot v = k^n \text{ for some } v \in k^n\}/\operatorname{Gl}_n(k)\end{aligned}$$

where one uses

$$\begin{pmatrix} Z^{-1} & 0 & 0 \\ 0 & Z^{-1} & 0 \\ 0 & 0 & Z^{-1} \end{pmatrix} \cdot \underbrace{\begin{pmatrix} 0 & Z & -Y \\ -Z & 0 & X \\ Y & -X & 0 \end{pmatrix}}_{M_A(X,Y,Z)^t} = \underbrace{\begin{pmatrix} 0 & \mathbb{I} & -\mathbb{Y} \\ -\mathbb{I} & 0 & \mathbb{X} \\ \mathbb{Y} & -\mathbb{X} & 0 \end{pmatrix}}_{M_A(\mathbb{X},\mathbb{Y},\mathbb{I})^t}$$

$$\begin{pmatrix} \mathbb{X} & \mathbb{Y} & \mathbb{I} \\ 0 & \mathbb{I} & 0 \\ \mathbb{I} & 0 & 0 \end{pmatrix} \cdot \underbrace{\begin{pmatrix} 0 & \mathbb{I} & -\mathbb{Y} \\ -\mathbb{I} & 0 & \mathbb{X} \\ \mathbb{Y} & -\mathbb{X} & 0 \end{pmatrix}}_{M_A(\mathbb{X},\mathbb{Y},\mathbb{I})^t} = \begin{pmatrix} 0 & 0 & [\mathbb{Y},\mathbb{X}] \\ -\mathbb{I} & 0 & \mathbb{X} \\ 0 & \mathbb{I} & -\mathbb{Y} \end{pmatrix}$$

The homogenized Weyl algebra

In [16] Berest and Wilson gave a new proof of Theorem 2.1.1 using noncommutative algebraic geometry [10, 80]. That such an approach should be possible was in fact anticipated very early by Le Bruyn who in [51] already came very close to proving Theorem 2.1.1. Let us indicate which methods are used.

Consider the homogenized Weyl algebra $H \cong \operatorname{Rees}(A_1)$ from Example 1.9.2 and put $\mathbb{P}^2_{\mathrm{q}} = \operatorname{Proj}(H)$. The relation $A_1 = H/(z-1)H$ gives a close interaction between right A_1-modules and graded right H-modules. Indeed, under the equivalence $\operatorname{Rees}(-) : \operatorname{Filt}(A_1) \to \operatorname{GrMod}(H)$ of §1.11 the isoclasses of right ideals of A_1 are in bijection with the set $R(H)$ of z-torsion free reflexive rank one modules. Under the exact quotient functor $\pi : \operatorname{GrMod}(H) \to \operatorname{Tails}(H)$ we then obtain a bijection with the set $\mathcal{R}(\mathbb{P}^2_{\mathrm{q}})$ of isoclasses of objects $\mathcal{I}$ on $\mathbb{P}^2_{\mathrm{q}} = \operatorname{Proj}(H)$ for which its restriction to $C = E_{\mathrm{red}}$ is trivial i.e. $u^*\mathcal{I} = \mathcal{O}_{\mathbb{P}^1}$. An object $\mathcal{I} \in \mathcal{R}(\mathbb{P}^2_{\mathrm{q}})$ is now determined by a representation M of the quiver Δ which is the same as the earlier quiver Δ except the relations now reflect the relations in H

$$H^1(\mathbb{P}^2_{\mathrm{q}},\mathcal{I}(-2)) \overset{X}{\underset{Z}{\overset{\longrightarrow}{\underset{\longrightarrow}{\xrightarrow{Y}}}}} H^1(\mathbb{P}^2_{\mathrm{q}},\mathcal{I}(-1)) \overset{X'}{\underset{Z'}{\overset{\longrightarrow}{\underset{\longrightarrow}{\xrightarrow{Y'}}}}} H^1(\mathbb{P}^2_{\mathrm{q}},\mathcal{I})$$

where

$$\begin{pmatrix} X' & Y' & Z' \end{pmatrix} \cdot \underbrace{\begin{pmatrix} 0 & Z & -Y \\ -Z & 0 & X \\ Y & -X & -Z \end{pmatrix}}_{M_H(X,Y,Z)^t} = 0$$

In addition, $\underline{\dim} M = (n,n,n-1)$ and the map Z is an isomorphism and Z' is surjective. As before the representation M is determined by the three linear maps on the left. The result is that the right ideals of the first Weyl algebra are in bijection with the objects of the category

$$\coprod_n \{(X,Y,Z) \in \mathrm{Rep}_{(n,n)}(\Delta^0) \mid Z \text{ isomorphism}, \ \operatorname{rank} M_H(X,Y,Z) \le 2n+1\} / \operatorname{Gl}_n(k)$$

And indeed, putting $\mathbb{X} = Z^{-1}X$, $\mathbb{Y} = Z^{-1}Y$ we recover (2.7)

$$\begin{aligned} &= \coprod_n \{(\mathbb{X},\mathbb{Y},\mathbb{I}) \in \mathrm{Rep}_{(n,n)}(\Delta^0) \mid \operatorname{rank} M_H(\mathbb{X},\mathbb{Y},\mathbb{I}) \le 2n+1\} / \operatorname{Gl}_n(k) \\ &= \coprod_n \{(\mathbb{X},\mathbb{Y},\mathbb{I}) \in \mathrm{Rep}_{(n,n)}(\Delta^0) \mid \operatorname{rank}([\mathbb{Y},\mathbb{X}] - \mathbb{I}) \le 1\} / \operatorname{Gl}_n(k) = \coprod_n C_n \end{aligned}$$

where one uses

$$\begin{pmatrix} Z^{-1} & 0 & 0 \\ 0 & Z^{-1} & 0 \\ 0 & 0 & Z^{-1} \end{pmatrix} \cdot \underbrace{\begin{pmatrix} 0 & Z & -Y \\ -Z & 0 & X \\ Y & -X & -Z \end{pmatrix}}_{M_H(X,Y,Z)} = \underbrace{\begin{pmatrix} 0 & \mathbb{I} & -\mathbb{Y} \\ -\mathbb{I} & 0 & \mathbb{X} \\ \mathbb{Y} & -\mathbb{X} & -\mathbb{I} \end{pmatrix}}_{M_H(\mathbb{X},\mathbb{Y},\mathbb{I})^t}$$

$$\begin{pmatrix} \mathbb{X} & \mathbb{Y} & \mathbb{I} \\ 0 & \mathbb{I} & 0 \\ \mathbb{I} & 0 & 0 \end{pmatrix} \cdot \underbrace{\begin{pmatrix} 0 & \mathbb{I} & -\mathbb{Y} \\ -\mathbb{I} & 0 & \mathbb{X} \\ \mathbb{Y} & -\mathbb{X} & -\mathbb{I} \end{pmatrix}}_{M_H(\mathbb{X},\mathbb{Y},\mathbb{I})^t} = \begin{pmatrix} 0 & 0 & [\mathbb{Y},\mathbb{X}] - \mathbb{I} \\ -\mathbb{I} & 0 & \mathbb{X} \\ 0 & \mathbb{I} & -\mathbb{Y} \end{pmatrix}$$

Quadratic Artin-Schelter algebras

Let A be a quadratic Artin-Schelter algebra with associated geometric data $(C,\sigma,\mathcal{O}_C)$. Let $R(A)$ be the set of reflexive graded right A-modules of rank one, considered up to isomorphism and shift of grading. In the linear case this set is trivially $\{A\}$, see Proposition 2.2.13 below. In the elliptic case we will prove the following result in §2.4.7.

Theorem 1. *Assume k is uncountable. Let A be an elliptic quadratic Artin-Schelter algebra for which the associated automorphism σ has infinite order. There exist smooth locally closed varieties D_n of dimension $2n$ such that the set $R(A)$ is in natural bijection with $\coprod_n D_n$.*

Moreover D_n has the following description (for $n > 1$)

$$D_n = \{F = (X, Y, Z) \in \operatorname{Rep}_{(n,n)}(\Delta^0) \mid F \text{ is } \theta\text{-stable and } \operatorname{rank} M_A(X, Y, Z) \leq 2n + 1\} / \operatorname{Gl}_n(k) \quad (2.9)$$

where $\theta = (-1, 1)$ and M_A is the matrix as defined in §1.9.3. It follows D_n is a closed set of the quasi-affine variety consisting of the θ-stable representations in $\operatorname{Rep}_{(n,n)}(\Delta^0)$. For a description of D_1 we refer to Corollary 2.4.5. In particular D_0 is a point and D_1 is the complement of C under a natural embedding in $\mathbb{P}^2$.

Remark 2.1.3. In [60] Nevins and Stafford proved a result similar to Theorem 1, although without an explicit description of D_n. It turns out D_n is an open subset in a projective variety $\operatorname{Hilb}_n(\mathbb{P}^2_q)$ of dimension $2n$, which is an analogue of the classical Hilbert scheme of points on the projective plane $\mathbb{P}^2$. We will come back on this in Chapter 3.

Remark 2.1.4. In fact D_n is connected. This will follow from (the proof of) Theorem 5 and Proposition 3.3.6 in Chapter 3. See also [60].

Remark 2.1.5. Note Theorem 1 also applies for the homogenized Weyl algebra and in that case it follows from the description of D_n above that $D_n = C_n$ for all n.

In the Sklyanin case we have in addition (see Theorem 2.4.24)

Theorem 2. *Let A be a three dimensional Sklyanin algebra for which σ has infinite order. Then the varieties D_n in Theorem 1 are affine.*

Our proof of Theorem 2 is as follows. We will show in Theorem 2.4.24 that D_n has the alternative description

$$D_n = \{F = (X, Y, Z) \in \operatorname{Rep}_{(n,n)}(\Delta^0) \mid F \perp V \text{ and } \operatorname{rank} M_A(X, Y, Z) \leq 2n + 1\} / \operatorname{Gl}_n(k) \quad (2.10)$$

Here V is a fixed representation of Δ^0 with dimension vector $\underline{\dim} V = (6, 3)$, independent of $F \in D_n$. In particular there is some freedom in choosing V. From the description (2.10) it follows D_n is a closed subset of $\varphi_V \neq 0$ so it is affine.

Remark 2.1.6. Our proof of Theorems 1 is similar in spirit to the proof of Theorem 2.1.1. However it is substantially more involved. The reason for this is that the proofs for the Weyl algebra rely heavily on the fact the homogenized Weyl algebra H contains a central element in degree one (namely z) while for generic A the lowest central element in A has degree three.

Remark 2.1.7. The reader will notice Theorem 2 is weaker than Theorem 2.1.1 but this is probably unavoidable. Although (2.10) is a fairly succinct description of the varieties D_n it is not as explicit as (2.6), (2.7). And very likely D_n can also not be viewed in a natural way as the orbit of a group. This is also motivated by the fact that, in the notations of Theorem 2, $\mathrm{Aut}(A)$ is a finite group (see [60, Proposition 2.10]).

We also have a result in §2.4.9 below, that describes the elements of $R(A)$ by means of filtrations.

Theorem 3. *Assume k is uncountable. Let A be an elliptic quadratic Artin-Schelter algebra and assume σ has infinite order. Let $I \in R(A)$. Then there exists an $m \in \mathbb{N}$ together with a monomorphism $I(-m) \hookrightarrow A$ such that there exists a filtration of reflexive graded right A-modules of rank one*

$$A = M_0 \supset M_1 \supset \cdots \supset M_u = I(-m)$$

with the property the M_i/M_{i+1} are shifted line modules, up to finite length modules.

Remark 2.1.8. By Proposition 2.2.13 below this result is (trivially) true in case A is a linear quadratic Artin-Schelter algebra.

Remark 2.1.9. Dropping the hypothesis k is uncountable, Theorems 1 and 3 remain true for three dimensional Sklyanin algebras for which σ has infinite order. See Remark 2.4.14 below for this.

2.1.3 Stable vectorbundles on the projective plane

Finally, we would like to point out another analogy.

Let A be a quadratic Artin-Schelter algebra. As we will see in §2.2 reflexive modules over A give rise to certain objects on $\mathbb{P}^2_{\mathrm{q}}$, which we will call "vector bundles". In the commutative case this terminology coincides with the classical notion of a vector bundle. A vector bundle of rank one will be called a line bundle. Furthermore to any object $\mathcal{M}$ on $\mathbb{P}^2_{\mathrm{q}}$ we can associate two integers $c_1(\mathcal{M})$, $c_2(\mathcal{M})$ which are the analogues of the first and second Chern class in the commutative case. In this terminology the variety D_n from Theorem 1 parameterizes the linebundles $\mathcal{I}$ on $\mathbb{P}^2_{\mathrm{q}}$ for which $c_1(\mathcal{M}) = 0$, $c_2(\mathcal{M}) = n$.

As there are no nontrivial line bundles on the projective plane $\mathbb{P}^2$ it is difficult (and perhaps even unreasonable) to see the analogy of Theorem 1 with the commutative case. However there are plenty vector bundles of rank two on $\mathbb{P}^2$. It was shown by Hulek [44] that (for $n \geq 2$) the moduli space $\mathcal{M}(2, 0, n)$ of stable rank 2 vector bundles on $\mathbb{P}^2$ with first Chern class zero and second Chern class n is given by

$$\mathcal{M}(2,0,n) = \{F = (X, Y, Z) \in \mathrm{Rep}_{(n,n)}(\Delta^0) \mid F \text{ is } \theta\text{-stable and } \\ \operatorname{rank} M_A(X, Y, Z) \leq 2n + 2\} / \mathrm{Gl}_n(k)$$

As every line bundle is (trivially) stable, the analogy between $\mathcal{M}(2, 0, n)$ and the description (2.9) of D_n is very clear.

Remark 2.1.10. Hulek used an equivalent definition of a stable representation, namely properly stable. That these two notions are equivalent is easily seen and is left as an exercise for the reader.

2.2 From reflexive ideals to normalized line bundles bundles

For the rest of this chapter, A will denote a quadratic Artin-Schelter algebra as defined in §1.9. We will use the notations as discussed in Chapter 1, so we write $\mathbb{P}^2_q = \operatorname{Proj} A$, $\operatorname{Qcoh}(\mathbb{P}^2_q) = \operatorname{Tails}(A)$, $\operatorname{coh}(\mathbb{P}^2_q) = \operatorname{tails}(A)$, $\pi A = \mathcal{O}$. We also write $(E, \sigma, \mathcal{O}_E(1))$ and $(C, \sigma, \mathcal{O}_C(1))$ for the associated data.

Our first aim is to relate reflexive A-modules with certain objects on $\mathbb{P}^2_q$ (so-called vector bundles). Second, any shift of such a reflexive module remains reflexive and in the rank one case we will normalize this shift. The corresponding objects in $\operatorname{coh}(\mathbb{P}^2_q)$ will be called normalized line bundles on $\mathbb{P}^2_q$. Finally we will compute partially the cohomology of these normalized line bundles.

2.2.1 Torsion free and reflexive objects

A non-zero object M in $\operatorname{grmod}(A)$ is called *torsion* if M has rank zero and $M \neq 0$ is called *torsion free* if M contains no torsion subobject. This is the same as saying M is pure of GK-dimension three. In particular $M^* = M^\vee = \underline{\operatorname{Hom}}_A(M, A)$. We use the same terminology for objects in $\operatorname{coh}(\mathbb{P}^2_q)$. It is easy to see torsion free rank one objects in $\operatorname{grmod}(A)$ and $\operatorname{coh}(\mathbb{P}^2_q)$ are critical. The following lemma helps us to characterise torsion free objects.

Lemma 2.2.1. *For $0 \neq M \in \operatorname{grmod}(A)$ the following are equivalent*

1. *M is torsion free,*
2. *the canonical morphism $M \to M^{**}$ is injective,*
3. *$\operatorname{Hom}_A(N, M) = 0$ for all $N \in \operatorname{grmod}(A)$ of GK-dimension ≤ 2.*

For $0 \neq \mathcal{M} \in \operatorname{coh}(\mathbb{P}^2_q)$ the following are equivalent

1. *$\mathcal{M}$ is torsion free,*
2. *$\operatorname{Hom}_{\mathbb{P}^2_q}(\mathcal{N}, \mathcal{M}) = 0$ for all $\mathcal{N} \in \operatorname{coh}(\mathbb{P}^2_q)$ of dimension ≤ 1.*

Proof. The second part of the lemma (the part where $\mathcal{M} \in \operatorname{coh}(\mathbb{P}^2_q)$) follows from Lemma 1.9.10. So assume $0 \neq M \in \operatorname{grmod}(A)$. The equivalence (1) $\Leftrightarrow$ (3) is again clear by definition and Lemma 1.9.10. Further from [8, Proposition 2.40] we recall $M \to M^{**}$ is injective if and only if M is a first syzygy i.e. there is an exact sequence $0 \to M \to P$ for some projective (hence free) $P \in \operatorname{grmod}(A)$. Applying $\operatorname{Hom}_A(N, -)$

to such an exact sequence and bearing in mind that P is torsion free proves implication (2) ⇒ (3). Thus the lemma follows if we prove (3) ⇒ (2). Assuming $\operatorname{Hom}_A(N, M) = 0$ for all $N \in \operatorname{grmod}(A)$ of GK-dimension ≤ 2 it is clear $\operatorname{GKdim} M = 3$ (since $M \neq 0$). Hence $M^* = M^\vee$ and by Theorem 1.9.8(7) we conclude $\mu : M \to M^{**}$ is injective which means we are done. □

Next we point out the connection with graded right ideals of A. It is clear that any graded right ideal of A is torsion free of rank one. Up to shift of grading, the converse is also true.

Proposition 2.2.2. *Let $0 \neq I \in \operatorname{grmod}(A)$ be a torsion free of rank one. Then there is an integer m such that $I(-m)$ is isomorphic to a graded right ideal of A.*

Proof. By definition it is sufficient to prove there is an injective map $I(-m) \to A$ for some integer m. Consider the graded dual $I^* = \underline{\operatorname{Hom}}_A(I, A)$. By the fact $\operatorname{GKdim} I = 3$, Theorem 1.9.8(2) implies $I^* \neq 0$. Pick an integer m such that $(I^*)_m = \operatorname{Hom}_A(I(-m), A) \neq 0$. By Lemma 1.9.7(3) we are done. □

Remark 2.2.3. At this point we note the set of *all* graded right ideals of A is far too big to describe, as for any such ideal I we may construct numerous other closely related ideals by taking the kernel of any surjective map from I to a module of GK-dimension zero. The standard example is $A_{\geq 1}$, the kernel of the surjective map $A \to k$. This matter will be solved by considering to graded ideals of projective dimension one, or more restrictively, reflexive rank one modules.

Recall a non-zero A-module $M \in \operatorname{grmod}(A)$ is said to be *reflexive* if $M^{**} = M$ where $M^* = \underline{\operatorname{Hom}}_A(M, A)$ is the graded dual of M. An object $\mathcal{M} \in \operatorname{coh}(\mathbb{P}^2_q)$ is called *reflexive* (or a *vector bundle* on $\mathbb{P}^2_q$) if $\mathcal{M} = \pi M$ for some reflexive object $M \in \operatorname{grmod}(A)$. Vector bundles on $\mathbb{P}^2_q$ of rank one are called *line bundles* on $\mathbb{P}^2_q$. In case A is commutative the definitions above are equivalent to the standard ones.

Lemma 2.2.4. *For $0 \neq M \in \operatorname{grmod}(A)$ the following are equivalent*

1. *M is reflexive,*
2. *M is torsion free and $\operatorname{Ext}^1_A(N, M) = 0$ for all $N \in \operatorname{grmod}(A)$ of $\operatorname{GKdim} N \leq 1$.*

For $0 \neq \mathcal{M} \in \operatorname{coh}(\mathbb{P}^2_q)$ the following are equivalent

1. *$\mathcal{M}$ is a vector bundle on $\mathbb{P}^2_q$,*
2. *$\mathcal{M}$ is torsion free and $\operatorname{Ext}^1_{\mathbb{P}^2_q}(\mathcal{N}, \mathcal{M}) = 0$ for all $\mathcal{N} \in \operatorname{coh}(\mathbb{P}^2_q)$ of dimension 0.*

Proof. For the first part assume M is reflexive i.e. the canonical morphism $\mu : M \to M^{**}$ is an isomorphism. Then M is torsion free by Lemma 2.2.1. Assume by contradiction $\operatorname{Ext}^1_A(N, M) \neq 0$ for some $N \in \operatorname{grmod}(A)$, $\operatorname{GKdim} N \leq 1$. This means there is a non-split exact sequence

$$0 \to M \to M' \to N \to 0 \tag{2.11}$$

By Theorem 1.9.8(1) one has $\underline{\mathrm{Ext}}^i_A(N,A) = 0$ for $i \leq 1$. Hence we obtain $M'^* = M^*$ and thus $M = M^{**} = M'^{**}$. Thus the composition of $M \to M' \to M'^{**}$ is an isomorphism, implying that the first map splits. This contradicts the non-triviality of the extention (2.11).

For the other implication, let $M \in \mathrm{grmod}(A)$ be torsion free and $\mathrm{Ext}^1_A(N,M) = 0$ for all $N \in \mathrm{grmod}(A)$ of GK-dimension ≤ 1. By Theorem 1.9.8(7) the canonical map $\mu : M \to M^{**}$ is injective and we have an exact sequence

$$0 \to M \to M^{**} \to \operatorname{coker}\mu \to 0 \tag{2.12}$$

where $\mathrm{GKdim}(\operatorname{coker}\mu) \leq 1$. By assumption $\mathrm{Ext}^1_A(\operatorname{coker}\mu, M) = 0$ hence (2.12) splits. Theorem 1.9.8(5) implies M^{**} is pure of GK-dimension three and therefore $\operatorname{coker}\mu = 0$, proving M is reflexive.

For the second part of the lemma, the implication (1) ⇒ (2) follows from the first part and Lemma 2.2.1 using (1.2). To prove (2) ⇒ (1) we choose $M \in \mathrm{grmod}(A)$ such that $\pi M = \mathcal{M}$. We may assume M is torsion free. As above we have an exact sequence

$$0 \to M \to M^{**} \to \operatorname{coker}\mu \to 0 \tag{2.13}$$

where $\mathrm{GKdim}(\operatorname{coker}\mu) \leq 1$, and applying π yields

$$0 \to \mathcal{M} \to \pi M^{**} \to \mathcal{N} \to 0 \tag{2.14}$$

where $\mathcal{N} = \pi(\operatorname{coker}\mu)$. Now $\mathcal{N}$ must be zero, otherwise $\dim \mathcal{N} = 0$ and since $\mathrm{Ext}^1_{\mathbb{P}^2_q}(\mathcal{N},\mathcal{M}) = 0$ the sequence (2.14) would split, which is impossible because M^{**} is pure three dimensional. Hence $\mathcal{M} = \pi M = \pi M^{**}$ and thus $\mathcal{M}$ is reflexive completing the proof. □

Next we relate torsion free (and reflexive) objects in $\mathrm{grmod}(A)$ and $\mathrm{coh}(\mathbb{P}^2_q)$.

Proposition 2.2.5. *Assume $\mathcal{M} \in \mathrm{coh}(\mathbb{P}^2_q)$ is torsion free. Then $\omega\mathcal{M}$ is finitely generated torsion free and has projective dimension ≤ 1.*

Assume $M \in \mathrm{grmod}(A)$ be torsion free. Then M has projective dimension ≤ 1 if and only if $\omega\pi M = M$.

Proof. Assume $\mathcal{M} = \pi M$. Without loss of generality we may assume M is finitely generated and torsion free. It follows from localisation theory that $\omega\pi M$ is the largest extention N of M such that N/M is a union of finite length modules. Thus we have an exact sequence of the form

$$0 \to M \to \omega\mathcal{M} \to F \to 0 \tag{2.15}$$

where $F \in \mathrm{grmod}(A)$ has GK-dimension zero. On the other hand since M is pure it follows from Theorem 1.9.8 that $\mu : M \to M^{\vee\vee} = M^{**}$ is injective. Since M^{**} is reflexive, applying $\mathrm{Hom}_A(-, M^{**})$ to (2.15) yields $\mathrm{Hom}(\omega\mathcal{M}, M^{**}) \cong \mathrm{Hom}(M, M^{**})$. Thus there is a map $\psi \in \mathrm{Hom}_A(\omega\mathcal{M}, M^{**})$ such that the composition

$$M \subset \omega\mathcal{M} \xrightarrow{\psi} M^{**}$$

is equal to μ. We claim ψ is injective. Indeed, should this not be the case then $\ker\psi$ has GK-dimension three since $\omega\mathcal{M}$ is pure. But since μ is injective, this means $\ker\psi \subset \omega\mathcal{M}/M = F$, which is clearly a contradition. Thus $\omega\mathcal{M} \subset M^{**}$ and this shows $\omega\mathcal{M}$ is finitely generated and torsion free.

We now replace M by $\omega\mathcal{M}$. In particular $\underline{\mathrm{Ext}}^1_A(k, M) = 0$. Consider a minimal resolution of M

$$\cdots \to F_2 \to F_1 \to F_0 \to M \to 0$$

By applying to it the right exact functor $\underline{\mathrm{Ext}}^3_A(k, -)$ we see $\underline{\mathrm{Ext}}^1_A(k, M) = 0$ implies $F_2 = 0$ and hence M has projective dimension ≤ 1. This proves the first statement.

Second, assume M has projective dimension one. Then it follows from a projective resolution of M and the fact A is Gorenstein that $\underline{\mathrm{Ext}}^1_A(k, M) = 0$. Since $\mathrm{coker}(M \to \omega\pi M)$ is a union of finite dimensional modules this implies $\omega\pi M = M$.

Conversely, assume $\omega\pi M = M$. Observe $\pi M \in \mathrm{coh}(\mathbb{P}^2_{\mathrm{q}})$ is torsion free, which follows from the fact that M is torsion free and ω is left exact. It now follows from the first part that $M = \omega\pi M$ has projective dimension ≤ 1. □

Corollary 2.2.6. *The functors π and ω define inverse equivalences between the full subcategories of* $\mathrm{grmod}(A)$ *and* $\mathrm{coh}(\mathbb{P}^2_q)$ *with objects*

$$\{\textit{torsion free objects in } \mathrm{grmod}(A) \textit{ of projective dimension} \leq 1\}$$

and

$$\{\textit{torsion free objects in } \mathrm{coh}(\mathbb{P}^2_q)\}$$

which restricts to inverse equivalences π and ω between the full subcategories of $\mathrm{grmod}(A)$ *and* $\mathrm{coh}(\mathbb{P}^2_q)$ *with objects*

$$\{\textit{reflexive objects in } \mathrm{grmod}(A)\}$$

and

$$\{\textit{vector bundles on } \mathbb{P}^2_q\}$$

Proof. Follows from Lemma 1.9.7 and Proposition 2.2.5. □

Remark 2.2.7. Let $\mathrm{grmod}(A)_i$ be the full subcategory of $\mathrm{grmod}(A)$ consisting of modules of GK-dimension at most i. It is clear that $\mathrm{grmod}(A)_i$ is a Serre subcategory of $\mathrm{grmod}(A)$. Two modules in $\mathrm{grmod}(A)$ are called *i-equivalent* if they have the same GK-dimension and their images in the quotient category $\mathrm{grmod}(A)/\mathrm{grmod}(A)_i$ are isomorphic. It follows that for every graded right ideal I of A there is, up to isomorphism, a unique torsion free rank one A-module $J \in \mathrm{grmod}(A)$ of projective dimension one (namely $\omega\pi I$) for which I and J are 0-equivalent. Similarly, there is up to isomorphism a unique reflexive rank one A-module $J' \in \mathrm{grmod}(A)$ (namely I^{**}) for which I and J' are 1-equivalent.

2.2.2 The Grothendieck group and the Euler form for quantum planes

In this part we discuss a natural $\mathbb{Z}$-module basis for the Grothendieck group $K_0(\mathbb{P}^2_q)$ and determine the matrix representation of the Euler form χ with respect to this basis.

Let $\mathcal{M} \in \mathrm{coh}(\mathbb{P}^2_q)$. Thus $\mathcal{M} = \pi M$ for some $M \in \mathrm{grmod}(A)$. According to Lemma 1.9.10 we may write $h_M(t)$ in the standard form

$$h_M(t) = \frac{r}{(1-t)^3} + \frac{a}{(1-t)^2} + \frac{b}{1-t} + f(t) \tag{2.16}$$

where $r, a, b \in \mathbb{Z}$ and $f(t) \in \mathbb{Z}[t^{-1}, t]$ is a Laurent polynomial. The expansion of the characteristic polynomial $q_M(t) \in \mathbb{Z}[t, t^{-1}]$ in powers of $(1-t)$ is then given by

$$q_M(t) = r + a(1-t) + b(1-t)^2 + f(t)(1-t)^3$$

Now let P be a point module and S a line module over A and denote the corresponding objects in $\mathrm{coh}(\mathbb{P}^2_q)$ by $\mathcal{P}$ and $\mathcal{S}$. Since $q_A(t) = 1$, $q_S(t) = 1-t$ and $q_P(t) = (1-t)^2$ it follows from Theorem 1.7.1

$$[\mathcal{M}] = r[\mathcal{O}] + a[\mathcal{S}] + b[\mathcal{P}]$$

hence $\{[\mathcal{O}], [\mathcal{S}], [\mathcal{P}]\}$ is a $\mathbb{Z}$-module basis for $K_0(\mathbb{P}^2_q)$ (which does not depend on the particular choice of P and S). By Lemma 1.9.10 the multiplicity $e(\mathcal{M}) = e(M)$ is the first (leftmost) nonvanishing coordinate of $[\mathcal{M}]$ with respect to this basis. It also follows from (1.14)

$$[\mathcal{M}(l)] = r[\mathcal{O}] + (lr + a)[\mathcal{S}] + \left(\frac{1}{2}l(l+1)r + la + b\right)[\mathcal{P}] \tag{2.17}$$

for all integers l.

From now on we fix such a $\mathbb{Z}$-module basis $\mathcal{B} = \{[\mathcal{O}], [\mathcal{S}], [\mathcal{P}]\}$ for $K_0(\mathbb{P}^2_q)$. From the full cohomology modules of $\mathcal{O}$ (see Theorem 1.8.2) we deduce $\chi(\mathcal{O}, \mathcal{O}(l)) = (l+1)(l+2)/2$ and using minimal projective resolutions for $\mathcal{S}, \mathcal{P} \in \mathrm{coh}(\mathbb{P}^2_q)$ (apply π to (1.17)) one verifies the matrix representation of the Euler form χ for $\mathrm{coh}(\mathbb{P}^2_q)$ with respect to the basis $\mathcal{B}$ is given by

$$m(\chi)_{\mathcal{B}} = \begin{pmatrix} 1 & 1 & 1 \\ -2 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \tag{2.18}$$

2.2.3 Normalized rank one objects

Assume $I \in \mathrm{grmod}(A)$ has rank one. We say I is *normalized* if the coefficient a in (2.16) is zero. In that case we call $n = -b$ the *invariant* of I. For torsion free I we will prove later (Theorem 2.2.11) that this invariant n is actually positive. An object $\mathcal{I} \in \operatorname{coh}(\mathbb{P}^2_{\mathrm{q}})$ of rank one is called *normalized* if $\mathcal{I} = \pi I$ for some normalized rank one module $I \in \operatorname{grmod}(A)$. Equation (2.17) shows that if I has rank one then there is always a unique integer l for which $I(l)$ is normalized. Note if A is commutative i.e. $A = k[x, y, z]$ then $\mathcal{I} \in \operatorname{coh}(\mathbb{P}^2)$ of rank one is normalized if and only if the first Chern class $c_1(\mathcal{I}) = 0$. In that case, the invariant of $\mathcal{I}$ is given by the second Chern class $c_2(\mathcal{I})$.

Lemma 2.2.8. *Let $I \in \operatorname{grmod}(A)$. Then the following are equivalent.*

1. *I has rank one and is normalized with invariant n.*
2. *The Hilbert series of I has the form*
$$\frac{1}{(1-t)^3} - \frac{s(t)}{1-t} \tag{2.19}$$
for a Laurent polynomial $s(t) \in \mathbb{Z}[t, t^{-1}]$ with $s(1) = n$.
3. *$\dim_k A_m - \dim_k I_m = n$ for $m \gg 0$.*

Proof. Easy. □

It is easy to see Proposition 2.2.6 specializes to

Proposition 2.2.9. *The functors π and ω define inverse equivalences between the full subcategories of* $\operatorname{grmod}(A)$ *and* $\operatorname{coh}(\mathbb{P}^2_q)$ *with objects*

$$\operatorname{Hilb}_n(\mathbb{P}^2_{\mathrm{q}}) := \{\textit{normalized torsion free rank one objects in } \operatorname{grmod}(A) \textit{ of projective dimension one and invariant } n\}$$

and

$$\{\textit{normalized torsion free rank one objects in } \operatorname{coh}(\mathbb{P}^2_q) \textit{ with invariant } n\}$$

which for any integer n restricts to inverse equivalences π and ω between the full subcategories of $\operatorname{grmod}(A)$ *and* $\operatorname{coh}(\mathbb{P}^2_q)$ *with objects*

$$R_n(A) := \{\textit{normalized reflexive rank one objects in } \operatorname{grmod}(A) \textit{ with invariant } n\}$$

and

$$\mathcal{R}_n(\mathbb{P}^2_{\mathrm{q}}) := \{\textit{normalized line bundles on } \mathbb{P}^2_q \textit{ with invariant } n\}$$

Remark 2.2.10. 1. By Lemma 2.2.8 it is easy to see if A is commutative then the objects in $\operatorname{Hilb}_n(\mathbb{P}^2_q)$ are precisely the graded A-modules which occur as the graded ideals I_X for $X \in \operatorname{Hilb}_n(\mathbb{P}^2)$. It turns out [60] that $\operatorname{Hilb}_n(\mathbb{P}^2_q)$ is the correct generalization for the Hilbert scheme of points $\operatorname{Hilb}_n(\mathbb{P}^2)$ on the (commutative) projective plane $\mathbb{P}^2$. We will come back on this in Chapter 3.

2. Observe every non-zero morphism in $R_n(A)$ is an isomorphism. Thus $R_n(A)$ and $\mathcal{R}_n(\mathbb{P}^2_{\mathrm{q}})$ are in fact groupoids. Indeed, if $M, N \in R_n(A)$ and $0 \neq f : M \to N$ then f is injective by Lemma 1.9.7. Using the fact that M, N are normalized we find that the cokernel of f has (if non-zero) GK-dimension zero. Since M is reflexive, it has projective dimension one thus $\underline{\mathrm{Ext}}^1_A(k, M) = 0$ (see proof of Lemma 2.2.4). Thus $\operatorname{coker} f = 0$ and f is an isomorphism. Similarly a non-zero morphism in $\mathcal{R}_n(\mathbb{P}^2_{\mathrm{q}})$ is necesseraly an isomorphism.

3. We obtain a natural bijection between the elements of the set

$$R(A) = \{\text{reflexive rank one graded right } A\text{-modules}\}/ \text{ iso, shift}$$

and the isomorphism classes in the categories $\coprod_n R_n(A)$ and $\coprod_n \mathcal{R}_n(\mathbb{P}^2_{\mathrm{q}})$.

2.2.4 Cohomology of line bundles on quantum planes

In the next theorem we partially compute the cohomology of normalized line bundles.

Theorem 2.2.11. *Let $\mathcal{I} \in \operatorname{coh}(\mathbb{P}^2_{\mathrm{q}})$ be torsion free of rank one and normalized i.e. $[\mathcal{I}] = [\mathcal{O}] - n[\mathcal{P}]$ for some integer n. Assume $\mathcal{I} \not\cong \mathcal{O}$. Then*

1. $H^0(\mathbb{P}^2_{\mathrm{q}}, \mathcal{I}(l)) = 0$ *for* $l \leq 0$
   $H^2(\mathbb{P}^2_{\mathrm{q}}, \mathcal{I}(l)) = 0$ *for* $l \geq -2$
   $H^j(\mathbb{P}^2_{\mathrm{q}}, \mathcal{I}(l)) = 0$ *for* $j \geq 3$ *and for all integers* l

2. $\chi(\mathcal{O}, \mathcal{I}(l)) = \frac{1}{2}(l+1)(l+2) - n$ *for all integers* l

3. $\dim_k H^1(\mathbb{P}^2_{\mathrm{q}}, \mathcal{I}) = n - 1$
   $\dim_k H^1(\mathbb{P}^2_{\mathrm{q}}, \mathcal{I}(-1)) = n$
   $\dim_k H^1(\mathbb{P}^2_{\mathrm{q}}, \mathcal{I}(-2)) = n$

As a consequence, n is positive and non-zero.

If $\mathcal{I}$ is a line bundle i.e. $\mathcal{I} \in \mathcal{R}_n(\mathbb{P}^2_{\mathrm{q}})$ then we have in addition

$$H^2(\mathbb{P}^2_{\mathrm{q}}, \mathcal{I}(-3)) = 0 \text{ and } \dim_k H^1(\mathbb{P}^2_{\mathrm{q}}, \mathcal{I}(-3)) = n - 1.$$

Proof. That $H^j(\mathbb{P}^2_{\mathrm{q}}, \mathcal{I}) = 0$ for $j \geq 3$ follows from the fact that $\mathbb{P}^2_{\mathrm{q}}$ has cohomological dimension two (see Theorem 1.8.2).

To prove the rest of the current theorem we first let $l \leq 0$. Suppose f is a non-zero morphism in $\operatorname{Hom}_{\mathbb{P}^2_{\mathrm{q}}}(\mathcal{O}, \mathcal{I}(l))$. By Lemma 1.9.7 f is injective and from the exact sequence

$$0 \to \mathcal{O} \to \mathcal{I}(l) \to \operatorname{coker} f \to 0 \tag{2.20}$$

we get $[\operatorname{coker} f] = l[\mathcal{S}] + (l(l+1)/2 - n)[\mathcal{P}]$. By $\mathcal{I} \not\cong \mathcal{O}$, $\operatorname{coker} f \neq 0$. Hence $l \geq 0$, otherwise $e(\operatorname{coker} f) = l < 0$ which is impossible. Thus $l = 0$ and $[\operatorname{coker} f] = -n[\mathcal{P}]$.

We obtain $\dim(\operatorname{coker} f) = 0$. By Lemma 2.2.4 the exact sequence (2.20) splits hence $\mathcal{I}$ is not torsion free. A contradiction. We conclude $\operatorname{Hom}_{\mathbb{P}^2_q}(\mathcal{O}, \mathcal{I}(l)) = 0$ for $l \leq 0$.

Second, let $l \geq -2$. Serre duality (Theorem 1.10.5) yields

$$\operatorname{Ext}^2_{\mathbb{P}^2_q}(\mathcal{O}, \mathcal{I}(l))' \cong \operatorname{Hom}_{\mathbb{P}^2_q}(\mathcal{I}(l+3), \mathcal{O}).$$

If g is a non-zero morphism in $\operatorname{Hom}_{\mathbb{P}^2_q}(\mathcal{I}(l+3), \mathcal{O})$ then g is injective, and from the exact sequence

$$0 \to \mathcal{I}(l+3) \to \mathcal{O} \to \operatorname{coker} g \to 0 \tag{2.21}$$

we get $[\operatorname{coker} g] = u[\mathcal{S}] + v[\mathcal{P}]$ where $u = -(l+3)$ and $v = n - (l+3)(l+4)/2$. By Lemma 1.9.10 $u \geq 0$ but $l \geq -2$ implies $u < 0$. This yields a contradiction.

Assume now $l \geq -3$ and $\mathcal{I}$ reflexive. By the same reasoning as above we obtain $l = -3$ and thus the dimension of $\operatorname{coker} g$ is zero. By Lemma 2.2.4 it follows that (2.21) splits. But this contradicts the fact that $\mathcal{O}$ is torsion free.

For the second part, using (2.17) and (2.18) we obtain

$$\chi(\mathcal{O}, \mathcal{I}(l)) = \frac{1}{2}(l+1)(l+2) - n \tag{2.22}$$

for all integers l.

Finally, we combine the first two results of the theorem. If $-2 \leq l \leq 0$ (or $-3 \leq l \leq 0$ if $\mathcal{I}$ is reflexive) the first statement gives

$$\begin{aligned}\chi(\mathcal{O}, \mathcal{I}(l)) &= \dim_k H^0(\mathbb{P}^2_q, \mathcal{I}(l)) - \dim_k H^1(\mathbb{P}^2_q, \mathcal{I}(l)) + \dim_k H^2(\mathbb{P}^2_q, \mathcal{I}(l)) \\ &= -\dim_k H^1(\mathbb{P}^2_q, \mathcal{I}(l))\end{aligned}$$

and comparing with the expression (2.22) completes the proof. □

Using Theorem 2.2.11 the torsion free rank one graded right A-modules having invariant zero are easy to determine.

Corollary 2.2.12. *Let $I \in \operatorname{grmod}(A)$ be torsion free of rank one with invariant n. Then*

$$n = 0 \Leftrightarrow I \cong A(d) \text{ for some integer } d.$$

Proof. By Proposition 2.2.9 it is sufficient to prove the corresponding statement for $\pi I = \mathcal{I}$. If $\mathcal{I} \cong \mathcal{O}(d)$ then clearly $n = 0$. Assume conversely $n = 0$. We may assume $\mathcal{I}$ is normalized. If $\mathcal{I} \not\cong \mathcal{O}$ then by Theorem 2.2.11 $n > 0$. Since $n = 0$ we obtain $\mathcal{I} \cong \mathcal{O}$ by contraposition. □

As a consequence $R_0(A) = \{A\}$ and $\mathcal{R}_0(\mathbb{P}^2_q) = \{\mathcal{O}\}$.

2.2.5 Nonemptyness of $R_n(A)$ and $\mathcal{R}_n(\mathbb{P}^2_q)$

It follows from Theorem 2.2.11 that $\mathcal{R}_n(\mathbb{P}^2_q)$ is empty for $n < 0$. However we do not know yet if $\mathcal{R}_n(\mathbb{P}^2_q) \neq \emptyset$ for $n \geq 0$.

Let us restrict for a moment to the case where A is linear. Thus in the geometric data $(E, \sigma, \mathcal{O}_E(1))$ we have $E = \mathbb{P}^2$ and $\sigma \in \mathrm{Aut}(\mathbb{P}^2)$. It is a well-known fact there are no nontrivial line bundles on $\mathbb{P}^2$. By the category equivalence between $\mathrm{Tails}(A) \cong \mathrm{Tails}(B)$ and $\mathrm{Qcoh}(\mathbb{P}^2)$ we obtain

Proposition 2.2.13. *Assume A is linear and let $I \in \mathrm{grmod}(A)$ be reflexive or rank one. Then I is free i.e. $I = A(d)$ for some integer d. As a consequence $R_n(A) = \emptyset = \mathcal{R}_n(\mathbb{P}^2_q)$ for $n > 0$.*

In the elliptic case however we have

Proposition 2.2.14. *Let A be elliptic and assume σ has infinite order. Then $R_n(A)$ and $\mathcal{R}_n(\mathbb{P}^2_q)$ are nonempty for $n \geq 0$.*

Proof. Let $\mathcal{S} = \pi(A/uA)$ be a line object on $\mathbb{P}^2_q$ for which the line $\{u = 0\}$ in $\mathbb{P}^2$ is not a component of C. Writing $\mathcal{S}$ as the cokernel of a map $\mathcal{O}(-1) \to \mathcal{O}$ we find by Theorem 1.8.2 that if $n \geq -1$ then $H^0(\mathbb{P}^2_q, \mathcal{S}(n))$ has dimension $n + 1$. By [8] any non-zero subobject of a line object is a shifted line object; there exist at most three line objects $\mathcal{S}'$ such that $\mathcal{S}'(-1)$ is a subobject of $\mathcal{S}$ and furthermore any non-trivial subobject of $\mathcal{S}$ is a subobject of one of these three objects $\mathcal{S}'(-1)$. Hence if a non-zero morphism $f : \mathcal{O} \to \mathcal{S}(n)$ is not surjective then f induces a morphism $\mathcal{O} \to \mathcal{S}'(n-1)$ for such a line object $\mathcal{S}'(-1) \subset \mathcal{S}$ thus $f \in H^0(\mathbb{P}^2_q, \mathcal{S}'(n-1))$ which is n-dimensional. Hence if $n \geq 0$ then we may pick an epimorphism $f : \mathcal{O} \to \mathcal{S}(n)$ (a generic f will do). Put $\mathcal{I} = (\ker f)(1)$. We find $[\mathcal{I}(-1)] = [\mathcal{O}] - ([\mathcal{S}] + n[\mathcal{P}])$ and hence $[\mathcal{I}] = [\mathcal{O}] - n[\mathcal{P}]$. It is easy to see $\mathcal{I}$ is reflexive. Thus $\mathcal{I} \in \mathcal{R}_n(\mathbb{P}^2_q)$. □

Below we will show that for elliptic A where σ has infinite order, $\mathcal{R}_n(\mathbb{P}^2_q)$ is parametrized by a variety of dimension $2n$. The amount of freedom in the construction exhibited in the proof of Proposition 2.2.14 is less than or equal to $2(\text{choice of } \mathcal{S}) + n(\text{choice of } f)$ parameters, hence for $n > 2$ this construction can not possibly yield all elements of $\mathcal{R}_n(\mathbb{P}^2_q)$. In §2.4.9 we will exhibit a related construction which works for all n.

2.3 Restriction of line bundles to the divisor C

In this section A is an elliptic quadratic Artin-Schelter algebra, to which §1.9.4 we associate the geometric data $(C, \sigma, \mathcal{O}_C(1))$, the homogeneous coordinate ring $D = B(C, \sigma, \mathcal{O}_C(1))$ and the map of noncommutative schemes $u : C \to \mathbb{P}^2_q$. The dimension of objects in $\mathrm{grmod}(D)$ or $\mathrm{tails}(D)$ will be computed in $\mathrm{grmod}(A)$ or $\mathrm{tails}(A)$. The dimension of objects in $\mathrm{coh}(C)$ is the dimension of their support.

Lemma 2.3.1. *Assume A is an elliptic quadratic Artin-Schelter algebra.*

1. *If $M \in \operatorname{grmod}(D)$ is pure two dimensional then $\widetilde{M} \in \operatorname{coh}(C)$ is pure one dimensional.*

2. *If $\mathcal{N} \in \operatorname{coh}(C)$ is pure one dimensional then $\Gamma_*(\mathcal{N})$ is pure two dimensional.*

Proof. The indecomposable objects in $\operatorname{coh}(C)$ are vector bundles and finite length objects. Using Riemann-Roch it is easy to see that if $0 \neq \mathcal{U} \in \operatorname{coh}(C)$ then $\operatorname{GKdim} \Gamma_*(\mathcal{U}) = \dim \mathcal{U} + 1$. From this we deduce that if $V \in \operatorname{grmod}(D)$ is not in $\operatorname{tors}(D)$ then $\operatorname{GKdim} V = \dim \widetilde{V} + 1$. The lemma now easily follows. □

The following result was also proved in [60].

Proposition 2.3.2. *Assume A is an elliptic quadratic Artin-Schelter algebra.*

1. *If $\mathcal{M}$ is a vector bundle on $\mathbb{P}^2_{\mathrm{q}}$ then $L_j u^*\mathcal{M} = 0$ for $j > 0$ and $u^*\mathcal{M}$ is a vector bundle on C.*

2. *Assume σ has infinite order and $\mathcal{M} \in D^b(\operatorname{coh}(\mathbb{P}^2_{\mathrm{q}}))$ is such that $Lu^*\mathcal{M}$ is a vector bundle on C. Then $\mathcal{M}$ is a vector bundle on $\mathbb{P}^2_{\mathrm{q}}$.*

Proof.

1. We have $\mathcal{M} = \pi M$ where M is reflexive. In particular M is torsion free. By Lemma 1.9.20 it follows $L_j u^*\mathcal{M} = 0$ for $j > 0$ and $u^*\mathcal{M} = (M/Mh)^{\sim}$.

   Write $c = \deg h$. The torsionfreeness of M also implies that the multiplication map $M(-c) \xrightarrow{\cdot h} M$ is injective. Hence $M(-c) \cong Mh$ thus Mh is reflexive and $\operatorname{rank} M = \operatorname{rank} Mh$ which also gives $\operatorname{GKdim} M/Mh \leq 2$.

   If M/Mh contains a non-zero submodule N/Mh of GK-dimension ≤ 1 then it follows from the short exact sequence $0 \to Mh \to N \to N/Mh \to 0$ that N represents an element of $\operatorname{Ext}^1_A(N/Mh, Mh)$, which must be zero by Lemma 2.2.4. Thus $N/Mh \subset N \subset M$. This is impossible since M is torsion free. Hence M/Mh is pure of GK-dimension two. By the previous lemma it follows $(M/Mh)^{\sim}$ is a vector bundle.

2. It follows from Lemma 1.9.21 that $u^*H^j(\mathcal{M}) = 0$ for $j \neq 0$. Then it follows from Lemma 1.9.19 that $\mathcal{M} \in \operatorname{coh}(\mathbb{P}^2_{\mathrm{q}})$ and $L_1 u^*\mathcal{M} = 0$, using Lemma 1.9.21 again.

   Pick an object M in $\operatorname{grmod}(A)$ such that $\pi M = \mathcal{M}$. We may assume M contains no subobject in $\operatorname{tors}(A)$. By Lemma 1.9.20 we have $L_1 u^*\mathcal{M} = \ker(M(-c) \xrightarrow{\times h} M)^{\sim}$. Thus $\ker(M(-c) \xrightarrow{\times h} M) \in \operatorname{tors}(A)$. Since M contains no subobject in $\operatorname{tors}(A)$ it follows that M is h-torsion free. Furthermore by Lemma 2.3.1 $\Gamma_*(u^*\mathcal{M}) = \Gamma_*((M/Mh)^{\sim})$ is pure two dimensional. If T is the maximal submodule of M/Mh which is in $\operatorname{tors}(A)$ then since $(M/Mh)/T \subset \Gamma_*((M/Mh)^{\sim})$ we obtain $(M/Mh)/T$ is pure two dimensional.

We now claim M is pure three dimensional. Let N be the maximal submodule of M of dimension ≤ 2. Then $K = M/N$ is pure three dimensional and in particular h-torsion-free. Hence we have a short exact sequence

$$0 \to N/Nh \to M/Mh \to K/Kh \to 0$$

By the purity of $(M/Mh)/T$ it follows $N/Nh \subset T$ and hence $N/Nh \in \mathrm{tors}(A)$. It follows from Lemma 1.9.19 than $N \in \mathrm{tors}(A)$ and hence $N = 0$. This shows M is pure.

Put $Q = M^{**}/M$. Thus we obtain an exact sequence

$$0 \to \mathrm{Tor}_1^A(Q, D)^{\sim} \to (M \otimes_A D)^{\sim} \to (M^{**} \otimes_A D)^{\sim} \to (Q \otimes_A D)^{\sim} \to 0$$

By [8] we have $\mathrm{GKdim}\, Q \leq 1$. Thus we have $\mathrm{GKdim}\, \mathrm{Tor}_1^A(Q, D) \leq \mathrm{GKdim}\, Q \leq 1$. So by the proof of Lemma 2.3.1, $\dim \mathrm{Tor}_1^A(Q, D)^{\sim} \leq 0$. Since $(M \otimes_A D)^{\sim}$ is a vector bundle by hypotheses it contains no finite dimensional subobjects and we obtain $\mathrm{Tor}_1^A(Q, D)^{\sim} = 0$. Thus $\mathrm{Tor}_1^A(Q, D) \in \mathrm{tors}(A)$. Thus, in high degree, multiplication by h is an isomorphism on Q. But then by Lemma 1.9.19 $Q \in \mathrm{tors}(A)$. Hence $\mathcal{M} = \pi M = \pi M^{**}$ and thus $\mathcal{M}$ is reflexive.

□

Although some of the following results may be generalized [60], for the rest of this Section 2.3 we will restrict to the case where A is a three dimensional Sklyanin algebra $\mathrm{Skl}_3(a, b, c)$. Thus E is a smooth elliptic curve and fixing a group law on E the automorphism σ is a translation by some element $\xi \in C$ i.e. $\sigma p = p + \xi$ for all $p \in E$. Since E is reduced the geometric data $(E, \sigma, \mathcal{O}_E(1))$ and $(C, \sigma, \mathcal{O}_C(1))$ coincide and $g = h \in A_3$, $D = B = B(E, \sigma, \mathcal{O}_E(1))$.

The functor $u^* : \mathrm{coh}(\mathbb{P}^2_q) \to \mathrm{coh}(C)$ induces a group homomorphism

$$\begin{aligned} u^* : K_0(\mathbb{P}^2_q) &\to K_0(C) \\ [\mathcal{M}] &\mapsto \sum_j (-1)^j [L_j u^* \mathcal{M}] = [u^* \mathcal{M}] - [L_1 u^* \mathcal{M}] \end{aligned}$$

Recall the basis $\mathcal{B} = \{[\mathcal{O}], [\mathcal{S}], [\mathcal{Q}], [\mathcal{P}]\}$ for $K_0(X)$ from §2.2.2. The image of $\mathcal{B}$ under u^* is computed in the following

Lemma 2.3.3. *Assume A is a three dimensional Sklyanin algebra. We have*

$$\begin{aligned} u^*[\mathcal{O}] &= [\mathcal{O}_C] \\ u^*[\mathcal{S}] &= [\mathcal{O}_p] + [\mathcal{O}_q] + [\mathcal{O}_r] && p, q, r \in C \text{ arbitrary but collinear} \\ u^*[\mathcal{P}] &= [\mathcal{O}_p] - [\mathcal{O}_{\sigma^{-3}p}] && p \in C \text{ arbitrary} \end{aligned}$$

Proof. Since A is h-torsion free it follows $u^*[\mathcal{O}] = [u^*\mathcal{O}] = [\mathcal{O}_C]$. Similarly $u^*[\mathcal{S}] = [u^*\mathcal{S}]$. Write $\mathcal{S} = \pi(A/aA)$ for some $a \in A_1$. From [8] it follows that $u^*\mathcal{S} = \mathcal{O}_L$ where

L is the scheme-theoretic intersection $C \cap a$. Since L consists of three collinear points we obtain $[\mathcal{O}_L] = [\mathcal{O}_p] + [\mathcal{O}_q] + [\mathcal{O}_r]$ for some collinear points $p, q, r \in C$.

Finally, put $\mathcal{P} = \mathcal{N}_p$ where $p \in C$. Applying $N_p \otimes_A -$ to the short exact sequence of A-bimodules

$$0 \to A(-3) \xrightarrow{\cdot h} A \to D \to 0$$

we get the exact sequence

$$0 \to \operatorname{Tor}_1^A(N_p, D) \to N_p(-3) \xrightarrow{\cdot h} N_p \to N_p \otimes_A D \to 0$$

Now $N_p \otimes_A D \cong N_p/N_p h = N_p$ thus $u^*\mathcal{N}_p = \widetilde{N_p} = \mathcal{O}_p$. This also means $N_p(-3) \xrightarrow{\cdot h} N_p$ is the zero map. Thus $\operatorname{Tor}_1^A(N_p, D) = N_p(-3) = \left(N_{\sigma^{-3}p}\right)_{\geq 3}$ by (1.24). We conclude

$$L_1 u^*\mathcal{N}_p = \operatorname{Tor}_1^A(N_p, D)^{\sim} = \left(N_{\sigma^{-3}p}\right)^{\sim} = \mathcal{O}_{\sigma^{-3}p} \quad \square$$

From Lemma 2.3.3 we deduce for $\mathcal{M} \in \operatorname{coh}(\mathbb{P}^2_q)$ for which $[\mathcal{M}] = r[\mathcal{O}] + a[\mathcal{S}] + b[\mathcal{P}]$ we have

$$\operatorname{rank} u^*[\mathcal{M}] = r = \operatorname{rank} \mathcal{M} \text{ and } \deg u^*[\mathcal{M}] = 3a \tag{2.23}$$

Proposition 2.3.4. *Let A be a three dimensional Sklyanin algebra.*

1. *If $\mathcal{I}$ is a line bundle on $\mathbb{P}^2_q$ then $u^*\mathcal{I}$ is a line bundle on C, and $\mathcal{I}$ is normalized if and only if $\deg u^*\mathcal{I} = 0$.*
2. *If $\mathcal{I}$ is a normalized line bundle on $\mathbb{P}^2_q$ with invariant n then*

$$c_1(u^*\mathcal{I}) = \mathcal{O}((o) - (3n\xi))$$

where "o" is the origin for the group law.

Proof. 1. This follows from Proposition 2.3.2 and (2.23).

2. We have $[\mathcal{M}] = [\mathcal{O}] - n[\mathcal{P}]$. By Lemma 2.3.3 we obtain $[u^*\mathcal{M}] = [\mathcal{O}_C] - n[\mathcal{O}_p] + n[\mathcal{O}_{\sigma^{-3}p}]$. Hence $c_1(u^*\mathcal{M}) = \mathcal{O}(n(\sigma^{-3}p) - n(p))$. From $\sigma p = p + \xi$ we deduce $\sigma^{-3}p = p - 3\xi$. Thus $n(\sigma^{-3}p) - n(p)$ and $(o) - (3n\xi)$ are both divisors of degree zero which have the same sum for the group law. Hence they are linearly equivalent by [41, IV Thm 4.13B]. This finishes the proof. $\square$

Corollary 2.3.5. *Let A be a three dimensional Sklyanin algebra and assume σ has infinite order. Then the category*

$$\mathcal{R}(\mathbb{P}^2_q) = \coprod_n \mathcal{R}_n(\mathbb{P}^2_q) = \{\textit{normalized line bundles on } \mathbb{P}^2_q\}$$

is equivalent to the full subcategory of $\operatorname{coh}(\mathbb{P}^2_q)$ with objects

$$\{\mathcal{M} \in \operatorname{coh}(\mathbb{P}^2_q) \mid u^*\mathcal{M} \textit{ is a line bundle on } C \textit{ of degree zero}\}$$

Proof. Due to Proposition 2.3.4 it is sufficient to prove that if $\mathcal{M} \in \mathrm{coh}(\mathbb{P}^2_q)$ for which $u^*\mathcal{M} \in \mathrm{coh}(C)$ is a line bundle of degree zero, then $\mathcal{M}$ is a normalized line bundle on $\mathbb{P}^2_q$. Pick $M \in \mathrm{grmod}(A)$ for which $\pi M = \mathcal{M}$. By Proposition 2.3.2 it suffices to prove $Lu^*\mathcal{M} = u^*\mathcal{M}$ i.e. $L_1 u^*\mathcal{M} = 0$.

It is sufficient to prove M is torsion free, since it then follows M is h-torsion free whence $L_1u^*\mathcal{M} = \ker(M(-3) \xrightarrow{\times h} M)\tilde{} = 0$. So let us assume by contradiction M is not torsion free. Let $T \subset M$ the maximal torsion submodule of M. Thus $0 \neq M/T$ is torsion free. Applying u^* to $0 \to \pi T \to \mathcal{M} \to \pi(M/T) \to 0$ then gives the exact sequence $0 \to u^*\pi T \to u^*\mathcal{M} \to u^*\pi(M/T) \to 0$. Since $u^*\mathcal{M}$ is a line bundle on C, it is pure hence either $u^*\pi T$ is a line bundle or $u^*\pi T = 0$.

If $u^*\pi T$ would be a line bundle then $u^*\pi(M/T) = (M/T \otimes_A D)\tilde{}$ has rank zero. Thus $M/T \otimes_A D \in \mathrm{grmod}(D)$ has GK-dimension ≤ 1. But then $\mathrm{GKdim}\, M/T \leq 2$, a contradiction with the fact $M/T \in \mathrm{grmod}(A)$ is non-zero and torsion free. Thus $u^*\pi T = 0$ i.e. $(T/hT)\tilde{} = 0$. This means $\pi(T/hT) = 0$ hence $T/hT \in \mathrm{tors}(A)$. By Lemma 1.9.19 we deduce $T \in \mathrm{tors}(A)$ thus $T = 0$ since M contains no subobjects in $\mathrm{tors}(A)$. This ends the proof. □

Remark 2.3.6. As pointed out before, some of the results above may be generalized to other elliptic quadratic Artin-Schelter algebras. See also [60]. For example, if we consider the situation where $A = H$ is the homogenized Weyl algebra then one obtains as stated in [16]

- If $\mathcal{I}$ is a line bundle on $\mathbb{P}^2_q$ then $u^*\mathcal{I}$ is a line bundle on $C = \mathbb{P}^1$, and $\mathcal{I}$ is normalized if and only if $u^*\mathcal{I}$ has degree zero, i.e. if and only if $u^*\mathcal{I} \cong \mathcal{O}_{\mathbb{P}^1}$ (since $\mathrm{Pic}(\mathbb{P}^1) = \mathbb{Z}$).

- The category $\mathcal{R}(\mathbb{P}^2_q) = \coprod_n \mathcal{R}_n(\mathbb{P}^2_q)$ is equivalent to the full subcategory of $\mathrm{coh}(\mathbb{P}^2_q)$ with objects $\{\mathcal{M} \in \mathrm{coh}(\mathbb{P}^2_q) \mid u^*\mathcal{M} \cong \mathcal{O}_{\mathbb{P}^1}\}$.

2.4 From line bundles to quiver representations

Throughout this Section 2.4, A is a quadratic Artin-Schelter algebra. From §2.4.3 onwards we will furthermore assume A is elliptic (and for most results σ has infinite order). As usual we write $(C, \sigma, \mathcal{O}_C(1))$ for the geometric data associated to A and $u : C \to \mathbb{P}^2_q$ for the map of noncommutative schemes as defined in §1.9.4.

2.4.1 Generalized Beilinson equivalence

We set $\mathcal{E} = \mathcal{O}(2) \oplus \mathcal{O}(1) \oplus \mathcal{O}$ and

$$U = \mathrm{Hom}_{\mathbb{P}^2_q}(\mathcal{E}, \mathcal{E}) = \bigoplus_{i,j=0}^{2} \mathrm{Hom}_{\mathbb{P}^2_q}(\mathcal{O}(i), \mathcal{O}(j))$$

the algebra of endomorphisms of $\mathcal{E}$. We consider the left exact functor $\mathrm{Hom}_{\mathbb{P}^2_q}(\mathcal{E},-)$ which takes coherent sheaves on $\mathbb{P}^2_q$ to right U-modules.

Now $\mathrm{Hom}_{\mathbb{P}^2_q}(\mathcal{E},-)$ extends to a functor $\mathrm{RHom}_{\mathbb{P}^2_q}(\mathcal{E},-)$ on bounded derived categories

$$\mathrm{RHom}_{\mathbb{P}^2_q}(\mathcal{E},-) : \mathrm{D}^b(\mathrm{coh}(\mathbb{P}^2_q)) \to \mathrm{D}^b(\mathrm{mod}(U)). \tag{2.24}$$

This is done as follows: $\mathrm{Qcoh}(\mathbb{P}^2_q)$ has enough injectives and this yields a functor $\mathrm{RHom}_{\mathbb{P}^2_q}(\mathcal{E},-) : \mathrm{D}^b_{\mathrm{coh}(\mathbb{P}^2_q)}(\mathrm{Qcoh}(\mathbb{P}^2_q)) \to D^b_{\mathrm{mod}(U)}(\mathrm{Mod}(U))$. Now $\mathrm{coh}(\mathbb{P}^2_q)$ and $\mathrm{mod}(U)$ are noetherian abelian categories and this yields equivalences $\mathrm{D}^b(\mathrm{coh}(\mathbb{P}^2_q)) \cong \mathrm{D}^b_{\mathrm{coh}(\mathbb{P}^2_q)}(\mathrm{Qcoh}(\mathbb{P}^2_q))$ and $\mathrm{D}^b(\mathrm{mod}(U)) \cong D^b_{\mathrm{mod}(U)}(\mathrm{Mod}(U))$ (Lemma 1.1.2). The functor (2.24) is obtained by composing with these equivalences.

In a similar way as in [18, Theorem 6.2] one shows $\mathrm{RHom}_{\mathbb{P}^2_q}(\mathcal{E},-)$ is an equivalence of derived categories. The inverse functor is given by $- \overset{\mathbf{L}}{\otimes}_U \mathcal{E}$

$$\mathrm{D}^b(\mathrm{coh}(\mathbb{P}^2_q)) \underset{-\overset{\mathbf{L}}{\otimes}_U \mathcal{E}}{\overset{\mathrm{RHom}_{\mathbb{P}^2_q}(\mathcal{E},-)}{\rightleftarrows}} \mathrm{D}^b(\mathrm{mod}(U)) \tag{2.25}$$

For the commutative case $A = k[x,y,z]$ this derived equivalence is known as Beilinson equivalence [15]. Therefore we sometimes refer to (2.25) as *generalized* Beilinson equivalence.

For a non-negative integer i generalized Beilinson equivalence restricts to an equivalence between $\mathcal{X}_i$ and $\mathcal{Y}_i$ where $\mathcal{X}_i \subset \mathrm{coh}(\mathbb{P}^2_q)$ is the full subcategory with objects

$$\mathcal{X}_i = \{\mathcal{M} \in \mathrm{coh}(\mathbb{P}^2_q) \mid \mathrm{Ext}^j_{\mathbb{P}^2_q}(\mathcal{E},\mathcal{M}) = 0 \text{ for } j \neq i\}$$

and $\mathcal{Y}_i \subset \mathrm{mod}(U)$ the full subcategory with objects

$$\mathcal{Y}_i = \{M \in \mathrm{mod}(U) \mid \mathrm{Tor}^U_j(M,\mathcal{E}) = 0 \text{ for } j \neq i\}.$$

The inverse equivalences between these categories are given by $\mathrm{Ext}^i_{\mathbb{P}^2_q}(\mathcal{E},-)$ and $\mathrm{Tor}^U_i(-,\mathcal{E})$ (see for example [12, Theorem 3.2.1])

$$\mathcal{X}_i \underset{\mathrm{Tor}^U_i(-,\mathcal{E})}{\overset{\mathrm{Ext}^1_{\mathbb{P}^2_q}(\mathcal{E},-)}{\rightleftarrows}} \mathcal{Y}_i$$

It is easy to see $U \cong k\Delta/(R)$ where (Δ, R) is the quiver

$$-2 \overset{X_{-2}}{\underset{Z_{-2}}{\overset{Y_{-2}}{\rightrightarrows\!\!\to}}} -1 \overset{X_{-1}}{\underset{Z_{-1}}{\overset{Y_{-1}}{\rightrightarrows\!\!\to}}} 0 \tag{2.26}$$

with relations R reflecting the relations of A. If we write the relations of A as (1.18) then the relations R are given by

$$\begin{pmatrix} X_{-1} & Y_{-1} & Z_{-1} \end{pmatrix} \cdot M_A(X_{-2}, Y_{-2}, Z_{-2})^t = 0 \tag{2.27}$$

where $M_A(X_{-2}, Y_{-2}, Z_{-2})$ is obtained from the matrix M_A by replacing x, y, z by X_{-2}, Y_{-2}, Z_{-2}. For example, if $A = H$ is the homogenized Weyl algebra then R is given by

$$\begin{cases} Z_{-1}Y_{-2} = Y_{-1}Z_{-2} \\ X_{-1}Z_{-2} = Z_{-1}X_{-2} \\ Y_{-1}X_{-2} - X_{-1}Y_{-2} = Z_{-1}Z_{-2} \end{cases}$$

and case A is of type A the relations are given by

$$\begin{cases} aZ_{-1}Y_{-2} + bY_{-1}Z_{-2} + cX_{-1}X_{-2} = 0 \\ aX_{-1}Z_{-2} + bZ_{-1}X_{-2} + cY_{-1}Y_{-2} = 0 \\ aY_{-1}Z_{-2} + bX_{-1}Y_{-2} + cZ_{-1}Z_{-2} = 0 \end{cases}$$

Recall from §1.3 that representations of quivers are always assumed to satisfy the relations of the quiver. Since the category $\operatorname{Mod}(\Delta)$ of representations of Δ is equivalent to the category of right $k\Delta/(R)$-modules we deduce $\operatorname{Mod}(\Delta) \cong \operatorname{Mod}(U)$. From now on we write $\operatorname{Mod}(\Delta)$ instead of $\operatorname{Mod}(U)$.

One verifies the matrix representation of the Euler form $\chi : K_0(\Delta) \times K_0(\Delta) \to \mathbb{Z}$ with respect to the basis $\{S_{-2}, S_{-1}, S_0\}$ of $K_0(\Delta)$ is given by

$$\begin{pmatrix} 1 & -3 & 3 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{pmatrix} \tag{2.28}$$

Under the isomorphism $K_0(\Delta) \cong \mathbb{Z}^3$ we identify χ with the associated bilinear form on $\mathbb{Z}^3$.

2.4.2 Point and line representations

For further use we need to determine the representations of Δ corresponding to point and line objects on $\mathbb{P}^2_q$. Recall for any point $p \in C$ we write N_p for the corresponding point module over A, and $\mathcal{N}_p = \pi N_p$ for the point module on $\mathbb{P}^2_q$.

Lemma 2.4.1. *Let $p = (\alpha, \beta, \gamma) \in C$ and put $(\alpha_i, \beta_i, \gamma_i) = \sigma^i p$.*

1. *$H^j(\mathbb{P}^2_q, \mathcal{N}_p(m)) = 0$ for all integers m and $j > 0$. In particular $\mathcal{N}_p \in \mathcal{X}_0$.*
2. *$\dim_k (\omega\mathcal{N}_p)_m = 1$ for all m and $(\omega\mathcal{N}_p)_{\geq m}$ is a shifted point module for all integers m. In particular $(\omega\mathcal{N}_p)_{\geq 0} = N_p$.*
3. *$H^0(\mathbb{P}^2_q, \mathcal{N}_p(m)) = (\omega\mathcal{N}_p)_m$.*

4. *Write* $\mathrm{RHom}_{\mathbb{P}^2_q}(\mathcal{E},\mathcal{N}_p) = p$. *Then* $\underline{\dim} p = (1,1,1)$ *and* $p \in \mathrm{mod}(\Delta)$ *corresponds to*

$$k \overset{\alpha_{-2}}{\underset{\gamma_{-2}}{\overset{\beta_{-2}}{\rightrightarrows}}} k \overset{\alpha_{-1}}{\underset{\gamma_{-1}}{\overset{\beta_{-1}}{\rightrightarrows}}} k$$

Proof. 1. By Equation (1.24) it is sufficient to treat the case $m = 0$. We use Lemma 1.9.20 and the discussion before that. We have $\mathcal{N}_p = u_*\mathcal{O}_p$ and hence $\mathrm{Ext}^j_{\mathbb{P}^2_q}(\mathcal{O},\mathcal{N}_p) = \mathrm{Ext}^j_C(Lu^*\mathcal{O},\mathcal{O}_p) = \mathrm{Ext}^j_C(\mathcal{O}_C,\mathcal{O}_p) = 0$ for $j > 0$.

2. This is easy to check.

3. Use $\omega = \underline{\mathrm{Hom}}_{\mathrm{Tails}}(\mathcal{O}, -)$.

4. This follows from the previous step. □

Lemma 2.4.2. *Let* $\mathcal{S} = \pi(A/uA)$ *be a line object on* $\mathbb{P}^2_q$ *(where* $u \in A_1$*).*

1. $H^1(\mathbb{P}^2_q, \mathcal{S}(m)) \cong (A/Au)'_{-m-2}$ *for* $m \leq -1$

2. $H^j(\mathbb{P}^2_q, \mathcal{S}(m)) = 0$ *for* $m \leq -1$ *and* $j \neq 1$. *In particular* $\mathcal{S}(-1) \in \mathcal{X}_1$.

3. *If* $\eta \in A_1$ *then the induced linear map* $H^1(\mathbb{P}^2_q, \mathcal{S}(m)) \xrightarrow{\cdot\eta} H^1(\mathbb{P}^2_q, \mathcal{S}(m+1))$ *corresponds to* $(\eta\cdot)'$ *on* $(A/Au)'$.

4. *Write* $\mathrm{RHom}_{\mathbb{P}^2_q}(\mathcal{E},\mathcal{S}(-1)) = S[-1]$. *Then* $\underline{\dim} S = (2,1,0)$ *and* $S \in \mathrm{mod}(\Delta)$ *corresponds to*

$$(A/Au)'_1 \overset{(x\cdot)'}{\underset{(z\cdot)'}{\overset{(y\cdot)'}{\rightrightarrows}}} k \rightrightarrows 0 \tag{2.29}$$

Proof. That $H^0(\mathbb{P}^2_q, \mathcal{S}(m)) = 0$ follows by writing $\mathcal{S}$ as the cokernel of a map $\mathcal{O}(-1) \to \mathcal{O}$ and invoking Theorem 1.8.2. That $H^2(\mathbb{P}^2_q, \mathcal{S}(m)) = 0$ follows by Serre duality (Theorem 1.10.5). Using Theorem 1.8.2 we find

$$H^1(\mathbb{P}^2_q, \mathcal{S}(m)) = \ker(\mathrm{Ext}^2_{\mathbb{P}^2_q}(\mathcal{O}(-m),\mathcal{O}(-1)) \xrightarrow{(-,u\cdot)} \mathrm{Ext}^2_{\mathbb{P}^2_q}(\mathcal{O}(-m),\mathcal{O}))$$

By Serre duality this translates into

$$H^1(\mathbb{P}^2_q, \mathcal{S}(m)) = \ker(\mathrm{Hom}_{\mathbb{P}^2_q}(\mathcal{O}(-1),\mathcal{O}(-m-3))' \xrightarrow{(u\cdot,-)'} \mathrm{Hom}_{\mathbb{P}^2_q}(\mathcal{O},\mathcal{O}(-m-3))')$$

Dualizing yields indeed $H^1(\mathbb{P}^2_q, \mathcal{S}(m)) \cong (A/Au)'_{-m-2}$. That η acts in the indicated way follows by inspecting the appropriate commutative diagram. The final statement follows immediately. □

2.4.3 First description of $\mathcal{R}_n(\mathbb{P}^2_q)$

Recall from §2.2.3 that $\mathcal{R}_n(\mathbb{P}^2_q)$ is by definition the full subcategory of $\mathrm{coh}(\mathbb{P}^2_q)$ which objects are given by

$$\begin{aligned}\mathcal{R}_n(\mathbb{P}^2_q) &= \{\text{normalized line bundles on } \mathbb{P}^2_q \text{ with invariant } n\} \\ &= \{\mathcal{M} \in \mathrm{coh}(\mathbb{P}^2_q) \mid \mathcal{M} \text{ reflexive and } [\mathcal{M}] = [\mathcal{O}] - n[\mathcal{P}]\}\end{aligned}$$

By the discussion in §2.2.5 and Corollary 2.2.12 we may (and will) assume for the rest of §2.4 that A is elliptic and $n \geq 0$. Where needed we will furthermore assume σ has infinite order.

We would like to understand the image of $\mathcal{R}_n(\mathbb{P}^2_q)$ under the generalized Beilinson equivalence (2.25). Let $\mathcal{M}$ be an object of $\mathcal{R}_n(\mathbb{P}^2_q)$ and consider $\mathcal{M}$ as a complex in $\mathrm{D}^b(\mathrm{coh}(\mathbb{P}^2_q))$ of degree zero. Theorem 2.2.11 implies $\mathcal{M} \in \mathcal{X}_1$ thus the image of this complex is concentrated in degree one

$$\mathrm{RHom}_{\mathbb{P}^2_q}(\mathcal{E}, \mathcal{M}) = M[-1]$$

where $M = \mathrm{Ext}^1_{\mathbb{P}^2_q}(\mathcal{E}, \mathcal{M})$. Hence M is a representation of Δ. By functoriality, multiplication by $x \in A$ induces linear maps

$$H^1(\mathbb{P}^2_q, \mathcal{M}(-2)) \xrightarrow{M(X_{-2})\cdot} H^1(\mathbb{P}^2_q, \mathcal{M}(-1)) \xrightarrow{M(X_{-1})\cdot} H^1(\mathbb{P}^2_q, \mathcal{M})$$

and similar for multiplication with y, z hence M is determined by the following representation of Δ

$$H^1(\mathbb{P}^2_q, \mathcal{M}(-2)) \overset{M(X_{-2})\cdot}{\underset{M(Z_{-2})\cdot}{\xrightarrow[M(Y_{-2})\cdot]{}}} H^1(\mathbb{P}^2_q, \mathcal{M}(-1)) \overset{M(X_{-1})\cdot}{\underset{M(Z_{-1})\cdot}{\xrightarrow[M(Y_{-1})\cdot]{}}} H^1(\mathbb{P}^2_q, \mathcal{M})$$

We denote $\mathcal{C}_n(\Delta)$ for the image of $\mathcal{R}_n(\mathbb{P}^2_q)$ under the equivalence $\mathcal{X}_1 \cong \mathcal{Y}_1$.

Theorem 2.4.3. *Let A be an elliptic quadratic Artin-Schelter algebra where σ has infinite order. Let $n > 0$. Then there is an equivalence of categories*

$$\mathcal{R}_n(\mathbb{P}^2_q) \underset{\mathrm{Tor}^{\Delta}_1(-,\mathcal{E})}{\overset{\mathrm{Ext}^1_{\mathbb{P}^2_q}(\mathcal{E},-)}{\rightleftarrows}} \mathcal{C}_n(\Delta)$$

where

$$\mathcal{C}_n(\Delta) = \{M \in \mathrm{mod}(\Delta) \mid \underline{\dim} M = (n, n, n-1) \text{ and } \mathrm{Hom}_\Delta(M, p) = 0, \mathrm{Hom}_\Delta(p, M) = 0 \text{ for all } p \in C\}.$$

Proof. First, let $\mathcal{I}$ be an object of $\mathcal{R}_n(\mathbb{P}^2_q)$ and write $I = \mathrm{Ext}^1_{\mathbb{P}^2_q}(\mathcal{E}, \mathcal{I})$. That $\underline{\dim} I = (n, n, n-1)$ follows from Theorem 2.2.11. Further, let $p \in C$ and as in Lemma 2.4.1 we denote $p = \mathrm{Hom}_{\mathbb{P}^2_q}(\mathcal{E}, \mathcal{N}_p)$. Lemma 2.2.4 implies $\mathrm{Ext}^i_{\mathbb{P}^2_q}(\mathcal{I}, \mathcal{N}_p) = 0$ for $i \geq 1$ hence $\chi(\mathcal{I}, \mathcal{N}_p) = 1$ yields

$$k = \mathrm{RHom}_{\mathbb{P}^2_q}(\mathcal{I}, \mathcal{N}_p) \cong \mathrm{RHom}_\Delta(I[-1], p)$$

In particular $\mathrm{Hom}_\Delta(I, p) = H^1(\mathrm{RHom}_\Delta(I[-1], p)) = 0$.
Further we compute $\chi(\mathcal{N}_p, \mathcal{I}) = 1$ and again by Lemma 2.2.4

$$k[-2] = \mathrm{RHom}_{\mathbb{P}^2_q}(\mathcal{N}_p, \mathcal{I}) \cong \mathrm{RHom}_\Delta(p, I[-1])$$

and in particular $\mathrm{Hom}_\Delta(p, I) = 0$.

Conversely let $M \in \mathrm{mod}(\Delta)$ such that $\underline{\dim} M = (n, n, n-1)$ and $\mathrm{Hom}_\Delta(M, p) = \mathrm{Hom}_\Delta(p, M) = 0$ for all $p \in C$. By Corollary 1.10.6

$$\begin{aligned} H^2(\mathrm{RHom}_\Delta(M, p)) &= H^2(\mathrm{RHom}_{\mathbb{P}^2_q}(M \overset{\mathbf{L}}{\otimes}_\Delta \mathcal{E}, \mathcal{N}_p)) \\ &\cong H^0(\mathrm{RHom}_{\mathbb{P}^2_q}(\mathcal{N}_{\sigma^3 p}, M \overset{\mathbf{L}}{\otimes}_\Delta \mathcal{E})) = H^0(\mathrm{RHom}_\Delta(\sigma^3 p, M)) \end{aligned}$$

Thus $\mathrm{Hom}_\Delta(M, p) = \mathrm{Ext}^2_\Delta(M, p) = 0$ for all $p \in C$. Now $\mathrm{gldim}\,\mathrm{mod}(\Delta) = 2$ so we may compute $\dim_k \mathrm{Ext}^1_\Delta(M, p)$ using the Euler form on $\mathrm{mod}(\Delta)$. We obtain $\chi(p, M) = -1$ hence $\mathrm{Ext}^1_\Delta(M, p) = k$. In other words $\mathrm{RHom}_\Delta(M[-1], p) = k$.

Put $\mathcal{M} = M[-1] \overset{\mathbf{L}}{\otimes}_\Delta \mathcal{E}$. By the generalized Beilinson equivalence (2.25) we obtain $\mathrm{RHom}_{\mathbb{P}^2_q}(\mathcal{M}, \mathcal{N}_p) = k$, giving (by adjointness) $\mathrm{RHom}_C(Lu^*\mathcal{M}, \mathcal{O}_p) = k$. By Lemma 1.1.3 this implies $Lu^*\mathcal{M}$ is a line bundle on C. Hence by Proposition 2.3.2 $\mathcal{M}$ is a vector bundle on $\mathbb{P}^2_q$. In particular $M \in \mathcal{Y}_1$. What is left to check is that $\mathcal{M}$ is normalized of rank one. The derived equivalence (2.25) gives rise to group isomorphisms

$$\mu : K_0(\mathbb{P}^2_q) \to K_0(\Delta) : [\mathcal{N}] \mapsto \sum_i (-1)^i [\mathrm{Ext}^i_{\mathbb{P}^2_q}(\mathcal{E}, \mathcal{N})]$$
$$\nu : K_0(\Delta) \to K_0(\mathbb{P}^2_q) : [N] \mapsto \sum_i (-1)^i [\mathrm{Tor}^\Delta_i(N, \mathcal{E})]$$

inverse of each other, see for example [12, Proposition 3.2.3]. Using Lemmas 2.4.1 and 2.4.2 it is checked that the image of the basis $\{[\mathcal{O}], [\mathcal{S}], [\mathcal{P}]\}$ for $\mathbb{P}^2_q$ under μ is the $\mathbb{Z}$-basis $\{[S_0], -2[S_{-2}] - [S_{-1}], [S_{-2}] + [S_{-1}] + [S_0]\}$ for $K_0(\Delta)$. And since $M \in \mathcal{Y}_1$ we have $\nu[M] = -[\mathcal{M}]$. It is then easy to check $[\mathcal{M}] = [\mathcal{O}] - n[\mathcal{P}]$. We conclude $\mathcal{M}$ is a normalized line bundle on $\mathbb{P}^2_q$ i.e. $\mathcal{M} \in \mathcal{R}_n(\mathbb{P}^2_q)$. □

Remark 2.4.4. For the homogenized Weyl algebra it was shown in [16]

$$\begin{aligned} \mathcal{C}_n(\Delta) = \{M \in \mathrm{mod}(\Delta) \mid \underline{\dim} M = (n, n, n-1) \text{ and} \\ M(Z_{-2}) \text{ isomorphism and } M(Z_{-1}) \text{ surjective } \} \end{aligned} \tag{2.30}$$

and in fact, one may now show directly this is equivalent with the desciption given in Theorem 2.4.3.

2.4.4 Line bundles on $\mathbb{P}^2_q$ with invariant one

It is now easy to parametrize the line bundles on $\mathbb{P}^2_q$ with invariant one. For the homogenized Weyl algebra this result can also be deduced from [16].

Corollary 2.4.5. *Let A be an elliptic quadratic Artin-Schelter algebra where σ has infinite order. The normalized line bundles of invariant one $\mathcal{R}_1(\mathbb{P}^2_q)$ correspond to the objects in $\mathcal{C}_1(\Delta)$ which are the representations of the form*

$$k \overset{\alpha,\ \beta,\ \gamma}{\rightrightarrows} k \overset{0,\ 0,\ 0}{\rightrightarrows} 0 \tag{2.31}$$

for some $(\alpha, \beta, \gamma) \in \mathbb{P}^2 - C$

Proof. First let $F \in \mathcal{C}_1(\Delta)$. By Theorem 2.4.3 F is given by a representation as in (2.31) for some scalars $\alpha, \beta, \gamma \in k$. The condition $\mathrm{Hom}_\Delta(p, F) = 0$ for $p \in C$ implies $(\alpha, \beta, \gamma) \notin C$. With a little more thought we also have $(\alpha, \beta, \gamma) \neq (0, 0, 0)$. Conversely let F be as in (2.31) with $(\alpha, \beta, \gamma) \in \mathbb{P}^2 - C$. Then we immediately have $\mathrm{Hom}_\Delta(p, F) = \mathrm{Hom}_\Delta(F, p) = 0$ for $p \in C$. □

2.4.5 Induced Kronecker quiver representations

Although the category $\mathcal{C}_n(\Delta)$ has a fairly elementary description in Theorem 2.4.3 it is not so easy to handle. One may ask if one can simplify the description of $\mathcal{C}_n(\Delta)$ in the Sklyanin case as done in for the homogenized Weyl algebra, see (2.30), (2.6), (2.7). At this point we mention the insight of Le Bruyn [51] that the representations $M \in \mathcal{C}_n(\Delta)$ in the Weyl case are determined by the three most left maps, using an argument of Baer [12]. We will mimic this idea.

Below, A is an elliptic quadratic Artin-Schelter algebra. Let Δ^0 be the full subquiver of Δ consisting of the vertices $-2, -1$ and let $\mathrm{Res} : \mathrm{Mod}(\Delta) \to \mathrm{Mod}(\Delta^0)$ be the obvious restriction functor. Res has a left adjoint which we denote by Ind. If e is the sum of the vertices of Δ^0 then $\mathrm{Ind} = - \otimes_{k\Delta^0} ek\Delta$. Note $\mathrm{Res} \circ \mathrm{Ind} = \mathrm{id}$. If $M \in \mathrm{Mod}(\Delta)$ we will denote $\mathrm{Res}\, M$ by M^0. We have

Lemma 2.4.6. *Let $M \in \mathrm{mod}(\Delta)$. Then $M = \mathrm{Ind}\,\mathrm{Res}\, M$ if and only if $M \perp S_0$.*

Proof. First assume $M = \mathrm{Ind}\,\mathrm{Res}\, M$. Put $M^0 = \mathrm{Res}\, M$ and take a projective resolution

$$0 \to F_1^0 \to F_0^0 \to M^0 \to 0$$

Applying Ind we get a projective resolution of M of the form

$$0 \to S_0^a \to F_1 \to F_0 \to M \to 0$$

for some $a \in \mathbb{N}$ where $F_i = \mathrm{Ind}\, F_i^0$. By adjointness we have for all integers j

$$\mathrm{Ext}^j_\Delta(F_i, S_0) = \mathrm{Ext}^j_\Delta(\mathrm{Ind}\, F_i^0, S_0) = \mathrm{Ext}^j_{\Delta^0}(F_i^0, \mathrm{Res}\, S_0) = \mathrm{Ext}^j_{\Delta^0}(F_i^0, 0) = 0$$

which implies $\mathrm{Hom}_\Delta(M, S_0) = 0$ and $\mathrm{Ext}^1_\Delta(M, S_0) = 0$ which means $M \perp S_0$.

To prove the converse let $N = \mathrm{Ind}\,\mathrm{Res}\,M$. By adjointness we have a map $f : N \to M$ whose kernel K and cokernel C' are direct sums of S_0. We have $\mathrm{Hom}_\Delta(M, S_0) = 0$ and hence $\mathrm{Hom}_\Delta(C', S_0) = 0$. Thus $C' = 0$ and f is surjective. Applying $\mathrm{Hom}_\Delta(-, S_0)$ to the short exact sequence

$$0 \to K \to N \xrightarrow{f} M \to 0$$

and using $\mathrm{Hom}_\Delta(N, S_0) = 0$ (by adjointness) yields $\mathrm{Hom}_\Delta(K, S_0) = 0$ and hence $K = 0$. Thus f is an isomorphism and we are done. □

Recall from §1.1.1 the notation

$${}^\perp S_0 = \{M \in \mathrm{mod}(\Delta) \mid M \perp S_0\}$$

It is clear that ${}^\perp S_0$ is an abelian subcategory of $\mathrm{mod}(\Delta)$. Lemma 2.4.6 implies that the functors Res and Ind define inverse equivalences

$$\mathrm{mod}(\Delta) \supset {}^\perp S_0 \underset{\mathrm{Ind}}{\overset{\mathrm{Res}}{\rightleftarrows}} \mathrm{mod}(\Delta^0) \tag{2.32}$$

This means any $M \in {}^\perp S_0$ is totally determined by $\mathrm{Res}\,M$.

The following lemma was already observed by Le Bruyn [51] in the case of the homogenized Weyl algebra.

Lemma 2.4.7. *$\mathcal{C}_n(\Delta) \subset {}^\perp S_0$ for $n > 0$.*

Proof. It is sufficient to prove that if $\mathcal{M} \neq \mathcal{O}$ is a normalized line bundle on $\mathbb{P}^2_q$ and $M = \mathrm{Ext}^1_{\mathbb{P}^2_q}(\mathcal{E}, \mathcal{M})$ then $M \in {}^\perp S_0$. We have $\mathrm{RHom}_{\mathbb{P}^2_q}(\mathcal{E}, \mathcal{M}) = M[-1]$ and $\mathrm{RHom}_{\mathbb{P}^2_q}(\mathcal{E}, \mathcal{O}) = S_0$. Thus for all integers j

$$\mathrm{Ext}^j_{\mathbb{P}^2_q}(\mathcal{M}, \mathcal{O}) = \mathrm{Ext}^j_\Delta(M[-1], S_0) = \mathrm{Ext}^{j+1}_\Delta(M, S_0)$$

In particular $\mathrm{Hom}_\Delta(M, S_0) = 0$ and

$$\mathrm{Ext}^1_\Delta(M, S_0) = \mathrm{Hom}_{\mathbb{P}^2_q}(\mathcal{M}, \mathcal{O}) \cong H^2(\mathbb{P}^2_q, \mathcal{M}(-3))' = 0$$

where we have used Serre duality (Theorem 1.10.5) and Theorem 2.2.11. □

We also mention

Lemma 2.4.8. *Let $p \in C$ and let $\mathcal{S}$ a line object on $\mathbb{P}^2_q$.*

1. *$S = \mathrm{Ext}^1_{\mathbb{P}^2_q}(\mathcal{E}, \mathcal{S}(-1)) \in {}^\perp S_0$,*

2. *$p = \mathrm{Hom}_{\mathbb{P}^2_q}(\mathcal{E}, \mathcal{N}_p) \in {}^\perp S_0$,*

3. $\operatorname{Res} p$ *is* θ*-stable for* $\theta = (-1, 1)$.

Proof. The first two statements easy to verify. For the final part we have $(\underline{\dim} \operatorname{Res} p) \cdot \theta = (1, 1) \cdot (-1, 1) = 0$, so what remains to verify is $\underline{\dim} N \cdot \theta > 0$ for all nontrivial subrepresentations $N \subset \operatorname{Res} p$. For such $N \subset \operatorname{Res} p$ we either have $\underline{\dim} N = (1, 0)$ or $\underline{\dim} N = (0, 1)$. That $\underline{\dim} N = (1, 0)$ is impossible is seen by inspecting the appropriate commutative diagram. This finishes the proof. □

2.4.6 Stable representations

Let A be an elliptic quadratic Artin-Schelter algebra. We have seen $M \in \mathcal{C}_n(\Delta)$ is completely determined by its restriction $M^0 = \operatorname{Res} M \in \operatorname{mod}(\Delta^0)$. In case $A = H$ is the homogenized Weyl algebra we furthermore have $M^0(Z_{-2})$ is an isomorphism (see Remark 2.4.4). This is best understood by considering line objects on $\mathbb{P}^2_q$. We first note the following

Proposition 2.4.9. *Let* $\mathcal{S} = \pi(A/uA)$ *be a line object on* $\mathbb{P}^2_q$ *where* $u = \alpha x + \beta y + \gamma z \in A_1$ *and write* $S = \operatorname{Ext}^1_{\mathbb{P}^2_q}(\mathcal{E}, \mathcal{S}(-1)) \in \operatorname{mod}(\Delta)$. *Let* $n > 0$, $\mathcal{M} \in \mathcal{R}_n(\mathbb{P}^2_q)$ *and write* $M = \operatorname{Ext}^1_{\mathbb{P}^2_q}(\mathcal{E}, \mathcal{M}) \in \operatorname{mod}(\Delta)$. *Then the following are equivalent:*

1. $M^0 \perp S^0$
2. $\operatorname{Hom}_{\Delta^0}(M^0, S^0) = 0$
3. $\operatorname{Hom}_{\mathbb{P}^2_q}(\mathcal{M}, \mathcal{S}(-1)) = 0$
4. $\mathcal{M} \perp \mathcal{S}(-1)$
5. *The following linear map is an isomorphism*

$$f = \alpha M^0(X_{-2}) + \beta M^0(Y_{-2}) + \gamma M^0(Z_{-2}) : M_{-2} \to M_{-1}$$

Proof. Equivalence of (1) and (2): $M^0 \perp S^0$ implies $\operatorname{Hom}_{\Delta^0}(M^0, S^0) = 0$. Conversely, if $\operatorname{Hom}_{\Delta^0}(M^0, S^0) = 0$ then by computing $\chi(M^0, S^0) = 0$ we also have $\operatorname{Ext}^1_{\Delta^0}(M^0, S^0) = 0$.

Equivalence of (2) and (3): By

$$\begin{aligned}\operatorname{Hom}_{\Delta^0}(M^0, S^0) &= \operatorname{Hom}_{\Delta}(\operatorname{Ind} M^0, S) = \operatorname{Hom}_{\Delta}(M, S) = H^0(\operatorname{RHom}_{\Delta}(M, S)) \\ &\cong H^0(\operatorname{RHom}_{\mathbb{P}^2_q}(\mathcal{M}, \mathcal{S}(-1))) = \operatorname{Hom}_{\mathbb{P}^2_q}(\mathcal{M}, \mathcal{S}(-1))\end{aligned}$$

Equivalence of (3) and (4): We have $[\mathcal{M}] = [\mathcal{O}] - n[\mathcal{P}]$ and $[\mathcal{S}(-1)] = [\mathcal{S}] - [\mathcal{P}]$. An easy computation shows $\chi(\mathcal{M}, \mathcal{S}(-1)) = 0$. And by Serre duality $\operatorname{Ext}^2_{\mathbb{P}^2_q}(\mathcal{M}, \mathcal{S}(-1)) \cong \operatorname{Hom}_{\mathbb{P}^2_q}(\mathcal{S}(2), \mathcal{M})' = 0$ since $\mathcal{M}$ is reflexive hence torsion free. We conclude $\mathcal{M} \perp \mathcal{S}(-1)$ if and only if $\operatorname{Hom}_{\mathbb{P}^2_q}(\mathcal{M}, \mathcal{S}(-1)) = 0$.

Equivalence of (4) and (5): Applying $\mathrm{Hom}_{\mathbb{P}^2_q}(-,\mathcal{M})$ to a minimal resolution of $\mathcal{S}(2)$

$$0 \to \mathcal{O}(1) \to \mathcal{O}(2) \to \mathcal{S}(2) \to 0$$

gives

$$0 \to \mathrm{Ext}^1_{\mathbb{P}^2_q}(\mathcal{S}(2),\mathcal{M}) \to M_{-2} \xrightarrow{f} M_{-1} \to \mathrm{Ext}^2_{\mathbb{P}^2_q}(\mathcal{S}(2),\mathcal{M}) \to 0$$

where we have used Theorem 2.2.11. It is clear that the linear map f is given by $\alpha M^0(X_{-2}) + \beta M^0(Y_{-2}) + \gamma M^0(Z_{-2})$. Thus f is an isomorphism if and only if $\mathrm{Ext}^1_{\mathbb{P}^2_q}(\mathcal{S}(2),\mathcal{M}) = 0$ and $\mathrm{Ext}^2_{\mathbb{P}^2_q}(\mathcal{S}(2),\mathcal{M}) = 0$. Again using Serre duality this is equivalent with $\mathcal{M} \perp \mathcal{S}(-1)$. □

Remark 2.4.10. In case $A = H$ is the homogenized Weyl algebra we recover the argument $M^0(Z_{-2})$ is an isomorphism (see Remark 2.4.4), as it was found in [16]. For the line object $\mathcal{S} = \pi(H/zH)$ we deduce

$$\mathrm{RHom}_{\mathbb{P}^2_q}(\mathcal{M},\mathcal{S}(-1)) = \mathrm{RHom}_{\mathbb{P}^2_q}(\mathcal{M}, u_*\mathcal{O}_{\mathbb{P}^1}(-1)) \cong \mathrm{RHom}_{\mathbb{P}^1}(Lu^*\mathcal{M}, \mathcal{O}_{\mathbb{P}^1}(-1))$$

and since $Lu^*\mathcal{M} = \mathcal{O}_{\mathbb{P}^1}$ we obtain

$$\mathrm{Hom}_{\mathbb{P}^2_q}(\mathcal{M},\mathcal{S}(-1)) \cong \mathrm{Hom}_{\mathbb{P}^1}(\mathcal{O}_{\mathbb{P}^1}, \mathcal{O}_{\mathbb{P}^1}(-1)) = 0$$

As a consequence the representations in $\mathcal{C}_n(\Delta)$ are θ-semistable for some $\theta \in \mathbb{Z}^2$. Since $\chi(-, \underline{\dim} S^0) = - \cdot (-1,1)$we may take $\theta = (-1,1)$.

Inspired by the previous remark one might try to find, for general elliptic A, a particular line object $\mathcal{S}$ on $\mathbb{P}^2_q$ for which $\mathrm{Hom}_{\mathbb{P}^2}(\mathcal{M},\mathcal{S}(-1))$ is zero for all $\mathcal{M} \in \mathcal{R}_n(\mathbb{P}^2_q)$. We did not manage to find such a line object which is independent of $\mathcal{M}$. However we were able to prove that for a fixed normalized line bundle $\mathcal{M}$ on $\mathbb{P}^2_q$ there is a line object (which depends on $\mathcal{M}$) such that $\mathrm{Hom}_{\mathbb{P}^2_q}(\mathcal{M},\mathcal{S}(-1)) = 0$.

Proposition 2.4.11. *Assume k is uncountable and σ has infinite order. Let $n > 0$ and $\mathcal{I} \in \mathcal{R}_n(\mathbb{P}^2_q)$. Then the set of line objects $\mathcal{S}$ such that $\mathrm{Hom}_{\mathbb{P}^2_q}(\mathcal{I},\mathcal{S}(-1)) \neq 0$ is a curve of degree n in $\mathbb{P}(A_1)$. In particular this set is non-empty.*

Proof. It follows from Proposition 2.4.9 that $\mathrm{Hom}_{\mathbb{P}^2_q}(\mathcal{I},\mathcal{S}(-1)) \neq 0$ if and only if $\det f = 0$. This is a homogeneous equation in (α,β,γ) and we have to show it is not identically zero, i.e. we have to show there is at least one $\mathcal{S}$ such that $\mathrm{Hom}_{\mathbb{P}^2_q}(\mathcal{I},\mathcal{S}(-1)) = 0$. This follows from Lemma 2.4.12 and Lemma 2.4.13 below. □

Lemma 2.4.12. *Assume k is uncountable and σ has infinite order. Let $n \geq 0$ and $\mathcal{I} \in \mathcal{R}_n(\mathbb{P}^2_q)$. Let $p \in C$. Then, modulo zero dimensional objects, there exist at most n different line objects $\mathcal{S}$ such that $\mathrm{Hom}_{\mathbb{P}^2_q}(\mathcal{I},\mathcal{S}(-1)) \neq 0$ and such that $\mathrm{Hom}_{\mathbb{P}^2_q}(\mathcal{S},\mathcal{N}_p) \neq 0$.*

Proof. We use induction on n. Writing $\mathcal{S}$ as the cokernel of a map $\mathcal{O}(-1) \to \mathcal{O}$ we deduce by Theorem 1.8.2 that $\mathrm{Hom}_{\mathbb{P}^2_q}(\mathcal{O}, \mathcal{S}(-1)) = 0$. So the case $n = 0$ is clear by Corollary 2.2.12. Assume $n > 0$. Let $(\mathcal{S}_i)_{i=1,\dots,m}$ be the different line objects (modulo zero dimensional objects) satisfying $\mathrm{Hom}_{\mathbb{P}^2_q}(\mathcal{I}, \mathcal{S}_i(-1)) \neq 0$ and $\mathrm{Hom}_{\mathbb{P}^2_q}(\mathcal{S}_i, \mathcal{N}_p) \neq 0$. If $m = 0$ then we are done. So assume $m > 0$. Let $\mathcal{S}'_i(-1)$ be the kernel of a non-trivial map $\mathcal{S}_i \to \mathcal{N}_p$. It is proved in [8, Proposition 6.24] there is some point object $\mathcal{N}_{p'}$ such that $\mathrm{Hom}_{\mathbb{P}^2_q}(\mathcal{S}'_i, \mathcal{N}_{p'}) \neq 0$ for all i. Let $\mathcal{I}'(-1)$ be the kernel of a non-trivial map $\mathcal{I} \to \mathcal{S}_1(-1)$. The subobjects of line objects are shifted line objects and hence the image of $\mathcal{I}$ in $\mathcal{S}_1(-1)$ is a shifted line object. We find $[\mathcal{I}'] = [\mathcal{O}] - (n-b)[\mathcal{P}]$ with $b \geq 1$ and Lemma 2.2.4 implies $\mathcal{I}'$ is a normalized line bundle with invariant $\leq n-1$.

By Serre duality, Lemma 2.2.4 and Lemma 2.2.1 we find

$$\mathrm{Ext}^1_{\mathbb{P}^2_q}(\mathcal{I}, \mathcal{N}_p(-1)) = \mathrm{Ext}^1_{\mathbb{P}^2_q}(\mathcal{N}_p(-1), \mathcal{I}(-3))' = 0$$
$$\mathrm{Ext}^2_{\mathbb{P}^2_q}(\mathcal{I}, \mathcal{N}_p(-1)) = \mathrm{Hom}_{\mathbb{P}^2_q}(\mathcal{N}_p(-1), \mathcal{I}(-3))' = 0$$

and since $\chi(\mathcal{I}, \mathcal{N}_p(-1)) = 1$ we deduce $\dim_k \mathrm{Hom}_{\mathbb{P}^2_q}(\mathcal{I}, \mathcal{N}_p(-1)) = 1$. Hence for all i the composition $\mathcal{I} \to \mathcal{S}_i(-1) \to \mathcal{N}_p(-1)$ is a scalar multiple of the composition $\mathcal{I} \to \mathcal{S}_1(-1) \to \mathcal{N}_p(-1)$. Therefore the composition $\mathcal{I}'(-1) \to \mathcal{I} \to \mathcal{S}_i(-1)$ maps $\mathcal{I}'(-1)$ to $\mathcal{S}'_i(-2)$. We claim for $i > 1$ this map must be non-zero. If not then there is a non-trivial map $\mathcal{I}/\mathcal{I}'(-1) \to \mathcal{S}_i(-1)$ and since $\mathcal{I}/\mathcal{I}'(-1)$ is also subobject of $\mathcal{S}_1(-1)$ it follows that $\mathcal{S}_1$ and $\mathcal{S}_i$ have a common subobject. But this is impossible since $\mathcal{S}_1$ and $\mathcal{S}_i$ are different modulo zero dimensional objects.

Hence $\mathrm{Hom}_{\mathbb{P}^2_q}(\mathcal{I}', \mathcal{S}'_i(-1)) \neq 0$ for $i = 2, \dots, m$. Since the $\mathcal{S}'_i$ are still different modulo zero dimensional objects, we obtain $m - 1 \leq n - 1$ and hence $m \leq n$. □

The next lemma is easily proved for generic A. For general A one needs a more subtle treatment.

Lemma 2.4.13. *Assume k is uncountable and σ has infinite order. Let $p \in C$. Then, modulo zero dimensional objects, there exist infinitely many line objects $\mathcal{S}$ such that $\mathrm{Hom}_{\mathbb{P}^2_q}(\mathcal{S}, \mathcal{N}_p) \neq 0$.*

Proof. Let $p = (\alpha, \beta, \gamma) \in C$. We will prove the lemma in six steps.

Step 1. Let $d \in \mathbb{N}$ and let $\mathcal{S}, \mathcal{S}'$ be two line objects for which $\mathcal{S}'(-d) \subset \mathcal{S}$. Then there is a filtration

$$\mathcal{S}'(-d) = \mathcal{S}_d(-d) \subset \mathcal{S}_{d-1}(-d+1) \subset \cdots \subset \mathcal{S}_1(-1) \subset \mathcal{S}_0 = \mathcal{S}$$

where $\mathcal{S}_i$ are line objects and the successive quotients are point objects on $\mathbb{P}^2_q$. This is proved by observing that the zero dimensional object $\mathcal{N} = \mathcal{S}/\mathcal{S}'(-d)$ of multiplicity d maps surjectively to a point object (by the fact that σ has infinite order, see [8]).

Step 2. Let $\mathcal{A}$ denote the set of isoclasses of line objects $\mathcal{S}$ on $\mathbb{P}^2_q$ such that $\mathrm{Hom}_{\mathbb{P}^2_q}(\mathcal{S}, \mathcal{N}_p) \neq 0$. Then $\mathcal{A}$ is an uncountable set. Indeed, for any $\mathcal{S} = \pi(A/uA)$

for which $u = \lambda x + \mu y + \nu z \in A_1$ it is easy to see $\mathrm{Hom}_{\mathbb{P}^2_q}(\mathcal{S}, \mathcal{N}_p) \neq 0$ if and only if $\overline{u}(p) = 0$ i.e. if and only if $\lambda \alpha + \mu\beta + \nu\gamma = 0$. Moreover two line objects $\pi(A/uA)$ and $\pi(A/u'A)$ are isomorphic if and only if $u = \rho u'$ for some $\rho \in k$. Since we assume k is an uncountable field we derive $\mathcal{A}$ is uncountable.

Step 3. Let $\mathcal{B} \subset \mathcal{A}$ consist of the isoclasses of line objects $\mathcal{S} = \pi(A/uA)$ such that the line $\{u = 0\}$ in $\mathbb{P}^2$ is not a component of C. This means A/uA is h-torsion free i.e. u does not divide h. Then $\mathcal{B}$ is uncountable since we only exclude at most three line objects.

Step 4. For any $\mathcal{S} \in \mathcal{B}$ there are, up to isomorphism, only finitely many points $p \in C$ for which $\mathrm{Hom}_{\mathbb{P}^2_q}(\mathcal{S}, \mathcal{N}_p) \neq 0$ or $\mathrm{Ext}^1_{\mathbb{P}^2_q}(\mathcal{N}_p, \mathcal{S}(-1)) \neq 0$. Indeed, it follows from [8] there are at most three different point objects $\mathcal{N}_p$ on $\mathbb{P}^2_q$ for which $\mathrm{Hom}_{\mathbb{P}^2_q}(\mathcal{S}, \mathcal{N}_p) \neq 0$. In that case $\mathrm{Hom}_{\mathbb{P}^2_q}(\mathcal{S}, \mathcal{N}_p) = k$ which is seen by applying $\mathrm{Hom}_{\mathbb{P}^2_q}(-, \mathcal{N}_p)$ to a standard resolution of $\mathcal{S}$. For the second part, Serre duality implies

$$\mathrm{Ext}^1_{\mathbb{P}^2_q}(\mathcal{N}_p, \mathcal{S}(-1)) = \mathrm{Ext}^1_{\mathbb{P}^2_q}(\mathcal{S}, \mathcal{N}_{p'})', \quad 0 = \mathrm{Hom}_{\mathbb{P}^2_q}(\mathcal{N}_p, \mathcal{S}(-1)) = \mathrm{Ext}^2_{\mathbb{P}^2_q}(\mathcal{S}, \mathcal{N}_{p'})'$$

for a suitable point $p' \in C$. By $\chi(\mathcal{S}, \mathcal{N}_{p'}) = 0$ and the first part of Step 4 we deduce there are only finitely many points $p' \in C$ for which $\mathrm{Ext}^1_{\mathbb{P}^2_q}(\mathcal{S}, \mathcal{N}_{p'}) \neq 0$. Hence there are only finitely many points $p \in C$ for which $\mathrm{Ext}^1_{\mathbb{P}^2_q}(\mathcal{N}_p, \mathcal{S}(-1)) \neq 0$.

Step 5. For any $\mathcal{S}_i \in \mathcal{B}$ and $d \in \mathbb{N}$ the following subset of $\mathcal{B}$ is finite

$$\mathcal{V}_d(\mathcal{S}_i) = \{\mathcal{S} \in \mathcal{B} \mid \mathcal{S}'(-d) \subset \mathcal{S} \text{ for a line object } \mathcal{S}' \text{ for which } \mathcal{S}'(-d) \subset \mathcal{S}_i\}.$$

We will prove this for $d = 1$, for general d the same arguments may be used combined with Step 1. Let $\mathcal{S}'(-1) \subset \mathcal{S}_i$. Note $\mathcal{S}' \in \mathcal{B}$. Clearly any line object $\mathcal{S}$ on $\mathbb{P}^2_q$ for which $\mathcal{S}'(-1) \subset \mathcal{S}$ holds is represented by an element of $\mathrm{Ext}^1_{\mathbb{P}^2_q}(\mathcal{P}, \mathcal{S}'(-1))$ for some point object $\mathcal{N}_p$, and two such line objects $\mathcal{S}$ are isomorphic if and only if the corresponding extentions only differ by a scalar. By Step 4 and its proof there are only finitely many such $\mathcal{S}$, up to isomorphism.

Step 6. There exist infinitely many line objects $\mathcal{S}_0, \mathcal{S}_1, \mathcal{S}_2, \ldots$ such that $\mathrm{Hom}_{\mathbb{P}^2_q}(\mathcal{S}_i, \mathcal{N}_p) \neq 0$ and $\mathcal{S}_i$, $\mathcal{S}_j$ do not have a common subobject for all $j < i$. Indeed, choose $\mathcal{S}_0 \in \mathcal{B}$ arbitrary and having $\mathcal{S}_0, \mathcal{S}_1, \ldots, \mathcal{S}_{i-1}$ we pick $\mathcal{S}_i$ as an element of $\mathcal{B}$ which does not appear in the countable subset

$$\bigcup_{d \in \mathbb{N}, j < i} \mathcal{V}_d(\mathcal{S}_j)$$

Subobjects of line objects are shifted line objects hence Step 6 follows.

As saying that two line objects are different modulo zero dimensional objects is saying that they do not have a common subobject, this means we have proved the lemma. ∎

Remark 2.4.14. The proof of Lemma 2.4.13 requires k to be uncountable, and this is the only place in Chapter 2 where we need this hypothesis on k. However in case A is a three dimensional Sklyanin algebra Lemma 2.4.13 may be proved much simpler by observing [8] that for any line object $\mathcal{S} = \pi(A/uA)$ containing a shifted line object $\mathcal{S}' = \pi(A/u'A)$ we have

$$\operatorname{div}(\overline{u'}) = \sigma^a p + \sigma^b q + \sigma^{-a-b} r \quad \text{for some } a, b \in \mathbb{Z}$$

where we have written $\operatorname{div}(\overline{u}) = p + q + r$. Therefore if A is a three dimensional Sklyanin algebra we may drop the hypothesis k is uncountable.

Before we come to the main result of this part we need some more lemmas.

Lemma 2.4.15. *Assume $0 \neq F, G \in \operatorname{mod}(\Delta^0)$ are θ-semistable for $\theta = (-1, 1)$.*

1. *If G is θ-stable then every non-zero map in $\operatorname{Hom}_{\Delta^0}(F, G)$ is surjective.*
2. *If F is θ-stable then every non-zero map in $\operatorname{Hom}_{\Delta^0}(F, G)$ is injective.*

Proof. Since F and G are both θ-semistable and non-zero we may write $\underline{\dim} F = (m, m) \neq 0$, $\underline{\dim} G = (n, n) \neq 0$ for some non-negative integers m, n.

First, let G be θ-stable and let $0 \neq f \in \operatorname{Hom}_{\Delta^0}(F, G)$ be such a non-zero map. Assume by contradiction $\operatorname{im} f \neq G$. Write $\underline{\dim} \operatorname{im} f = (a, b)$. Since $\operatorname{im} f \subset G$ is a non-trivial subrepresentation (since $f \neq 0$), it follows from the θ-stability of G

$$0 < \theta \cdot \underline{\dim} \operatorname{im} f = (-1, 1) \cdot (a, b) = b - a$$

On the other hand, $\ker f \neq 0$, otherwise $(a, b) = (m, m)$ which contradicts $0 < b - a$. Also $\ker f \neq F$, since $f \neq 0$. Therefore, by the θ-semistability of F, we obtain

$$0 \leq \theta \cdot \underline{\dim} \ker f = (-1, 1) \cdot (m - a, m - b) = a - b$$

yielding the desired contradiction. Hence f is surjective, which proves the first part of the lemma. The second statement is is shown in an analogous way. □

Lemma 2.4.16. *Put $\theta = (-1, 1)$. Let $V \in \operatorname{mod}(\Delta^0)$ and assume the forms $- \cdot \theta$ and $\chi(-, \underline{\dim} V)$ are proportional. For any $F \in \operatorname{mod}(\Delta^0)$ for which $F \perp V$ we have*

1. *If $F' \subset F$ such that $\underline{\dim} F' \cdot \theta = 0$ then $F' \perp V$ and $F/F' \perp V$*
2. $\operatorname{Hom}_{\Delta^0}(F, \operatorname{Res} p) = \operatorname{Hom}_{\Delta^0}(\operatorname{Res} p, F) = 0$ *for all $p \in C$ for which $\operatorname{Res} p$ is not perpendicular to V.*

Proof. Using the Euler form on Δ it is easy to see $\underline{\dim} V = (2l, l)$ for some $l \in \mathbb{N}$. We may assume $V \neq 0$. Let $F \perp V$. This implies F is θ-semistable and $\underline{\dim} F = (n, n)$ for some $n \in \mathbb{N}$. Since result trivially holds for $F = 0$ we may assume $n > 0$.

1. Let $F' \subset F$ such that $\underline{\dim} F' \cdot \theta = 0$. Thus $\underline{\dim} F' = (m,m)$ and $\underline{\dim} F/F' = (n-m, n-m)$ for some $m \leq n$. Applying $\mathrm{Hom}_{\Delta^0}(-,V)$ to the short exact sequence $0 \to F' \to F \to F/F' \to 0$ in $\mathrm{mod}(\Delta^0)$ gives the long exact sequence

$$\begin{aligned} 0 &\to \mathrm{Hom}_{\Delta^0}(F/F',V) \to \mathrm{Hom}_{\Delta^0}(F,V) \to \mathrm{Hom}_{\Delta^0}(F',V) \\ &\to \mathrm{Ext}^1_{\Delta^0}(F/F',V) \to \mathrm{Ext}^1_{\Delta^0}(F,V) \to \mathrm{Ext}^1_{\Delta^0}(F',V) \to 0 \end{aligned}$$

   Since $F \perp V$ we deduce $\mathrm{Hom}_{\Delta^0}(F/F',V) = 0$ and $\mathrm{Ext}^1_{\Delta^0}(F',V) = 0$. Computations show $\chi(F',V) = \chi(F/F',V) = 0$ which yield $\mathrm{Ext}^1_{\Delta^0}(F/F',V) = 0$ and $\mathrm{Hom}_{\Delta^0}(F',V) = 0$. We conclude $F' \perp V$ and $F/F' \perp V$.

2. Let $p \in C$ for which $\mathrm{Res}\, p$ is not perpendicular to V. Recall from Lemma 2.4.8 that $\mathrm{Res}\, p$ is θ-stable.

   First, if $\mathrm{Hom}_{\Delta^0}(F, \mathrm{Res}\, p) \neq 0$ then by Lemma 2.4.15 there is an epimorphism $F \to \mathrm{Res}\, p$. By part 1 we obtain $\mathrm{Res}\, p \perp V$, contradiction.

   Second, if $\mathrm{Hom}_{\Delta^0}(\mathrm{Res}\, p, F) \neq 0$ then by Lemma 2.4.15 there is an injective map $\mathrm{Res}\, p \to F$. By part 1 of the current lemma we obtain $\mathrm{Res}\, p \perp V$, again a contradiction. This finishes the proof. □

Lemma 2.4.17. *Assume σ has infinite order. Let $N \in \mathrm{mod}(\Delta^0)$ with dimension vector (n,n) where $n > 0$. If $\mathrm{Hom}_{\Delta^0}(N, \mathrm{Res}\, p) = \mathrm{Hom}_{\Delta^0}(\mathrm{Res}\, p, N) = 0$ for all $p \in C$ then $\dim_k(\mathrm{Ind}\, N)_0 \leq n-1$.*

Proof. Assume the lemma is false. Thus $\dim(\mathrm{Ind}\, N)_0 = d \geq n$. In case $d = n$ we put $W = \mathrm{Ind}\, N$. Otherwise we have $S_0^{d-n} \subset \mathrm{Ind}\, N$ where

$$S_0^{d-n} = \underbrace{S_0 \oplus \cdots \oplus S_0}_{d-n \text{ times}}$$

and we let $W = (\mathrm{Ind}\, N)/S_0^{d-n}$ be the quotient. In either case we have a surjective map $\mathrm{Ind}\, N \to W$ where $\underline{\dim} W = (n,n,n)$. Note $\mathrm{Res}\, W = N$. We will consider $W \overset{\mathbf{L}}{\otimes}_\Gamma \mathcal{E} \in D^b(\mathrm{coh}(\mathbb{P}^2_q))$ and $Li^*(W \overset{\mathbf{L}}{\otimes}_\Gamma \mathcal{E}) \in D^b(\mathrm{coh}(C))$. We have for $p \in C$

$$\mathrm{Ext}^j_C(Lu^*(W \overset{\mathbf{L}}{\otimes}_\Delta \mathcal{E}), \mathcal{O}_p) = \mathrm{Ext}^j_{\mathbb{P}^2_q}(W \overset{\mathbf{L}}{\otimes}_\Delta \mathcal{E}, u_*\mathcal{O}_p) = \mathrm{Ext}^j_\Delta(W,p) \tag{2.33}$$

By applying $\mathrm{Hom}_\Delta(-,p)$ to $\mathrm{Ind}\, N \to W \to 0$ we have

$$\mathrm{Hom}_\Delta(W,p) \subset \mathrm{Hom}_\Delta(\mathrm{Ind}\, N, p) = \mathrm{Hom}_{\Delta^0}(N, \mathrm{Res}\, p) = 0$$

and by Serre duality on $\mathbb{P}^2_q$ and adjointness

$$\mathrm{Ext}^2_\Delta(W,p) = \mathrm{Hom}_\Delta(\tilde{p}, W)' = \mathrm{Hom}_{\Delta^0}(\mathrm{Res}\, \tilde{p}, N)' = 0$$

for some $\tilde{p} \in C$. Since $\chi(W,p) = 0$ we conclude also $\mathrm{Ext}^1_\Delta(W,p) = 0$. It follows from (2.33) that $Lu^*(W \overset{\mathbf{L}}{\otimes}_\Delta \mathcal{E}) = 0$. It is easy to see this implies $W \overset{\mathbf{L}}{\otimes}_\Delta \mathcal{E} = 0$ where we have used Lemma 1.9.19. By the derived equivalence $D^b(\mathrm{coh}(\mathbb{P}^2_q)) \cong D^b(\mathrm{mod}(\Delta))$ we obtain $W = 0$ which is a contradiction. □

We now come to the main result of this section.

Theorem 2.4.18. *Assume k is uncountable. Let A be an elliptic quadratic Artin-Schelter algebra for which σ has infinite order. Let $n > 0$. If $M \in \mathcal{C}_n(\Delta)$ then M^0 is θ-stable for $\theta = (-1, 1)$.*

Proof. Pick $M \in \mathcal{C}_n(\Delta)$. Thus there is an object $\mathcal{M} \in \mathcal{R}_n(\mathbb{P}^2_q)$ for which $M[-1] = \mathrm{RHom}_{\mathbb{P}^2_q}(\mathcal{E}, \mathcal{M})$. We write $F = M^0$.

It follows from Propositions 2.4.9 and 2.4.11 that there exists a line object $\mathcal{S}$ on $\mathbb{P}^2_q$ such that $F \perp S^0$, where $S[-1] = \mathrm{RHom}_{\mathbb{P}^2_q}(\mathcal{E}, \mathcal{S}(-1))$. This shows F is θ-semistable and as in Remark 2.4.10 it is checked we may take $\theta = (-1, 1)$.

Since F is θ-semistable there is a subrepresentation $F' \subsetneq F$ such that F/F' is θ-stable. We will prove F' is necessarily zero, from which the result will follow. So assume by contradiction $F' \neq 0$. Since F/F' is θ-stable we have $\theta \cdot \underline{\dim} F/F' = 0$ thus we may put $\underline{\dim} F/F' = (n - m, n - m)$ where $0 < m < n$. It follows that $\underline{\dim} F' = (m, m)$. Note F' is θ-semistable.

We now claim that for all $p \in C$ we have

$$\mathrm{Hom}_{\Delta^0}(F', \operatorname{Res} p) = \mathrm{Hom}_{\Delta^0}(\operatorname{Res} p, F') = 0$$
$$\mathrm{Hom}_{\Delta^0}(F/F', \operatorname{Res} p) = \mathrm{Hom}_{\Delta^0}(\operatorname{Res} p, F/F') = 0$$

To prove this, pick any $p \in C$. Recall $\mathcal{N}_p(-2) = \mathcal{N}_{\sigma^2 p}$. It follows from Lemmas 2.4.12 and 2.4.13 there exists a line object $\mathcal{S}'$ on $\mathbb{P}^2_q$ such that $\mathrm{Hom}_{\mathbb{P}^2_q}(\mathcal{S}', \mathcal{N}_{\sigma^2 p}) \neq 0$ and $\mathrm{Hom}_{\mathbb{P}^2_q}(\mathcal{M}, \mathcal{S}'(-1)) = 0$. Writing $S'[-1] = \mathrm{RHom}_{\mathbb{P}^2_q}(\mathcal{E}, \mathcal{S}'(-1))$ and using Serre duality on $\mathbb{P}^2_q$ this becomes $\mathrm{Ext}^1_\Delta(p, S') \neq 0$ and $F \perp S'$ by Proposition 2.4.9. In particular $\operatorname{Res} p$ is *not* perpendicular to $\operatorname{Res} S'$. The claim above now follows from Lemma 2.4.16.

Combining the above with Lemma 2.4.17 we see $\dim_k(\operatorname{Ind} F/F')_0 \leq n - m - 1$ and $\dim_k(\operatorname{Ind} F')_0 \leq m - 1$. Application of the right exact functor Ind on

$$0 \to F' \to F \to F/F' \to 0$$

yields

$$\ldots \to \operatorname{Ind} F' \to \operatorname{Ind} F \to \operatorname{Ind} F/F' \to 0$$

hence

$$\begin{aligned} n - 1 = \dim_k(\operatorname{Ind} F)_0 &\leq \dim_k(\operatorname{Ind} F')_0 + \dim_k(\operatorname{Ind} F/F')_0 \\ &\leq (m - 1) + (n - m - 1) = n - 2 \end{aligned}$$

which is absurd. Thus $F' = 0$ and we conclude F is θ-stable. □

Remark 2.4.19. In case A is a three dimensional Sklyanin algebra and σ has infinite order we do not need the hypothesis k is uncountable in Theorem 2.4.18. This is because we may prove Proposition 2.4.11 without the additional hypothesis on k, using the proof of Lemma 2.4.12 and Remark 2.4.14.

2.4.7 Second description of $\mathcal{R}_n(\mathbb{P}^2_q)$ and proof of Theorem 1

Recall from Lemma 2.4.7 that $\mathcal{C}_n(\Delta) \subset {}^{\perp}S_0$ for $n > 0$. We denote $\mathcal{D}_n(\Delta^0)$ for the image of $\mathcal{C}_n(\Delta)$ under the equivalence ${}^{\perp}S_0 \cong \operatorname{mod}(\Delta^0)$ of (2.32).

Theorem 2.4.20. *Assumee k is uncountable. Let A be an elliptic quadratic Artin-Schelter algebra for which σ has infinite order. Let $\theta = (-1, 1)$ and $n > 1$. Then there is an equivalence of categories*

$$\mathcal{C}_n(\Delta) \underset{\text{Ind}}{\overset{\text{Res}}{\rightleftarrows}} \mathcal{D}_n(\Delta^0)$$

where

$$\mathcal{D}_n(\Delta^0) = \{F \in \operatorname{mod}(\Delta^0) \mid \underline{\dim} F = (n, n), F \textit{ is } \theta\textit{-stable}, \ \dim_k(\operatorname{Ind} F)_0 \geq n-1\}.$$

Proof. Below we use often implicitly the equivalence $\mathcal{C}_n(\Delta) \cong \mathcal{R}_n(\mathbb{P}^2_q)$ from Theorem 2.4.3. We break the proof into five steps.

Step 1. $\operatorname{Res}(\mathcal{C}_n(\Delta)) \subset \mathcal{D}_n(\Delta^0)$. This follows from Theorem 2.4.18, Lemma 2.4.6 and Theorem 2.2.11.

Step 2. If $F \in \mathcal{D}_n(\Delta^0)$ then $\operatorname{Hom}_{\Delta^0}(F, \operatorname{Res} p) = \operatorname{Hom}_{\Delta^0}(\operatorname{Res} p, F) = 0$ for all $p \in C$. Indeed, by Lemma 2.4.8 and Lemma 2.4.15 any non-zero morphism would yield an isomorphism $F \cong \operatorname{Res} p$ hence $n = 1$, contradicting the assumption $n > 1$.

Step 3. If $F \in \mathcal{D}_n(\Delta^0)$ then $\operatorname{Hom}_{\Delta}(\operatorname{Ind} F, p) = \operatorname{Hom}_{\Delta}(p, \operatorname{Ind} F) = 0$ for all $p \in E$. This follows $0 = \operatorname{Hom}_{\Delta^0}(F, \operatorname{Res} p) = \operatorname{Hom}_{\Delta}(\operatorname{Ind} F, p)$ and

$$0 = \operatorname{Hom}_{\Delta^0}(\operatorname{Res} p, F) = \operatorname{Hom}_{\Delta^0}(\operatorname{Res} p, \operatorname{Res} \operatorname{Ind} F) = \operatorname{Hom}_{\Delta}(\operatorname{Ind} \operatorname{Res} p, \operatorname{Ind} F)$$

where we have used Step 2 and $\operatorname{Ind} \operatorname{Res} p = p$ by Lemma 2.4.8.

Step 4. $\operatorname{Ind}(\mathcal{D}_n(\Delta^0)) \subset \mathcal{C}_n(\Delta)$. Let $F \in \mathcal{D}_n(\Delta^0)$. Combining Step 2 with Lemma 2.4.17 it follows that $\underline{\dim} \operatorname{Ind} F = (n, n, n-1)$. Now Step 3 shows $\operatorname{Ind} F \in \mathcal{C}_n(\Delta)$.

Step 5. Ind and Res are inverses to each other. To prove this we only need to show $\operatorname{Ind} \operatorname{Res} F = F$ for $F \in \mathcal{C}_n(\Delta)$. This follows from Lemmas 2.4.6 and 2.4.7. □

Let $\alpha = (n, n)$ and put (for $n > 1$)

$$\begin{aligned} \widetilde{D}_n &= \{F \in \operatorname{Rep}_\alpha(\Delta^0) \mid F \in \mathcal{D}_n(\Delta^0)\} \\ &= \{F \in \operatorname{Rep}_\alpha(\Delta^0) \mid F \text{ is } \theta\text{-stable}, \ \dim(\operatorname{Ind} F)_0 \geq n-1\}. \end{aligned} \tag{2.34}$$

Clearly $\widetilde{D}_n$ is a closed subset of the dense open subset of $\operatorname{Rep}_\alpha(\Delta^0)$ consisting of all θ-stable representations. Hence $\widetilde{D}_n$ is locally closed.

Put $D_n = \widetilde{D}_n /\!\!/ \operatorname{Gl}_\alpha(k)$.

Theorem 2.4.21. *Assume k is uncountable. Let A be an elliptic quadratic Artin-Schelter algebra for which σ has infinite order. Then for $n \in \mathbb{N}$ there exists a smooth locally closed variety D_n of dimension $2n$ such that the isomorphism classes in $\mathcal{D}_n(\Delta^0)$ (and hence in $\mathcal{R}_n(\mathbb{P}^2_q)$) are in natural bijection with the points in D_n.*

Proof. For $n = 0$ or 1 we refer to Corollaries 2.2.12, 2.4.5 to see that we may take a point for D_0 and $\mathbb{P}^2 - C$ for D_1. So we may assume $n > 1$ throughout this proof.

Since all representations in $\widetilde{D}_n$ are stable, all $\mathrm{Gl}_\alpha(k)$-orbits on $\widetilde{D}_n$ are closed and so D_n is really the orbit space for the $\mathrm{Gl}_\alpha(k)$ action on $\widetilde{D}_n$. This proves the isomorphism classes in $\mathcal{D}_n(\Delta^0)$ are in natural bijection with the points in D_n.

To prove D_n is smooth it suffices to prove $\widetilde{D}_n$ is smooth (this follows for example using the Luna slice theorem [54]).

We first estimate the dimension of $\widetilde{D}_n$. We write the equations of A in the usual form $M_A(xyz)^t$. Given $n \times n$-matrices X, Y, Z let $M_A(X, Y, Z)$ be obtained from M_A by replacing (x, y, z) by X, Y, Z (thus $M_A(X, Y, Z)$ is a $3n \times 3n$-matrix). Then $\widetilde{D}_n$ has the following alternative description:

$$\widetilde{D}_n = \{(X, Y, Z) \in M_n(k)^3 \mid (X, Y, Z) \text{ is } \theta\text{-stable and } \operatorname{rank} M_A(X, Y, Z) \leq 2n + 1\}.$$

By Proposition 2.2.14 $\widetilde{D}_n$ is non-empty. The triples (X, Y, Z) for which the associated respresentation is stable are a dense open subset of $M_n(k)^3$ and hence they represent a quasi-variety of dimension $3n^2$. Imposing $M_A(X, Y, Z)$ should have corank $\geq n - 1$ represents $(n-1)^2$ independent conditions. So the irreducible components of $\widetilde{D}_n$ have dimension $\geq 3n^2 - (n - 1)^2$.

Define $\widetilde{C}_n$ by

$$\{G \in \mathrm{Rep}_{\widetilde{\alpha}}(\Delta) \mid G \cong \operatorname{Ind}\operatorname{Res} G, \operatorname{Res} G \in \widetilde{D}_n\}$$

where $\widetilde{\alpha} = (n, n, n - 1)$ (as usual we assume the points of $\mathrm{Rep}_{\widetilde{\alpha}}(\Delta)$ to satisfy the relation imposed on Δ).

To extend $F \in \widetilde{D}_n$ to a point in $\widetilde{C}_n$ we need to choose a basis in $(\operatorname{Ind} F)_0$. Thus $\widetilde{C}_n$ is a principal $\mathrm{Gl}_{n-1}(k)$ fiber bundle over $\widetilde{D}_n$. In particular $\widetilde{C}_n$ is smooth if and only $\widetilde{D}_n$ is smooth and the irreducible components of $\widetilde{C}_n$ have dimension $\geq 3n^2 - (n-1)^2 + (n-1)^2 = 3n^2$. Note by the description of $\mathcal{C}_n(\Delta)$ in Theorem 2.4.3 it follows that $\widetilde{C}_n$ is an open subset of $\mathrm{Rep}(\Delta, \widetilde{\alpha})$.

Let $x \in \widetilde{C}_n$. The stabilizer of x consists of scalars thus if we put $G = \mathrm{Gl}(\widetilde{\alpha})/k^*$ then we have inclusions $\mathrm{Lie}(G) \subset T_x(\widetilde{C}_n) = T_x(\mathrm{Rep}(\Delta, \widetilde{\alpha}))$. Voigt in [82, Ch. 2, §3.4] has shown there is a natural inclusion $T_x(\mathrm{Rep}(\Delta, \widetilde{\alpha}))/\mathrm{Lie}(G) \hookrightarrow \mathrm{Ext}^1_\Delta(x, x)$ (Voigt actually obtains an isomorphism since he is not assuming his representation spaces to be reduced). Now x corresponds to some normalized line bundle $\mathcal{H}$ on $\mathbb{P}^2_q$ and we have $\mathrm{Ext}^1_\Delta(x, x) = \mathrm{Ext}^1_{\mathbb{P}^2_q}(\mathcal{H}, \mathcal{H})$. An easy computation shows $\chi(\mathcal{H}, \mathcal{H}) = \chi(x, x) = 1 - 2n$. We have $\mathrm{Hom}_{\mathbb{P}^2_q}(\mathcal{H}, \mathcal{H}) = k$ and by Serre duality $\mathrm{Ext}^2_{\mathbb{P}^2_q}(\mathcal{H}, \mathcal{H}) = \mathrm{Hom}_{\mathbb{P}^2_q}(\mathcal{H}, \mathcal{H}(-3))' = 0$. Thus $\dim_k \mathrm{Ext}^1_{\mathbb{P}^2_q}(\mathcal{H}, \mathcal{H}) = 2n$.

Hence we obtain $3n^2 \leq \dim T_x(\widetilde{C}_n) \leq 2n + \dim G = 2n + 2n^2 + (n-1)^2 - 1 = 3n^2$. Thus $\dim T_x(\widetilde{C}_n) = 3n^2$ is constant and hence $\widetilde{C}_n$ is smooth. We also obtain $\dim \widetilde{D}_n = 3n^2 - (n-1)^2$.

The dimension of D_n is equal to $\dim \widetilde{D}_n - \dim \mathrm{Gl}_\alpha(k) + 1 = 3n^2 - (n-1)^2 - 2n^2 + 1 = 2n$. This finishes the proof. □

Proof of Theorem 1. By Theorem 2.4.21. For $n \leq 1$ Theorem 1 follows from Corollary 2.2.12 and Corollary 2.4.5. □

2.4.8 Description of the varieties D_n for Sklyanin algebras and proof of Theorem 2

In §2.4.6 we have tried to generalize the results of the homogenized Weyl algebra [16] by looking for a line object $\mathcal{S}$ on $\mathbb{P}^2_q$ for which $M^0 \perp S^0$ *for all* $M \in \mathcal{C}_n(\Delta)$. Although we did not succeed in doing this, there is another interpretation. For the homogenized Weyl algebra the restriction $u^*\mathcal{M} = \mathcal{O}_{\mathbb{P}^1}$ translated into $M^0 \perp S^0$, see Remark 2.4.10. For the Sklyanin case we now have

Lemma 2.4.22. *Let A be a three dimensional Sklyanin algebra for which σ has infinite order. There exists $V \in \mathrm{mod}(\Delta^0)$ with $\underline{\dim} V = (6,3)$ such that*

1. *for all $M \in \mathcal{C}_n(\mathbb{P}^2_q)$ we have $M^0 \perp V$, and*
2. *if $p \in C$ then $\mathrm{Res}\, p$ is not perpendicular to V.*

Proof. 1. Pick a degree zero line bundle $\mathcal{U}$ on C which is not of the form $\mathcal{O}((o) - (3n\xi))$ for $n \in \mathbb{N}$ (where o, ξ are as in Proposition 2.3.4). Let $\mathcal{M} \in \mathcal{R}_n(\mathbb{P}^2_q)$. Then we have by adjointness $\mathrm{RHom}_{\mathbb{P}^2_q}(\mathcal{M}, u_*\mathcal{U}) = \mathrm{RHom}_C(Lu^*\mathcal{M}, \mathcal{U})$. By Proposition 2.3.4 we have $Lu^*\mathcal{M} = \mathcal{O}((o) - (3n\xi))$. We conclude by Serre duality for C that $\mathrm{RHom}_{\mathbb{P}^2_q}(\mathcal{M}, u_*\mathcal{U}) = 0$. Now put $M = \mathrm{Ext}^1_{\mathbb{P}^2_q}(\mathcal{E}, \mathcal{M})$ and $U' = \mathrm{RHom}_{\mathbb{P}^2_q}(\mathcal{E}, u_*\mathcal{U})$. We obtain $\mathrm{RHom}_\Delta(M[-1], U') = 0$.

What is U'? By adjointness we have $\mathrm{RHom}_{\mathbb{P}^2_q}(\mathcal{E}, u_*\mathcal{U}) = \mathrm{RHom}_C(Lu^*\mathcal{E}, \mathcal{U})$. Since restriction to C commutes with the shift functor, it follows from (1.22)

$$Lu^*\mathcal{E} = \mathcal{O}_C(2) \oplus \mathcal{O}_C(1) \oplus \mathcal{O}_C = \left(\sigma_*^2(\mathcal{L}) \otimes_C \sigma_*(\mathcal{L})\right) \oplus \sigma_*\mathcal{L} \oplus \mathcal{O}_C$$

By Riemann-Roch and Serre duality $U' = U[-1]$ where $\underline{\dim} U = (6,3,0)$. Put $V = \mathrm{Res}\, U$. Thus $\underline{\dim} V = (6,3)$. Replacing M with a projective resolution it is easy to see $\mathrm{RHom}_\Delta(M, U) = \mathrm{RHom}_{\Delta^0}(M^0, V)$. It follows that $\mathrm{Hom}_{\Delta^0}(M^0, V) = 0$ and $\mathrm{Ext}^1_{\Delta^0}(M^0, V) = 0$.

2. Put $Q = \mathrm{Res}\, p$ for $p \in C$. Then

$$\begin{aligned}\mathrm{RHom}_{\Delta^0}(Q, V) &= \mathrm{RHom}_\Delta(p, U) = \mathrm{RHom}_\Delta(p[-1], U') \\ &= \mathrm{RHom}_{\mathbb{P}^2_q}(u_*\mathcal{O}_p[-1], u_*\mathcal{U}) = \mathrm{RHom}_C(Lu^*u_*\mathcal{O}_p[-1], \mathcal{U})\end{aligned}$$

Now $Lu^*u_*\mathcal{O}_p[-1]$ is a non-zero complex whose homology has finite length. It is easy to deduce from this $\mathrm{RHom}_C(Lu^*u_*\mathcal{O}_p[-1], \mathcal{U}) \neq 0$. Hence we are done. □

Remark 2.4.23. Note the choice $\mathcal{U} = \mathcal{O}_C$ in the previous proof would fail since $\mathrm{Hom}_{\mathbb{P}^2_q}(\mathcal{E}, u_*\mathcal{O}_C) \neq 0$, $\mathrm{Ext}^1_{\mathbb{P}^2_q}(\mathcal{E}, u_*\mathcal{O}_C) \neq 0$ hence $\mathrm{RHom}_{\mathbb{P}^2_q}(\mathcal{E}, u_*\mathcal{O}_C)$ is *not* concentrated in a single degree. Thus the image of $u_*\mathcal{O}_C$ under the generalized Beilinson equivalence cannot be identified with an object in $\mathrm{mod}(\Delta)$.

Theorem 2.4.24. *Let A be a three dimensional Sklyanin algebra for which σ has infinite order. Let $V \in \mathrm{mod}(\Delta^0)$ be as in Lemma 2.4.22. Let $n \in \mathbb{N}$. The isomorphism classes in $R_n(A)$ are in natural bijection with the points in the smooth affine variety D_n of dimension $2n$ where*

$$D_n = \{F = (X, Y, Z) \in M_n(k)^3 \mid F \perp V, \quad \mathrm{rank}\begin{pmatrix} cX & aZ & bY \\ bZ & cY & aX \\ aY & bX & cZ \end{pmatrix} \leq 2n + 1\} / \mathrm{Gl}_\alpha(k)$$

Proof. For $n = 0$ or 1 we refer to Corollaries 2.2.12, 2.4.5 to see that D_n has the description as in the statement of the current theorem. So we may assume $n > 1$ throughout this proof.

It is sufficient to show $\mathcal{D}_n(\Delta^0)$ has the alternative description

$$\mathcal{D}'_n(\Delta^0) := \{F \in \mathrm{mod}(\Delta^0) \mid \underline{\dim} F = (n, n), F \perp V, \dim_k(\mathrm{Ind}\, F)_0 \geq n - 1\}.$$

Indeed, if $\mathcal{D}_n(\Delta^0) = \mathcal{D}'_n(\Delta^0)$ we then have

$$\begin{aligned} \widetilde{D}_n &= \{F \in \mathrm{Rep}_\alpha(\Delta^0) \mid F \in \mathcal{D}_n(\Delta^0)\} \\ &= \{F \in \mathrm{Rep}_\alpha(\Delta^0) \mid \phi_V(F) \neq 0, \dim_k(\mathrm{Ind}\, F)_0 \geq n - 1\} \end{aligned}$$

from which we see $\widetilde{D}_n$ is a closed subset of $\{\phi_V \neq 0\}$ so in particular $\widetilde{D}_n$ is affine. This means $D_n = \widetilde{D}_n / \mathrm{Gl}(\alpha)$ is an affine variety. Theorem 2.4.21 further implies D_n is smooth of dimension $2n$ which points are in natural bijection with the isomorphism classes in $\mathcal{R}_n(\mathbb{P}^2_q)$ whence in $R_n(A)$ by §2.2.3. Moreover, as in the proof of Theorem 2.4.21, $\widetilde{D}_n$ has the alternative description

$$\widetilde{D}_n = \{F = (X, Y, Z) \in \mathrm{Rep}_\alpha(\Delta^0) \mid F \perp V \text{ and } \mathrm{rank}\, M_A(X, Y, Z) \leq 2n + 1\}.$$

Explicitely writing down M_A by (1.9), (1.18) yields the desired description of D_n.

So to prove the current theorem it remains to prove $\mathcal{D}_n(\Delta^0) = \mathcal{D}'_n(\Delta^0)$. We will do this by showing that the functors Res and Ind define inverse equivalences between $\mathcal{C}_n(\Delta)$ and $\mathcal{D}'_n(\Delta^0)$.

Step 1. $\mathrm{Res}(\mathcal{C}_n(\Delta)) \subset \mathcal{D}'_n(\Delta^0)$. This follows from Lemmas 2.4.7 and 2.4.22.

Step 2. $\mathrm{Ind}(\mathcal{D}'_n(\Delta^0)) \subset \mathcal{C}_n(\Delta)$. Let $F \in \mathcal{D}'_n(\Delta^0)$. Combining the Lemmas 2.4.22, 2.4.16 and 2.4.17 we obtain $\dim(\mathrm{Ind}\, F)_0 = n-1$. It remains to show that for $p \in C$ we have $\mathrm{Hom}_\Delta(\mathrm{Ind}\, F, p) = \mathrm{Hom}_\Delta(p, \mathrm{Ind}\, F) = 0$. By Lemma 2.4.8 we have $p = \mathrm{Ind}\,\mathrm{Res}\, p$. Thus $\mathrm{Hom}_\Delta(\mathrm{Ind}\, F, p) = \mathrm{Hom}_{\Delta^0}(F, \mathrm{Res}\, p) = 0$ and similarly

$$\mathrm{Hom}_\Delta(p, \mathrm{Ind}\, F) = \mathrm{Hom}_{\Delta^0}(\mathrm{Res}\, p, \mathrm{Res}\,\mathrm{Ind}\, F) = \mathrm{Hom}_{\Delta^0}(\mathrm{Res}\, p, F) = 0$$

where we have used Lemma 2.4.16 again.

Step 3. Ind and Res are inverses to each other. This follows from Lemma 2.4.6 and Lemma 2.4.7. □

We conclude with

Proof of Theorem 2. Follows from Theorem 2.4.24. □

2.4.9 Filtrations of line bundles and proof of Theorem 3

Let A be an elliptic quadratic Artin-Schelter algebra for which σ has infinite order. The following lemma shows how to reduce the invariant of a line bundle.

Lemma 2.4.25. *Assume k is uncountable and σ has infinite order. Let $n > 0$ and $\mathcal{I} \in \mathcal{R}_n(\mathbb{P}^2_q)$. Then there exists a line object $\mathcal{S}$ on $\mathbb{P}^2_q$ such that $\mathrm{Ext}^1_{\mathbb{P}^2_q}(\mathcal{S}(1), \mathcal{I}(-1)) \neq 0$. If $\mathcal{J} = \pi J$ is the middle term of a corresponding non-trivial extension and $\mathcal{J}^{**} = \pi J^{**}$ then $\mathcal{J}^{**} \in \mathcal{R}_m(\mathbb{P}^2_q)$ with $m < n$. Furthermore $\mathcal{J}^{**}/\mathcal{I}(-1)$ is a shifted line object.*

Proof. Serre duality gives

$$\mathrm{Ext}^1_{\mathbb{P}^2_q}(\mathcal{S}(1), \mathcal{I}(-1)) = \mathrm{Ext}^1_{\mathbb{P}^2_q}(\mathcal{I}(-1), \mathcal{S}(-2))' = \mathrm{Ext}^1_{\mathbb{P}^2_q}(\mathcal{I}, \mathcal{S}(-1))'$$

Also using Serre duality we deduce $\mathrm{Ext}^2_{\mathbb{P}^2_q}(\mathcal{I}, \mathcal{S}(-1)) = 0$. Then a simple computation using the Euler form shows $\dim_k \mathrm{Hom}_{\mathbb{P}^2_q}(\mathcal{I}, \mathcal{S}(-1)) = \dim_k \mathrm{Ext}^1_{\mathbb{P}^2_q}(\mathcal{I}, \mathcal{S}(-1))$. Hence it follows from Proposition 2.4.11 there exist $\mathcal{S}$ such that $\mathrm{Ext}^1_{\mathbb{P}^2_q}(\mathcal{S}(1), \mathcal{I}(-1)) \neq 0$.

Now let $\mathcal{J} = \pi J$ be the middle term of a non-trivial extension of $\mathcal{I}(-1)$ by $\mathcal{S}(1)$. Then we have $[\mathcal{J}] = [\mathcal{O}] - [\mathcal{S}] - n[\mathcal{P}] + [\mathcal{S}] + [\mathcal{P}] = [\mathcal{O}] - (n-1)[\mathcal{P}]$.

We claim $\mathcal{J}$ is torsion free. Assume this is not the case and let $\mathcal{F} \subset \mathcal{J}$ be a maximal subobject of $\mathcal{J}$ of dimension ≤ 1. So $\mathcal{F} \neq 0$. Since $\mathcal{I}$ is torsion free we have $\mathcal{F} \cap \mathcal{I}(-1) = 0$. So we may consider $\mathcal{F}$ as a subobject of $\mathcal{S}(1)$. Hence we obtain an extention

$$0 \to \mathcal{I}(-1) \to \mathcal{J}/\mathcal{F} \to \mathcal{S}(1)/\mathcal{F} \to 0 \tag{2.35}$$

According to Lemma 2.2.4 this extension is split. But this means $\mathcal{S}(1)/\mathcal{F}$ is a subobject of $\mathcal{J}/\mathcal{F}$ of dimension ≤ 1, contradicting the maximality of $\mathcal{F}$.

It follows from [8] that $\mathrm{GKdim}\, J^{**}/J \leq 1$. Thus $\mathcal{J}^{**}/\mathcal{J} = b[\mathcal{P}]$ for some $b \geq 0$ by Lemma 1.9.10. Hence $[\mathcal{J}^{**}] = [\mathcal{O}] - (n-1-b)[\mathcal{P}]$.

Let $\mathcal{S}' = \mathcal{J}^{**}/\mathcal{I}(-1)$. Then by Lemma 2.2.4 $\mathcal{S}'$ is pure and furthermore we have $e(\mathcal{S}') = 1$. We now claim this implies $\mathcal{S}'$ is a shifted line object on $\mathbb{P}^2_q$. By Lemma 1.9.7(2) we may pick an object $S' \in \operatorname{grmod}(A)$ which is pure of GK-dimension two and for which $\pi S' = \mathcal{S}'$. By Theorem 1.9.8 the canonical map $\mu : S' \to S'^{\vee\vee}$ is injective with cokernel of GK-dimension zero (if non-zero). Hence $\pi S' = \pi S'^{\vee\vee} = \mathcal{S}'$. It now follows from [8, Proposition 6.2] that $\mathcal{S}'$ is a shifted line object on $\mathbb{P}^2_q$. This ends the proof. □

We can now prove another main result.

Theorem 2.4.26. *Let A be an elliptic quadratic Artin-Schelter algebra and assume σ has infinite order. Let $n \geq 0$ and $\mathcal{I} \in \mathcal{R}_n(\mathbb{P}^2_q)$. Then there exists an integer m, $0 \leq m \leq n$ together with a monomorphism $\mathcal{I}(-m) \hookrightarrow \mathcal{O}$ such that there exists a filtration of line bundles on $\mathbb{P}^2_q$*

$$\mathcal{O} = \mathcal{M}_0 \supset \mathcal{M}_1 \supset \cdots \supset \mathcal{M}_u = \mathcal{I}(-m)$$

with the property the $\mathcal{M}_i/\mathcal{M}_{i+1}$ are shifted line objects.

Proof. This follows easily from Lemma 2.4.25 and Corollary 2.2.12. □

Proof of Theorem 3. By Theorem 2.4.26 and the equivalence $R(A) = \coprod_{n\in N} \mathcal{R}_n$ from §2.2.3. □

Chapter 3

Hilbert series of ideals of quadratic Artin-Schelter algebras

All results in this chapter are published in [28]. We determine the possible Hilbert functions of graded rank one torsion free modules over quadratic three dimensional Artin-Schelter regular algebras. It turns out that, as in the commutative case, they are related to Castelnuovo functions. From this we obtain an intrinsic proof that the space of torsion free rank one modules on a quantum projective plane $\mathbb{P}^2_q$ is connected. A different proof of this fact, based on deformation theoretic methods and the known commutative case has recently been given by Nevins and Stafford [60]. For the Weyl algebra it was proved by Wilson [84].

3.1 Introduction and main results

Put $A = k[x, y, z]$. We view A as the homogeneous coordinate ring of $\mathbb{P}^2$. Let $\mathrm{Hilb}_n(\mathbb{P}^2)$ be the Hilbert scheme of zero-dimensional subschemes of degree n in $\mathbb{P}^2$. It is well-known that this is a smooth connected projective variety of dimension $2n$.

Let $X \in \mathrm{Hilb}_n(\mathbb{P}^2)$ and let $\mathcal{I}_X \subset \mathcal{O}_{\mathbb{P}^2}$ be the ideal sheaf of X. Let I_X be the graded ideal associated to X

$$I_X = \Gamma_*(\mathbb{P}^2, \mathcal{I}_X) = \oplus_l \Gamma(\mathbb{P}^2, \mathcal{I}_X(l))$$

The graded ring $A(X) = A/I_X$ is the homogeneous coordinate ring of X. Let h_X be its Hilbert function:

$$h_X : \mathbb{N} \to \mathbb{N} : m \mapsto \dim_k A(X)_m$$

The function h_X is of considerable interest in classical algebraic geometry as $h_X(m)$ gives the number of conditions for a plane curve of degree m to contain X. It is easy to see that $h_X(m) = n$ for $m \gg 0$, but for small values of m the situation is more complicated (see Example 3.1.2 below).

A characterization of all possible Hilbert functions of graded ideals of $k[x_1, \ldots, x_n]$ was given by Macaulay in [55]. Apparently it was Castelnuovo who first recognized the utility of the difference function (see [26])

$$s_X(m) = h_X(m) - h_X(m-1)$$

Since h_X is constant in high degree one has $s_X(m) = 0$ for $m \gg 0$. It turns out s_X is a so-called *Castelnuovo function* [26] which by definition has the form

$$s(0) = 1, s(1) = 2, \ldots, s(\sigma - 1) = \sigma \text{ and } s(\sigma - 1) \geq s(\sigma) \geq s(\sigma + 1) \geq \cdots \geq 0 \quad (3.1)$$

for some integer $\sigma \geq 0$. The *height* of $s(t)$ is defined as $\max\{s_i\}$.

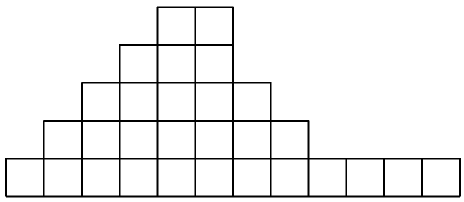
It is convenient to visualize a Castelnuovo function using the graph of the staircase function

$$F_s : \mathbb{R} \to \mathbb{N} : x \mapsto s(\lfloor x \rfloor)$$

and to divide the area under this graph in unit cases. We will call the result a *Castelnuovo diagram*. The *weight* of a Castelnuovo function is the sum of its values, i.e. the number of cases in the diagram.

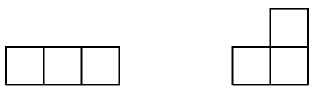
In the sequel we identify a function $f : \mathbb{Z} \to \mathbb{C}$ with its generating function $f(t) = \sum_n f(n)t^n$. We refer to $f(t)$ as a polynomial or a series depending on whether the support of f is finite or not.

Example 3.1.1. $s(t) = 1 + 2t + 3t^2 + 4t^3 + 5t^4 + 5t^5 + 3t^6 + 2t^7 + t^8 + t^9 + t^{10} + t^{11}$ is a Castelnuovo polynomial of weight 29. The corresponding diagram is



It is known [26, 34, 37] that a function h is of the form h_X for $X \in \mathrm{Hilb}_n(\mathbb{P}^2)$ if and only of $h(m) = 0$ for $m < 0$ and $h(m) - h(m-1)$ is a Castelnuovo function of weight n.

Example 3.1.2. Assume $n = 3$. In that case there are two Castelnuovo diagrams



These distinguish whether the points in X are collinear or not. The corresponding Hilbert functions are

$$1, 2, 3, 3, 3, 3, \dots \text{ and } 1, 3, 3, 3, 3, 3, \dots$$

where, as expected, a difference occurs in degree one.

Our aim in this chapter is to generalize the above results to quadratic Artin-Schelter algebras. The Hilbert scheme $\mathrm{Hilb}_n(\mathbb{P}^2_q)$ was constructed in [60]. The definition of $\mathrm{Hilb}_n(\mathbb{P}^2_q)$ is not entirely straightforward since in general $\mathbb{P}^2_q$ will have very few zero-dimensional noncommutative subschemes (see [67]), so a different approach is needed. It turns out that the correct generalization is to define $\mathrm{Hilb}_n(\mathbb{P}^2_q)$ as in Proposition 2.2.9, i.e. as the scheme parametrizing the torsion free graded A-modules I of projective dimension one which are normalized

$$h_A(m) - h_I(m) = \dim_k A_m - \dim_k I_m = n \quad \text{for} \quad m \gg 0$$

(in particular I has rank one as A-module, see Lemma 2.2.8). It is easy to see that if A is commutative then this condition singles out precisely the graded A-modules which occur as I_X for $X \in \mathrm{Hilb}_n(\mathbb{P}^2)$.

The following theorem is the main result of this chapter.

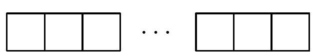
Theorem 4. *Let A be a quadratic Artin-Schelter algebra. There is a bijective correspondence between Castelnuovo polynomials $s(t)$ of weight n and Hilbert series $h_I(t)$ of objects in* $\mathrm{Hilb}_n(\mathbb{P}^2_q)$*, given by*

$$h_I(t) = \frac{1}{(1-t)^3} - \frac{s(t)}{1-t} \tag{3.2}$$

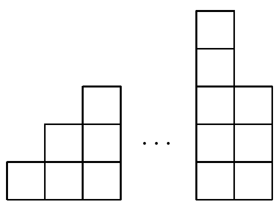
Remark 3.1.3. By shifting the rows in a Castelnuovo diagram in such a way they are left aligned one sees that the number of diagrams of a given weight is equal to the number of partitions of n with distinct parts. It is well-known that this is also equal to the number of partitions of n with odd parts [4].

Remark 3.1.4. For the benefit of the reader we have included in Appendix C the list of Castelnuovo diagrams of weight up to six, as well as some associated data. See also Appendix E.

From Theorem 4 one easily deduces there is a unique maximal Hilbert series $h_{\max}(t)$ and a unique minimal Hilbert series $h_{\min}(t)$ for objects in $\mathrm{Hilb}_n(\mathbb{P}^2_q)$. These correspond to the Castelnuovo diagrams



and



We will also prove:

Theorem 5. $\mathrm{Hilb}_n(\mathbb{P}^2_q)$ *is connected.*

This result was recently proved for almost all A by Nevins and Stafford [60], using deformation theoretic methods and the known commutative case. In the case where A is the homogenization of the first Weyl algebra this result was also proved by Wilson in [84].

We now outline our proof of Theorem 5. For a Hilbert series $h(t)$ as in (3.2) define

$$\mathrm{Hilb}_h(\mathbb{P}^2_q) = \{I \in \mathrm{Hilb}_n(\mathbb{P}^2_q) \mid h_I(t) = h(t)\}$$

Clearly

$$\mathrm{Hilb}_n(\mathbb{P}^2_q) = \bigcup_h \mathrm{Hilb}_h(\mathbb{P}^2_q) \tag{3.3}$$

We show below (Theorem 3.5.1) that (3.3) yields a stratification of $\mathrm{Hilb}_n(\mathbb{P}^2)$ into non-empty smooth connected locally closed subvarieties. In the commutative case this was shown by Gotzmann [36]. Our proof however is entirely different and seems easier.

Furthermore there is a formula for $\dim \mathrm{Hilb}_h(\mathbb{P}^2_q)$ in terms of h (see Corollary 3.5.12 below). From that formula it follows there is a *unique stratum of maximal dimension* in (3.3), (which corresponds to $h = h_{\min}$). In other words $\mathrm{Hilb}_n(\mathbb{P}^2_q)$ contains a dense open connected subvariety. This clearly implies that it is connected.

To finish this introduction let us indicate how we prove Theorem 4. Let M be a torsion free graded A-module of projective dimension one (so we do *not* require M to have rank one). Thus M has a minimal resolution of the form

$$0 \to \oplus_i A(-i)^{b_i} \to \oplus_i A(-i)^{a_i} \to M \to 0 \tag{3.4}$$

where $(a_i), (b_i)$ are finite supported sequences of non-negative integers. These numbers are called the *Betti numbers* of M. It follows the characteristic polynomial $q_M(t)$ is is given by $\sum_i (a_i - b_i)t^i$ and equation (1.4) now gives a relation between the Hilbert series and the Betti numbers of M

$$h_M(t) = \frac{\sum_i (a_i - b_i)t^i}{(1-t)^3} \tag{3.5}$$

So the Betti numbers determine the Hilbert series of M but the converse is not true as some a_i and b_i may be both non-zero at the same time (see e.g. Example 3.1.7 below).

Theorem 4 is an easy corollary of the following more refined result.

Theorem 6. *Let A be a quadratic Artin-Schelter algebra. Let $0 \neq q(t) \in \mathbb{Z}[t^{-1}, t]$ be a Laurent polynomial such that $q_\sigma t^\sigma$ is the lowest non-zero term of q. Then a finitely supported sequence (a_i) of integers occurs among the Betti numbers $(a_i), (b_i)$ of a torsion free graded A-module of projective dimension one with Hilbert series $q(t)/(1-t)^3$ if and only if*

1. $a_l = 0$ *for* $l < \sigma$.
2. $a_\sigma = q_\sigma > 0$.
3. $\max(q_l, 0) \leq a_l < \sum_{i \leq l} q_i$ *for* $l > \sigma$.

This theorem is a natural complement to (3.5) as it bounds the Betti numbers in terms of the Hilbert series.

In Proposition 3.3.6 below we show that if A is elliptic and σ has infinite order, the graded A-module whose existence is asserted in Theorem 6 can actually be chosen to be reflexive. This means it corresponds to a vector bundle on $\mathbb{P}^2_q$.

Corollary 3.1.5. *A Laurent series* $h(t) = q(t)/(1-t)^3 \in \mathbb{Z}((t))$ *occurs as the Hilbert series of a graded torsion free A-module of projective dimension one if and only if for some* $\sigma \in \mathbb{Z}$

$$\sum_{i \leq l} q_i \quad \begin{cases} > 0 & \text{for } l \geq \sigma \\ 0 & \text{for } l < \sigma \end{cases} \tag{3.6}$$

I.e. if and only if

$$q(t)/(1-t) = (1-t)^2 h(t) = \sum_{l \geq \sigma} p_l t^l \tag{3.7}$$

with $p_l > 0$ *for all* $l \geq \sigma$.

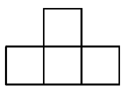
In the rank one case Theorem 6 has the following

Corollary 3.1.6. *Let* $h(t) = 1/(1-t)^3 - s(t)/(1-t)$ *where* $s(t)$ *is a Castelnuovo polynomial and let* $\sigma = \max_i s_i$ *(this is the same* σ *as in* (G.2)*). Then the number of minimal resolutions for an object in* $\mathrm{Hilb}_h(\mathbb{P}^2_q)$ *is equal to*

$$\prod_{l > \sigma} [1 + \min(s_{l-1} - s_l, s_{l-2} - s_{l-1})]$$

This number is bigger than one if and only if there are two consecutive downward jumps in the coefficients of $s(t)$.

Example 3.1.7. Assume $I \in \mathrm{Hilb}_n(\mathbb{P}^2_q)$ has Castelnuovo diagram



By Corollary 3.1.6 we expect two different minimal resolutions for I. It follows from Theorem 6 these are given by

$$0 \to A(-4) \to A(-2)^2 \to I \to 0 \tag{3.8}$$

$$0 \to A(-3) \oplus A(-4) \to A(-2)^2 \oplus A(-3) \to I \to 0 \tag{3.9}$$

In the commutative case (3.8) corresponds to 4 point in general position and (3.9) corresponds to a configuration of 4 points among which exactly 3 are collinear.

Remark 3.1.8. Let M be a torsion free graded A-module of projective dimension one and let its Hilbert series be equal to $q(t)/(1-t)^3$. Then Theorem 6 yields the constraint $0 \leq a_l < q(1)$ for $l \gg 0$ and it is easy to see $q(1)$ is equal to the rank of M. Hence if M has rank one then there are only a finite number of possibilities for its Betti numbers but this is never the case for higher rank.

It follows that in the case of rank > 1 the torsion free modules M of projective dimension one with fixed Hilbert series are not parametrized by a finite number of algebraic varieties. This is to be expected as we have not imposed any stability conditions on M.

3.2 Notations and conventions

Except for §3.5.1 which is about moduli spaces, a point of a reduced scheme of finite type over k is a closed point and we confuse such schemes with their set of k-points.

Some results in this chapter are for rank one modules and others are for arbitrary rank. To make the distinction clear we usually denote rank one modules by the letter I and arbitrary rank modules by the letter M.

Recall from Lemma 2.2.8 that for $I \in \operatorname{grmod}(A)$ has rank one and is normalized with invariant n if and only if the Hilbert series of I has the form

$$h_I(t) = \frac{1}{(1-t)^3} - \frac{s(t)}{1-t}$$

for a Laurent polynomial $s(t) \in \mathbb{Z}[t, t^{-1}]$ with $s(1) = n$. In that case we write $s_I(t) = s(t)$. We also put $s_{\mathcal{I}}(t) = s_{\omega\mathcal{I}}(t)$.

3.3 Proof of Theorem 6

3.3.1 Preliminaries

Throughout the rest of this chapter, A will be a quadratic Artin-Schelter algebra and $\mathbb{P}^2_q = \operatorname{Proj} A$ is the associated quantum projective plane.

We will need several equivalent versions of the conditions (1-3) in the statement of Theorem 6. One of those versions is in terms of "ladders".

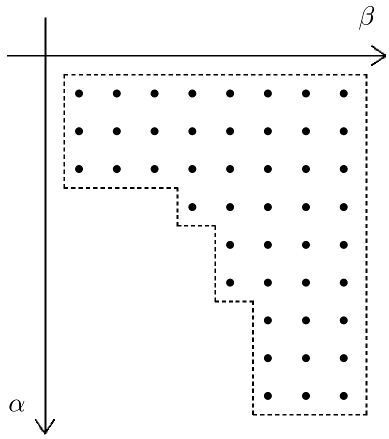
For positive integers m, n consider the rectangle

$$R_{m,n} = [1, m] \times [1, n] = \{(\alpha, \beta) \mid 1 \leq \alpha \leq m, 1 \leq \beta \leq n\} \subset \mathbb{Z}^2$$

A subset $L \subset R_{m,n}$ is called a *ladder* if

$$\forall (\alpha, \beta) \in R_{m,n} : (\alpha, \beta) \notin L \Rightarrow (\alpha + 1, \beta), (\alpha, \beta - 1) \notin L$$

Example 3.3.1. The ladder below is indicated with a dotted line.



Let $(a_i), (b_i)$ be finitely supported sequences of non-negative integers. We associate a sequence $S(c)$ of length $\sum_i c_i$ to a finitely supported sequence (c_i) as follows

$$\dots, \underbrace{i-1, \dots, i-1}_{c_{i-1}\text{ times}}, \underbrace{i, \dots, i}_{c_i\text{ times}}, \underbrace{i+1, \dots, i+1}_{c_{i+1}\text{ times}}, \dots$$

where by convention the left most non-zero entry of $S(c)$ has index one.

Let $m = \sum_i a_i$, $n = \sum_i b_i$ and put $R = [1, m] \times [1, n]$. We associate a ladder to $(a_i), (b_i)$ as follows

$$L_{a,b} = \{(\alpha, \beta) \in R \mid S(a)_\alpha < S(b)_\beta\} \tag{3.10}$$

Lemma 3.3.2. *Let $(a_i), (b_i)$ be finitely supported sequences of integers and put $q_i = a_i - b_i$. The following sets of conditions are equivalent.*

1. *Let q_σ be the lowest non-zero q_i.*
   (a) $a_l = 0$ *for* $l < \sigma$.
   (b) $a_\sigma = q_\sigma > 0$.
   (c) $\max(q_l, 0) \le a_l < \sum_{i \le l} q_i$ *for* $l > \sigma$.
2. *Let a_σ be the lowest non-zero a_i.*
   (a) *The $(a_i), (b_i)$ are non-negative.*
   (b) $b_i = 0$ *for* $i \le \sigma$
   (c) $\sum_{i \le l} b_i < \sum_{i < l} a_i$ *for* $l > \sigma$
3. *Put $m = \sum_i a_i$, $n = \sum_i b_i$.*
   (a) *The $(a_i), (b_i)$ are non-negative.*
   (b) $n < m$.

(c) $\forall(\alpha,\beta)\in R:\beta\geq\alpha-1\Rightarrow(\alpha,\beta)\in L_{a,b}$.

Proof. The equivalence between (1) and (2) as well as the equivalence between (2) and (3) is easy to see. We leave the details to the reader. □

3.3.2 Proof that the conditions in Theorem 6 are necessary

We will show the equivalent conditions given in Lemma 3.3.2(2) are necessary. The method for the proof has already been used in [8] and also by Ajitabh in [1]. Assume $M\in\operatorname{grmod}(A)$ is torsion free of projective dimension one and consider the minimal projective resolution of M.

$$0\to\oplus_i A(-i)^{b_i}\to\oplus_i A(-i)^{a_i}\to M\to 0 \tag{3.11}$$

There is nothing to prove for (2a) so we discuss (2b)(2c). Since (3.11) is a minimal resolution, it contains for all integers l a subcomplex of the form

$$\oplus_{i\leq l}A(-i)^{b_i}\xrightarrow{\phi_l}\oplus_{i<l}A(-i)^{a_i}$$

The fact ϕ_l must be injective implies

$$\sum_{i\leq l}b_i\leq\sum_{i<l}a_i$$

In particular, if we take $l=\sigma$ this already shows $b_i=0$ for $i\leq\sigma$ which proves (2b). Finally, to prove (2c), assume there is some $l>\sigma$ such that $\sum_{i\leq l}b_i=\sum_{i<l}a_i$. This means $\operatorname{coker}\phi_l$ is torsion and different from zero. Note that $\oplus_{i<l}A(-i)^{a_i}$ is not zero since $l>\sigma$. We have a map

$$\operatorname{coker}\phi_l\to M$$

which must be zero since M is assumed to be torsion free. But this implies that $\oplus_{i<l}A(-i)^{a_i}\to M$ is the zero map, which is obviously impossible given the minimality of our chosen resolution (3.11). Thus we obtain

$$\sum_{i\leq l}b_i<\sum_{i<l}a_i$$

which completes the proof.

3.3.3 Proof that the conditions in Theorem 6 are sufficient

We will assume the equivalent conditions given in Lemma 3.3.2(3) hold. Thus we fix finitely supported sequences (a_i), (b_i) of non-negative integers such that $n=\sum_i b_i<m=\sum_i a_i$ and we assume in addition the ladder condition (3c) is true.

Our proof of the converse of Theorem 6 is a suitably adapted version of [23, p468]. It is based on a series of observations, the first one of which is the next lemma.

Lemma 3.3.3. *If $M \in \mathrm{grmod}(A)$ has a resolution (not necessarily minimal)*

$$0 \to \oplus_i A(-i)^{b_i} \xrightarrow{\phi} \oplus_i A(-i)^{a_i} \to M \to 0$$

such that the restriction

$$u^*(\pi\phi) : \oplus_i \mathcal{O}_C(-i)^{b_i} \to \oplus_i \mathcal{O}_C(-i)^{a_i}$$

has maximal rank at every point in C then M is torsion free.

Proof. Assume M is not torsion free and $u^*(\pi\phi)$ has the stated property. This means $u^*(\pi\phi)$ is an injective map whose cokernel $u^*\pi M$ is a vector bundle on C.

Let T be the torsion submodule of M. Note first that M cannot have a submodule of GK-dimension ≤ 1 as $\mathrm{Ext}^1_A(-, A)$ is zero on modules of GK-dimension ≤ 1 [8]. Hence T has pure GK-dimension two.

If T contains h-torsion then $\mathrm{Tor}_1^A(D, M)$ is not zero and in fact has GK-dimension two. Thus $u^*(\pi\phi)$ is not injective, yielding a contradiction.

Assume now T is h-torsion free. In that case T/Th is a submodule of GK-dimension one of M/Mh. And hence $u^*\pi T$ is a submodule of dimension zero of $u^*\pi M$ which is again a contradiction. □

Now note that the map

$$\mathrm{Hom}_A(\oplus_i A(-i)^{b_i}, \oplus_i A(-i)^{a_i}) \to \\ \mathrm{Hom}_C(\oplus_i \mathcal{O}_C(-i)^{b_i}, \oplus_i \mathcal{O}_C(-i)^{a_i}) : \phi \mapsto u^*(\pi\phi) \qquad (3.12)$$

is surjective. Let H be the linear subspace of $\mathrm{Hom}_C(\oplus_i \mathcal{O}_C(-i)^{b_i}, \oplus_i \mathcal{O}_C(-i)^{a_i})$ whose elements are such that the projections on $\mathrm{Hom}_C(\mathcal{O}_C(-i)^{b_i}, \mathcal{O}_C(-i)^{a_i})$ are zero for all i. If we can find $N \in H$ of maximal rank in every point then an arbitrary lifting of N under (3.12) yields a torsion free A-module with Betti numbers (a_i), (b_i).

The elements of H are given by matrices $(h_{\alpha\beta})_{\alpha\beta}$ for $(\alpha, \beta) \in L_{a,b}$ where $L_{a,b}$ is as in (3.10) and where the $h_{\alpha\beta}$ are elements of suitable non-zero $\mathrm{Hom}_C(\mathcal{O}_C(-i), \mathcal{O}_C(-j))$. We will look for N in the linear subspace ${}^0H$ of H given by those matrices where $h_{\alpha\beta} = 0$ for $\beta \neq \alpha, \alpha - 1$.

To find N we use the next observation.

Lemma 3.3.4. *For $p \in C$ and $N \in {}^0H$ let N_p be the restriction of N to p and write*

$${}^0H_p = \{N \in {}^0H \mid N_p \text{ has non-maximal rank}\}$$

If

$$\mathrm{codim}_{{}^0H}\, {}^0H_p \geq 2 \text{ for all } p \in C \qquad (3.13)$$

then there exists an N in ${}^0H$ which has maximal rank everywhere.

Proof. Assume (3.13) holds. Since $({}^0H_p)_p$ is a one-dimensional family of subvarieties of codimension ≥ 2 in ${}^0H$ it is intuitively clear their union cannot be the whole of ${}^0H$, proving the lemma.

To make this idea precise let $\mathcal{E}_1$, $\mathcal{E}_0$ be the pullbacks of the vector bundles $\oplus_i \mathcal{O}_C(-i)^{b_i}$, $\oplus_i \mathcal{O}_C(-i)^{a_i}$ to ${}^0H \times C$ and let $\mathcal{N} : \mathcal{E}_1 \to \mathcal{E}_0$ be the vector bundle map which is equal to N_p in the point $(N, p) \in {}^0H \times C$. Let ${}^0\mathcal{H} \subset H \times C$ be the locus of points x in ${}^0H \times C$ where $\mathcal{N}_x$ has non-maximal rank. It is well-known and easy to see that ${}^0\mathcal{H}$ is closed in ${}^0H \times C$. A more down to earth description of ${}^0\mathcal{H}$ is

$${}^0\mathcal{H} = \{(N, p) \in {}^0H \times C \mid N_p \text{ has non-maximal rank}\}$$

By considering the fibers of the projection ${}^0H \times C \to C$ we see ${}^0\mathcal{H}$ has codimension ≤ 2 in ${}^0H \times C$. Hence its projection on ${}^0H$, which is $\bigcup_p {}^0H_p$, has codimension ≥ 1. □

Fix a point $p \in C$ and fix basis elements for the one-dimensional vector spaces $\mathcal{O}_C(-i)_p$. Let $\mathbb{L}$ be the vector space associated to the ladder $L_{a,b}$ (see (3.10)) as follows

$$\mathbb{L} = \{A \in M_{m \times n}(k) \mid A_{\alpha\beta} = 0 \text{ for } (\alpha, \beta) \notin L\}$$

and let ${}^0\mathbb{L}$ be the subspace defined by $A_{\alpha\beta} = 0$ for $\beta \neq \alpha, \alpha - 1$. Then there is a surjective linear map

$$\phi_p : {}^0H \to {}^0\mathbb{L} : N \mapsto N_p$$

Let V be the matrices of non-maximal rank in ${}^0\mathbb{L}$. We have

$${}^0H_p = \phi_p^{-1}(V)$$

Now by looking at the two topmost $n \times n$-submatrices we see that for a matrix in ${}^0\mathbb{L}$ to not have maximal rank both the diagonals $\beta = \alpha$ and $\beta = \alpha - 1$ must contain a zero (this is not sufficient). Using condition 3.3.2(3c) we see V has codimension ≥ 2 and so the same holds for ${}^0H_p$. This means we are done.

Remark 3.3.5. It is easy to see that the actual torsion free module constructed in this section is the direct sum of a free module and a module of rank one.

3.3.4 A refinement

Proposition 3.3.6. *Assume A is a elliptic and that in the geometric data $(E, \mathcal{O}_E(1), \sigma)$ associated to A, σ has infinite order. Then the graded A-module whose existence is asserted in Theorem 6 can be chosen to be reflexive.*

Proof. The modules that are constructed in the proof of Theorem 6 satisfy the criterion given in Proposition 2.3.2, hence they are reflexive. □

3.4 Proof of other properties of Hilbert series

Proof of Corollary 3.1.5. It is easy to see that the conditions (1-3) in Theorem 6 have a solution for (a_i) if and only if (3.6) is true. The equivalence of (3.6) and (3.7) is clear. □

Proof of Theorem 4. Let $h(t)$ is a Hilbert series of the form (2.19). Thus $h(t) = q(t)/(1-t)^3$ where $q(t) = 1-(1-t)^2 s(t)$ and hence $q(t)/(1-t) = 1/(1-t)-(1-t)s(t)$. Thus (3.7) is equivalent to $(1-t)s(t)$ being of the form

$$(1-t)s(t) = 1 + t + t^2 + \cdots + t^{\sigma-1} + d_\sigma t^\sigma + \cdots$$

where $d_i \leq 0$ for $i \geq \sigma$. Multiplying by $1/(1-t) = 1 + t + t^2 + \cdots$ shows this is equivalent to $s(t)$ being a Castelnuovo polynomial. □

Proof of Corollary 3.1.6. The number of solutions to the conditions (1-3) in the statement of Theorem 6 is

$$\prod_{l>\sigma}\left(\left(\sum_{i\leq l} q_i\right) - \max(q_l, 0)\right) = \prod_{l>\sigma} \min\left(\sum_{i<l} q_i, \sum_{i\leq l} q_i\right)$$

Noting $\sum_{i\leq l} q_i = 1 + s_{l-1} - s_l$ finishes the proof. □

Convention 3.4.1. *Below we will call a formal power series of the form*

$$\frac{1}{(1-t)^3} - \frac{s(t)}{1-t}$$

where $s(t)$ is a Castelnuovo polynomial of weight n an admissible Hilbert series *of weight n.*

3.5 The stratification by Hilbert series

In this section we will prove the following result.

Theorem 3.5.1. *There is a (weak) stratification into smooth, non-empty connected locally closed sets*

$$\mathrm{Hilb}_n(\mathbb{P}^2_q) = \bigcup_h \mathrm{Hilb}_h(\mathbb{P}^2_q) \tag{3.14}$$

where the union runs over the (finite set) of admissible Hilbert series of weight n and where the points in $\mathrm{Hilb}_h(\mathbb{P}^2_q)$ represents the points in $\mathrm{Hilb}_n(\mathbb{P}^2_q)$ corresponding to objects with Hilbert series h.

Furthermore we have

$$\overline{\mathrm{Hilb}_h(\mathbb{P}^2_q)} \subset \bigcup_{h'\geq h} \mathrm{Hilb}_{h'}(\mathbb{P}^2_q) \tag{3.15}$$

In the decomposition (3.14) *there is a unique stratum of maximal dimension* $2n$ *which corresponds to the Hilbert series* $h_{min}(t)$ *(see §3.1).*

That the strata are non-empty is Theorem 4. The rest of Theorem 3.5.1 will be a consequence of Lemma 3.5.5, Corollary 3.5.12 and Proposition 3.5.15 below.

We refer to (3.14) as a "weak" stratification (for an ordinary stratification one would require the inclusions in (3.15) to be equalities, which is generally not the case).

In the commutative case Theorem 3.15 was proved by Gotzmann [36]. It is not clear to us that Gotzmann's method can be generalized to the noncommutative case. In any case, the reader will notice that our proof is substantially different.

Proof of Theorem 5. This is now clear from Theorem 3.5.1. □

It follows from Theorem 6 that given a Hilbert series $h(t) = q(t)/(1-t)^3$ there is a unique legal choice of Betti numbers $(a_i)_i$, $(b_i)_i$ such that a_i and b_i are not both non-zero for all i. Namely

$$(a_i, b_i) = \begin{cases} (q_i, 0) & \text{if } q_i \geq 0 \\ (0, -q_i) & \text{otherwise} \end{cases} \tag{3.16}$$

We call this the *minimal Betti numbers* associated to h.

We have some extra information on the strata $\mathrm{Hilb}_h(\mathbb{P}^2_q)$. Define $\mathrm{Hilb}_h(\mathbb{P}^2_q)^{\min}$ as the subset of $\mathrm{Hilb}_h(\mathbb{P}^2_q)$ consisting of objects with minimal Betti numbers.

Proposition 3.5.2. $\mathrm{Hilb}_h(\mathbb{P}^2_q)^{\min}$ *is open in* $\mathrm{Hilb}_h(\mathbb{P}^2_q)$

This is proved in §3.5.1 below.

Assume A is elliptic and that in the geometric data $(E, \mathcal{O}_E(1), \sigma)$ associated to A, σ has infinite order. Let $\mathrm{Hilb}_n(\mathbb{P}^2_q)^{\mathrm{inv}}$ be the reflexive objects in $\mathrm{Hilb}_n(\mathbb{P}^2_q)$. This is an open subset (see [60, Theorem 8.11] or Theorem 7 in Chapter 4 below).

Proposition 3.5.3. *For all admissible Hilbert series* h *with weight* n *we have*

$$\mathrm{Hilb}_h(\mathbb{P}^2_q) \cap \mathrm{Hilb}_n(\mathbb{P}^2_q)^{\mathrm{inv}} \neq \emptyset$$

Proof. This is a special case of Proposition 3.3.6. □

Remark 3.5.4. Consider the Hilbert scheme of points $\mathrm{Hilb}_n(\mathbb{P}^2)$ in the projective plane $\mathbb{P}^2$. The inclusion relation between the closures of the strata of $\mathrm{Hilb}_n(\mathbb{P}^2)$ has been a subject of interest in [21, 24, 25, 43]. Although in general the precise inclusion relation is still unknown, the special case where the Hilbert series of the strata are as close as possible is completely settled (see [38] and Chapter 5 below). It is a natural to consider the same question for the varieties $\mathrm{Hilb}_n(\mathbb{P}^2_q)$, where one may use the same techniques as in Chapter 5.

3.5.1 Moduli spaces

In this section "points" of schemes will be not necessarily closed. We will consider functors from the category of noetherian k-algebras Noeth/k to the category of sets. For $R \in \mathrm{Noeth}/k$ we write $(-)_R$ for the base extension $-\otimes R$. If x is a (not necessarily closed) point in $\operatorname{Spec} R$ then we write $(-)_x$ for the base extension $-\otimes_R k(x)$. We put $\mathbb{P}^2_{q,R} = \operatorname{Proj} A_R$.

It follows from [6, Prop. 4.9(1) and 4.13] that A is strongly noetherian so A_R is still noetherian. Furthermore it follows from [11, Prop. C6] that A_R satisfies the χ-condition and finally by [11, Cor. C7] $\Gamma(\mathbb{P}^2_{q,R}, -)$ has cohomological dimension two.

An R-family of objects in $\operatorname{coh}(\mathbb{P}^2_q)$ or $\operatorname{grmod}(A)$ is by definition an R-flat object [11] in these categories.

For $n \in \mathbb{N}$ let $\mathcal{H}ilb_n(\mathbb{P}^2_q)(R)$ be the R-families of objects $\mathcal{I}$ in $\operatorname{coh}(\mathbb{P}^2_q)$, *modulo Zariski local isomorphism on* $\operatorname{Spec} R$, with the property that for any map $x \in \operatorname{Spec} R$, $\mathcal{I}_x$ is torsion free normalized of rank one in $\operatorname{coh}(\mathbb{P}^2_{q,k(x)})$.

The main result of [60] is that $\mathcal{H}ilb_n(\mathbb{P}^2_q)$ is represented by a smooth scheme $\mathrm{Hilb}_n(\mathbb{P}^2_q)$ of dimension $2n$ (see also Chapter 2 for a special case, treated with a different method which yields some extra information).

Warning. *The reader will now notice that the set* $\mathrm{Hilb}_n(\mathbb{P}^2_q) = \mathcal{H}ilb_n(\mathbb{P}^2_q)(k)$ *parametrizes objects in* $\operatorname{coh}(\mathbb{P}^2_q)$ *rather than in* $\operatorname{grmod}(A)$ *as was the case in Proposition 2.2.9. However by Corollary 2.2.6 the new point of view is equivalent to the old one.*

If $h(t)$ is a admissible Hilbert series of weight n then $\mathcal{H}ilb_h(\mathbb{P}^2_q)(R)$ is the set of R-families of torsion free graded A-modules which have Hilbert series h and which have projective dimension one, *modulo local isomorphism on* $\operatorname{Spec} R$. The map π defines a map

$$\pi(R) : \mathcal{H}ilb_h(\mathbb{P}^2_q)(R) \to \mathcal{H}ilb_n(\mathbb{P}^2_q)(R) : I \mapsto \pi I$$

Below we will write $\mathcal{I}^u$ for a universal family on $\mathrm{Hilb}_n(\mathbb{P}^2_q)$. This is a sheaf of graded $\mathcal{O}_{\mathrm{Hilb}_n(\mathbb{P}^2_q)} \otimes A$-modules on $\mathrm{Hilb}_n(\mathbb{P}^2_q)$.

Lemma 3.5.5. *The map* $\pi(k)$ *is an injection which identifies* $\mathcal{H}ilb_h(\mathbb{P}^2_q)(k)$ *with*

$$\{x \in \mathcal{H}ilb_n(\mathbb{P}^2_q)(k) \mid h_{\mathcal{I}^u_x} = h\}$$

This is a locally closed subset of $\mathcal{H}ilb_n(\mathbb{P}^2_q)(k)$. *Furthermore*

$$\overline{\mathcal{H}ilb_h(\mathbb{P}^2_q)(k)} \subset \bigcup_{h' \geq h} \mathcal{H}ilb_{h'}(\mathbb{P}^2_q)(k) \tag{3.17}$$

Proof. The fact that $\pi(k)$ is an injection and does the required identification follows from Corollary 2.2.6.

For any $N \geq 0$ we have by Corollary B.3

$$\mathcal{H}ilb_{h,N}(\mathbb{P}_q^2)(k) = \{x \in \mathcal{H}ilb_n(\mathbb{P}_q^2)(k) \mid h_{\mathcal{I}_x^u}(n) = h(n) \text{ for } n \leq N\}$$

is locally closed in $\mathcal{H}ilb_n(\mathbb{P}_q^2)(k)$. By Theorem 4 we know that only a finite number of Hilbert series occur for objects in $\mathcal{H}ilb_n(\mathbb{P}_q^2)(k)$. Thus $\mathcal{H}ilb_{h,N}(\mathbb{P}_q^2)(k) = \mathcal{H}ilb_h(\mathbb{P}_q^2)(k)$ for $N \gg 0$. (3.17) also follows easily from semi-continuity. □

Now let $\mathrm{Hilb}_h(\mathbb{P}_q^2)$ be the reduced locally closed subscheme of $\mathrm{Hilb}_n(\mathbb{P}_q^2)$ whose closed points are given by $\mathcal{H}ilb_h(\mathbb{P}_q^2)(k)$. We then have the following result.

Proposition 3.5.6. $\mathrm{Hilb}_h(\mathbb{P}_q^2)$ *represents the functor* $\mathcal{H}ilb_h(\mathbb{P}_q^2)$.

Before proving this proposition we need some technical results. The following is proved in [60]. For the convenience of the reader we put the proof here.

Lemma 3.5.7. *Assume $\mathcal{I}$, $\mathcal{J}$ are R-families of objects in* $\mathrm{coh}(\mathbb{P}_q^2)$ *with the property that for any map $x \in \operatorname{Spec} R$, $\mathcal{I}_x$ is torsion free of rank one in* $\mathrm{coh}(\mathbb{P}^2_{q,k(x)})$. *Then $\mathcal{I}$, $\mathcal{J}$ represent the same object in $\mathcal{H}ilb_n(\mathbb{P}_q^2)(R)$ if and only if there is an invertible module $\mathfrak{l}$ in* $\mathrm{Mod}(R)$ *such that*

$$\mathcal{J} = \mathfrak{l} \otimes_R \mathcal{I}$$

Proof. Let $\mathcal{I}$ be as in the statement of the lemma. We first claim that the natural map

$$R \to \mathrm{End}(\mathcal{I}) \tag{3.18}$$

is an isomorphism. Assume first $f \neq 0$ is in the kernel of (3.18). Then the flatness of $\mathcal{I}$ implies $\mathcal{I} \otimes_R Rf = 0$. This implies $\mathcal{I}_x = \mathcal{I} \otimes_R k(x) = 0$ for some $x \in \operatorname{Spec} R$ and this is a contradiction since by definition $\mathcal{I}_x \neq 0$.

It is easy to see (3.18) is surjective (in fact an isomorphism) when R is a field. It follows that for all $x \in \operatorname{Spec} R$

$$\mathrm{End}(\mathcal{I}) \otimes_R k(x) \to \mathrm{End}(\mathcal{I} \otimes_R k(x))$$

is surjective. Then it follows from base change (see [60, Thm 4.3(1)(4)]) that $\mathrm{End}(\mathcal{I}) \otimes_R k(x)$ is one dimensional and hence (3.18) is surjective by Nakayama's lemma.

Now let $\mathcal{I}$, $\mathcal{J}$ be as in the statement of the lemma and assume they represent the same element of $\mathcal{H}ilb_n(\mathbb{P}_q^2)(R)$, i.e. they are locally isomorphic. Put

$$\mathfrak{l} = \mathrm{Hom}(\mathcal{I}, \mathcal{J})$$

It is easy to see $\mathfrak{l}$ has the required properties since this may be checked locally on $\operatorname{Spec} R$ and then we may invoke the isomorphism 3.18. □

Lemma 3.5.8. *Assume R is finitely generated and let P_0, P_1 be finitely generated graded free A_R-modules. Let $N \in \mathrm{Hom}_A(P_1, P_0)$. Then*

$$V = \{x \in \operatorname{Spec} R \mid N_x \textit{ is injective with torsion free cokernel}\}$$

is open. Furthermore the restriction of $\operatorname{coker} N$ *to V is R-flat.*

Proof. We first note that the formation of V is compatible with base change. It is sufficient to prove this for an extension of fields. The key point is that if $K \subset L$ is an extension of fields and $M \in \mathrm{grmod}(A_K)$ then M is torsion free if and only if M_L is torsion free. This follows from the fact that if D is the graded quotient field of A_K then M is torsion free if and only if the map $M \to M \otimes_{A_K} D$ is injective.

To prove openness of V we may now assume by [6, Theorem 0.5] that R is a Dedekind domain (not necessarily finitely generated).

Assume $K = \ker N \neq 0$. Since $\operatorname{gldim} R = 1$ we deduce that the map $K \to P_1$ is degree wise split. Hence N_x is never injective and the set V is empty.

So we assume $K = 0$ and we let $C = \operatorname{coker} N$. Let T_0 be the R-torsion part of C. Since A_R is noetherian T_0 is finitely generated. We may decompose T_0 degree wise according to the maximal ideals of R. Since it is clear this yields a decomposition of T_0 as A_R-module it follows there can be only a finite number of points in the support of T_0 as R-module.

If $x \in \operatorname{Spec} R$ is in the support of T_0 as R-module then $\operatorname{Tor}_1^R(C, k(x)) \neq 0$ and hence N_x is not injective. Therefore $x \notin V$. By considering an affine covering of the complement of the support of T_0 as R-module we reduce to the case where C is torsion free as R-module.

Let η be the generic point of $\operatorname{Spec} R$ and assume C_η has a non-zero torsion submodule T_η. Put $T = T_\eta \cap C$. Since R is Dedekind the map $T \to C$ is degree wise split. Hence $T_{k(x)} \subset C_{k(x)}$ and so $C_{k(x)}$ will always have torsion. Thus V is empty.

Assume $T_\eta = 0$. It is now sufficient to construct an non-empty open U in $\operatorname{Spec} R$ such that $U \subset V$. We have an embedding $C \subset C^{**}$. Let Q be the maximal A_R submodule of C^{**} containing C such that Q/C is R-torsion. Since Q/C is finitely generated it is supported on a finite number of closed points of $\operatorname{Spec} R$ and we can get rid of those by considering an affine open of the complement of those points.

Thus we may assume C^{**}/C is R-torsion free. Under this hypothesis we will prove C_x is torsion free for all closed points $x \in \operatorname{Spec} R$. Since we now have an injection $C_x \to (C^{**})_x$ it is sufficient to prove $(C^{**})_x$ is torsion free. To this end we way assume R is a discrete valuation ring and x is the closed point of $\operatorname{Spec} R$.

Let Π be the uniformizing element of R and let T_1 be the torsion submodule of $(C^{**})_x$. Assume $T_1 \neq 0$ and let Q be its inverse image in C^{**}. Thus we have an exact sequence

$$0 \to \Pi C^{**} \to Q \to T_1 \to 0 \tag{3.19}$$

which is cannot be split since otherwise $T_1 \subset C^{**}$ which is impossible.

We now apply $(-)^*$ to (3.19). Using $\underline{\mathrm{Ext}}^1_{A_R}(T_1, A_R) = \underline{\mathrm{Hom}}_{A_x}(T_1, A_x) = 0$ we deduce $Q^* = C^{***} = C^*$. Applying $(-)^*$ again we deduce $Q^{**} = C^{**}$ and hence the map $Q \to Q^{**} \cong C^{**}$ gives a splitting of (3.19), which is a contradiction. This finishes the proof of the openness of V.

The flatness assertion may be checked locally. So we may assume R is a local ring with closed point x and $x \in V$. Thus for any m we have a map between free R-modules $(P_1)_m \to (P_0)_m$ which remains injective when tensored with $k(x)$. A standard application of Nakayama's lemma then yields the map is split, and hence its cokernel is projective. ☐

Lemma 3.5.9. *Assume $I \in \mathcal{H}ilb_h(\mathbb{P}_q^2)(R)$ and $x \in \operatorname{Spec} R$. Then there exist:*

1. *an element $r \in R$ with $r(x) \neq 0$;*
2. *a polynomial ring $S = k[x_1, \ldots, x_n]$;*
3. *a point $y \in \operatorname{Spec} S$;*
4. *an element $s \in S$ with $s(y) \neq 0$;*
5. *a homomorphism of rings $\phi : S_s \to R_r$ such that $\phi(x) = y$ (where we also have written ϕ for the dual map $\operatorname{Spec} R_r \to \operatorname{Spec} S_s$);*
6. *an object $I^{(0)}$ in $\mathcal{H}ilb_h(\mathbb{P}_q^2)(S_s)$*

such that $I^{(0)} \otimes_{S_s} R_r = I \otimes_S S_s$.

Proof. By hypotheses I has a presentation

$$0 \to P_1 \to P_0 \to I \to 0$$

where P_0, P_1 are finitely graded projective A_R-modules. It is classical that we have $P_0 \cong \mathfrak{p}_0 \otimes_R A$, $P_1 \cong \mathfrak{p}_1 \otimes_R A$ where $\mathfrak{p}_0$, $\mathfrak{p}_1$ are finitely generated graded projective R-modules. By localizing R at an element which is non-zero in x we may assume P_0, P_1 are graded free A_R-modules. After doing this N is given by a $p \times q$-matrix with coefficients in A_R for certain p, q.

Then by choosing a k-basis for A and writing out the entries of N in terms of this basis with coefficients in R we may construct a polynomial ring $S = k[x_1, \ldots, x_n]$ together with a morphism $S \to R$ and a $p \times q$-matrix $N^{(0)}$ over A_S such that N is obtained by base-extension from $N^{(0)}$. Thus I is obtained by base-extension from the cokernel $I^{(0)}$ of a map

$$N^{(0)} : P_1^{(0)} \longrightarrow P_0^{(0)}$$

where $P_1^{(0)}$, $P_0^{(0)}$ are graded free A_S-modules. Let y be the image of x in $\operatorname{Spec} S$. By construction we have $I_x = I_y^{(0)} \otimes_{k(y)} k(x)$. From this it easily follows that $I_y^{(0)} \in \mathcal{H}ilb_h(k(y))$.

The module $I^{(0)}$ will not in general satisfy the requirements of the lemma but it follows from Lemma 3.5.8 that this will be the case after inverting a suitable element in S non-zero in y. This finishes the proof. ☐

Proof of Proposition 3.5.6. Let $R \in \operatorname{Noeth}/k$. We will construct inverse bijections

$$\Phi(R) : \mathcal{H}ilb_h(\mathbb{P}_q^2)(R) \to \operatorname{Hom}(\operatorname{Spec} R, \operatorname{Hilb}_h(\mathbb{P}_q^2))$$
$$\Psi(R) : \operatorname{Hom}(\operatorname{Spec} R, \operatorname{Hilb}_h(\mathbb{P}_q^2)) \to \mathcal{H}ilb_h(\mathbb{P}_q^2)(R)$$

We start with Ψ. For $w \in \operatorname{Hom}(\operatorname{Spec} R, \operatorname{Hilb}_h(\mathbb{P}_q^2))$ we put

$$\Psi(R)(w) = \omega(\mathcal{I}_R^u) = \bigoplus_m \Gamma(\mathbb{P}^2_{q,R}, \mathcal{I}_R^u(m))$$

We need to show $\omega(\mathcal{I}_R^u) \in \mathcal{H}ilb_h(\mathbb{P}_q^2)(R)$. It is clear this can be done Zariski locally on $\operatorname{Spec} R$. Therefore we may assume w factors as

$$\operatorname{Spec} R \to \operatorname{Spec} S \to \operatorname{Hilb}_h(\mathbb{P}_q^2)$$

where $\operatorname{Spec} S$ is an affine open subset of $\operatorname{Hilb}_h(\mathbb{P}_q^2)$.

Now by Lemma 3.5.5

$$\operatorname{Spec} S \to \mathbb{N} : x \mapsto \dim_k \Gamma(\mathbb{P}^2_{q,x}, \mathcal{I}_x^u(m))$$

has constant value $h(m)$ and hence by Corollary B.4 below $\Gamma(\mathbb{P}^2_{q,S}, \mathcal{I}_S^u(m))$ is a projective S-module and furthermore by [11, Lemma C6.6]

$$\Gamma(\mathbb{P}^2_{q,x}, \mathcal{I}_x^u(m)) = \Gamma(\mathbb{P}^2_{q,S}, \mathcal{I}_S^u(m)) \otimes_S k(x)$$
$$\Gamma(\mathbb{P}^2_{q,R}, \mathcal{I}^u(m)_R) = \Gamma(\mathbb{P}^2_{q,S}, \mathcal{I}_S^u(m)) \otimes_S R$$

for $x \in \operatorname{Spec} S$. We deduce $\omega(\mathcal{I}_S^u)$ is flat and furthermore

$$\omega(\mathcal{I}_S^u)_x = \omega(\mathcal{I}_x^u), \quad \omega(\mathcal{I}_S^u)_R = \omega(\mathcal{I}_R^u)$$

Using the first equation we deduce from Corollary 2.2.6 and Nakayama's lemma that $\omega(\mathcal{I}_S^u)$ has projective dimension one. Thus $\omega(\mathcal{I}_S^u) \in \mathcal{H}ilb_h(\mathbb{P}_q^2)(S)$. From the second equation we then deduce $\omega(\mathcal{I}_R^u) \in \mathcal{H}ilb_h(\mathbb{P}_q^2)(R)$.

Now we define Φ. Let $I \in \mathcal{H}ilb_h(R)$. We define $\Phi(R)(I)$ as the map $w : \operatorname{Spec} R \to \operatorname{Hilb}_n(\mathbb{P}_q^2)$ corresponding to πI. I.e. formally

$$\pi I = \mathcal{I}_w^u \otimes_R \mathfrak{l}$$

where $\mathfrak{l}$ is an invertible R module and where this time we have made the base change map w explicit in the notation. We need to show $\operatorname{im} w$ lies in $\operatorname{Hilb}_h(\mathbb{P}_q^2)$. Again we may do this locally on $\operatorname{Spec} R$. Thus by Lemma 3.5.9 we may assume there is a map $\theta : S \to R$ where S is integral and finitely generated over k and I is obtained from $I^{(0)} \in \mathcal{H}ilb_h(S)$ by base change. Let $v : \operatorname{Spec} S \to \operatorname{Hilb}_n(\mathbb{P}_q^2)$ be the map corresponding to $I^{(0)}$. An elementary computation shows $v\theta = w$. In other words it is sufficient to check $\operatorname{im} v \subset \operatorname{Hilb}_h(\mathbb{P}_q^2)$. But since S is integral of finite type over k it suffices to check this for k-points. But then it follows from Lemma 3.5.5.

We leave to the reader the purely formal computation that Φ and Ψ are each others inverse. □

Proof of Proposition 3.5.2. Let $(a_i)_i$, $(b_i)_i$ be minimal Betti numbers corresponding to h. Let I^u be the universal family on $\operatorname{Hilb}_h(\mathbb{P}_q^2)$. Then it is easy to see

$$\operatorname{Hilb}_h(\mathbb{P}_q^2)^{\min} = \{x \in \operatorname{Hilb}_h(\mathbb{P}_q^2) \mid \forall i : \dim_{k(x)}(I_x^u \otimes_{A_x} k(x))_i = a_i\}.$$

It follows from Lemma B.1 below that this defines an open subset. □

3.5.2 Dimensions

Below a point will again be be closed point.

Lemma 3.5.10. *Let $I \in \mathrm{Hilb}_h(\mathbb{P}^2_q)$. Then canonically*

$$T_I(\mathrm{Hilb}_h(\mathbb{P}^2_q)) \cong \mathrm{Ext}^1_A(I, I)$$

Proof. If $\mathcal{F}$ is a functor from (certain) rings to sets and $x \in \mathcal{F}(k)$ then the tangent space $T_x(\mathcal{F})$ is by definition the inverse image of x under the map

$$\mathcal{F}(k[\epsilon]/(\epsilon^2)) \to \mathcal{F}(k)$$

which as usual is canonically a k-vector space. If $\mathcal{F}$ is represented by a scheme F then of course $T_x(\mathcal{F}) = T_x(F)$.

The proposition follows from the fact that if $I \in \mathcal{H}ilb_h(\mathbb{P}^2_q)(k)$ then the tangent space $T_I(\mathcal{H}ilb_h(\mathbb{P}^2_q))$ is canonically identified with $\mathrm{Ext}^1_A(I, I)$ (see [11, Prop. E1.1]). □

We now express $\dim_k \mathrm{Ext}^1_A(I, I)$ in terms of $s_I(t)$.

Proposition 3.5.11. *Let $I \in \mathrm{Hilb}_h(\mathbb{P}^2)$ and assume $I \neq A$. Let $s_I(t)$ be the Castelnuovo polynomial of I. Then we have*

$$\dim_k \mathrm{Ext}^1_A(I, I) = 1 + n + c$$

where n is the invariant of I and c is the constant term of

$$(t^{-1} - t^{-2}) s_I(t^{-1}) s_I(t) \tag{3.20}$$

In particular this dimension is independent of I.

Corollary 3.5.12. $\mathrm{Hilb}_h(\mathbb{P}^2_q)$ *is smooth of dimension $1 + n + c$ where c is as in the previous theorem.*

Proof. This follows from the fact that the tangent spaces of $\mathrm{Hilb}_h(\mathbb{P}^2_q)$ have constant dimension $1 + n + c$. □

Proof of Proposition 3.5.11. We start with the following observation.

$$\sum_i (-1)^i h_{\underline{\mathrm{Ext}}^i_A(M,N)}(t) = h_M(t^{-1}) h_N(t) (1 - t^{-1})^3$$

for $M, N \in \mathrm{grmod}(A)$. This follows from the fact that both sides a additive on short exact sequences, and they are equal for $M = A(-i)$, $N = A(-j)$. Alternatively, see [74, Lemma 2.3].

Applying this with $M = N = I$ and using $\operatorname{pd} I = 1$, $\operatorname{Hom}_A(I,I) = k$ we obtain $\dim_k \operatorname{Ext}^1_A(I,I)$ is the constant term of

$$\begin{aligned}1 - h(t^{-1})h(t)(1-t^{-1})^3 &= 1 - (1-t^{-1})^3\left(\frac{1}{(1-t^{-1})^3} - \frac{s(t^{-1})}{1-t^{-1}}\right)\left(\frac{1}{(1-t)^3} - \frac{s(t)}{1-t}\right)\\ &= 1 - \frac{1}{(1-t)^3} + \frac{s(t)}{1-t} + \frac{t^{-2}s(t^{-1})}{1-t} - t^{-2}(1-t)s(t^{-1})s(t)\end{aligned}$$

(where we dropped the index "I"). Introducing the known constant terms finishes the proof. □

Corollary 3.5.13. *Let $\mathcal{I} \in \operatorname{Hilb}_n(\mathbb{P}^2_q)$ and $I = \omega\mathcal{I}$. Then*

$$\min(n+2, 2n) \leq \dim_k \operatorname{Ext}^1_A(I,I) \leq 2n \tag{3.21}$$

with equality on the left if and only if $h_I(t) = h_{\max}(t)$ and equality on the right if and only if $h_I(t) = h_{\min}(t)$.

Proof. Since the case $n = 0$ is obvious we assume below $n \geq 1$. We compute the constant term of (3.20). Put $s(t) = s_I(t) = \sum s_i t^i$. Thus the sought constant term is the difference between the coefficient of t and the coefficient of t^2 in $s(t^{-1})s(t)$. This difference is

$$\sum_{j-i=1} s_i s_j - \sum_{j-i=2} s_i s_j$$

which may be rewritten as

$$\sum_j s_{j+1}s_j - \sum_j s_{j+2}s_j = \sum_j s_j s_{j-1} - \sum_j s_j s_{j-2} = \sum_j s_j(s_{j-1} - s_{j-2})$$

Now we always have $s_{j-1} - s_{j-2} \leq 1$ and $s_{-1} - s_{-2} = 0$. Thus

$$\sum_j s_j(s_{j-1} - s_{j-2}) \leq -1 + \sum_j s_j = n - 1$$

which implies $\dim_k \operatorname{Ext}^1_A(I,I) \leq 2n$ by Proposition 3.5.11, and we will clearly have equality if and only if $s_{j-1} - s_{j-2} = 1$ for $j > 0$ and $s_j \neq 0$. This is equivalent to $s(t)$ being of the form

$$1 + 2t + 3t^2 + \cdots + (u-1)t^u + vt^{u+1}$$

for some integers $u > 0$ and $v \geq 0$. This in turn is equivalent with $h_I(t)$ being equal to $h_{\min}(t)$. This proves the upper bound of (3.21).

Now we prove the lower bound. Since $s_I(t)$ is a Castelnuovo polynomial it has the form

$$s(t) = 1 + 2t + 3t^2 + \ldots + \sigma t^{\sigma-1} + s_\sigma t^\sigma + s_{\sigma+1}t^{\sigma+1} + \ldots$$

where

$$\sigma \geq s_\sigma \geq s_{\sigma+1} \geq \ldots$$

We obtain

$$\begin{aligned} c &= \sum_j s_j(s_{j+1} - s_{j+2}) \\ &= -(1 + 2 + 3 + \ldots + (\sigma - 2)) + \sum_{j \geq \sigma - 2} s_j(s_{j+1} - s_{j+2}) \end{aligned}$$

We denote the subsequence obtained by dropping the zeroes from the sequence of non-negative integers $(s_{j+1} - s_{j+2})_{j \geq \sigma - 2}$ by $e_1, e_2, \ldots, e_r$. Note $\sum_i e_i = \sigma$. We get

$$\begin{aligned} c \geq &- (1 + 2 + 3 + \ldots + (\sigma - 2)) \\ &+ (\sigma - \delta)e_1 + (\sigma - e_1)e_2 + \ldots + (\sigma - e_1 - \ldots - e_{r-1})e_r \end{aligned}$$

where $\delta = 1$ if $s_\sigma < \sigma$ and 0 otherwise. Now we have

$$\begin{aligned} (\sigma - e_1 - \cdots - e_{r-1})e_r &= e_r e_r \geq 1 + \cdots + e_r \\ (\sigma - e_1 - \cdots - e_{r-2})e_{r-1} &= (e_{r-1} + e_r)e_{r-1} \geq (1 + e_r) + \cdots + (e_{r-1} + e_r) \\ &\vdots \\ (\sigma - e_1)e_2 &= (e_2 + \cdots + e_r)e_2 \geq (1 + e_3 + \cdots + e_r) + \cdots + (e_2 + \cdots + e_r) \\ \sigma e_1 &= (e_1 + \cdots + e_r)e_1 \geq (1 + e_2 + \cdots + e_r) + \cdots + (e_1 + \cdots + e_r) \end{aligned}$$

hence

$$c \geq 2\sigma - 1 - \delta e_1$$

Hence $c \geq 0$ and $c = 0$ if and only if $\sigma = 1$, $r = 1$ and $\delta = 1$, so if and only if $s_I(t) = 1$. In that case, the invariant n of I is 1. If $n > 1$ then $c \geq 1$ which proves the lower bound of (3.21) by Proposition 3.5.11. Clearly $c = 1$ if and only if $\sigma = 1$ and $r = 1$, which is equivalent with $h_I(t)$ being equal to $h_{\max}(t)$. □

Remark 3.5.14. The fact $\dim_k \operatorname{Ext}^1_A(I, I) \leq 2n$ can be shown directly. Indeed from the formula (1.2)

$$\operatorname{Ext}^1_{\operatorname{Tails}(A)}(\mathcal{I}, \mathcal{I}) \cong \varinjlim \operatorname{Ext}^1_A(I_{\geq n}, I)$$

and from $\operatorname{Ext}^1_A(k, I) = 0$ we obtain an injection

$$\operatorname{Ext}^1_A(I, I) \hookrightarrow \operatorname{Ext}^1_{\operatorname{Tails}(A)}(\mathcal{I}, \mathcal{I})$$

and the right hand side is the tangent space $\mathcal{I}$ in the smooth variety $\operatorname{Hilb}_n(\mathbb{P}^2_q)$ which has dimension $2n$.

3.5.3 Connectedness

In this section we prove

Proposition 3.5.15. *Assume h is an admissible Hilbert polynomial. Then any two points in $\mathrm{Hilb}_h(\mathbb{P}^2_q)$ can be connected using an open subset of an affine line.*

Proof. Let $I, J \in \mathrm{Hilb}_h(\mathbb{P}^2_q)$. Then I, J have resolutions

$$0 \to \oplus_i A(-i)^{b_i} \to \oplus_i A(-i)^{a_i} \to I \to 0$$
$$0 \to \oplus_i A(-i)^{d_i} \to \oplus_i A(-i)^{c_i} \to J \to 0$$

where $a_i - b_i = c_i - d_i$. Adding terms of the form $A(-j) \xrightarrow{\mathrm{id}} A(-j)$ we may change these resolutions to have the following form

$$0 \to \oplus_i A(-i)^{f_i} \xrightarrow{M} \oplus_i A(-i)^{e_i} \to I \to 0$$
$$0 \to \oplus_i A(-i)^{f_i} \xrightarrow{N} \oplus_i A(-i)^{e_i} \to J \to 0$$

for matrices $M, N \in H = \mathrm{Hom}_A(\oplus_i A(-i)^{e_i}, \oplus_i A(-i)^{f_i})$. Let $L \subset H$ be the line through M and N. Then by Lemma 3.5.8 an open set of L defines points in $\mathrm{Hilb}_h(\mathbb{P}^2_q)$. This finishes the proof. □

Chapter 4

Modules of GK-dimension one over quadratic Artin-Schelter algebras

In this short chapter we would like to point out a connection between the previous two chapters and the study of graded right modules of GK-dimension one over quadratic Artin-Schelter algebras. Up to finite length, such a module is presented by a Cohen-Macaulay module of GK-dimension one. It turns out there is a natural correspondence between such cyclic modules and the boundary $\mathrm{Hilb}_n(\mathbb{P}^2) \setminus \mathrm{Hilb}_n(\mathbb{P}^2)^{\mathrm{inv}}$.

These results were found in collaboration with S.P. Smith.

4.1 Introduction

Let A be a quadratic Artin-Schelter algebra, and write $\mathbb{P}^2_{\mathrm{q}} = \operatorname{Proj} A$. We begin this introduction by pointing out a correspondence between certain modules of GK-dimension three and certain modules of GK-dimension one.

In the previous chapters we have discussed the Hilbert scheme of points $\mathrm{Hilb}_n(\mathbb{P}^2_{\mathrm{q}})$ and its subset $\mathcal{R}_n(\mathbb{P}^2_{\mathrm{q}}) = \mathrm{Hilb}_n(\mathbb{P}^2_{\mathrm{q}})^{\mathrm{inv}}$ consisting of the reflexive objects. We have seen there are two distinguished situations (see Proposition 2.2.13 and Theorem 1)

- A is linear. Then $\mathrm{Hilb}_n(\mathbb{P}^2_{\mathrm{q}})^{\mathrm{inv}} = \emptyset$ for all $n > 0$.
- A is elliptic and σ has infinite order. Then $\mathrm{Hilb}_n(\mathbb{P}^2_{\mathrm{q}})^{\mathrm{inv}}$ is a locally closed variety of dimension $2n$. It was shown in [60] that $\mathrm{Hilb}_n(\mathbb{P}^2_{\mathrm{q}})^{\mathrm{inv}}$ is a dense open subset of $\mathrm{Hilb}_n(\mathbb{P}^2_{\mathrm{q}})$.

First, assume A is linear. Let $n > 0$. From Proposition 2.2.13 we deduce $I^{**} = A$ for any $I \in \mathrm{Hilb}_n(\mathbb{P}^2_{\mathrm{q}})$, implying $I \subset A$. By considering A/I it easily follows that

$\mathrm{Hilb}_n(\mathbb{P}^2_q)$ parameterizes the objects $N \in \mathrm{grmod}(A)$ for which

1. N has GK-dimension one
2. N is cyclic and generated in degree zero
3. N is Cohen-Macaulay

In particular the Hilbert series of N is of the form $h_N(t) = s(t)(1-t)^{-1}$ where $s(t)$ Castelnuovo polynomial of weight n, see for §3.1. A minimal resolution of N is of the form

$$0 \to \oplus_i A(-i)^{b_i} \to \oplus_i A(-i)^{a_i} \to A \to N \to 0$$

In the commutative case i.e. $A = k[x,y,z]$ these objects N are exactly the coordinate rings $A(X)$ of zero dimensional subschemes of degree n on $\mathbb{P}^2$, parameterized by the classical Hilbert scheme of points $\mathrm{Hilb}_n(\mathbb{P}^2)$ on $\mathbb{P}^2$.

Let us assume for the rest of this introduction A is elliptic and σ has infinite order. It is a natural question to describe the boundary $\mathrm{Hilb}_n(\mathbb{P}^2_q) \setminus \mathrm{Hilb}_n(\mathbb{P}^2_q)^{\mathrm{inv}}$. In particular one could ask to describe the objects $I \in \mathrm{Hilb}_n(\mathbb{P}^2_q)$ for which $I^{**} = A$. i.e. the objects which are "as far as reflexive as possible". Similar as in the linear case, this question is equivalent to the description of the objects $N \in \mathrm{grmod}(A)$ of has GK-dimension one, cyclic, generated in degree zero and Cohen-Macaulay. See Corollary 4.2.5 below for a more exact statement.

Under the assumptions on A, the critical GK-1 modules of multiplicity one are exactly the shifted point modules [8]. We will show in next section this leads to

Proposition 4.1.1. *Let $I \in \mathrm{Hilb}_n(\mathbb{P}^2_q)$. There is a filtration*

$$I = I_0 \subset I_1 \subset \cdots \subset I_d = I^{**}$$

such that each $I_i \in \mathrm{Hilb}_{n-i}(\mathbb{P}^2_q)$ and each quotient I_i/I_{i-1} is a shifted point module.

The integer d appearing in Proposition 4.1.1 is uniquely determined by I. We will write $d(I) = d$ and refer to it as the *defect* of I. It is natural to define for $0 \le d \le n$ the following subsets in $\mathrm{Hilb}_n(\mathbb{P}^2_q)$

$$\mathrm{Hilb}^d_n(\mathbb{P}^2_q) = \{I \in \mathrm{Hilb}_n(\mathbb{P}^2_q) \mid d(I) = d\},$$
$$\mathrm{Hilb}^{\ge d}_n(\mathbb{P}^2_q) = \{I \in \mathrm{Hilb}_n(\mathbb{P}^2_q) \mid d(I) \ge d\}.$$

Clearly

$$\mathrm{Hilb}_n(\mathbb{P}^2_q) = \bigcup_{0 \le d \le n} \mathrm{Hilb}^d_n(\mathbb{P}^2_q),$$
$$\mathrm{Hilb}^n_n(\mathbb{P}^2_q) = \mathrm{Hilb}^{\ge n}_n(\mathbb{P}^2_q) \subset \mathrm{Hilb}^{\ge n-1}_n(\mathbb{P}^2_q) \subset \cdots \subset \mathrm{Hilb}^{\ge 0}_n(\mathbb{P}^2_q) = \mathrm{Hilb}_n(\mathbb{P}^2_q).$$

Example 4.1.2. Let $n = 1$. By Theorem 6 of Chapter 3, an object $I \in \mathrm{Hilb}_1(\mathbb{P}^2_q)$ has a minimal resolution of the form

$$0 \longrightarrow A(-2) \xrightarrow{\begin{pmatrix} l \\ l' \end{pmatrix}\cdot} A(-1)^2 \longrightarrow I \longrightarrow 0$$

where $l, l' \in A_1$. Conversely, any choice of two linearly independent forms l, l' defines an object $I \in \mathrm{Hilb}_1(\mathbb{P}^2_q)$. There are two distinguished cases.

Case 1. There is no $p \in C$ such that $\bar{l}(p) = \bar{l}(p) = 0$. By Proposition 2.3.2 and Lemma 3.3.3 the module I is reflexive.

Case 2. There exists a $p \in C$ such that $\bar{l}(p) = \bar{l}(p) = 0$. Then I is not reflexive, hence $I^{**} = A$. By [1] the quotient A/I is the point module $N_{\sigma^2 p}$.

We obtain $\mathrm{Hilb}_1(\mathbb{P}^2_q) = \mathbb{P}^2$, $\mathrm{Hilb}^1_1(\mathbb{P}^2_q) = \mathrm{Hilb}^{\geq 1}_1(\mathbb{P}^2_q) = C$ and $\mathrm{Hilb}^{\mathrm{inv}}_1(\mathbb{P}^2_q) = \mathbb{P}^2 \setminus C$. See also Corollary 2.4.5.

In general, $\mathrm{Hilb}^d_n(\mathbb{P}^2_q)$ is more difficult to understand. As a first step, we provide

Theorem 7. *Assume A is elliptic and σ has infinite order. Let $n \geq 0$ and $0 \leq d \leq n$. We have*

1. $\mathrm{Hilb}^d_n(\mathbb{P}^2_q)$ *is non-empty,*
2. $\mathrm{Hilb}^{\geq d}_n(\mathbb{P}^2_q) \subset \mathrm{Hilb}_n(\mathbb{P}^2_q)$ *is a projective variety of dimension $2n - d$.*

The first part of Theorem 7 will be a consequence of Theorem 8 below. The second statement in Theorem 7 will be proved in §4.4. Theorem 7 implies the boundary $\mathrm{Hilb}_n(\mathbb{P}^2_q) \setminus \mathrm{Hilb}_n(\mathbb{P}^2_q)^{\mathrm{inv}} = \mathrm{Hilb}^{\geq 1}_n(\mathbb{P}^2_q)$ is a projective variety of dimension $2n - 1$. In particular $\mathrm{Hilb}_n(\mathbb{P}^2_q)^{\mathrm{inv}}$ is open, which was already proved in [60].

The next feature might be to determine all possible Hilbert series of objects in $\mathrm{Hilb}^d_n(\mathbb{P}^2_q)$. Theorem 4 from Chapter 3 implies

Proposition 4.1.3. *Assume A is elliptic and σ has infinite order. Let $n \geq 0$ and $0 \leq d \leq n$.*

1. *If $I \in \mathrm{Hilb}^d_n(\mathbb{P}^2_q)$ then there is a Castelnuovo polynomial $s(t)$ of weight n such that*
$$h_I(t) = \frac{1}{(1-t)^3} - \frac{s(t)}{1-t} \tag{4.1}$$
2. *If $d = 0$ then (4.1) gives a bijective correspondence between Castelnuovo polynomials $s(t)$ of weight n and Hilbert series $h_I(t)$ of objects in $I \in \mathrm{Hilb}^0_n(\mathbb{P}^2_q)$.*

It is natural to wonder if the correspondence in Proposition 4.1.3 is bijective for all d. We were only able to answer this question very partially. Recall from §3.1 there is a unique maximal Hilbert series $h_{\max}(t)$ for objects in $\mathrm{Hilb}_n(\mathbb{P}^2_q)$. This correspond to the Castelnuovo diagram

An object $I \in \mathrm{Hilb}_{h_{\max}}(\mathbb{P}^2_{\mathrm{q}})$ has a minimal projective resolution of the form

$$0 \to A(-n-1) \oplus A(-n) \oplus A(-1) \to I \to 0$$

In the commutative case, i.e. $A = k[x, y, z]$ then this corresponds to the case where all n points in $\mathbb{P}^2$ are collinear. Also recall from Proposition 3.5.11 the stratum $\mathrm{Hilb}_{h_{\max}}(\mathbb{P}^2_{\mathrm{q}})$ in $\mathrm{Hilb}_n(\mathbb{P}^2_{\mathrm{q}})$ has (minimal) dimension $n+1$. In Section §4.3 we will prove

Theorem 8. *Assume A is elliptic and σ has infinite order. Let $n \geq 0$ and $0 \leq d \leq n$. Then there is an object $I \in \mathrm{Hilb}^d_n(\mathbb{P}^2_{\mathrm{q}})$ with Hilbert series $h_I(t) = h_{\max}(t)$.*

Our proof of Theorem 8 is based on the observation that for $n, d \geq 1$ and for an object $J \in \mathrm{Hilb}^{d-1}_{n-1}(\mathbb{P}^2_{\mathrm{q}})$ for which $h_J(t) = h_{\max}(t)$ there is a point module N_p over A for which I maps surjectively to $N_p(1-n)$. Taking the kernel of such a map yields an object $I \in \mathrm{Hilb}^d_n(\mathbb{P}^2_{\mathrm{q}})$ with $h_I(t) = h_{\max}(t)$.

Remark 4.1.4. Consider the stratification $\mathrm{Hilb}_n(\mathbb{P}^2_q) = \bigcup_h \mathrm{Hilb}_h(\mathbb{P}^2_q)$ of Chapter 3. Theorem 8 assures that of all strata, the stratum with *minimal* dimension passes through all $\mathrm{Hilb}^d_n(\mathbb{P}^2_{\mathrm{q}})$. The author is quite convinced that in fact all possible minimal resolutions for objects in $\mathrm{Hilb}^0_n(\mathbb{P}^2_{\mathrm{q}})$ occur for objects in $\mathrm{Hilb}^d_n(\mathbb{P}^2_{\mathrm{q}})$. It would be interesting to do a more extensive investigation.

4.2 Filtrations

In this part A will be an elliptic quadratic Artin-Schelter algebra.

Lemma 4.2.1. *Let $n > 0$ and $I \in \mathrm{Hilb}_n(\mathbb{P}^2_{\mathrm{q}})$. Then the following are equivalent:*

1. $\mathrm{Hom}_A(I, A) \neq 0$,
2. *There is a (non-split) exact sequence $0 \to I \to A \to N \to 0$ where $N \in \mathrm{grmod}(A)$ is Cohen-Macaulay of GK-dimension one,*
3. $I^* = A$,
4. $I^{**} = A$.

Proof. Recall $I \in \mathrm{grmod}(A)$ is an object in $\mathrm{Hilb}_n(\mathbb{P}^2_q)$ if I is torsion free, $\mathrm{pd}\, I = 1$ and the Hilbert series of I is of the form

$$h_I(t) = \frac{1}{(1-t)^3} - \frac{n}{1-t} + f(t)$$

for some $f(t) \in \mathbb{Z}[t, t^{-1}]$. From this it is easy to see (1) implies (2). Note we have used $n > 0$.

To prove (2) $\Rightarrow$ (3), applying $\underline{\mathrm{Hom}}_A(-, A)$ to the exact sequence

$$0 \to I \to A \to N \to 0$$

gives the long exact sequence of graded left A-modules

$$0 \to N^* \to A^* \to I^* \to \underline{\mathrm{Ext}}^1_A(N, A) \to \dots$$

Since $\mathrm{GKdim}\, N = 1$ we have $\underline{\mathrm{Ext}}^i_A(N, A) = 0$ for $i \leq 1$ and using $A^* = A$ we obtain $I^* = A$, as required.

It is trivial that (3) implies (4), so in order to finish the proof we assume (4) holds and prove (1). But this follows directly from Lemma 2.2.1. □

Remark 4.2.2. As discussed in §2.2, for any graded right ideal I of projective dimension one there is an unique integer l such that $I(l) \in \mathrm{Hilb}_n(\mathbb{P}^2_q)$, and I is normalized if and only if $l = 0$. As a consequence of the previous lemma, the normalized graded right ideals I of A of projective dimension one are exactly the objects in $\mathrm{Hilb}_n(\mathbb{P}^2_q)$ for which $I^{**} = A$.

Before we come to the proof of Proposition 4.1.1 we need two more lemmas.

Lemma 4.2.3. *Let $I \in \mathrm{Hilb}_n(\mathbb{P}^2_q)$, $J \in \mathrm{Hilb}_m(\mathbb{P}^2_q)$ such that $I \subsetneq J$. Then J/I is Cohen-Macaulay of GK-dimension one and multiplicity $n - m$.*

Proof. The fact that I and J are normalized implies $\mathrm{GKdim}(I/J) \leq 1$. Since I has projective dimension one, $\underline{\mathrm{Ext}}^1_A(k, I) = 0$. From this we deduce J/I is pure of GK-dimension one. Taking Hilbert series shows $e(J/I) = n - m$. □

The following result is well-known.

Lemma 4.2.4. *Let $N \in \mathrm{grmod}(A)$ be pure of GK-dimension one. Then there is a filtration $0 = N_0 \subset N_1 \subset \dots \subset N_r = N$ such that the quotients N_{i+1}/N_i are critical of GK-dimension one.*

Proof. Choose a submodule $D \subset N$ maximal such that $\mathrm{GKdim}(N/D) = 1$. Then N/D is critical of GK-dimension one. By repeating the arguments we find a chain of submodules for which the successive quotiens are critical of GK-dimension one. Since N has finite multiplicity this chain is finite. □

Proof of Proposition 4.1.1. Recall from [8] that since σ has infinite order, the critical modules of GK-dimension one are exactly the shifted point modules over A.

Let $r = e(I^{**}/I)$ and put $I_r = I^{**}$. By Theorem 1.9.8(7) it follows I_r is normalized with invariant $n - r$ hence $I_r \in \mathrm{Hilb}_{n-r}(\mathbb{P}^2_q)$.

By Lemma 4.2.3 and Lemma 4.2.4 there is a filtration $0 = N_0 \subset N_1 \subset \cdots \subset N_r = I_r/I$ where each quotient is a critical module of GK-dimension one, hence a shifted point module. Let I_{r-1} be the kernel of the surjective composition $I_r \to N_r \to N_r/N_{r-1}$. Then $I \subset I_{r-1}$ and by taking Hilbert series it follows $I_{r-1} \in \mathrm{Hilb}_{n-r+1}(\mathbb{P}^2_q)$. Moreover $e(I_{r-1}/I) = r - 1$. The statement is then shown by downwards induction on r. □

Corollary 4.2.5. *Assume A is elliptic and σ has infinite order. Let $n \geq 0$ and $0 \leq d \leq n$. Then $\mathrm{Hilb}_n^{\geq d}(\mathbb{P}^2_q) \subset \mathrm{Hilb}_n(\mathbb{P}^2_q)$ is closed.*

The points of the projective variety $\mathrm{Hilb}_n^n(\mathbb{P}^2_q)$ are in natural bijection with the isomorphism classes of the full subcategory of $\mathrm{grmod}(A)$ *which objects are of GK-dimension one, cyclic, generated in degree zero and Cohen-Macaulay.*

Proof. Introduce the subsets

$$\mathrm{Hilb}_n^{\leq d}(\mathbb{P}^2_q) = \{I \in \mathrm{Hilb}_n(\mathbb{P}^2_q) \mid d(I) \leq d\}$$

By Proposition 4.1.1 it is easy to see for $d < n$ we have the alternative description

$$\mathrm{Hilb}_n^{\leq d}(\mathbb{P}^2_q) = \{I \in \mathrm{Hilb}_n(\mathbb{P}^2_q) \mid \mathrm{Hom}_A(I, J) = 0 \text{ for all } J \in \mathrm{Hilb}_{n-d-1}(\mathbb{P}^2_q)\}$$

from which we deduce $\mathrm{Hilb}_n^{\leq d}(\mathbb{P}^2_q)$ is open in $\mathrm{Hilb}_n(\mathbb{P}^2_q)$. Thus $\mathrm{Hilb}_n^{\geq d}(\mathbb{P}^2_q) \subset \mathrm{Hilb}_n(\mathbb{P}^2_q)$ is closed. Thus $\mathrm{Hilb}_n^{\geq d}(\mathbb{P}^2_q)$ is a projective variety. The rest of the statement follows from Lemma 4.2.1. □

Remark 4.2.6. Note that Lemma 4.2.1 provides the alternative description

$$\mathrm{Hilb}_n^n(\mathbb{P}^2_q) = \mathrm{Hilb}_n(\mathbb{P}^2_q) \setminus \mathrm{Hilb}_n^{\leq n-1}(\mathbb{P}^2_q) = \{I \in \mathrm{Hilb}_n(\mathbb{P}^2_q) \mid \mathrm{Hom}_A(I, A) \neq 0\}$$

from which we immediately deduce $\mathrm{Hilb}_n^n(\mathbb{P}^2_q)$ is closed.

Remark 4.2.7. It would be interesting to see if the previous corollary relates to [76] in which the authors classified graded right A-modules of GK-dimension one, up to modules of finite length.

4.3 Proof of Theorem 8

We now come to the proof of Theorem 8. Recall from the introduction

$$h_{\max}(t) = h_A(t) - \frac{s(t)}{1-t}$$

where $s(t) = 1 + t + t^2 + \cdots + t^{n-1}$ for $n > 0$ and $s(t) = 0$ for $n = 0$. We need to show that for $n \geq 0$ and $0 \leq d \leq n$ there exists an object $I \in \mathrm{Hilb}_n^d(\mathbb{P}^2_{\mathrm{q}})$ for which $h_I(t) = h_{\max}(t)$. The statement is trivially true for $n = 0$ (take $I = A$) and by Proposition 4.1.1 the assertion is true for $d = 0$. So we will assume $d > 0$ for the rest of the proof. We present the proof by induction on n.

For $n = 1$ the result follows from Example 4.1.2.

Let $n > 1$ and $1 \leq d \leq n$. By the induction hypothesis we may choose an object $J \in \mathrm{Hilb}_{n-1}^{d-1}(\mathbb{P}^2_{\mathrm{q}})$ for which $h_J = h_{\max}$. It follows from (3.5) and Theorem 6 that J has a minimal resolution of the form

$$0 \longrightarrow A(-n) \xrightarrow{\begin{pmatrix} u \\ v \end{pmatrix} \cdot} A(-1) \oplus A(1-n) \longrightarrow J \longrightarrow 0$$

where $u \in A_{n-1}$ and $v \in A_1$. Observe neither u nor v are be zero due to the torsion freeness of J. Applying $\mathrm{Hom}_A(-, N_q(l))$ for any $q \in C$ and $l \in \mathbb{Z}$ yields an exact sequence of k-vector spaces

$$0 \to \underline{\mathrm{Hom}}_A(J, N_q)_l \to N_q(1)_l \oplus N_q(n-1)_l \xrightarrow{\begin{pmatrix} \overline{u}(\sigma^{l+1}q) & \overline{v}(\sigma^{n+l-1}q) \end{pmatrix} \cdot} N_q(n)_l \tag{4.2}$$

where we identify $(N_q)_m = k$ for $m \geq 0$. Now we pick a point $p \in C$ such that $\overline{v}(p) = 0$. By putting $q = \sigma^{1-n-l}p$ in (4.2) we obtain

$$\dim_k \underline{\mathrm{Hom}}_A(J, N_{\sigma^{1-n-l}p})_l = \begin{cases} 0 & \text{if } l < 1-n \\ 1 & \text{if } 1-n \leq l \leq -2 \end{cases} \tag{4.3}$$

In particular we may pick a non-zero map $f : J \to N_p(1-n)$. Now f is surjective, since otherwise $\operatorname{im} f$ is contained in the unique maximal submodule of N_p, namely $N_{\sigma p}(-1)$. But then $\mathrm{Hom}_A(J, N_{\sigma p}(-1)) \neq 0$, a contradiction to (4.3).

Let I be kernel of f. Thus

$$0 \to I \to J \to N_p(1-n) \to 0 \tag{4.4}$$

We now claim $I \in \mathrm{Hilb}_n(\mathbb{P}^2_{\mathrm{q}})$. It is clear I is torsion free of rank one. Application of $\underline{\mathrm{Hom}}_A(k, -)$ shows $\underline{\mathrm{Ext}}^1_A(k, I) \subset \underline{\mathrm{Ext}}^1_A(k, J) = 0$ hence I has projective dimension one (see the proof of Proposition 2.2.5). And Lemma 2.2.8(3) implies I is normalized with invariant n.

Taking the dual of (4.4) we obtain $I^* = J^*$ and it is easy to see $e(J^{**}/I) = e(J^{**}/J) + 1$. Hence $d(I) = d(J) + 1$ and we obtain $I \in \mathrm{Hilb}_n^d(\mathbb{P}^2_{\mathrm{q}})$. Finally we take Hilbert series of (4.4), giving

$$\begin{aligned} h_I(t) &= h_J(t) - h_{N_p(1-n)}(t) \\ &= h_A(t) - \frac{1 + t + \cdots + t^{n-2}}{1-t} - \frac{t^{n-1}}{1-t} \\ &= h_A(t) - \frac{1 + t + \cdots + t^{n-1}}{1-t} \end{aligned}$$

We conclude $I \in \mathrm{Hilb}_n^d(\mathbb{P}^2_{\mathrm{q}})$ and $h_I(t) = h_{\max}(t)$. This ends the proof of Theorem 8.

4.4 Proof of Theorem 7

By Theorem 8 and Corollary 4.2.5 it remains to prove that the projective variety $\mathrm{Hilb}_n^{\geq d}(\mathbb{P}^2_{\mathrm{q}})$ has dimension $2n - d$.

Define

$$H = \{(I,p) \mid \mathrm{Ext}^1_{\mathbb{P}^2_{\mathrm{q}}}(\mathcal{N}_p, \mathcal{I}) = 0\} \subset \mathrm{Hilb}_n^{\geq d} \times C$$

where we write $\mathcal{I} = \pi I$. It follows that H is closed in $\mathrm{Hilb}_n^{\geq d} \times C$. Computation of the Euler form shows that for such $(I,p) \in H$ we have $\mathrm{Hom}_{\mathbb{P}^2_{\mathrm{q}}}(\mathcal{I}, \mathcal{N}_p) = k$. Moreover such a non-zero map $\mathcal{I} \to \mathcal{N}_p$ is surjective, and its kernel $\mathcal{J}$ is up to isomorphism uniquely determined by I and p. Further, it is easy to see $\mathcal{J}$ determines an object in $\mathrm{Hilb}_n^{\geq d+1}$. Thus we have a well-defined map

$$\phi : H \to \mathrm{Hilb}_{n+1}^{\geq d+1} : (I,p) \mapsto \ker(\mathcal{I} \to \mathcal{N}_p)$$

which is onto. Since for an object $\mathcal{J} \in \mathrm{Hilb}_{n+1}(\mathbb{P}^2_{\mathrm{q}})$ there are only finitely many points $p \in C$ for which $\mathrm{Ext}^1_{\mathbb{P}^2_{\mathrm{q}}}(\mathcal{N}_p, \mathcal{J}) \neq 0$ it follows that all fibres of ϕ are finite. Hence $\dim \mathrm{Hilb}_{n+1}^{\geq d+1} = \dim \mathrm{Hilb}_n^{\geq d} + \dim C$. By induction and the fact $\dim \mathrm{Hilb}_n^{\geq 0} = 2n$ this means we are done.

Chapter 5

Incidence between strata on the Hilbert scheme of points on the projective plane

The Hilbert scheme $\mathrm{Hilb}_n(\mathbb{P}^2)$ of n points in the projective plane $\mathbb{P}^2$ has a natural stratification obtained from the associated Hilbert series (see Chapter 3). In general, the precise inclusion relation between the closures of the strata is still unknown.

In [38] Guerimand studied this problem for strata whose Hilbert series are as close as possible. Preimposing a certain technical condition he obtained necessary and sufficient conditions for the incidence of such strata.

In this chapter we present a new approach, based on deformation theory, to Guerimand's result. This allows us to show the technical condition is not necessary.

Presented results in this chapter are submitted [29].

5.1 Introduction and main result

Throughout this chapter $A = k[x, y, z]$ is the commutative polynomial ring in three variables. We will use the notations as in §3.1 of Chapter 3. In particular $\mathrm{Hilb}_n(\mathbb{P}^2)$ will be the Hilbert scheme parametrizing zero-dimensional subschemes of length n in $\mathbb{P}^2$. Recall this is a smooth connected projective variety of dimension $2n$.

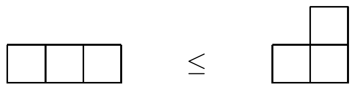
Associated to $X \in \mathrm{Hilb}_n(\mathbb{P}^2)$ is its Hilbert function h_X and its Castelnuovo function s_X of weight n. The corresponding Castelnuovo diagram F_{s_X} will also be refered to as s_X.

We refer to a series $\varphi \in \mathbb{Z}((t))$ for which $\varphi = h_X$ for some $X \in \mathrm{Hilb}_n(\mathbb{P}^2)$ as a *Hilbert function of degree n*. The set of all Hilbert functions of degree n (or equivalently the set of all Castelnuovo diagrams of weight n) will be denoted by Γ_n.

For $\varphi, \psi \in \Gamma_n$ we have $\psi(t) - \varphi(t)$ is a polynomial, and we write $\varphi \leq \psi$ if its coefficients are non-negative. In this way $\leq$ becomes a partial ordering on Γ_n and we call the associated directed graph the *Hilbert graph*, also denoted by Γ_n.

If $s, s' \in \Gamma_n$ are Castelnuovo diagrams such that $s \leq s'$ then it is easy to see s' is obtained from s by making a number of squares "jump to the left" while, at each step, preserving the Castelnuovo property.

Example 5.1.1. There are two Castelnuovo diagrams of weight 3.



These distinguish whether three points are collinear or not. The corresponding Hilbert functions are $1, 2, 3, 3, 3, 3, \ldots$ and $1, 3, 3, 3, 3, 3, \ldots$.

Remark 5.1.2. The number of Castelnuovo diagrams with weight n is equal to the number of partitions of n with distinct parts (or equivalently the number of partitions of n with odd parts), see Remark 3.1.3. In Appendix C there is a table of Castelnuovo diagrams of weight up to 6 as well as some associated data. The Hilbert graph is rather trivial for low values of n. The case $n = 17$ is more typical, see Appendix F, where we discuss Hilbert graphs in more detail.

Hilbert functions provide a natural stratification of the Hilbert scheme. For any Hilbert function ψ of degree n one defines a smooth connected subscheme [36] H_ψ of $\mathrm{Hilb}_n(\mathbb{P}^2)$ by

$$H_\psi = \{X \in \mathrm{Hilb}_n(\mathbb{P}^2) \mid h_X = \psi\}.$$

(see also Chapter 3). The family $\{H_\psi\}_{\psi \in \Gamma_n}$ forms a stratification of $\mathrm{Hilb}_n(\mathbb{P}^2)$ in the sense

$$\overline{H_\psi} \subset \bigcup_{\varphi \leq \psi} H_\varphi = \{X \in \mathrm{Hilb}_n(\mathbb{P}^2_{\mathrm{q}}) \mid h_X \leq \psi\}.$$

It follows that if $H_\varphi \subset \overline{H_\psi}$ then $\varphi \leq \psi$. The converse implication is in general false and it is still an open problem to find necessary and sufficient conditions for the existence of an inclusion $H_\varphi \subset \overline{H_\psi}$ [21, 24, 25, 43]. This problem is sometimes referred to as the *incidence problem*.

Guerimand in his PhD-thesis [38] introduced two additional necessary conditions for incidence of strata which we now discuss.

$$\text{The } \textit{dimension condition}\text{: } \dim H_\varphi < \dim H_\psi \tag{5.1}$$

This criterion can be used effectively since there are formulas for $\dim H_\psi$, see [36] and Chapter 3.

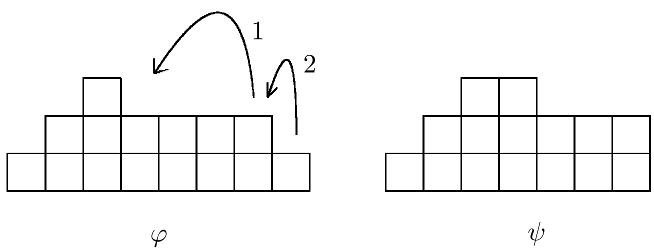
The *tangent function* t_φ of a Hilbert function $\varphi \in \Gamma_n$ is defined as the Hilbert function of $\mathcal{I}_X \otimes_{\mathbb{P}^2} \mathcal{T}_{\mathbb{P}^2}$, where $X \in H_\varphi$ is generic. Semi-continuity yields:

$$\text{The } \textit{tangent condition}\text{: } t_\varphi \geq t_\psi \tag{5.2}$$

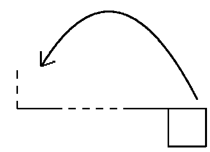
Again it is possible to compute t_ψ from ψ (see [38, Lemme 2.2.4] and also Proposition 5.3.4 below).

Let us say a pair of Hilbert functions (φ, ψ) of degree n has *length zero* if $\varphi < \psi$ and there are no Hilbert functions τ of degree n such that $\varphi < \tau < \psi$.[1] It is easy to see (φ, ψ) has length zero if and only if the Castelnuovo diagram of ψ can be obtained from that of φ by making a minimal movevement to the left of one square [38, Proposition 2.1.7].

Example 5.1.3. Although in the following pair s_ψ is obtained from s_φ by moving one square, it is not length zero since s_φ may be obtained from s_ψ by first doing movement 1 and then 2.



In general a movement of a square by one column is always length zero. A movement by more than one column is length zero if and only if it is of the form



(5.3)

The dotted lines represent zero or more squares.

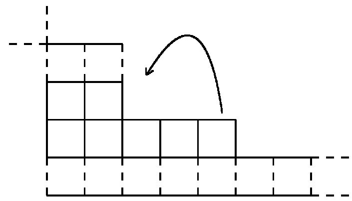
The following theorem is the main result of this chapter.

Theorem 9. *Assume (φ, ψ) has length zero. Then $H_\varphi \subset \overline{H_\psi}$ if and only if the dimension condition and the tangent condition hold.*

[1] This is a minor deviation of Guerimand's definition.

This result may be translated into a purely combinatorial (albeit technical) criterion for the existence of an inclusion $H_\varphi \subset \overline{H_\psi}$ (see Appendix D).

Guerimand proved Theorem 9 under the additional hypothesis (φ, ψ) is not of "type zero". A pair of Hilbert series (φ, ψ) has *type zero* if it is obtained by moving the indicated square in the diagram below.[2]

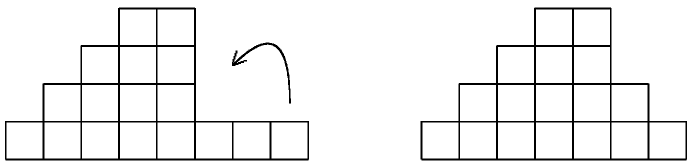


The dotted lines represent zero or more squares.

From the results in Appendix D one immediately deduces

Proposition 5.1.4. *Let φ, ψ be Hilbert functions of degree n such that (φ, ψ) has type zero. Then $H_\varphi \subset \overline{H_\psi}$.*

Remark 5.1.5. The smallest, previously open, incidence problem of type zero seems to be



$\varphi = 1, 3, 6, 10, 14, 15, 16, 17, 17, \ldots \qquad \psi = 1, 3, 6, 10, 14, 16, 17, 17, \ldots$

(see [38, Exemple A.4.2]).

Remark 5.1.6. Theorem 9 if false without the condition of (φ, ψ) being of length zero. See [38, Exemple A.2.1].

We became interested in the incidence problem while they were studying the deformations of the Hilbert schemes of $\mathbb{P}^2$ which come from noncommutative geometry, see [60] and Chapter 2 and 3 above.

It seems that the geometric methods of Guerimand do not apply in a noncommutative context and therefore we developed an alternative approach to the incidence problem based on deformation theory (see §5.2). In this approach the type zero condition turned out to be unnecessary. For this reason we have decided to write down our results first in a purely commutative setting. In forthcoming work we hope to describe the corresponding noncommutative theory.

[2] It is easy to see that this definition of type zero is equivalent to the one in [38].

5.2 Outline of the proof of the main theorem

5.2.1 Generic Betti numbers

Let $X \in \mathrm{Hilb}_n(\mathbb{P}^2)$. The graded ideal I_X associated to X admits a minimal free resolution of the form

$$0 \to \oplus_i A(-i)^{b_i} \to \oplus_i A(-i)^{a_i} \to I_X \to 0 \tag{5.4}$$

where $(a_i), (b_i)$ are sequences of non-negative integers which have finite support, called the *graded Betti numbers* of I_X (and X). They are related to the Hilbert series of I_X as

$$h_{I_X}(t) = h_A(t) \sum_i (a_i - b_i) t^i = \frac{\sum_i (a_i - b_i) t^i}{(1-t)^3} \tag{5.5}$$

So the Betti numbers determine the Hilbert series of I_X. For generic X (in a stratum H_ψ) the converse is true since in that case a_i and b_i are not both non-zero. We will call such $(a_i)_i$, $(b_i)_i$ *generic Betti numbers.*

5.2.2 Four sets of conditions

We fix a pair of Hilbert series (φ, ψ) of length zero. Thus for the associated Castelnuovo functions we have

$$s_\psi(t) = s_\varphi(t) + t^u - t^{v+1} \tag{5.6}$$

for some integers $0 < u \leq v$. To prove Theorem 9 we will show that 4 sets of conditions on (φ, ψ) are equivalent.

Condition A. $H_\varphi \subset \overline{H_\psi}$.

Condition B. *The dimension and the tangent condition hold for* (φ, ψ).

Let $(a_i)_i$ and $(b_i)_i$ be the generic Betti numbers associated to φ. The next technical condition restricts the values of the Betti numbers for $i = u,\ u+1,\ v+2,\ v+3$.

Condition C. $a_u \neq 0$, $b_{v+3} \neq 0$ *and*

$$\begin{cases} b_{u+1} \leq a_u \leq b_{u+1} + 1 \text{ and } b_{v+3} = a_{v+2} & \\ \text{or} & \text{if } v = u+1 \\ a_u = b_{u+1} + 1 \text{ and } b_{v+3} = a_{v+2} - 1 & \\ & \\ a_u = b_{u+1} + 1 \text{ and } b_{v+3} = a_{v+2} & \text{if } v \geq u+2 \end{cases}$$

The last condition is of homological nature. Let $I \subset A$ be a graded ideal corresponding to a generic point of H_φ. Put

$$\hat{A} = \begin{pmatrix} A & A \\ 0 & A \end{pmatrix}$$

For an ideal $J \subset I$ put

$$\hat{J} = \begin{pmatrix} J & I \end{pmatrix}$$

This is a right $\hat{A}$-module.

Condition D. *There exists an ideal $J \subset I$, $h_J = h_A - \psi$ such that*

$$\dim_k \operatorname{Ext}^1_{\hat{A}}(\hat{J}, \hat{J}) < \dim_k \operatorname{Ext}^1_A(J, J)$$

In the sequel we will verify the implications

$$A \Rightarrow B \Rightarrow C \Rightarrow D \Rightarrow A$$

Here the implication $A \Rightarrow B$ is clear and the implication $B \Rightarrow C$ is purely combinatorial.

The implication $C \Rightarrow D$ is based on the observation that I/J must be a so-called truncated point module (see §5.4.1 below). This allows us to construct the projective resolution of J from that of I and in this way we can compute $\dim_k \operatorname{Ext}^1_A(J, J)$. To compute $\operatorname{Ext}^1_{\hat{A}}(\hat{J}, \hat{J})$ we view it as the tangent space to the moduli-space of pairs (J, I).

The implication $D \Rightarrow A$ uses elementary deformation theory. Assume D holds. Starting from some $\zeta \in \operatorname{Ext}^1_A(J, J)$ (which we view as a first order deformation of J), not in the image of $\operatorname{Ext}^1_{\hat{A}}(\hat{J}, \hat{J})$ we construct a one-parameter family of ideals J_θ such that $J_0 = J$ and $\operatorname{pd} J_\theta = 1$ for $\theta \neq 0$. Since I and $J = J_0$ have the same image in $\operatorname{Hilb}_n(\mathbb{P}^2)$, this shows H_φ is indeed in the closure of H_ψ.

5.3 The implication $B \Rightarrow C$

In this section we translate the length zero condition, the dimension condition and the tangent condition in terms of Betti numbers. As a result we obtain that Condition B implies Condition C.

To make the connection between Betti numbers and Castelnuovo diagrams we frequently use the identities

$$\sum_{i \leq l} (a_i - b_i) = 1 + s_{l-1} - s_l \quad \text{if} \quad l \geq 0 \tag{5.7}$$

$$a_l - b_l = -s_l + 2s_{l-1} - s_{l-2} \quad \text{if} \quad l > 0 \tag{5.8}$$

Throughout we fix a pair of Hilbert functions (φ, ψ) of degree n and length zero and we let $s = s_\varphi$, $\tilde{s} = s_\psi$ be the corresponding Castelnuovo diagrams. Thus we have

$$\psi(t) = \varphi(t) + t^u + t^{u+1} + \cdots + t^v \tag{5.9}$$

and

$$\tilde{s} = s + t^u - t^{v+1} \tag{5.10}$$

for some $0 < u \leq v$.

The corresponding generic Betti numbers (cfr §5.2.1) are written as $(a_i), (b_i)$ resp. $(\tilde{a}_i), (\tilde{b}_i)$. We also write

$$\sigma = \min\{i \mid s_i \geq s_{i+1}\} = \min\{i \mid a_i > 0\}$$
$$\tilde{\sigma} = \min\{i \mid \tilde{s}_i \geq \tilde{s}_{i+1}\} = \min\{i \mid \tilde{a}_i > 0\}$$

5.3.1 Translation of the length zero condition

The proof of the following result is left to the reader.

Proposition 5.3.1. *If $v \geq u+1$ then we have*

i	...	u	$u+1$	$u+2$	...	$v+1$	$v+2$	$v+3$	...
a_i	...	$*$	0	0	...	0	$*$	$*$	...
b_i	...	$*$	$*$	0	...	0	0	$*$	...

where

$$a_u \leq b_{u+1} + 1,\ a_{v+2} > 0,\ b_{v+3} \leq a_{v+2}.$$

This result is based on the identity (5.8). The zeroes among the Betti numbers are caused by the "plateau" in s between the u'th and the $v+1$'th column (see (5.3)).

5.3.2 Translation of the dimension condition

The following result allows us to compare the dimensions of the strata H_φ and H_ψ.

Proposition 5.3.2. *One has*

$$\dim H_\psi = \dim H_\varphi + \sum_{i=u}^{v}(a_i - b_i) - \sum_{i=u+3}^{v+3}(a_i - b_i) + e \tag{5.11}$$

and

$$\begin{aligned}\dim H_\psi = \dim H_\varphi &- s_{u-2} + s_{u-1} + s_{u+1} - s_{u+2} \\ &+ s_{v-1} - s_v - s_{v+2} + s_{v+3} + e\end{aligned} \tag{5.12}$$

where

$$e = \begin{cases} -1 & \text{if } v = u \\ 1 & \text{if } v = u+1 \\ 0 & \text{if } v \geq u+2 \end{cases}$$

Proof. The proof uses only (5.10). One has the formula from Proposition 3.5.11

$$\dim H_\varphi = 1 + n + c_\varphi$$

where c_φ is the constant term of

$$f_\varphi(t) = (t^{-1} - t^{-2})s_\varphi(t^{-1})s_\varphi(t)$$

We find

$$\begin{aligned}
f_\psi(t) &= (t^{-1} - t^{-2})s_\psi(t^{-1})s_\psi(t) \\
&= (t^{-1} - t^{-2})(s_\varphi(t^{-1}) + t^{-u} - t^{-v-1})(s_\varphi(t) + t^u - t^{v+1}) \\
&= (t^{-1} - t^{-2})\Big(\sum_i s_i t^{-i} + t^{-u} - t^{-v-1}\Big)\Big(\sum_j s_j t^j + t^u - t^{v+1}\Big) \\
&= f_\varphi(t) + (t^{-1} - t^{-2})\Big(\sum_i s_i t^{u-i} - \sum_i s_i t^{v+1-i} \\
&\qquad + \sum_j s_j t^{j-u} - \sum_j s_j t^{j-v-1} - t^{v+1-u} - t^{u-v-1} + 2\Big)
\end{aligned}$$

Taking constant terms we obtain (5.12). Applying (5.7) finishes the proof. □

We obtain the following rather strong consequence of the dimension condition.

Corollary 5.3.3. *If $v \geq u + 2$ then*

$$\dim H_\varphi < \dim H_\psi \Leftrightarrow a_u = b_{u+1} + 1 \text{ and } a_{v+2} = b_{v+3}$$

and if this is the case then we have in addition

$$\dim H_\psi = \dim H_\varphi + 1 \text{ and } u = \sigma,\ a_u > 0\ a_{v+2} = b_{v+3} > 0$$

Proof. Due to Proposition 5.3.1 we have $s_{u+1} = s_{u+2}$ and $s_{v-1} = s_v$ so (5.12) becomes

$$\dim H_\varphi < \dim H_\psi \Leftrightarrow (s_{u-2} - s_{u-1}) + (s_{v+2} - s_{v+3}) < 0$$

We have $1 \leq \sigma \leq u$, which implies $s_{v+2} \geq s_{v+3}$, and either $s_{u-2} \geq s_{u-1}$ or $s_{u-1} = s_{u-2} + 1$. From this it is easy to see we have $(s_{u-2} - s_{u-1}) + (s_{v+2} - s_{v+3}) < 0$ if and only if $s_{u-1} = s_{u-2} + 1$ and $s_{v+2} = s_{v+3}$.

First assume this is the case. Then it follows from (5.7) and Proposition 5.3.1 that $\sigma = u$ hence $a_u > 0$, $b_u = 0$. Equation (5.7) together with $s_u = s_{u+1}$ gives $\sum_{i \leq u+1}(a_i - b_i) = 1$ and since $a_{u+1} = 0$ (see Proposition 5.3.1) we have $a_u = b_{u+1} + 1$. Further, (5.7) together with $s_{v+2} = s_{v+3}$ gives $\sum_{i \leq v+3}(a_i - b_i) = 1$. Combined with $\sum_{i \leq u+1}(a_i - b_i) = 1$ and Proposition 5.3.1 we get $a_{v+2} + (a_{v+3} - b_{v+3}) = 0$ where $a_{v+2} > 0$. This gives $a_{v+2} = b_{v+3} > 0$.

Conversely, assume $a_u = b_{u+1} + 1$ and $a_{v+2} = b_{v+3}$. Observe Proposition 5.3.1 implies $s_u = s_{u+1}$ and $a_{u+1} = 0$, so using (5.7) yields

$$1 = \sum_{i \leq u+1}(a_i - b_i) = \sum_{i \leq u-1}(a_i - b_i) + a_u - b_{u+1}$$

Since we assumed $a_u = b_{u+1} + 1$, we find $\sum_{i\le u-1}(a_i - b_i) = 0$ and using (5.7) again we get $s_{u-2}+1 = s_{u-1}$. Next, the fact $s_v = s_{v+1}$ (see Proposition 5.3.1) together with (5.7) yields $\sum_{i\le v+1}(a_i-b_i) = 1$. In combination with equation (5.7) for $l = v+3$ and Proposition 5.3.1 we get $s_{v+2} - s_{v+3} = a_{v+2} + (a_{v+3} - b_{v+3}) = 0$. Since we assumed $a_{v+2} = b_{v+3}$ this implies $s_{v+2} - s_{v+3} = a_{v+3}$. Further, since $b_{v+3} = a_{v+2} > 0$ (see Proposition 5.3.1) we have $a_{v+3} = 0$. We conclude $s_{u-2} + 1 = s_{u-1}$ and $s_{v+2} = s_{v+3}$ which finishes the proof. □

5.3.3 Translation of the tangent condition

Recall from §5.1 that the tangent function t_φ is the Hilbert function of $\mathcal{I}_X \otimes_{\mathbb{P}^2} \mathcal{T}_{\mathbb{P}^2}$ for $X \in H_\varphi$ generic.

Proposition 5.3.4. *(See also [38, Lemme 2.2.24]) We have*

$$t_\varphi(t) = h_{\mathcal{T}_{\mathbb{P}^2}}(t) - (3t^{-1} - 1)\varphi(t) + \sum_i b_{i+3}t^i \tag{5.13}$$

Proof. From the exact sequence

$$0 \to \mathcal{T}_{\mathbb{P}^2} \to \mathcal{O}(2)^3 \to \mathcal{O}(3) \to 0$$

we deduce

$$H^1(\mathbb{P}^2, \mathcal{T}_{\mathbb{P}^2}(n)) = \begin{cases} k & \text{if } n = -3 \\ 0 & \text{otherwise} \end{cases} \tag{5.14}$$

Let $\mathcal{I} = \mathcal{I}_X$ (X generic) and consider the associated resolution.

$$0 \to \oplus_j \mathcal{O}(-j)^{b_j} \to \oplus_i \mathcal{O}(-i)^{a_i} \to \mathcal{I} \to 0$$

Tensoring with $\mathcal{T}_{\mathbb{P}^2}(n)$ and applying the long exact sequence for $H^*(\mathbb{P}^2, -)$ we obtain an exact sequence

$$\begin{aligned} 0 \to \oplus_j \Gamma(\mathbb{P}^2, \mathcal{T}_{\mathbb{P}^2}(n-j)^{b_j}) \to \oplus_i \Gamma(\mathbb{P}^2, \mathcal{T}_{\mathbb{P}^2}(n-i)^{a_i}) \to \Gamma(\mathbb{P}^2, \mathcal{I} \otimes_{\mathbb{P}^2} \mathcal{T}_{\mathbb{P}^2}(n)) \to \\ \oplus_j H^1(\mathbb{P}^2, \mathcal{T}_{\mathbb{P}^2}(n-j)^{b_j}) \to \oplus_i H^1(\mathbb{P}^2, \mathcal{T}_{\mathbb{P}^2}(n-i)^{a_i}) \end{aligned}$$

It follows from (5.14) that the rightmost arrow is zero. This easily yields the required formula. □

Remark 5.3.5. The previous proposition has an easy generalization which is perhaps useful and which is proved in the same way. Let M be the second syzygy of a finite dimensional graded A-module F and let $\mathcal{M}$ be the associated coherent sheaf. Write $h_M(t) = q_M(t)/(1-t)^3$. Then the Hilbert series of $\mathcal{I}_X \otimes_{\mathbb{P}^2} \mathcal{M}$ is given by

$$q_M(t)h_{I_X}(t) + h_{\mathrm{Tor}_1^A(F, I_X)}(t)$$

The case where $\mathcal{M}$ is the tangent bundle corresponds to $F = k(3)$.

Proposition 5.3.6. *We have*

1. $t_\psi(l) \le t_\varphi(l)$ *for* $l \ne u-3, v$
2. $t_\psi \le t_\varphi \Leftrightarrow a_u \ne 0$ *and* $b_{v+3} \ne 0$

Proof. The proof uses only (5.10). Comparing (5.13) for φ and ψ gives

$$t_\varphi(t) - t_\psi(t) = 3t^{u-1} + 2(t^u + t^{u+1} + \ldots + t^{v-1}) - t^v + \sum_i (b_{i+3} - \tilde{b}_{i+3})t^i \quad (5.15)$$

where we have used (5.9). In order to prove the statements, we have to estimate the polynomial $\sum_i (b_{i+3} - \tilde{b}_{i+3})t^i$. For this, substituting (5.5) for φ and ψ in (5.9) gives

$$\begin{aligned}\sum_i (\tilde{a}_i - \tilde{b}_i)t^i &= \sum_i (a_i - b_i)t^i - (t^u - t^{v+1})(1-t)^2 \\ &= \sum_i (a_i - b_i)t^i - t^u + 2t^{u+1} - t^{u+2} + t^{v+1} - 2t^{v+2} + t^{v+3}\end{aligned}$$

hence

$$\begin{aligned}
\tilde{a}_u - \tilde{b}_u &= a_u - b_u - 1 \\
\tilde{a}_{u+1} - \tilde{b}_{u+1} &= a_{u+1} - b_{u+1} + \begin{cases} 3 & \text{if } v = u \\ 2 & \text{if } v \ge u+1 \end{cases} \\
\tilde{a}_{u+2} - \tilde{b}_{u+2} &= a_{u+2} - b_{u+2} + \begin{cases} -3 & \text{if } v = u \\ 0 & \text{if } v = u+1 \\ -1 & \text{if } v \ge u+2 \end{cases} \\
\tilde{a}_{v+1} - \tilde{b}_{v+1} &= a_{v+1} - b_{v+1} + \begin{cases} 3 & \text{if } v = u \\ 0 & \text{if } v = u+1 \\ 1 & \text{if } v \ge u+2 \end{cases} \\
\tilde{a}_{v+2} - \tilde{b}_{v+2} &= a_{v+2} - b_{v+2} + \begin{cases} -3 & \text{if } v = u \\ -2 & \text{if } v \ge u+1 \end{cases} \\
\tilde{a}_{v+3} - \tilde{b}_{v+3} &= a_{v+3} - b_{v+3} + 1 \\
\tilde{a}_l - \tilde{b}_l &= a_l - b_l \quad \text{if } l \notin \{u, u+1, u+2, v+1, v+2, v+3\}
\end{aligned} \quad (5.16)$$

To obtain information about the differences $b_{i+3} - \tilde{b}_{i+3}$, we observe that for $c \ge 0$ and for all integers l we have

$$\begin{aligned} \tilde{a}_l - \tilde{b}_l = a_l - b_l + c &\Rightarrow \tilde{b}_l \le b_l \\ \tilde{a}_l - \tilde{b}_l = a_l - b_l - c &\Rightarrow \tilde{b}_l \le b_l + c \end{aligned} \quad (5.17)$$

Indeed, first let $\tilde{a}_l - \tilde{b}_l = a_l - b_l + c$. In case $0 \le b_l \le c$ then $0 = \tilde{b}_l \le b_l$. And in case $c < b_l$ then $\tilde{b}_l = b_l - c \le b_l$.

Second, let $\tilde{a}_l - \tilde{b}_l = a_l - b_l - c$. In case $0 \leq a_l \leq c$ then $\tilde{a}_l = 0$ hence $\tilde{b}_l = b_l + c - a_l \leq b_l + c$. And in case $c < a_l$ then $0 = \tilde{b}_l \leq c = b_l + c$. So this proves (5.17). Applying (5.17) to (5.16) yields

$$\begin{aligned} \tilde{b}_u &\leq b_u + 1 \\ \tilde{b}_{u+1} &\leq b_{u+1} \\ \tilde{b}_{u+2} &\leq b_{u+2} + \begin{cases} 3 & \text{if } v = u \\ 0 & \text{if } v = u+1 \\ 1 & \text{if } v \geq u+2 \end{cases} \\ \tilde{b}_{v+1} &\leq b_{v+1} \\ \tilde{b}_{v+2} &\leq b_{v+2} + \begin{cases} 3 & \text{if } v = u \\ 2 & \text{if } v \geq u+1 \end{cases} \\ \tilde{b}_{v+3} &\leq b_{v+3} \\ \tilde{b}_l &\leq b_l \text{ if } l \notin \{u, u+1, u+2, v+1, v+2, v+3\} \end{aligned} \tag{5.18}$$

Now we are able to prove the first statement. Combining (5.18) and (5.15) gives

$$t_\varphi(t) - t_\psi(t) \geq \begin{cases} -t^{u-3} - t^v & \text{if } v = u \\ -t^{u-3} + 3t^{u-1} - t^v & \text{if } v = u+1 \\ -t^{u-3} + 2(t^{u-1} + t^u + \ldots + t^{v-2}) - t^v & \text{if } v \geq u+2 \end{cases} \tag{5.19}$$

and therefore $t_\varphi(t) - t_\psi(t) \geq -t^{u-3} - t^v$ which concludes the proof of the first statement.

For the second part, assume $t_\psi \leq t_\varphi$. Equation (5.15) implies

$$\begin{aligned} \tilde{b}_u &\leq b_u \\ \tilde{b}_{v+3} &\leq b_{v+3} - 1 \end{aligned} \tag{5.20}$$

Since $\tilde{b}_{v+3} \geq 0$ we clearly have $b_{v+3} > 0$. Assume, by contradiction, $a_u = 0$. From (5.16) we have $\tilde{a}_u - \tilde{b}_u = a_u - b_u - 1$ hence $\tilde{a}_u = 0$ and $\tilde{b}_u = b_u + 1$. But this gives a contradiction with (5.20). Therefore

$$t_\psi \leq t_\varphi \Rightarrow a_u > 0 \text{ and } b_{v+3} > 0$$

To prove the converse let $a_u > 0$ and $b_{v+3} > 0$. Due to the first part we only need to prove $t_\psi(u-3) \leq t_\varphi(u-3)$ and $t_\psi(v) \leq t_\varphi(v)$. Equation (5.15) gives us

$$\begin{aligned} t_\varphi(u-3) - t_\psi(u-3) &= b_u - \tilde{b}_u \\ t_\varphi(v) - t_\psi(v) &= b_{v+3} - \tilde{b}_{v+3} - 1 \end{aligned} \tag{5.21}$$

while from (5.16) we have

$$\begin{aligned} \tilde{a}_u - \tilde{b}_u &= a_u - b_u - 1 \\ \tilde{a}_{v+3} - \tilde{b}_{v+3} &= a_{v+3} - b_{v+3} + 1 \end{aligned}$$

Since $a_u > 0$, $b_{v+3} > 0$ we have $b_u = 0$, $a_{v+3} = 0$ hence

$$\tilde{a}_u - \tilde{b}_u = a_u - 1$$
$$\tilde{a}_{v+3} - \tilde{b}_{v+3} = -b_{v+3} + 1$$

which implies $\tilde{a}_u - \tilde{b}_u \geq 0$, $\tilde{a}_{v+3} - \tilde{b}_{v+3} \leq 0$ hence $\tilde{b}_u = 0$, $\tilde{a}_{v+3} = 0$. Thus $b_u = \tilde{b}_u = 0$ and $\tilde{b}_{v+3} = b_{v+3} - 1$. Combining with (5.21) this proves $t_\varphi(u-3) = t_\psi(u-3)$ and $t_\varphi(v) = t_\psi(v)$, finishing the proof. □

5.3.4 Combining everything

In this section we prove Condition B implies Condition C. So assume Condition B holds.

Since the tangent condition holds we have by Proposition 5.3.6

$$a_u \neq 0 \qquad \text{and} \qquad b_{v+3} \neq 0$$

This means there is nothing to prove if $u = v$. We discuss the two remaining cases.

Case 3. $v = u + 1$

The fact $a_u \neq 0$, $b_{v+3} \neq 0$ implies $b_u = 0$, $a_{v+3} = 0$. Proposition 5.3.2 combined with Proposition 5.3.1 now gives

$$\dim H_\psi = \dim H_\varphi + a_u - b_{u+1} - a_{v+2} + b_{v+3} + 1$$

Hence $0 \leq (a_u - b_{u+1}) + (b_{v+3} - a_{v+2})$. But Proposition 5.3.1 also states $a_u \leq b_{u+1} + 1$, $a_{v+2} > 0$ and $b_{v+3} \leq a_{v+2}$. Therefore either we have

$$b_{u+1} \leq a_u \leq b_{u+1} + 1 \text{ and } b_{v+3} = a_{v+2}$$

or

$$a_u = b_{u+1} + 1 \text{ and } b_{v+3} = a_{v+2} - 1$$

Case 4. $v \geq u + 2$

It follows from Corollary 5.3.3

$$a_u = b_{u+1} + 1 \text{ and } a_{v+2} = b_{v+3}$$

This finishes the proof.

Remark 5.3.7. The reader will have noticed that our proof of the implication $B \Rightarrow C$ is rather involved. being purely combinatorial it can be checked directly for individual n. Using a computer we have verified the equivalence of B and C for $n \leq 70$, see Appendix E Remark E.2.1. As another independent verification we have a direct proof of the implication $C \Rightarrow B$ (i.e. without going through the other conditions).

Remark 5.3.8. The reader may observe that in case $v = u$ we have

$$t_\psi \leq t_\varphi \Rightarrow \dim H_\varphi < \dim H_\psi \tag{5.22}$$

while if $v \geq u + 2$ we have

$$\dim H_\varphi < \dim H_\psi \Rightarrow t_\psi \leq t_\varphi \tag{5.23}$$

It is easy to construct counter examples which show that the reverse implications do not hold, and neither (5.22) nor (5.23) is valid in case $v = u + 1$.

5.4 The implication $C \Rightarrow D$

In this section (φ, ψ) will have the same meaning as in §5.3 and we also keep the associated notations.

5.4.1 Truncated point modules

A truncated point module of length m is a graded A-module generated in degree zero with Hilbert series $1 + t + \cdots + t^{m-1}$.

If F is a truncated point module of length > 1 then there are two independent homogeneous linear forms l_1, l_2 vanishing on F and their intersection defines a point $p \in \mathbb{P}^2$. Similarly as in §1.9.3 we may choose basis vectors $e_i \in F_i$ such that

$$xe_i = x_p e_{i+1}, \qquad ye_i = y_p e_{i+1}, \quad ze_i = z_p e_{i+1}$$

where (x_p, y_p, z_p) is a set of homogeneous coordinates of p. It follows that if $f \in A$ is homogeneous of degree d and $i + d \leq m - 1$ then

$$fe_i = f_p e_{i+d}$$

where $(-)_p$ stands for evaluation in p (with respect to the homogeneous coordinates (x_p, y_p, z_p)).

If $G = \oplus_i A(-i)^{c_i}$ then we have

$$\operatorname{Hom}_A(G, F) = \oplus_{0 \leq i \leq m-1} F_i^{c_i} \cong k^{\sum_{0 \leq i \leq m-1} c_i} \tag{5.24}$$

where the last identification is made using the basis $(e_i)_i$ introduced above.

In the sequel we will need the minimal projective resolution of a truncated point module F of length m. It is easy to see it is given by

$$0 \to A(-m-2) \xrightarrow[f_3]{\begin{pmatrix} l_1 \\ l_2 \\ \rho \end{pmatrix} \cdot} A(-m-1)^2 \oplus A(-2) \xrightarrow[f_2]{\begin{pmatrix} 0 & -\rho & l_2 \\ \rho & 0 & -l_1 \\ -l_2 & l_1 & 0 \end{pmatrix} \cdot} A(-1)^2 \oplus A(-m) \xrightarrow[f_1]{(l_1 \quad l_2 \quad \rho) \cdot} A \to F \to 0 \tag{5.25}$$

where l_1, l_2 are the linear forms vanishing on F and ρ is a form of degree m such that $\rho_p \neq 0$ for the point p corresponding to F. Without loss of generality we may and we will assume $\rho_p = 1$.

5.4.2 A complex whose homology is J

In this section I is a graded ideal corresponding to a generic point in H_φ. The following lemma gives the connection between truncated point modules and Condition D.

Lemma 5.4.1. *If an ideal $J \subset I$ has Hilbert series $h_A - \psi$ then I/J is a (shifted by grading) truncated point module of length $v + 1 - u$.*

Proof. Since $F = I/J$ has the correct (shifted) Hilbert function, it is sufficient to show F is generated in degree u.

If $v = u$ then there is nothing to prove. If $v \geq u + 1$ then by Proposition 5.3.1 the generators of I are in degrees $\leq u$ and $\geq v + 2$. Since F lives in degrees $u, \dots, v$ this proves what we want. □

Let J, F be as in the previous lemma. Below we will need a complex whose homology is J. We write the minimal resolution of F as

$$0 \to G_3 \xrightarrow{f_3} G_2 \xrightarrow{f_2} G_1 \xrightarrow{f_1} G_0 \to F \to 0$$

where the maps f_i are as in (5.25), and the minimal resolution of I as

$$0 \to F_1 \to F_0 \to I \to 0$$

The map $I \to F$ induces a map of projective resolutions

$$\begin{array}{ccccccccccccc} & & & & 0 & \longrightarrow & F_1 & \xrightarrow{M} & F_0 & \longrightarrow & I & \longrightarrow & 0 \\ & & & & & & \downarrow{\scriptstyle\gamma_1} & & \downarrow{\scriptstyle\gamma_0} & & \downarrow & & \\ 0 & \longrightarrow & G_3 & \xrightarrow{f_3} & G_2 & \xrightarrow{f_2} & G_1 & \xrightarrow{f_1} & G_0 & \xrightarrow{f_0} & F & \longrightarrow & 0 \end{array} \tag{5.26}$$

Taking cones yields that J is the homology at $G_1 \oplus F_0$ of the following complex

$$0 \to G_3 \xrightarrow{\begin{pmatrix} f_3 \\ 0 \end{pmatrix}} G_2 \oplus F_1 \xrightarrow{\begin{pmatrix} f_2 & \gamma_1 \\ 0 & -M \end{pmatrix}} G_1 \oplus F_0 \xrightarrow{\begin{pmatrix} f_1 & \gamma_0 \end{pmatrix}} G_0 \to 0 \tag{5.27}$$

Note the rightmost map is split here. By selecting an explicit splitting we may construct a free resolution of J, but it will be convenient not to do this.

For use below we note the map $J \to I$ is obtained from taking homology of the following map of complexes.

$$\begin{array}{ccccccccc} 0 \longrightarrow G_3 & \xrightarrow{\begin{pmatrix} f_3 \\ 0 \end{pmatrix}} & G_2 \oplus F_1 & \xrightarrow{\begin{pmatrix} f_2 & \gamma_1 \\ 0 & -M \end{pmatrix}} & G_1 \oplus F_0 & \xrightarrow{\begin{pmatrix} f_1 & \gamma_0 \end{pmatrix}} & G_0 \longrightarrow 0 \\ & & \downarrow{\scriptstyle\begin{pmatrix} 0 & -1 \end{pmatrix}} & & \downarrow{\scriptstyle\begin{pmatrix} 0 & 1 \end{pmatrix}} & & \\ & 0 \longrightarrow & F_1 & \xrightarrow[M]{} & F_0 & \longrightarrow 0 & \end{array} \tag{5.28}$$

5.4.3 The Hilbert scheme of an ideal

In this section I is a graded ideal corresponding to a generic point in H_φ.

Let $\mathcal{V}$ be the Hilbert scheme of graded quotients F of I with Hilbert series $t^u + \cdots + t^v$. To see that $\mathcal{V}$ exists one may realize it as a closed subscheme of

$$\operatorname{Proj} S(I_u \oplus \cdots \oplus I_v)$$

where SV is the symmetric algebra of a vector space V. Alternatively see [11].

We will give an explicit description of $\mathcal{V}$ by equations. Here and below we use the following convention: if N is a matrix with coefficients in A representing a map $\oplus_j A(-j)^{d_j} \to \oplus_i A(-i)^{c_i}$ then $N(p,q)$ stands for the submatrix of N representing the induced map $A(-q)^{d_q} \to A(-p)^{c_p}$.

We now distinguish two cases.

- $v = u$. In this case clearly $\mathcal{V} \cong \mathbb{P}^{a_u - 1}$.

- $v \geq u + 1$. Let $F \in \mathcal{V}$ and let $p \in \mathbb{P}^2$ be the associated point. Let $(e_i)_i$ be a basis for F as in §5.4.1. The map $I \to F$ defines a map

  $$\lambda : A(-u)^{a_u} \to F$$

  such that the composition

  $$A(-u-1)^{b_{u+1}} \xrightarrow{M(u,u+1)\cdot} A(-u)^{a_u} \to F \tag{5.29}$$

  is zero. We may view λ as a scalar row vector as in (5.24). The fact that (5.29) has zero composition then translates into the condition

  $$\lambda \cdot M(u, u+1)_p = 0 \tag{5.30}$$

  It is easy to see this procedure is reversible and the equations (5.30) define $\mathcal{V}$ as a subscheme of $\mathbb{P}^{a_u-1} \times \mathbb{P}^2$.

Proposition 5.4.2. *Assume Condition C holds. Then $\mathcal{V}$ is smooth and*

$$\dim \mathcal{V} = \begin{cases} a_u - 1 & \text{if } v = u \\ a_u + 1 - b_{u+1} & \text{if } v \geq u+1 \end{cases}$$

Proof. The case $v = u$ is clear so assume $v \geq u + 1$. If we look carefully at (5.30) then we see it describes $\mathcal{V}$ as the zeroes of b_{u+1} generic sections in the very ample line bundle $\mathcal{O}_{\mathbb{P}^{a_u-1}}(1) \boxtimes \mathcal{O}_{\mathbb{P}^2}(1)$ on $\mathbb{P}^{a_u-1} \times \mathbb{P}^2$. It follows from Condition C that $b_{u+1} \leq \dim(\mathbb{P}^2 \times \mathbb{P}^{a_u-1}) = a_u + 1$. Hence by Bertini (see [41]) we deduce $\mathcal{V}$ is smooth of dimension $a_u + 1 - b_{u+1}$. □

5.4.4 Estimating the dimension of $\operatorname{Ext}^1_A(J,J)$

In this section I is a graded ideal corresponding to a generic point of H_φ. We prove the following result

Proposition 5.4.3. *Assume Condition C holds. Then there exists $F \in \mathcal{V}$ such that for $J = \ker(I \to F)$ we have*

$$\dim_k \operatorname{Ext}^1_A(J,J) \geq \begin{cases} \dim H_\psi + a_{v+3} = \dim H_\psi & \text{if } v = u \\ \dim H_\psi + a_{v+2} - b_{v+3} + 1 & \text{if } v = u+1 \\ \dim H_\psi + a_{v+2} - b_{v+3} + 2 = \dim H_\psi + 2 & \text{if } v \geq u+2 \end{cases} \quad (5.31)$$

It will become clear from the proof below that in case $v \geq u+1$ the righthand side of (5.31) is one higher than the expected dimension.

Below let $J \subset I$ be an arbitrary ideal such that $h_J = h_A - \psi$. Put $F = I/J$.

Proposition 5.4.4. *We have*

$$\dim_k \operatorname{Ext}^1_A(J,J) = \dim H_\psi + \dim_k \operatorname{Hom}_A(J, F(-3))$$

Proof. For $M, N \in \operatorname{gr} A$ write

$$\chi(M,N) = \sum_i (-1)^i \dim_k \operatorname{Ext}^i_A(M,N)$$

Clearly $\chi(M,N)$ only depends on the Hilbert series of M, N. Hence, taking J' to be an arbitrary point in H_ψ we have

$$\chi(J,J) = \chi(J',J') = 1 - \dim_k \operatorname{Ext}^1_A(J',J') = 1 - \dim H_\psi$$

where in the third equality we have used $\operatorname{Ext}^1_A(J',J')$ is the tangent space to H_ψ, see Lemma 3.5.10.

Since J has no socle we have $\operatorname{pd} J \leq 2$. Therefore $\operatorname{Ext}^i_A(J,J) = 0$ for $i \geq 3$. It follows that

$$\begin{aligned} \dim_k \operatorname{Ext}^1_A(J,J) &= -\chi(J,J) + 1 + \dim_k \operatorname{Ext}^2_A(J,J) \\ &= \dim H_\psi + \dim_k \operatorname{Ext}^3_A(F,J) \end{aligned}$$

By the approriate version of Serre duality we have

$$\operatorname{Ext}^3_A(F,J) = \operatorname{Hom}_A(J, F \otimes \omega_A)' = \operatorname{Hom}_A(J, F(-3))'$$

This finishes the proof. □

Proof of Proposition 5.4.3. It follows from the previous result that we need to control $\dim_k \operatorname{Hom}_A(J, F(-3))$. Of course we assume throughout Condition C holds and we also use Proposition 5.3.1.

Case 1. Assume $v = u$. For degree reasons any extension between F and $F(-3)$ must be split. Thus we have $\operatorname{Hom}_A(F, F(-3)) = \operatorname{Ext}^1_A(F, F(-3)) = 0$. Applying $\operatorname{Hom}_A(-, F(-3))$ to

$$0 \to J \to I \to F \to 0$$

we find

$$\operatorname{Hom}_A(J, F(-3)) = \operatorname{Hom}_A(I, F(-3))$$

Hence

$$\dim_k \operatorname{Hom}_A(J, F(-3)) = a_{v+3} = 0$$

Case 2. Assume $v = u + 1$. As in the previous case we find $\operatorname{Hom}_A(J, F(-3)) = \operatorname{Hom}_A(I, F(-3))$.

Thus a map $J \to F(-3)$ is now given (using Proposition 5.3.1) by a map

$$\beta : A(-v-2)^{a_{v+2}} \to F(-3)$$

(identified with a scalar vector as in (5.24)) such that the composition

$$A(-v-3)^{b_{v+3}} \xrightarrow{M(v+2,v+3)} A(-v-2)^{a_{v+2}} \xrightarrow{\beta} F(-3)$$

is zero. This translates into the condition

$$\beta \cdot M(v+2, v+3)_p = 0 \tag{5.32}$$

where p is the point corresponding to F. Now $M(v+2, v+3)$ is a $a_{v+2} \times b_{v+3}$ matrix. Since $b_{v+3} \leq a_{v+2}$ (by Proposition 5.3.1) we would expect (5.32) to have $a_{v+2} - b_{v+3}$ independent solutions. To have more, $M(v+2, v+3)$ has to have non-maximal rank. I.e. there should be a non-zero solution to the equation

$$M(v+2, v+3)_p \cdot \delta = 0 \tag{5.33}$$

This should be combined with (see (5.30))

$$\lambda \cdot M(u, u+1)_p = 0 \tag{5.34}$$

We view (5.33) and (5.34) as a system of $a_{v+2}+b_{u+1}$ equations in $\mathbb{P}^{a_u-1} \times \mathbb{P}^2 \times \mathbb{P}^{b_{v+3}-1}$.

Since (Condition C)

$$a_{v+2} + b_{u+1} \leq \dim(\mathbb{P}^{a_u-1} \times \mathbb{P}^2 \times \mathbb{P}^{b_{v+3}-1}) = a_u + b_{v+3}$$

the system (5.33)(5.34) has a solution provided the divisors in $\mathbb{P}^{a_u-1} \times \mathbb{P}^2 \times \mathbb{P}^{b_{v+3}-1}$ determined by the equations of the system have non-zero intersection product.

Let r, s, t be the hyperplane sections in $\mathbb{P}^{a_u-1}$, $\mathbb{P}^2$ and $\mathbb{P}^{b_{v+3}-1}$ respectively. The Chow ring of $\mathbb{P}^{a_u-1} \times \mathbb{P}^2 \times \mathbb{P}^{b_{v+3}-1}$ is given by

$$\mathbb{Z}[r, s, t]/(r^{a_u}, s^3, t^{b_{v+3}}) \tag{5.35}$$

The intersection product we have to compute is

$$(s+t)^{a_{v+2}}(r+s)^{b_{u+1}}$$

This product contains the terms

$$t^{a_{v+2}-2}s^2r^{b_{u+1}}$$
$$t^{a_{v+2}-1}s^2r^{b_{u+1}-1}$$
$$t^{a_{v+2}}s^2r^{b_{u+1}-2}$$

at least one of which is non-zero in (5.35) (using Condition C).

Case 3. Now assume $v \geq u+2$. We compute $\mathrm{Hom}_A(J, F(-3))$ as the homology of $\mathrm{Hom}_A((\text{eq.}5.27), F(-3))$. Since $G_0 = A(-u)$ we have $\mathrm{Hom}_A(G_0, F(-3)) = 0$ and hence a map $J \to F(-3)$ is given by a map

$$G_1 \oplus F_0 \to F(-3)$$

such that the composition

$$G_2 \oplus F_1 \xrightarrow{\begin{pmatrix} f_2 & \gamma_1 \\ 0 & -M \end{pmatrix}} G_1 \oplus F_0 \to F(-3)$$

is zero.

Introducing the explicit form of $(G_i)_i$, $(f_i)_i$ given by (5.25), and using Proposition 5.3.1 we find that a map $J \to F(-3)$ is given by a pair of maps

$$\mu : A(-v-1) \to F(-3)$$
$$\beta : A(-v-2)^{a_{v+2}} \to F(-3)$$

(identified with scalar vectors as in (5.24)) such that the composition

$$A(-v-2)^2 \oplus A(-v-3)^{b_{v+3}} \xrightarrow{\begin{pmatrix} -l_2 & l_1 & \gamma_1(v+1,v+3) \\ 0 & 0 & -M(v+2,v+3) \end{pmatrix}} A(-v-1) \oplus A(-v-2)^{a_{v+2}} \xrightarrow{\begin{pmatrix} \mu & \beta \end{pmatrix}} F$$

is zero.

Let p be the point associated to F. Since $(l_1)_p = (l_2)_p = 0$ we obtain the conditions

$$\begin{pmatrix} \mu & \beta \end{pmatrix} \begin{pmatrix} \gamma_1(v+1,v+3)_p \\ M(v+2,v+3)_p \end{pmatrix} = 0 \tag{5.36}$$

To use this we have to know what $\gamma_1(v+1, v+3)$ is. From the commutative diagram (5.26) we obtain the identity

$$\rho \cdot \gamma_1(v+1,v+3) = \lambda \cdot M(u,v+3)$$

where $\lambda = \gamma_0(u,u)$. Evaluation in p yields

$$\gamma_1(v+1, v+3)_p = \lambda \cdot M(u, v+3)_p$$

so (5.36) is equivalent to

$$\begin{pmatrix} \mu & \beta \end{pmatrix} \begin{pmatrix} \lambda \cdot M(u, v+3)_p \\ M(v+2, v+3)_p \end{pmatrix} = 0$$

Now $\begin{pmatrix} \lambda \cdot M(u, v+3)_p \\ M(v+2, v+3)_p \end{pmatrix}$ is a $(a_{v+2}+1) \times b_{v+3}$ matrix. Since $b_{v+3} < a_{v+2}+1$ (Proposition 5.3.1) we would expect (5.36) to have $a_{v+2}+1-b_{v+3}$ independent solutions. To have more, $\begin{pmatrix} \lambda \cdot M(u, v+3)_p \\ M(v+2, v+3)_p \end{pmatrix}$ has to have non-maximal rank. I.e. there should be a non-zero solution to the equation

$$\begin{pmatrix} \lambda \cdot M(u, v+3)_p \\ M(v+2, v+3)_p \end{pmatrix} \cdot \delta = 0$$

which may be broken up into two sets of equations

$$\lambda \cdot M(u, v+3)_p \cdot \delta = 0 \tag{5.37}$$

$$M(v+2, v+3)_p \cdot \delta = 0 \tag{5.38}$$

and we also still have

$$\lambda \cdot M(u, u+1)_p = 0 \tag{5.39}$$

We view (5.37)(5.38) and (5.39) as a system of $1+a_{v+2}+b_{u+1}$ equations in the variety $\mathbb{P}^{a_u-1} \times \mathbb{P}^2 \times \mathbb{P}^{b_{v+3}-1}$. Since (Condition C)

$$1 + a_{v+2} + b_{u+1} = \dim(\mathbb{P}^{a_u-1} \times \mathbb{P}^2 \times \mathbb{P}^{b_{v+3}-1}) = a_u + b_{v+3}$$

the existence of a solution can be decided numerically. The intersection product we have to compute is

$$(r+s+t)(s+t)^{a_{v+2}}(r+s)^{b_{u+1}}$$

This product contains the term

$$s^2 t^{a_{v+2}-1} r^{b_{u+1}}$$

which is non-zero in the Chow ring (using Condition C). □

5.4.5 Estimating the dimension of $\mathrm{Ext}^1_{\hat{A}}(\hat{J}, \hat{J})$

In this section we prove the following result.

Proposition 5.4.5. *Assume Condition C holds. Let $I \in H_\varphi$ be generic and let J be as in Condition D. Then*

$$\dim_k \operatorname{Ext}^1_{\hat{A}}(\hat{J}, \hat{J}) \leq \begin{cases} \dim H_\varphi + a_u - 1 & \text{if } v = u \\ \dim H_\varphi + a_u + 1 - b_{u+1} & \text{if } v \geq u + 1 \end{cases} \tag{5.40}$$

Proof. It has been shown in Chapter 3 that H_φ is the moduli-space of ideals of A of projective dimension one which have Hilbert series φ. Let $\tilde{I} \subset A_{H_\varphi}$ be the corresponding universal bundle. Let $\mathcal{M}$ be the moduli space of pairs (J, I) such that $I \in H_\varphi$ and $h_J = h_A - \psi$. To show that $\mathcal{M}$ exists on may realize it as a closed subscheme of

$$\underline{\operatorname{Proj}}\, S_{H_\varphi}(\tilde{I}_u \oplus \cdots \oplus \tilde{I}_v)$$

Sending (J, I) to I defines a map $q : \mathcal{M} \to H_\varphi$. We have an exact sequence

$$0 \to T_{(J,I)} q^{-1} I \to T_{(J,I)} \mathcal{M} \to T_I H_\varphi \tag{5.41}$$

Assume now I is generic and put $\mathcal{V} = q^{-1} I$ as above. By Proposition 5.4.2 we know $\mathcal{V}$ is smooth. Hence

$$\dim T_{(J,I)} \mathcal{M} \leq \dim \mathcal{V} + \dim H_\varphi$$

Applying Proposition 5.4.2 again, it follows that for I generic the dimension of $T_{(J,I)}\mathcal{M}$ is bounded by the right hand side of (5.40).

Since $\operatorname{Ext}^1_{\hat{A}}(\hat{J}, \hat{J})$ is the tangent space of $\mathcal{M}$ at (J, I) for $\hat{J} = (J \;\; I)$ this finishes the proof. □

Remark 5.4.6. It is not hard to see that (5.40) is actually an equality. This follows from the easily proved fact that the map q is generically smooth.

5.4.6 Tying things together

Combining the results of the previous two sections we see that if Condition C holds we have for a suitable choice of J

$$\begin{aligned} &\dim_k \operatorname{Ext}^1_A(J, J) - \dim_k \operatorname{Ext}^1_{\hat{A}}(\hat{J}, \hat{J}) \\ &\qquad \geq \begin{cases} \dim H_\psi - \dim H_\varphi + a_{v+3} - a_u + 1 & \text{if } v = u \\ \dim H_\psi - \dim H_\varphi + a_{v+2} - b_{v+3} - a_u + b_{u+1} & \text{if } v = u + 1 \\ \dim H_\psi - \dim H_\varphi + a_{v+2} - b_{v+3} - a_u + b_{u+1} + 1 & \text{if } v \geq u + 2 \end{cases} \end{aligned}$$

We may combine this with Proposition 5.3.2 which works out as (using Proposition 5.3.1)

$$\dim H_\psi - \dim H_\varphi = \begin{cases} a_u + b_{v+3} - 1 & \text{if } v = u \\ a_u - b_{u+1} - a_{v+2} + b_{v+3} + 1 & \text{if } v = u + 1 \\ a_u - b_{u+1} - a_{v+2} + b_{v+3} & \text{if } v \geq u + 2 \end{cases}$$

We then obtain

$$\dim_k \operatorname{Ext}^1_A(J,J) - \dim_k \operatorname{Ext}^1_{\hat{A}}(\hat{J},\hat{J}) \geq \begin{cases} b_{v+3} & \text{if } v = u \\ 1 & \text{if } v \geq u+1 \end{cases}$$

Hence in all cases we obtain a strictly positive result. This finishes the proof that Condition C implies Condition D.

Remark 5.4.7. As in Remark 5.3.7 it is possible to prove directly the converse implication $D \Rightarrow C$.

5.5 The implication $D \Rightarrow A$

In this section (φ, ψ) will have the same meaning as in §5.3 and we also keep the associated notations. We assume Condition D holds. Let I be a graded ideal corresponding to a generic point in H_φ. According to Condition D there exists an ideal $J \subset I$ with $h_J = h_A - \psi$ such that there is an $\eta \in \operatorname{Ext}^1_A(J,J)$ which is not in the image of $\operatorname{Ext}^1_{\hat{A}}(\hat{J},\hat{J})$.

We identify η with a one parameter deformation J' of J. I.e. J' is a flat $A[\epsilon]$-module where $\epsilon^2 = 0$ such that $J' \otimes_{k[\epsilon]} k \cong J$ and such that the short exact sequence

$$0 \to J \xrightarrow{\epsilon\cdot} J' \to J \to 0$$

corresponds to η.

In §5.4.2 we have written J as the homology of a complex. It follows for example from (the dual version of) [53, Thm 3.9], or directly, that J' is the homology of a complex of the form

$$0 \to G_3[\epsilon] \xrightarrow{\begin{pmatrix} f_3' \\ P\epsilon \end{pmatrix}} G_2[\epsilon] \oplus F_1[\epsilon] \xrightarrow{\begin{pmatrix} f_2' & \gamma_1' \\ Q\epsilon & -M' \end{pmatrix}} G_1[\epsilon] \oplus F_0[\epsilon] \xrightarrow{\begin{pmatrix} f_1' & \gamma_0' \end{pmatrix}} G_0[\epsilon] \to 0 \tag{5.42}$$

where for a matrix U over A, U' means a lift of U to $A[\epsilon]$. Recall $G_3 = A(-v-3)$.

Lemma 5.5.1. *We have* $P(v+3, v+3) \neq 0$.

Proof. Assume on the contrary $P(v+3, v+3) = 0$. Using Proposition 5.3.1 it follows that P has its image in $F_{11} = \oplus_{j \leq u+1} A(-j)^{b_j}$.

The fact that (5.42) is a complex implies $Qf_3 = MP$. Thus we have a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & G_3 & \xrightarrow{f_3} & G_2 & \xrightarrow{f_2} & G_1 & \xrightarrow{f_1} & G_0 \\ & & {\scriptstyle P_1}\big\downarrow & & \big\downarrow{\scriptstyle Q} & & & & \\ & & F_{11} & \xrightarrow{M_{11}} & F_0 & & & & \\ & & {\scriptstyle P_2}\big\downarrow & & \big\| & & & & \\ & & F_1 & \xrightarrow[M]{} & F_0 & & & & \end{array}$$

where P_2 is the inclusion and $M_{11} = MP_2$, $P = P_2P_1$. Put

$$D = \operatorname{coker}(F_{11} \to F_1)$$

Then (P_1, Q) represents an element of $\operatorname{Ext}^2_A(F, D) = \operatorname{Ext}^1_A(D, F(-3))' = 0$, where the last equality is for degree reasons.

It follows that there exist maps

$$R : G_1 \to F_0$$
$$T_1 : G_2 \to F_{11}$$

such that

$$Q = Rf_2 + M_{11}T_1$$
$$P_1 = T_1 f_3$$

Putting $T = P_2T_1$ we obtain

$$Q = Rf_2 + MT$$
$$P = Tf_3$$

We can now construct the following lifting of the commutative diagram (5.28):

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & G_3[\epsilon] & \xrightarrow{\begin{pmatrix} f_3' \\ P\epsilon \end{pmatrix}} & G_2[\epsilon] \oplus F_1[\epsilon] & \xrightarrow{\begin{pmatrix} f_2' & \gamma_1' \\ Q\epsilon & -M' \end{pmatrix}} & G_1[\epsilon] \oplus F_0[\epsilon] & \xrightarrow{\begin{pmatrix} f_1' & \gamma_0' \end{pmatrix}} & G_0[\epsilon] & \longrightarrow & 0 \\ & & & & \big\downarrow{\scriptstyle \begin{pmatrix} T\epsilon & -1 \end{pmatrix}} & & \big\downarrow{\scriptstyle \begin{pmatrix} -R\epsilon & 1 \end{pmatrix}} & & & & \\ & & 0 & \longrightarrow & F_1[\epsilon] & \xrightarrow[M' + R\gamma_1\epsilon]{} & F_0[\epsilon] & \longrightarrow & 0 & & \end{array}$$

Taking homology we see there is a first order deformation I' of I together with a lift of the inclusion $J \to I$ to a map $J' \to I'$. But this contradicts the assumption that η is not in the image of $\operatorname{Ext}^1_{\hat{A}}(\hat{J}, \hat{J})$. $\square$

In particular, Lemmma 5.5.1 implies $b_v + 3 \neq 0$. It will now be convenient to rearrange (5.42). Using the previous lemma and the fact that the rightmost map in (5.42) is split it follows that J' has a free resolution of the form

$$0 \to G_3[\epsilon] \xrightarrow{\begin{pmatrix} \epsilon \\ \alpha_0 + \alpha_1\epsilon \end{pmatrix}} G_3[\epsilon] \oplus H_1[\epsilon] \xrightarrow{\begin{pmatrix} \beta_0 + \beta_1\epsilon & \delta_0 + \delta_1\epsilon \end{pmatrix}} H_0[\epsilon] \to J' \to 0$$

which leads to the following equations

$$\begin{aligned} \delta_0\alpha_0 &= 0 \\ \beta_0 + \delta_1\alpha_0 + \delta_0\alpha_1 &= 0 \end{aligned}$$

Using these equations we can construct the following complex C_t over $A[t]$

$$0 \to G_3[t] \xrightarrow{\begin{pmatrix} t \\ \alpha_0 + \alpha_1 t \end{pmatrix}} G_3[t] \oplus H_1[t] \xrightarrow{\begin{pmatrix} \beta_0 - \delta_1\alpha_1 t & \delta_0 + \delta_1 t \end{pmatrix}} H_0[t]$$

For $\theta \in k$ put $C_\theta = C \otimes_{k[t]} k[t]/(t-\theta)$. Clearly C_0 is a resolution of J. By semi-continuity we find that for all but a finite number of θ, C_θ is the resolution of a rank one A-module J_θ. Furthermore we have $J_0 = J$ and $\operatorname{pd} J_\theta = 1$ for $\theta \neq 0$.

Let $\mathcal{J}_\theta$ be the rank one $\mathcal{O}_{\mathbb{P}^2}$-module corresponding to J_θ. $\mathcal{J}_\theta$ represents a point of H_ψ. Since I/J has finite length, $J_0 = J$ and I define the same object in $\mathrm{Hilb}_n(\mathbb{P}^2)$. Hence we have constructed a one parameter family of objects in $\mathrm{Hilb}_n(\mathbb{P}^2)$ connecting a generic object in H_φ to an object in H_ψ. This shows that indeed H_φ is in the closure of H_ψ. This completes the proof of the implication $D \Rightarrow A$.

Chapter 6

Ideals of cubic Artin-Schelter algebras

Let A be a three dimensional Artin-Schelter regular algebra, which is generated in degree one. As discussed in Chapter 1 there are two possibilities, either A is quadratic i.e. A has three generators and three defining homogeneous relations in degree two, or A is cubic i.e. it has two generators and two defining relations in degree three.

In Chapters 2 and 3 we have classified reflexive rank one graded right modules over (generic) quadratic Artin-Schelter algebras, and described their Hilbert series. It is a natural question to do the same for cubic Artin-Schelter algebras. In this chapter we do so. The ideals we use are quite the same, and therefore we will omit some of the proofs.

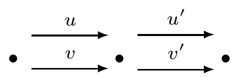
The results in Chapter 6 were obtained in collaboration with N. Marconnet and will appear in a submitted paper [30].

6.1 Introduction and main results

Let A be a cubic Artin-Schelter algebra. Similar as in the quadratic case it turns out that a torsion free rank one module $I \in \operatorname{grmod}(A)$ is determined by a quiver representation M of the quiver Γ

$$\bullet \underset{v}{\overset{u}{\rightrightarrows}} \bullet \underset{v'}{\overset{u'}{\rightrightarrows}} \bullet \underset{v''}{\overset{u''}{\rightrightarrows}} \bullet$$

whose relations are reflected by the defining relations of A. By partial computation of the homology groups $H^i(X, \pi I)$ it turns out that the dimension vector of this representation is given by $\underline{\dim} M = (n_o, n_e, n_o, n_e - 1)$ for some integers $n_e > 0$, $n_o \geq 0$. If furthermore I is reflexive then I is determined by a representation M^0 of the quiver Γ^0



obtained by deleting the rightmost vector space and maps in M.

Let $R(A)$ denote the set of reflexive graded right A-modules of rank one, considered up to isomorphism and shift of grading. Define the set

$$N = \{(n_e, n_o) \in \mathbb{N}^2 \mid n_e - (n_e - n_o)^2 \geq 0\}.$$

Our first result is similar to Theorem 1 of Chapter 2.

Theorem 10. *Let A be an elliptic cubic Artin-Schelter algebra for which σ has infinite order. Then for any $(n_e, n_o) \in N$ there exists a smooth locally closed variety $D_{(n_e,n_o)}$ of dimension $2(n_e - (n_e - n_o)^2)$ such that the set $R(A)$ is in natural bijection with $\coprod_{(n_e,n_o)\in N} D_{(n_e,n_o)}$.*

For $(n_e, n_o) \neq (1, 1)$ the variety $D_{(n_e,n_o)}$ has the following description

$$\begin{aligned} D_{(n_e,n_o)} = \{F = ((X, Y), (X', Y')) \in \operatorname{Rep}_{(n_o,n_e,n_o)}(\Gamma^0) \mid F \text{ is } \theta\text{-stable and} \\ \operatorname{rank} M_A(X, Y, X', Y') \leq 2n_o - (n_e - 1)\}/\operatorname{Gl}_{(n_o,n_e,n_o)}(k) \quad (6.1) \end{aligned}$$

where $\theta = (-1, 0, 1)$, M_A is the matrix as defined in (1.21) and the matrix $M_A(X, Y, X', Y') \in M_{2n_o\times 2n_o}(k)$ is obtained from M_A by replacing x^2, xy, yx, y^2 by $X'X, Y'X, X'Y, Y'Y$. It follows that $D_{(n_e,n_o)}$ is a closed set of the quasi-affine variety consisting of the θ-stable representations in $\operatorname{Rep}_{(n_o,n_e,n_o)}(\Gamma^0)$. For a description of $D_{(1,1)}$ we refer to Corollary 6.7.4. In particular $D_{(1,0)}$ is a point and $D_{(1,1)}$ is the complement of C under a natural embedding in $\mathbb{P}^1 \times \mathbb{P}^1$. In fact $D_{(n_e,n_o)}$ is a point whenever $n_e = (n_e - n_o)^2$.

In case A is of generic type A (see Example 1.9.3) we have in addition (compare to Theorem 2)

Theorem 11. *Let A be a cubic Artin-Schelter algebra of generic type A for which σ has infinite order. Then the varieties $D_{(n_e,n_o)}$ in Theorem 10 are affine.*

As for quadratic Artin-Schelter algebras our proof of Theorem 11 is as follows. We will show that $D_{(n_e,n_o)}$ has the alternative description

$$\begin{aligned} D_{(n_e,n_o)} = \{F = ((X, Y), (X', Y')) \in \operatorname{Rep}_{(n_o,n_e,n_o)}(\Gamma^0) \mid F \perp V \text{ and} \\ \operatorname{rank} M_A(X, Y, X', Y') \leq 2n_o - (n_e - 1)\}/\operatorname{Gl}_{n_o,n_e,n_o}(k) \quad (6.2) \end{aligned}$$

Here V is a fixed representation of Γ^0 with dimension vector $\underline{\dim} V = (6, 4, 2)$, independent of $F \in D_{(n_e,n_o)}$. In particular there is some freedom in choosing V. From the description (6.2) it follows that $D_{(n_e,n_o)}$ is a closed subset of $\varphi_V \neq 0$ so it is affine.

Finally, in Section 6.8 we describe the elements of $R(A)$ by means of filtrations.

Theorem 12. *Assume k is uncountable. Let A be an elliptic cubic Artin-Schelter algebra and assume σ has infinite order. Let $I \in R(A)$. Then there exists an $m \in \mathbb{N}$ together with a filtration of reflexive graded right A-modules of rank one*

$$I_0 \supset I_1 \supset \cdots \supset I_m = I$$

with the property that up to finite length modules the quotients are shifted conic modules i.e. modules of the form A/bA where $b \in A$ has degree two. Moreover I_0 admits a minimal resolution of the form

$$0 \to A(-c-1)^c \to A(-c)^{c+1} \to I_0 \to 0 \tag{6.3}$$

for some integer $c \geq 0$, and I_0 is up to isomorphism uniquely determined by c.

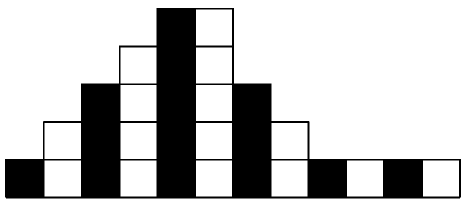
If A is linear it follows from Proposition 6.4.1 below that every reflexive graded right ideal of A admits a resolution of the form (6.3) (up to shift of grading). Hence Theorem 12 is trivially true for linear cubic Artin-Schelter algebras.

A crucial part of the proof of Theorem 10 consists in showing that the spaces $D_{(n_e,n_o)}$ are actually nonempty for $(n_e, n_o) \in N$. In contrast to quadratic Artin-Schelter algebras Chapter 2 and [60] this is not entirely straightforward. We will prove this by characterizing the appearing Hilbert series for objects in $R(A)$. In a very similar way as in [28] for quadratic Artin-Schelter algebras, we show in Section 6.3 that the Hilbert series of graded right A-ideals of projective dimension one are characterised by so-called *Castelnuovo polynomials* [26] $s(t) = \sum_{i=0}^n s_i t^i \in \mathbb{Z}[t]$ which are by definition of the form

$$s_0 = 1, s_1 = 2, \ldots, s_{\sigma-1} = \sigma \text{ and } s_{\sigma-1} \geq s_\sigma \geq s_{\sigma+1} \geq \cdots \geq 0$$

for some integer $\sigma \geq 0$. We refer to $\sum_i s_{2i}$ as the *even weight* of $s(t)$ and $\sum_i s_{2i+1}$ as the *odd weight* of $s(t)$.

Example 6.1.1. $s(t) = 1 + 2t + 3t^2 + 4t^3 + 5t^4 + 5t^5 + 3t^6 + 2t^7 + t^8 + t^9 + t^{10} + t^{11}$ is a Castelnuovo polynomial of even weight 14 and odd weight 15. The corresponding Castelnuovo diagram is (where the even columns are black)



Denote $X = \operatorname{Proj} A = (\mathbb{P}^1 \times \mathbb{P}^1)_q$. Write $\operatorname{Hilb}_{(n_e,n_o)}(X)$ for the groupoid of the torsion free graded right A-modules I of projective dimension one for which

$$h_A(m) - h_I(m) = \dim_k A_m - \dim_k I_m = \begin{cases} n_e & \text{for } m \text{ even} \\ n_o & \text{for } m \text{ odd} \end{cases} \quad \text{for } m \gg 0$$

(in particular I has rank one, see §6.2.3). Any graded right A-ideal I of projective dimension one admits an unique shift of grading $I(d)$ for which $I(d) \in \mathrm{Hilb}_{(n_e,n_o)}(X)$. Writing $R_{(n_e,n_o)}(A)$ for the full subcategory of $\mathrm{Hilb}_{(n_e,n_o)}(X)$ consisting of reflexive objects we have $R(A) = \coprod R_{(n_e,n_o)}(A)$, and in the setting of Theorem 1 the isoclasses of $R_{(n_e,n_o)}(A)$ are in natural bijection with the points of the variety $D_{(n_e,n_o)}$. In Section 6.3 below we prove the following analog of Theorem 4 of Chapter 3.

Theorem 13. *Let A be a cubic Artin-Schelter regular algebra. There is a bijective correspondence between Castelnuovo polynomials $s(t)$ of even weight n_e and odd weight n_o and Hilbert series $h_I(t)$ of objects I in $\mathrm{Hilb}_{(n_e,n_o)}(X)$, given by*

$$h_I(t) = \frac{1}{(1-t)^2(1-t^2)} - \frac{s(t)}{1-t^2}$$

Moreover if A is elliptic for which σ has infinite order this correspondence restricts to Hilbert series $h_I(t)$ of objects I in $R_{(n_e,n_o)}(A)$.

By shifting the rows in a Castelnuovo diagram in such a way they are left aligned one sees that the number of Castelnuovo diagrams of even weight n_e and odd weight n_o is equal to the number of partitions λ of $n_e + n_o$ with distinct parts, with the additional property that by putting a chessboard pattern on the Ferrers diagram of λ the number of black squares is equal to n_e and the number of white squares is equal to n_o. Anthony Henderson pointed out to us this number is given by the number of partitions of $n_e - (n_e - n_o)^2$. In particular the varieties $D_{(n_e,n_o)}$ in Theorem 1 are nonempty whenever $(n_e, n_o) \in N$. See Appendix G below.

Remark 6.1.2. In Appendix C we have included the list of Castelnuovo polynomials $s(t)$ of even weight $n_e \leq 3$ and odd weight $n_o \leq 3$, as well as some associated data.

As there exists no commutative cubic Artin-Schelter algebra A it seems difficult to compare our results with the commutative situation. However if A is a linear cubic Artin-Schelter algebra then $\operatorname{Proj} A$ is equivalent with the category of coherent sheaves on the quadric surface $\mathbb{P}^1 \times \mathbb{P}^1$. In Section 6.4 we discuss how the (classical) Hilbert scheme of points $\mathrm{Hilb}(\mathbb{P}^1 \times \mathbb{P}^1)$ parameterizes the objects in $\coprod_{(n_e,n_o)\in N} \mathrm{Hilb}_{(n_e,n_o)}(X)$ with the groupoid $\mathrm{Hilb}_{(n_e,n_o)}(X)$ as defined above.

Remark 6.1.3. For cubic Artin-Schelter algebras A we expect a similar treatment as in [60] to show $\mathrm{Hilb}_{(n_e,n_o)}(X)$ is a smooth projective variety of dimension $2(n_e - (n_e - n_o)^2)$. The authors are convinced that using the same methods as in the proof of Theorem 5 in Chapter 3 will lead to a proof that $\mathrm{Hilb}_{(n_e,n_o)}(X)$ is connected, hence also $D_{(n_e,n_o)}$ (for elliptic A for which σ has infinite order). We hope to come back on this in a forthcoming paper.

As an application, consider the enveloping algebra of the Heisenberg-Lie algebra

$$H_c = k\langle x, y, z\rangle/(yz - zy, xz - zx, xy - yx - z) = k\langle x, y\rangle/([y, [y, x]], [x, [x, y]])$$

where $[a, b] = ab - ba$. The graded algebra H_c is a cubic Artin-Schelter regular algebra. Consider the localisation $\Lambda = H_c[z^{-1}]$ of H_c at the powers of the central element

$z = xy - yx$ and its subalgebra Λ_0 of elements of degree zero. It was shown in [8] that $\Lambda_0 = A_1^{\langle\varphi\rangle}$, the algebra of invariants of the first Weyl algebra $A_1 = k\langle x, y\rangle/(xy-yx-1)$ under the automorphism φ defined by $\varphi(x) = -x$, $\varphi(y) = -y$. In Section 6.9 we deduce a classification of right ideals of $A_1^{\langle\varphi\rangle}$.

Theorem 14. *The set $R(A_1^{\langle\varphi\rangle})$ of isomorphism classes of right $A_1^{\langle\varphi\rangle}$-ideals is in natural bijection with the points of $\coprod_{(n_e,n_o)\in N} D_{(n_e,n_o)}$ where*

$$D_{(n_e,n_o)} = \{(\mathbb{X}, \mathbb{Y}, \mathbb{X}', \mathbb{Y}') \in M_{n_e\times n_o}(k)^2 \times M_{n_o\times n_e}(k)^2 \mid \mathbb{Y}'\mathbb{X} - \mathbb{X}'\mathbb{Y} = \mathbb{I} \text{ and } \operatorname{rank}(\mathbb{Y}\mathbb{X}' - \mathbb{X}\mathbb{Y}' - \mathbb{I}) \leq 1\}/\operatorname{Gl}_{n_e}(k) \times \operatorname{Gl}_{n_o}(k)$$

is a smooth affine variety $D_{(n_e,n_o)}$ of dimension $2(n_e - (n_e - n_o)^2)$.

Note $\operatorname{Gl}_{n_e}(k)\times\operatorname{Gl}_{n_o}(k)$ acts by conjugation $(g\mathbb{X}h^{-1}, g\mathbb{Y}h^{-1}, h\mathbb{X}'g^{-1}, h\mathbb{Y}'g^{-1})$. Comparing with the first Weyl algebra (see the introduction of this manuscript, or Theorem 2.1.1) it would be interesting to see if the orbits of $R(A_1^{\langle\varphi\rangle})$ under the automorphism group $\operatorname{Aut}(A_1^{\langle\varphi\rangle})$ are in bijection to $D_{(n_e,n_o)}$.

Remark 6.1.4. In case A is of generic type A or $A = H_c$ is the enveloping algebra of the Heisenberg-Lie algebra, Theorems 10 and 12 are proved without the hypothesis k is uncountable.

Most results in this chapter are obtained *mutatis mutandis* as for quadratic Artin-Schelter algebras in Chapters 2, 3 and to some extend [16, 52, 60]. However at some points the situation for cubic algebras is more complicated.

6.2 From reflexive ideals to normalized line bundles

Throughout A will be a cubic Artin-Schelter algebra as defined in §1.9. We will use the notations from Chapter 1, so we write $X = (\mathbb{P}^1 \times \mathbb{P}^1)_q = \operatorname{Proj} A$, $\operatorname{Qcoh}(X) = \operatorname{Tails}(A)$, $\operatorname{coh}(X) = \operatorname{tails}(A)$, $\pi A = \mathcal{O}$. We shall refer to X as a quantum quadric.

In this section our first aim is to relate reflexive A-modules with certain objects on X (so-called vector bundles). Any shift of such a reflexive module remains reflexive and in the rank one case we will normalize this shift. The corresponding objects in $\operatorname{coh}(X)$ will be called normalized line bundles. A helpful tool will be the choice of a suitable basis of the Grothendieck group $K_0(X)$. At the end of this section we will compute partially the cohomology of these normalized line bundles.

6.2.1 Torsion free and reflexive objects

An object $M \in \operatorname{grmod}(A)$ is *torsion free* if M is pure of maximal GK-dimension three. Recall M is called reflexive if $M^{**} = M$. Similarly an object $\mathcal{M} \in \operatorname{coh}(X)$ is *torsion free* if $\mathcal{M}$ is pure of maximal dimension two. An object $\mathcal{M} \in \operatorname{coh}(X)$ is called *reflexive* (or a *vector bundle* on X) if $\mathcal{M} = \pi M$ for some reflexive $M \in \operatorname{grmod}(A)$. We refer to a vector bundle of rank one as a *line bundle*.

The results of §2.2.1 together with their proofs remain valid. In particular we will need the following lemmas, see Lemma 2.2.4 and Corollary 2.2.6.

Lemma 6.2.1. *Let $\mathcal{M} \in \operatorname{coh}(X)$. Then $\mathcal{M}$ is a vector bundle on X if and only if $\mathcal{M}$ is torsion free and $\operatorname{Ext}^1_X(\mathcal{N}, \mathcal{M}) = 0$ for all $\mathcal{N} \in \operatorname{coh}(X)$ of dimension zero.*

Lemma 6.2.2. *The functors π and ω define inverse equivalences between the full subcategories of $\operatorname{grmod}(A)$ and $\operatorname{coh}(X)$ with objects*

$$\{\textit{torsion free objects in } \operatorname{grmod}(A) \textit{ of projective dimension one}\}$$

and

$$\{\textit{torsion free objects in } \operatorname{coh}(X)\}$$

Moreover this equivalence restricts to an equivalence between the full subcategories of $\operatorname{grmod}(A)$ and $\operatorname{coh}(X)$ with objects

$$\{\textit{reflexive objects in } \operatorname{grmod}(A)\} \quad \textit{and} \quad \{\textit{vector bundles on } X\}.$$

In this chapter we are interested in torsion free rank one modules of projective dimension one, or more restrictively, reflexive modules of rank one. Every graded right ideal of A is a torsion free rank one A-module. The following proposition shows that, up to shift of grading, the converse is also true.

Proposition 6.2.3. *Let $0 \neq I \in \operatorname{grmod}(A)$ be torsion free of rank one. Then there is an integer n such that $I(-n)$ is isomorphic to a graded right ideal of A.*

Proof. By $\operatorname{GKdim} I = 3$, Theorem 1.9.8 implies $I^* = \underline{\operatorname{Hom}}_A(I, A) \neq 0$. Thus $(I^*)_n = \operatorname{Hom}_A(I(-n), A) \neq 0$ for some integer n. By Lemma 1.9.7 we are done. □

Remark 6.2.4. The set of all graded right ideals is probably too large to describe, as for any ideal I we may construct numerous other closely related ideals by taking the kernel of any surjective map to a module of GK-dimension zero. We will restrict to graded ideals of projective dimension one (or more restrictively reflexive rank one modules). For such modules M we have $\underline{\operatorname{Ext}}^1_A(k, M) = 0$ and therefore M cannot appear as the kernel of such a surjective map.

6.2.2 The Grothendieck group and the Euler form for quantum quadrics

In this part we describe a natural $\mathbb{Z}$-module basis for the Grothendieck group $K_0(X)$ and determine the matrix representation of the Euler form χ with respect to this basis. To do so, it is convenient to start with a different basis of $K_0(X)$, corresponding to the standard basis of $\mathbb{Z}[t, t^{-1}]/(q_k(t))$ under the isomorphism of Theorem 1.7.1, and perform a base change afterwards.

Proposition 6.2.5. *The set $\mathcal{B} = \{[\mathcal{O}], [\mathcal{O}(-1)], [\mathcal{O}(-2)], [\mathcal{O}(-3)]\}$ is a $\mathbb{Z}$-module basis of $K_0(X)$. The matrix representations with respect to the basis $\mathcal{B}$ of the shift automorphism* sh *and the Euler form χ for $K_0(X)$ are given by*

$$m(\mathrm{sh})_{\mathcal{B}} = \begin{pmatrix} 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -2 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \quad m(\chi)_{\mathcal{B}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 4 & 2 & 1 & 0 \\ 6 & 4 & 2 & 1 \end{pmatrix}.$$

Proof. Let θ denote the isomorphism (1.5) of Theorem 1.7.1. Since $q_{A(-l)}(t) = t^l$ we have $\theta[\mathcal{O}(-l)] = \overline{t^l}$ for all integers l. As $\{\overline{1}, \overline{t}, \overline{t^2}, \overline{t^3}\}$ is a $\mathbb{Z}$-module basis for $\mathbb{Z}[t, t^{-1}]/(q_k(t)) = \mathbb{Z}[t, t^{-1}]/(1-t)^2(1-t^2)$ we deduce $\mathcal{B}$ is a basis for $K_0(X)$.

By $\mathrm{sh}[\mathcal{O}(l)] = [\mathcal{O}(l+1)]$ we find the last three columns of $m(\mathrm{sh})_{\mathcal{B}}$. Applying the exact functor π to a minimal resolution (1.6) of k_A yields the exact sequence

$$0 \to \mathcal{O}(-4) \to \mathcal{O}(-3)^2 \to \mathcal{O}(-1)^2 \to \mathcal{O} \to 0$$

from which we deduce $[\mathcal{O}(1)] = 2[\mathcal{O}] - 2[\mathcal{O}(-2)] + [\mathcal{O}(-3)]$, giving the first column of $m(\mathrm{sh})_{\mathcal{B}}$. Finally, Theorem 1.8.2 implies for all integers l

$$\chi(\mathcal{O}, \mathcal{O}(l)) = \dim_k A_l + \dim_k A_{-l-4} = \begin{cases} (l+2)^2/4 & \text{if } l \text{ is even} \\ (l+1)(l+3)/4 & \text{if } l \text{ is odd} \end{cases}$$

which allows one to compute the matrix $m(\chi)_{\mathcal{B}}$. This ends the proof. □

Proposition 6.2.6. *Let P be a point module, S a line module and Q a conic module over A. Denote the corresponding objects in $\mathrm{coh}(X)$ by $\mathcal{P}$, $\mathcal{S}$ and $\mathcal{Q}$. Then $\mathcal{B}' = \{[\mathcal{O}], [\mathcal{S}], [\mathcal{Q}], [\mathcal{P}]\}$ is a $\mathbb{Z}$-module basis of $K_0(X)$, which does not depend on the particular choice of S, Q and P. The matrix representations with respect to the basis $\mathcal{B}'$ of the shift automorphism* sh *and the Euler form χ for $K_0(X)$ are given by*

$$m(\mathrm{sh})_{\mathcal{B}'} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix}, \quad m(\chi)_{\mathcal{B}'} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ -1 & 0 & -1 & 0 \\ -3 & -1 & -2 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}. \tag{6.4}$$

Proof. 1. It follows from Theorem 1.7.1 that the class in $K_0(X)$ of an object πM only depends on the Hilbert series of M. Thus $[\mathcal{S}]$, $[\mathcal{Q}]$ and $[\mathcal{P}]$ are indeed independent of the particular choice of S, Q and P. Using a computation with Hilbert series we see that the images of $[\mathcal{O}], [\mathcal{S}], [\mathcal{Q}]$ and $[\mathcal{P}]$ under the isomorphism θ of Theorem 1.7.1 are respectively $\overline{1}$, $\overline{1-t}$, $\overline{1-t^2}$ and $\overline{(1-t)(1-t^2)}$, which form a $\mathbb{Z}$-module basis for $\mathbb{Z}[t, t^{-1}]/(1-t)^2(1-t^2)$. Hence $\mathcal{B}'$ is a basis for $K_0(X)$.

2. Using the isomorphism θ it follows from the previous part that $[\mathcal{S}] = [\mathcal{O}] - [\mathcal{O}(-1)]$, $[\mathcal{Q}] = [\mathcal{O}] - [\mathcal{O}(-2)]$ and $[\mathcal{P}] = [\mathcal{O}] - [\mathcal{O}(-1)] - [\mathcal{O}(-2)] + [\mathcal{O}(-3)]$. Hence the matrix of base change on $K_0(X)$ from $\mathcal{B}'$ to $\mathcal{B}$ is given by

$$m(\mathrm{id})_{\mathcal{B}'\mathcal{B}} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & -1 & 0 & -1 \\ 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

From the appropriate commutative diagram of $\mathbb{Z}$-modules we deduce that $m(\mathrm{sh})_{\mathcal{B}'}$ is equal to

$$\begin{aligned} & m(\mathrm{id})^{-1}_{\mathcal{B}'\mathcal{B}} \cdot m(\mathrm{sh})_{\mathcal{B}} \cdot m(\mathrm{id})_{\mathcal{B}'\mathcal{B}} \\ &= \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & -1 & 0 & -1 \\ 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -2 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & -1 & 0 & -1 \\ 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix} \end{aligned}$$

3. Again by standard linear algebra we find $m(\chi)_{\mathcal{B}'} = m(\mathrm{id})^t_{\mathcal{B}'\mathcal{B}} \cdot m(\chi)_{\mathcal{B}} \cdot m(\mathrm{id})_{\mathcal{B}'\mathcal{B}}$ which finishes the proof. □

From now on we fix such a $\mathbb{Z}$-module basis $\{[\mathcal{O}], [\mathcal{S}], [\mathcal{Q}], [\mathcal{P}]\}$ of $K_0(X)$. For any object $\mathcal{M} \in \mathrm{coh}(X)$ we may write

$$[\mathcal{M}] = r[\mathcal{O}] + a[\mathcal{S}] + b[\mathcal{Q}] + c[\mathcal{P}] \tag{6.5}$$

Writing $\mathcal{M} = \pi M$ where $M \in \mathrm{grmod}(A)$, equation (6.5) also follows directly from Lemma 1.9.11 i.e. we have

$$h_M(t) = \frac{r}{(1-t)^2(1-t^2)} + \frac{a}{(1-t)(1-t^2)} + \frac{b}{(1-t)^2} + \frac{c}{1-t} + f(t). \tag{6.6}$$

for some Laurent polynomial $f(t) \in \mathbb{Z}[t, t^{-1}]$. Note $r = \operatorname{rank} M = \operatorname{rank} \mathcal{M}$. By computing the powers of the matrix $m(\mathrm{sh})_{\mathcal{B}'}$ in Proposition 6.2.6 we deduce for any integer l

$$\begin{aligned} [\mathcal{M}(2l)] &= r[\mathcal{O}] + a[\mathcal{S}] + (lr + b)[\mathcal{Q}] + (l((l+1)r + a + 2b) + c)[\mathcal{P}] \\ [\mathcal{M}(2l-1)] &= r[\mathcal{O}] - (r+a)[\mathcal{S}] + (lr + a + b)[\mathcal{Q}] + (l(lr + a + 2b) - b + c)[\mathcal{P}] \end{aligned} \tag{6.7}$$

6.2.3 Normalized rank one objects

Any shift l of a torsion free rank one graded right A-module I gives rise to a torsion free rank one object $\mathcal{I}(l) = \pi I(l)$ on X. We will now normalize this shift. Our choice is motivated by the analogue of Lemma 2.2.8.

Proposition 6.2.7. *Let $I \in \mathrm{grmod}(A)$, set $\mathcal{I} = \pi I$ and write $[\mathcal{I}] = r[\mathcal{O}] + a[\mathcal{S}] + b[\mathcal{Q}] + c[\mathcal{P}]$. Then the following are equivalent.*

1. *There exist integers n_e, n_o such that for $l \gg 0$ we have*

$$\dim_k A_l - \dim_k I_l = \begin{cases} n_e & \textit{if } l \textit{ is even,} \\ n_o & \textit{if } l \textit{ is odd.} \end{cases}$$

2. *The Hilbert series of I is of the form*

$$h_I(t) = h_A(t) - \frac{s(t)}{1-t^2}$$

for a Laurent polynomial $s(t) \in \mathbb{Z}[t, t^{-1}]$.

3. *I has rank one and $a = -2b$.*

If these conditions hold then $s(1) = b - 2c$, $s(-1) = b$ and $n_e = b - c$, $n_o = -c$.

Proof. By (6.6) we may write

$$h_I(t) = \frac{r}{(1-t)^2(1-t^2)} + \frac{a + b(1+t)}{(1-t)(1-t^2)} + \frac{c(1+t) + f(t)(1-t^2)}{1-t^2}$$

for some $f(t) \in \mathbb{Z}[t, t^{-1}]$. Thus the second and the third statement are equivalent, and in that case $s(t) = b - c(1+t) - f(t)(1-t^2)$. Moreover, for $l \gg 0$ we obtain

$$\dim_k A_l - \dim_k I_l = \begin{cases} (1-r)(l+2)^2/4 - a(l/2+1) - b(l+1) - c & \text{for } l \text{ even} \\ (1-r)(l+1)(l+3)/4 - a(l+1)/2 - b(l+1) - c & \text{for } l \text{ odd} \end{cases}$$

from which we deduce the equivalence of (1) and (3), proving what we want. □

We will call a torsion free rank one object in $\mathrm{grmod}(A)$ *normalized* if it satisfies the equivalent conditions of Proposition 6.2.7. Similarly, a torsion free rank one object $\mathcal{I}$ in $\mathrm{coh}(X)$ is *normalized* if $[\mathcal{I}]$ is of the form

$$[\mathcal{I}] = [\mathcal{O}] - 2b[\mathcal{S}] + b[\mathcal{Q}] + c[\mathcal{P}]$$

for some integers $b, c \in \mathbb{Z}$. We refer to $(n_e, n_o) = (b - c, -c)$ as the *invariants* of I and $\mathcal{I}$ and call n_e the *even invariant* and n_o the *odd invariant* of I and $\mathcal{I}$. We will prove in Theorem 6.2.11 below that n_e and n_o are actually positive and characterize the appearing invariants (n_e, n_o) in Section 6.3.

Lemma 6.2.8. *Let $I \in \operatorname{grmod}(A)$ be torsion free of rank one and set $\mathcal{I} = \pi I$. Then there is a unique integer d for which $I(d)$ (and hence $\mathcal{I}(d)$) is normalized.*

Proof. Easy by (6.7). ∎

By Lemma 6.2.2 the functors π and ω define inverse equivalences between the full subcategories of $\operatorname{grmod}(A)$ and $\operatorname{coh}(X)$ with objects

$$\begin{aligned}\operatorname{Hilb}_{(n_e,n_o)}(X) := \{&\text{normalized torsion free rank one objects in } \operatorname{grmod}(A)\\ &\text{of projective dimension one and invariants } (n_e,n_o)\}\end{aligned}$$

and

$$\{\text{normalized torsion free rank one objects in } \operatorname{coh}(X) \text{ with invariants } (n_e,n_o)\}.$$

Remark 6.2.9. We expect $\coprod_{(n_e,n_o)} \operatorname{Hilb}_{(n_e,n_o)}(X)$ to be the correct generalization of the usual Hilbert scheme of points on $\mathbb{P}^1 \times \mathbb{P}^1$. In case A is linear then $\coprod_{(n_e,n_o)} \operatorname{Hilb}_{(n_e,n_o)}(X)$ coincides with the Hilbert scheme of points on $\mathbb{P}^1 \times \mathbb{P}^1$, see §6.4.2 below.

This equivalence restricts to an equivalence between the full subcategories of $\operatorname{grmod}(A)$ and $\operatorname{coh}(X)$ with objects

$$\begin{aligned}R_{(n_e,n_o)}(A) := \{&\text{normalized reflexive rank one objects in } \operatorname{grmod}(A)\\ &\text{with invariants } (n_e,n_o)\}\end{aligned}$$

and

$$\mathcal{R}_{(n_e,n_o)}(X) := \{\text{normalized line bundles on } X \text{ with invariants } (n_e,n_o)\}.$$

We obtain a natural bijection between the set $R(A)$ of reflexive rank one graded right A-modules considered up to isomorphism and shift, and the isomorphism classes in the categories $\coprod_{(n_e,n_o)} R_{(n_e,n_o)}(A)$ and $\coprod_{(n_e,n_o)} \mathcal{R}_{(n_e,n_o)}(X)$.

Remark 6.2.10. It is easy to see that the categories $R_{(n_e,n_o)}(A)$ and $\mathcal{R}_{(n_e,n_o)}(X)$ are groupoids, i.e. all non-zero morphisms are isomorphisms.

6.2.4 Cohomology of line bundles on quantum quadrics

The next theorem describes partially the cohomology of normalized line bundles.

Theorem 6.2.11. *Let $\mathcal{I} \in \operatorname{coh}(X)$ be torsion free of rank one and normalized i.e.*

$$[\mathcal{I}] = [\mathcal{O}] - 2(n_e - n_o)[\mathcal{S}] + (n_e - n_o)[\mathcal{Q}] - n_o[\mathcal{P}]$$

for some integers n_e, n_o. Assume $\mathcal{I}$ is not isomorphic to $\mathcal{O}$. Then

1. $H^0(X, \mathcal{I}(l)) = 0$ *for* $l \leq 0$
   $H^2(X, \mathcal{I}(l)) = 0$ *for* $l \geq -3$
   $H^j(X, \mathcal{I}(l)) = 0$ *for* $j \geq 3$ *and for all integers* l

2. $\chi(\mathcal{O}, \mathcal{I}(l)) = \begin{cases} (l+2)^2/4 - n_e & \text{if } l \in \mathbb{Z} \text{ is even} \\ (l+1)(l+3)/4 - n_o & \text{if } l \in \mathbb{Z} \text{ is odd} \end{cases}$

3. $\dim_k H^1(X, \mathcal{I}) = n_e - 1$
   $\dim_k H^1(X, \mathcal{I}(-1)) = n_o$
   $\dim_k H^1(X, \mathcal{I}(-2)) = n_e$
   $\dim_k H^1(X, \mathcal{I}(-3)) = n_o$

As a consequence, $n_e > 0$ *and* $n_o \geq 0$*. If* $\mathcal{I}$ *is a line bundle i.e.* $\mathcal{I} \in \mathcal{R}_{(n_e,n_o)}(X)$ *then we have in addition*

$$H^2(X, \mathcal{I}(-4)) = 0 \text{ and } \dim_k H^1(X, \mathcal{I}(-4)) = n_e - 1.$$

Proof. That $H^j(X, \mathcal{I}(l)) = 0$ for $j \geq 3$ and for all integers l follows from $\operatorname{cd} X = 2$, see Theorem 1.8.2. The rest of the first statement is proved in a similar way as Theorem 2.2.11(1). See also the proof of the final statement below.

For the second part, compute $\chi(\mathcal{O}, \mathcal{I}(l))$ using (6.7) and the matrix representation $m(\chi)_{\mathcal{B}'}$ from Proposition 6.2.6.

Combining the first two statements together with (2.18) yields the third part.

Finally, assume $\mathcal{I}$ is reflexive. By Theorem 1.10.5 (Serre duality) we have $H^2(X, \mathcal{I}(-4)) = \operatorname{Ext}^2_X(\mathcal{O}, \mathcal{I}(-4)) \cong \operatorname{Hom}_X(\mathcal{I}, \mathcal{O})'$. Assume by contradiction there is a non-zero morphism $f : \mathcal{I} \to \mathcal{O}$. As $\mathcal{I}$ is critical, f is injective and we compute $[\operatorname{coker} f] = 2(n_e - n_o)[\mathcal{S}] - (n_e - n_o)[\mathcal{Q}] + n_o[\mathcal{P}]$. By (6.5) and Lemma 1.9.11 we deduce $e_1(\operatorname{coker} f) = 0$ hence $\dim \operatorname{coker} f = 0$. Note $\operatorname{coker} f \neq 0$ by the assumption $\mathcal{I} \ncong \mathcal{O}$. Since $\mathcal{I}$ is reflexive, $\operatorname{Ext}^1_X(\operatorname{coker} f, \mathcal{I}) = 0$ thus the exact sequence $0 \to \mathcal{I} \to \mathcal{O} \to \operatorname{coker} f \to 0$ splits, contradicting the fact that $\mathcal{O}$ is torsion free. □

Corollary 6.2.12. *Let* $I \in \operatorname{grmod}(A)$ *be torsion free of rank one with invariants* (n_e, n_o)*. Then* $(n_e, n_o) = (0, 0)$ *if and only if* $\mathcal{I} \cong \mathcal{O}(d)$ *for some integer* d*.*

Proof. If $\mathcal{I} \cong \mathcal{O}(d)$ then $[\mathcal{I}(-d)] = [\mathcal{O}]$ hence $n_e = n_o = 0$. Assume conversely $(n_e, n_o) = (0, 0)$. We may assume $\mathcal{I}$ is normalized. If $\mathcal{I} \ncong \mathcal{O}$ then Theorem 6.2.11 implies $n_e > 0$. Since $n_e = 0$ we obtain $\mathcal{I} \cong \mathcal{O}$ by contraposition. □

At this point one may be tempted to think there are two independent parameters $n_e, n_o \in \mathbb{N}$ associated to an object in $\operatorname{Hilb}_{(n_e,n_o)}(X)$. However

Lemma 6.2.13. *Let* $I \in \operatorname{grmod}(A)$ *be torsion free of rank one with invariants* (n_e, n_o) *and write* $\mathcal{I} = \pi I$*. Then* $\dim_k \operatorname{Ext}^1_X(\mathcal{I}, \mathcal{I}) = 2(n_e - (n_e - n_o)^2) \geq 0$*.*

Proof. We may clearly assume $\mathcal{I}$ is normalized and by Proposition 6.2.6 we easily find $\chi(\mathcal{I}, \mathcal{I}) = 1 - 2(n_e - (n_e - n_o)^2)$. As $\mathcal{I}$ is critical we have $\operatorname{Hom}_X(\mathcal{I}, \mathcal{I}) = k$. Hence it will be sufficient to prove $\operatorname{Ext}^2_X(\mathcal{I}, \mathcal{I}) = 0$. Serre duality implies $\operatorname{Ext}^2_X(\mathcal{I}, \mathcal{I}) \cong$

$\mathrm{Hom}_X(\mathcal{I},\mathcal{I}(-4))'$. Thus assume by contradiction there is a non-zero morphism $f : \mathcal{I} \to \mathcal{I}(-4)$. Then f is injective and using (6.7) we have $[\mathcal{I}(-4)] = [\mathcal{O}] - 2(n_e - n_o)[\mathcal{S}] + (n_e - n_o - 2)[\mathcal{Q}] + (2 - n_o)[\mathcal{P}]$ hence $[\operatorname{coker} f] = -2[\mathcal{Q}] + 2[\mathcal{P}]$. By (6.5) and Lemma 1.9.11 we deduce $e_2(\operatorname{coker} f) < 0$ which is absurd. □

As a consequence if $\mathrm{Hilb}_{(n_e,n_o)}(X) \neq \emptyset$ for some integers n_e, n_o then $n_e \geq 0$, $n_o \geq 0$ and $n_e - (n_e - n_o)^2 \geq 0$. The converse will be proved in the next section.

6.3 Hilbert series of ideals and proof of Theorem 13

Let A be a quadratic or cubic Artin-Schelter algebra and let M be a torsion free graded right A-module of projective dimension one (so we do not require M to have rank one). Thus M has a minimal resolution of the form

$$0 \to \oplus_i A(-i)^{b_i} \to \oplus_i A(-i)^{a_i} \to M \to 0$$

where $(a_i), (b_i)$ are finitely supported sequences of non-negative integers. These numbers are called the *Betti numbers* of M. It is easy to see that the characteristic polynomial of M is given by $q_M(t) = \sum_i (a_i - b_i)t^i$. So by (1.4) the Betti numbers determine the Hilbert series of M, but the converse is not true.

For quadratic A the appearing Betti numbers were characterised in Chapter 3. The same technique may be used to obtain the same characterisation for cubic A. The result is

Proposition 6.3.1. *Let $(a_i), (b_i)$ be finitely supported sequences of non-negative integers. Let a_σ be the lowest non-zero a_i and put $r = \sum_i (a_i - b_i)$. Then the following are equivalent.*

1. *$(a_i), (b_i)$ are the Betti numbers of a torsion free graded right module of projective dimension one and rank r over a quadratic Artin-Schelter algebra,*
2. *$(a_i), (b_i)$ are the Betti numbers of a torsion free graded right module of projective dimension one and rank r over a cubic Artin-Schelter algebra,*
3. *$b_i = 0$ for $i \leq \sigma$ and $\sum_{i \leq l} b_i < \sum_{i<l} a_i$ for $l > \sigma$.*

Moreover if A is elliptic and σ has infinite order, these modules can be chosen to be reflexive.

Assume for the rest of Section 6.3 A is a cubic Artin-Schelter algebra. The previous proposition allows us to describe the Hilbert series of objects in $\mathrm{Hilb}_{(n_e,n_o)}(X)$. Recall from the introduction a Castelnuovo polynomial [26] $s(t) = \sum_{i=0}^n s_i t^i \in \mathbb{Z}[t]$ is by definition of the form

$$s_0 = 1, s_1 = 2, \ldots, s_{\sigma-1} = \sigma \text{ and } s_{\sigma-1} \geq s_\sigma \geq s_{\sigma+1} \geq \cdots \geq 0 \tag{6.8}$$

for some integer $\sigma \geq 0$. We refer to $\sum_i s_{2i}$ as the *even weight* of s and $\sum_i s_{2i+1}$ as the *odd weight* of $s(t)$. We may now prove Theorem 13.

Proof of Theorem 13. First, let us assume $I \in \mathrm{Hilb}_{(n_e,n_o)}(X)$ for some integers n_e, n_o. By Proposition 6.2.7 we may assume that the Hilbert series of I has the form

$$h_I(t) = \frac{1}{(1-t)^2(1-t^2)} - \frac{s(t)}{1-t^2}$$

for a Laurent polynomial $s(t) \in \mathbb{Z}[t, t^{-1}]$. We deduce $q_I(t)/(1-t) = h_I(t)(1-t)(1-t^2) = 1/(1-t) - s(t)(1-t)$. Writing $q_I(t) = \sum_i q_i t^i$ it is easy to see Proposition 6.3.1(3) is equivalent with

$$\sum_{i \le l} q_i \begin{cases} = 0 & \text{for } l < \sigma \\ > 0 & \text{for } l \ge \sigma \end{cases}$$

from which we deduce $s(t)(1-t)$ is of the form

$$s(t)(1-t) = 1 + t + t^2 + \cdots + t^{\sigma-1} + d_\sigma t^\sigma + d_{\sigma+1} t^{\sigma+1} + \ldots$$

where $d_i \le 0$ for $i \ge \sigma$. Multiplying by $1/(1-t) = 1 + t + t^2 + \ldots$ shows this is equivalent to $s(t)$ being a Castelnuovo polynomial. According to Proposition 6.2.7, $(s(1) + s(-1))/2 = n_e$ and $(s(1) - s(-1))/2 = n_o$ thus $s(t)$ has even weight n_e and odd weight n_o.

The converse statement is easily checked. □

As an application we may now prove nonemptyness for $R_{(n_e,n_o)}(A)$. As in the introduction we define

$$N = \{(n_e, n_o) \in \mathbb{N}^2 \mid n_e - (n_e - n_o)^2 \ge 0\}. \tag{6.9}$$

As done in Appendix G it is a simple exercise to check

$$N = \{(k^2 + l, k(k+1) + l) \mid k, l \in \mathbb{N}\} \cup \{((k+1)^2 + l, k(k+1) + l) \mid k, l \in \mathbb{N}\}. \tag{6.10}$$

Proposition 6.3.2. *Let n_e,n_o be any integers. Then $\mathrm{Hilb}_{(n_e,n_o)}(X)$ is nonempty if and only if $(n_e, n_o) \in N$.*
If A is elliptic and σ has infinite order then $R_{(n_e,n_o)}(A)$ whence $\mathcal{R}_{(n_e,n_o)}(X)$ is nonempty if and only if $(n_e, n_o) \in N$.

Proof. Assume $(n_e, n_o) \in N$. Due to Theorem 13 it will be sufficient to show there exists a Castelnuovo polynomial $s(t)$ for which the even resp. odd weight of $s(t)$ is equal to n_e resp. n_o. Shifting the rows in any Castelnuovo diagram in such a way they are left aligned induces a bijective correspondence between Castelnuovo functions s and partitions λ of $n = s(1)$ with distinct parts. For any partition λ we put a chess colouring on the Ferrers graph of λ, and write $b(\lambda)$ resp. $w(\lambda)$ for the number of black resp. white unit squares. By Theorem G.1 in Appendix G below there exists a partition λ in distinct parts for which $b(\lambda) = n_e$ and $w(\lambda) = n_o$ if and only if $(n_e, n_o) \in N$. □

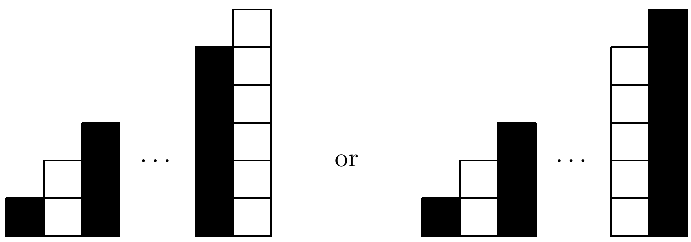
For $(n_e, n_o) \in N$ there is an unique integer $l \geq 0$ with the property (see (6.10))

$$(n_e - l, n_o - l) \in N \text{ and } (n_e - l - 1, n_o - l - 1) \notin N. \tag{6.11}$$

One verifies $(n_e - l', n_o - l') \notin N$ for all $l' > l$. By (6.10) we distinguish

Case 1. $(n_e - l, n_o - l) = (k^2, k(k+1))$ for $k \in \mathbb{N}$. The Castelnuovo polynomial of an object in $\mathrm{Hilb}_{(n_e-l,n_o-l)}(X)$ is $s(t) = 1 + 2t + 3t^2 + \cdots + (v-1)t^v + vt^{v+1}$ where v is even. Thus the Castelnuovo diagram is triangular and ends with a white column.

Case 2. $(n_e - l, n_o - l) = ((k+1)^2, k(k+1))$ for $k \in \mathbb{N}$. Then the Castelnuovo polynomial of an object in $\mathrm{Hilb}_{(n_e-l,n_o-l)}(X)$ is $s(t) = 1+2t+3t^2+\cdots+(v-1)t^v+vt^{v+1}$ where v is odd. The Castelnuovo diagram is triangular and ends with a black column.



case 1 case 2

The next proposition shows that not only the Hilbert series but also the Betti numbers of an object in $\mathrm{Hilb}_{(n_e-l,n_o-l)}(X)$ are fully determined.

Proposition 6.3.3. *Let* $(n_e, n_o) \in N$ *and let* $l \geq 0$ *be as in* (6.11). *Let* $I_0 \in \mathrm{Hilb}_{(n_e-l,n_o-l)}(X)$. *Then* I_0 *has a minimal resolution of the form*

$$0 \to A(-c-1)^c \to A(-c)^{c+1} \to I_0 \to 0$$

where

$$c = \begin{cases} 2k & \text{if } (n_e - l, n_o - l) = (k^2, k(k+1)) \\ 2k+1 & \text{if } (n_e - l, n_o - l) = ((k+1)^2, k(k+1)) \end{cases}$$

Proof. By Proposition 6.3.1 and same arguments as in the proof of Theorem 13. □

Remark 6.3.4. In the notations of the previous proposition one may compute $\dim_k \mathrm{Ext}^1_A(I_0, I_0) = 0$ which indicates that up to isomorphism $\mathrm{Hilb}_{(n_e-l,n_o-l)}(X) = R_{(n_e-l,n_o-l)}(A)$ consist of only one object. See also §6.4.1 below for linear A and the proof of Theorem 4 in Section 6.8 for generic elliptic A.

6.4 Ideals of linear cubic Artin-Schelter algebras

In this section we let A be a linear cubic Artin-Schelter algebra. As Tails(A) is equivalent to $\mathrm{Qcoh}(\mathbb{P}^1 \times \mathbb{P}^1)$ line bundles on $X = \operatorname{Proj} A$ are determined by line bundles on $\mathbb{P}^1 \times \mathbb{P}^1$. We will briefly recall the description of these objects which will lead to a characterisation of the set $R(A)$ of reflexive rank one modules over A, see Proposition 6.4.1. We will end with a discussion on the Hilbert scheme of points.

Let $Y = \mathbb{P}^1 \times \mathbb{P}^1$ denote the quadric surface. Consider for any integers m, n the canonical line bundle $\mathcal{O}_Y(m,n) = \mathcal{O}_{\mathbb{P}^1}(m) \boxtimes \mathcal{O}_{\mathbb{P}^1}(n)$. It is well-known that the map $\mathrm{Pic}(Y) \to \mathbb{Z} \oplus \mathbb{Z} : \mathcal{O}_Y(m,n) \mapsto (m,n)$ is a group isomorphism i.e. the objects $\mathcal{O}_Y(m,n)$ are the only reflexive rank one sheaves on $\mathbb{P}^1 \times \mathbb{P}^1$. Note there are short exact sequences on $\mathrm{coh}(Y)$

$$\begin{aligned} 0 \to \mathcal{O}_Y(m, n-1) \to \mathcal{O}_Y(m,n)^2 \to \mathcal{O}_Y(m, n+1) \to 0 \\ 0 \to \mathcal{O}_Y(m-1, n) \to \mathcal{O}_Y(m,n)^2 \to \mathcal{O}_Y(m+1, n) \to 0 \end{aligned} \tag{6.12}$$

for all integers m, n.

6.4.1 Line bundles

As usual we put $X = \operatorname{Proj} A$ and $\mathcal{O}_X = \mathcal{O}$. In [79] it is shown there is an equivalence of categories $\mathrm{Qcoh}(Y) \cong \mathrm{Qcoh}(X)$ such that $\mathcal{O}_Y(k,k)$ corresponds to $\mathcal{O}_X(2k)$ and $\mathcal{O}_Y(k,k+1)$ corresponds to $\mathcal{O}_X(2k+1)$. See also [69, §11.3]. Further, for any integers m, n we denote the image of $\mathcal{O}_Y(m,n)$ under the equivalence $\mathrm{Qcoh}(Y) \cong \mathrm{Qcoh}(X)$ as $\mathcal{O}(m,n)$. Clearly these objects $\mathcal{O}(m,n) \in \mathrm{coh}(X)$ are the only line bundles on X.

From (6.12) we compute the class of $\mathcal{O}(m,n)$ in $K_0(X)$

$$[\mathcal{O}(m,n)] = [\mathcal{O}] + (m-n)[\mathcal{S}] + n[\mathcal{Q}] + n(m+1)[\mathcal{P}]$$

for all $m, n \in \mathbb{Z}$. Using (6.7) we obtain

$$\mathcal{O}(m,n)(2k) = \mathcal{O}(m+k, n+k), \quad \mathcal{O}(m,n)(2k+1) = \mathcal{O}(n+k, m+k+1)$$

for all $m, n, k \in \mathbb{Z}$. By (6.7) it is easy to see $\mathcal{O}(m,n)(-m-n) = \mathcal{O}(u,-u)$ is a normalized line bundle where

$$u = \begin{cases} (m-n)/2 & \text{if } m-n \text{ is even} \\ (n-m-1)/2 & \text{if } m-n \text{ is odd} \end{cases}$$

Since $[\mathcal{O}(u,-u)] = [\mathcal{O}] + 2u[\mathcal{S}] - u[\mathcal{Q}] - u(u+1)[\mathcal{P}]$ the invariants (n_e, n_o) of $\mathcal{O}(u,-u)$ are given by $(n_e, n_o) = (u^2, u(u+1))$. Either $k = u \geq 0$ or $k = -u-1 \geq 0$. These two possibilities correspond to Cases 1 and 2 of Section 6.3. In particular $\mathcal{R}_{(n_e,n_o)}(X)$ is nonempty if and only if (n_e, n_o) is $(k^2, k(k+1))$ or $((k+1)^2, k(k+1))$ for some integer $k \geq 0$ and in that case $\mathcal{R}_{(n_e,n_o)}(X) = \{\mathcal{O}(n_o - n_e, n_e - n_o)\}$.

Proposition 6.3.3 implies that a minimal resolution for $\mathcal{O}(m,n)$ is of the form

$$0 \to \mathcal{O}(2n-1)^{m-n} \to \mathcal{O}(2n)^{m-n+1} \to \mathcal{O}(m,n) \to 0 \quad \text{if } m \geq n,$$
$$0 \to \mathcal{O}(2m)^{n-m-1} \to \mathcal{O}(2m+1)^{n-m} \to \mathcal{O}(m,n) \to 0 \quad \text{if } m < n.$$

We have shown

Proposition 6.4.1. *Assume A is linear and let $I \in \operatorname{grmod}(A)$ be a reflexive graded right ideal of A. Then I has a minimal resolution of the form*

$$0 \to A(-c-1)^c \to A(-c)^{c+1} \to I(d) \to 0 \tag{6.13}$$

for some integers d and c. As a consequence $R_{(n_e,n_o)}(A) = \emptyset = \mathcal{R}_{(n_e,n_0)}(X)$ unless $n_e = (n_e - n_o)^2$ i.e. $(n_e,n_o) = ((k+1)^2, k(k+1))$ or $(n_e,n_o) = (k^2, k(k+1))$ for some $k \in \mathbb{N}$.

6.4.2 Hilbert scheme of points

The Hilbert scheme of points for $Y = \mathbb{P}^1 \times \mathbb{P}^1$, which we will denote by $\operatorname{Hilb}(Y)$, parameterizes the torsion free rank one sheaves on Y up to shifting. By the category equivalence $\operatorname{Qcoh}(Y) \cong \operatorname{Qcoh}(X)$ where $X = \operatorname{Proj} A$ we see $\operatorname{Hilb}(Y)$ also parameterizes the torsion free rank one objects on X up to shifting. Let $\mathcal{I} \in \operatorname{coh}(X)$ be such an object. Put $\mathcal{I} = \pi I$ where $I \in \operatorname{grmod}(A)$. Thus $\mathcal{I}^{**} := \pi I^{**}$ is a line bundle on X of rank one hence $\mathcal{I}^{**} \cong \mathcal{O}(m,n)$ for some integers m,n. By [8, Corollary 4.2] there is an exact sequence

$$0 \to \mathcal{I} \to \mathcal{I}^{**} \to \mathcal{N} \to 0$$

where $\mathcal{N} \in \operatorname{coh}(X)$ is a zero dimensional object of some degree $l \geq 0$. Since $\mathcal{N}$ admits a filtration by point objects on X we have $[\mathcal{N}] = l[\mathcal{P}]$. Also $\mathcal{I}^{**}(d) \cong O(u,-u)$ for some $d,u \in \mathbb{Z}$. Computing the class of $\mathcal{I}(d)$ in $K_0(X)$ we find

$$[\mathcal{I}(d)] = [\mathcal{O}] + 2u[\mathcal{S}] - u[\mathcal{Q}] - (u(u+1) - l)\,[\mathcal{P}]$$

from which we deduce $\mathcal{I}(d) \in \operatorname{Hilb}_{(n_e,n_o)}(X)$, as defined in §6.2.3, where $(n_e,n_o) = (u^2 + l, u(u+1) + l)$. Again we separate

Case 1. $u \geq 0$. Put $k = u$. Then $(n_e,n_o) = (k^2 + l, k(k+1) + l)$ where $k,l \in \mathbb{N}$.

Case 2. $u < 0$. Put $k = -u-1$. Then $(n_e,n_o) = ((k+1)^2 + l, k(k+1) + l)$ where $k,l \in \mathbb{N}$.

Remark 6.4.2. By the above discussion we may associate invariants $(n_e,n_o) \in N = \{(n_e,n_o) \in \mathbb{N}^2 \mid n_e - (n_e - n_o)^2 \geq 0\}$ to any object in $\operatorname{Hilb}(Y)$. Let $\operatorname{Hilb}_{(n_e,n_o)}(Y)$ denote the associated parameter space. The dimension of $\operatorname{Hilb}_{(n_e,n_o)}(Y)$ may be deduced as follows. Given $(n_e,n_o) \in N$ fixes $l \in \mathbb{N}$ and $u \in \mathbb{Z}$ as above. The number of parameters to choose $\mathcal{O}(u,-u)$ is zero. On the other hand, to choose a point in $\mathbb{P}^1 \times \mathbb{P}^1$ we have two parameters. Thus to pick a zero-dimensional subsheaf $\mathcal{N}$ of degree l we have $2l$ parameters since such $\mathcal{N}$ admits a filtration of length l in points of $\mathbb{P}^1 \times \mathbb{P}^1$. Hence the freedom of choice in a normalized torsion free rank one sheaf $\mathcal{I}$ is $2l$. Hence $\dim \mathrm{Hilb}_{(n_e,n_o)}(Y) = 2l$. Since $l = n_e - (n_e - n_o)^2$ we have $\dim \mathrm{Hilb}_{(n_e,n_o)}(Y) = 2\left(n_e - (n_e - n_o)^2\right)$.

6.5 Some results on line and conic objects

In this section we gather some additional results on line objects and conic objects on quantum quadrics which will be used later on. These results are obtained by using similar techniques as in [1, 8].

Let A be a cubic Artin-Schelter algebra. We use the notations of §1.9.4. In particular $(E, \sigma, \mathcal{O}_E(1))$, $B = B(E, \sigma, \mathcal{O}_E(1))$, $(C, \sigma, \mathcal{O}_C(1))$, $D = B(C, \sigma, \mathcal{O}_C(1)) = \Gamma_*(\mathcal{O}_C)$, g and h will have their usual meaning. Recall the isomorphism of k-algebras $A/hA \xrightarrow{\cong} D : a \mapsto \overline{a}$. The dimension of objects in $\mathrm{grmod}(B)$, $\mathrm{grmod}(D)$ or $\mathrm{tails}(B)$, $\mathrm{tails}(D)$ will be computed in $\mathrm{grmod}(A)$ or $\mathrm{tails}(A)$. We begin with

Lemma 6.5.1. *Let $w \in A_d$ for some integer $d \geq 1$ and put $W = A/wA$, $\mathcal{W} = \pi W$.*

1. *Let $p \in C$. Then $\mathrm{Hom}_X(\mathcal{W}, \mathcal{N}_p) \neq 0$ if and only if $\overline{w}(p) = 0$.*
2. *$\dim_k \mathrm{Hom}_X(\mathcal{W}, \mathcal{N}_p) \leq 1$ for all $p \in C$.*

Proof. Firstly, if $f : W \to N_p$ is non-zero then $\pi f : \mathcal{W} \to \mathcal{N}_p$ is non-zero since N_p is socle-free i.e. $\underline{\mathrm{Hom}}_A(k, N_p) = 0$. Conversely, $\mathrm{Hom}_X(\mathcal{W}, \mathcal{N}_p) \neq 0$ implies $\mathrm{Hom}_A(W, N_p) \neq 0$. Indeed, a non-zero map $g : \mathcal{W} \to \mathcal{N}_p$ yields a surjective map $\omega g : W \to (\omega\mathcal{N}_p)_{\geq n}$ for $n \gg 0$. Now $(\omega\mathcal{N}_p)_{\geq n} = N_{\sigma^n p}(-n) \subset N_p$, which yields a non-zero map $W \to N_p$.

So to prove the first statement it is sufficient to show $\mathrm{Hom}_A(W, N_p) \neq 0$ if and only if $\overline{w}(p) = 0$. This is proved in a similar way as [1]. For convenience we shortly repeat the arguments. Writing down resolutions for W, N_p we see there is a non-zero map $f : W \to N_p$ if and only if we may find (non-zero) maps f_0, f_1 making the following diagram commutative

$$\begin{array}{ccccccccccc} & & 0 & \longrightarrow & A(-d) & \xrightarrow{w\cdot} & A & \longrightarrow & W & \longrightarrow & 0 \\ & & & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_0} & & & & \\ 0 & \longrightarrow & A(-3) & \longrightarrow & A(-1) \oplus A(-2) & \longrightarrow & A & \xrightarrow{\theta} & N_p & \longrightarrow & 0 \end{array}$$

The resolutions being projective, this is equivalent with saying there is a non-zero map f_0 such that $\theta \circ f_0 \circ w = 0$, i.e. $\overline{w}(p) = 0$.

The second part is shown by applying $\mathrm{Hom}_X(-, \mathcal{N}_p)$ to the short exact sequence $0 \to \mathcal{O}(-d) \to \mathcal{O} \to \mathcal{W} \to 0$ and bearing in mind $\mathrm{Hom}_X(\mathcal{O}, \mathcal{N}_p) = k$. ∎

Remark 6.5.2. It follows from the first part of the previous lemma there exists at least one $p \in C$ for which $\operatorname{Hom}_X(\mathcal{W}, \mathcal{N}_p) \neq 0$. Moreover any such non-zero morphism is surjective since point objects are simple objects in $\operatorname{coh}(X)$.

6.5.1 Line objects

Let $u = \lambda x + \mu y \in A_1$. Then $\overline{u} \in D_1 = H^0(C, \mathcal{O}_C(1))$ and a point $p = (p_1, p_2) \in C$ vanishes at $\overline{u}$ i.e. $\overline{u}(p) = 0$ if and only if $p_1 = (-\mu, \lambda) \in \mathbb{P}^1$. We have shown

Lemma 6.5.3. *Let $p \in C$. There exists, up to isomorphism, a unique line object $\mathcal{S}$ on X for which* $\operatorname{Hom}_X(\mathcal{S}, \mathcal{N}_p) \neq 0$.

In case A is elliptic then E is a divisor of bidegree $(2, 2)$ which means that for $u \in A_1$ the line $\{u = 0\} \times \mathbb{P}^1$ meets C in at most two different points p, q.

For general A we call two different points $p, q \in C$ *collinear* if $\overline{l}(p) = \overline{l}(q) = 0$ for some global section in $\overline{l} \in H^0(C, \mathcal{O}_C(1)) = D_1$. It follows from the previous discussion that $\operatorname{pr}_1 p = \operatorname{pr}_1 q$.

6.5.2 Conic objects

We now deduce

Lemma 6.5.4. *Let p, q, r be three distinct points in C. There exists, up to isomorphism, a unique conic object $\mathcal{Q}$ on X for which* $\operatorname{Hom}_X(\mathcal{Q}, \mathcal{N}_p) \neq 0$, $\operatorname{Hom}_X(\mathcal{Q}, \mathcal{N}_q) \neq 0$ *and* $\operatorname{Hom}_X(\mathcal{Q}, \mathcal{N}_r) \neq 0$.

Proof. Due to Lemma 6.5.1 it will be sufficient to prove there exists, up to scalar multiplication, a unique quadratic form $v \in A_2$ for which $\overline{v}(p) = \overline{v}(q) = \overline{v}(r) = 0$.

Writing $v = \lambda_1 x^2 + \lambda_2 xy + \lambda_3 yx + \lambda_4 y^2$ where $\lambda_i \in k$ and $p = ((\alpha, \beta), (\alpha', \beta')) \in C \subset \mathbb{P}^1 \times \mathbb{P}^1$, we see $\overline{v}(p) = 0$ if and only if $\lambda_1 \alpha\alpha' + \lambda_2 \alpha\beta' + \lambda_3 \beta\alpha' + \lambda_4 \beta\beta' = 0$. The condition $\overline{v}(p) = \overline{v}(q) = \overline{v}(r) = 0$ then translates to a system of three linear equations in $\lambda_1, \ldots, \lambda_4$, which admits a non-trivial solution. Moreover, this solution is unique (up to scalar multiplication) unless all maximal minors are zero, which implies that at least two points of p, q, r coincide. □

Subobjects of line objects on X are shifted line objects [8]. We may prove a similar result for conic objects.

Lemma 6.5.5. *Let $\mathcal{Q}$ be a conic object and $p \in C$. Assume* $\operatorname{Hom}_X(\mathcal{Q}, \mathcal{N}_p) \neq 0$.

1. *The kernel of a non-zero map $\mathcal{Q} \to \mathcal{N}_p$ is a shifted conic object $\mathcal{Q}'(-1)$.*

2. *Assume A is elliptic and σ has infinite order. If in addition $\mathcal{Q}$ is critical then all subobjects $\mathcal{Q}$ are shifted critical conic objects.*

Proof. Firstly, let f denote such a non-zero map $\mathcal{Q} \to \mathcal{N}_p$. Since $\mathcal{N}_p$ is simple, f is surjective. Putting $\mathcal{Q} = \pi Q$ where Q is a conic module over A it is sufficient to show that the kernel of a surjective map $Q \to (N_p)_{\geq n}$ is of the form $Q'(-1)$ for some conic object Q'. This is done by taking the cone of the induced map between resolutions of Q and $(N_p)_{\geq n}$.

Secondly, as $\mathcal{Q}$ is critical, any quotient of $\mathcal{Q}$ has dimension zero and since σ has infinite order such a quotient admits a filtration by shifted point objects on X, see [8]. By the first part this completes the proof. □

We will also need the dual statement of the previous result.

Lemma 6.5.6. *Let $\mathcal{Q}$ be a conic object and $p \in C$. Assume $\operatorname{Ext}^1_X(\mathcal{N}_p, \mathcal{Q}) \neq 0$.*

1. *The middle term of a non-zero extension in $\operatorname{Ext}^1_X(\mathcal{N}_p, \mathcal{Q})$ is a shifted conic object $\mathcal{Q}'(1)$.*
2. *Assume A is elliptic and σ has infinite order. Then any extension of $\mathcal{Q}$ by a zero dimensional object is a shifted conic object.*

Proof. Again the second statement is clear from the first one thus it suffices to prove the first part. Put $\mathcal{Q} = \pi Q$ where Q is a conic module over A. Let $\mathcal{J}$ denote the middle term of a non-trivial extension i.e. $0 \to \mathcal{Q} \to \mathcal{J} \to \mathcal{N}_p \to 0$. It is easy to see $\mathcal{J}$ is pure and $\omega\mathcal{J} \in \operatorname{coh}(X)$ has projective dimension one, see for example (the proof of) [28, Proposition 3.4.1]. Put $J = \omega\mathcal{J}$. Application of ω gives a short exact sequence

$$0 \to Q \to J \to (N_p)_{\geq n} \to 0 \tag{6.14}$$

Applying $\underline{\operatorname{Hom}}_A(-, A)$ on (6.14) yields $0 \to J^\vee \to Q^\vee \to ((N_p)_{\geq n})^\vee \to 0$. As $((N_p)_{\geq n})^\vee$ is a shifted point module and $Q^\vee$ is a shifted conic module it follows from Lemma 6.5.5 that $J^\vee$ is also a shifted conic module. Hence the same is true for $J^{\vee\vee}$. Consideration of Hilbert series shows $J^{\vee\vee} = Q'(1)$ for some conic module Q' over A. Since $\omega\mathcal{J}$ is Cohen-Macaulay Theorem 1.9.8 implies $\pi J^{\vee\vee} = \pi J = \mathcal{J}$. This finishes the proof. □

Remark 6.5.7. Lemmas 6.5.5 and 6.5.6 are in contrast with the situation for quadratic Artin-Schelter algebras [1, §4] where a non-zero map $A/vA \to N_p$ (where $v \in A_2$ and $p \in C$) will yield an exact sequence $0 \to Q'(-1) \to A/vA \to N_p \to 0$ for which Q' has a resolution of the form $0 \to A(-1)^2 \to A^2 \to Q' \to 0$.

Let $\mathcal{Z}$ denote the full subcategory of $\operatorname{coh}(X)$ whose objects consist of zero dimensional objects of $\operatorname{coh}(X)$. $\mathcal{Z}$ is a Serre subcategory of $\operatorname{coh}(X)$, see for example [83]. We say $\mathcal{M}, \mathcal{N} \in \operatorname{coh}(X)$ are *equivalent up to zero dimensional objects* if their images in the quotient category $\operatorname{coh}(X)/\mathcal{Z}$ are isomorphic. We say $\mathcal{M}$ and $\mathcal{N}$ are *different modulo zero dimensional objects* if they are not equivalent up to zero dimensional objects. Using Lemmas 6.5.5 and 6.5.6 one proves

Lemma 6.5.8. *Assume A is elliptic and σ has infinite order. Then two critical conic objects on X are equivalent up to zero dimensional objects if and only if they have a common subobject.*

We now come to a key result which we will need in §6.7.7 below.

Lemma 6.5.9. *Assume k is uncountable, A is elliptic and σ has infinite order. Let $p, p' \in C$ for which $p, p', \sigma p, \sigma p'$ are pairwise different and non-collinear. Then, modulo zero dimensional objects, there exist infinitely many critical conic objects $\mathcal{Q}$ for which $\operatorname{Hom}_X(\mathcal{Q}, \mathcal{N}_p) \neq 0$ and $\operatorname{Hom}_X(\mathcal{Q}, \mathcal{N}_{p'}) \neq 0$.*

Proof. Write $p = ((\alpha_0, \beta_0), (\alpha_1, \beta_1)) \in C$. We prove the lemma in seven steps.

Step 1. Let $d \in \mathbb{N}$ and let $\mathcal{Q}, \mathcal{Q}'$ be two critical objects for which $\mathcal{Q}'(-d) \subset \mathcal{Q}$. Then there is a filtration $\mathcal{Q}'(-d) = \mathcal{Q}_d(-d) \subset \mathcal{Q}_{d-1}(-d+1) \subset \cdots \subset \mathcal{Q}_1(-1) \subset \mathcal{Q}_0 = \mathcal{Q}$ where the $\mathcal{Q}_i$ are critical conic objects and the successive quotients are point objects on X. This follows from the proof of Lemma 6.5.5.

Step 2. Up to isomorphism there are uncountably many conic objects $\mathcal{Q}$ on X for which $\operatorname{Hom}_X(\mathcal{Q}, \mathcal{N}_p) \neq 0$, $\operatorname{Hom}_X(\mathcal{Q}, \mathcal{N}_{p'}) \neq 0$. See the proof of Lemma 6.5.4.

Step 3. Let $\mathcal{A}$ denote the set of isoclasses of critical conic objects $\mathcal{Q}$ for which $\operatorname{Hom}_X(\mathcal{Q}, \mathcal{N}_p) \neq 0$, $\operatorname{Hom}_X(\mathcal{Q}, \mathcal{N}_{p'}) \neq 0$. Then $\mathcal{A}$ is an uncountable set. By the previous step it is sufficient to show there are only finitely many non-critical conic objects $\mathcal{Q}$ on X for which $\operatorname{Hom}_X(\mathcal{Q}, \mathcal{N}_p) \neq 0$, $\operatorname{Hom}_X(\mathcal{Q}, \mathcal{N}_{p'}) \neq 0$. For such an object $\mathcal{Q}$ it is easy to see there exists an exact sequence

$$0 \to \mathcal{S}'(-1) \to \mathcal{Q} \to \mathcal{S} \to 0 \tag{6.15}$$

for some line objects $\mathcal{S}, \mathcal{S}'$ on X. Also, $\dim_k \operatorname{Ext}^1_X(\mathcal{S}, \mathcal{S}'(-1)) \leq 1$ hence $\mathcal{Q}$ is, up to isomorphism, fully determined by $\mathcal{S}$ and $\mathcal{S}'$. We deduce from (6.15) that either $\operatorname{Hom}_X(\mathcal{S}, \mathcal{N}_p) \neq 0$, $\operatorname{Hom}_X(\mathcal{S}', \mathcal{N}_{\sigma p}) \neq 0$ or $\operatorname{Hom}_X(\mathcal{S}, \mathcal{N}_{p'}) \neq 0$, $\operatorname{Hom}_X(\mathcal{S}', \mathcal{N}_{\sigma p'}) \neq 0$. By Lemma 6.5.3 this means there are at most two non-critical conic objects for which $\operatorname{Hom}_X(\mathcal{Q}, \mathcal{N}_p) \neq 0$ and $\operatorname{Hom}_X(\mathcal{Q}, \mathcal{N}_{p'}) \neq 0$.

Step 4. Let $\mathcal{B} \subset \mathcal{A}$ denote the set of conic objects $\mathcal{Q} = \pi Q$ for which Q is h-torsion free. Then $\mathcal{B}$ is uncountable. Indeed, writing $Q = A/vA$ we find Q is h-torsion free (meaning multiplication by h is injective) unless $\overline{v} : \mathcal{O}_C(-2) \to \mathcal{O}_C$ is not injective. Hence $\overline{v} = 0$ which means that v and h have a common divisor. As v is not a product of linear forms, v divides h. Up to scalar multiplication there are only finitely many possibilities for such v.

Step 5. For any $\mathcal{Q} \in \mathcal{B}$ there are, up to isomorphism, only finitely many points objects $\mathcal{N}_p$ for which $\operatorname{Hom}_X(\mathcal{Q}, \mathcal{N}_p) \neq 0$ or $\operatorname{Ext}^1_X(\mathcal{N}_p, \mathcal{Q}(-1)) \neq 0$. To show this, write $\mathcal{Q} = \pi(A/vA)$ and $Q = A/vA$. Since v does not divide h, it does not divide g thus Q is also g-torsion free. Thus Q/gQ is a B-module of GK-dimension one so $(Q/gQ)\tilde{}$ is a finite dimensional $\mathcal{O}_E$-module. Writing $\overline{v}_g$ for the image of v in B this implies there are only finitely many points $p \in E$ such that $\overline{v}_g(p) = 0$. By the same methods used in the proof of Lemma 6.5.1 one may show there are finitely many point objects $\mathcal{N}_p$ on X for which $\operatorname{Hom}_X(\mathcal{Q}, \mathcal{N}_p) \neq 0$.

For the second part, Serre duality implies $\operatorname{Ext}^i_X(\mathcal{N}_p, \mathcal{Q}(-1)) \cong \operatorname{Ext}^{2-i}_X(\mathcal{Q}, \hat{\mathcal{N}_p})'$ for $i = 0, 1, 2$ and a suitable point object $\hat{\mathcal{N}_p}$ on X. By $\chi(\mathcal{Q}, \hat{\mathcal{N}_p}) = 0$, Lemma 6.5.1(2) and the first part of Step 5 we are done.

Step 6. For any $\mathcal{Q}_i \in \mathcal{B}$ and any integer $d \geq 0$ the following subset of $\mathcal{B}$ is finite

$$\mathcal{V}_d(\mathcal{Q}_i) = \{\mathcal{Q} \in \mathcal{B} \mid \mathcal{Q}'(-d) \subset \mathcal{Q} \text{ for a conic object } \mathcal{Q}' \text{ for which } \mathcal{Q}'(-d) \subset \mathcal{Q}_i\}$$

We will prove this for $d = 1$, for general d the same arguments may be used combined with Step 1. Let $\mathcal{Q}'(-1) \subset \mathcal{Q}_i$. Note $\mathcal{Q}' \in \mathcal{B}$. Clearly any conic object $\mathcal{Q}$ on X for which $\mathcal{Q}'(-1) \subset \mathcal{Q}$ holds is represented by an element of $\operatorname{Ext}^1_X(\mathcal{N}_p, \mathcal{Q}'(-1))$ for some point object $\mathcal{N}_p$, and two such conic objects $\mathcal{Q}$ are isomorphic if and only if the corresponding extensions only differ by a scalar. By Step 5 and its proof there are only finitely many such $\mathcal{Q}$, up to isomorphism.

Step 7. There exist infinitely many critical conic objects $\mathcal{Q}_0, \mathcal{Q}_1, \mathcal{Q}_2, \ldots$ for which $\operatorname{Hom}_X(\mathcal{Q}_i, \mathcal{N}_p) \neq 0$, $\operatorname{Hom}_X(\mathcal{Q}_i, \mathcal{N}_{p'}) \neq 0$ and $\mathcal{Q}_i$, $\mathcal{Q}_j$ do not have a common subobject for all $j < i$. Indeed, choose $\mathcal{Q}_0 \in \mathcal{B}$ arbitrary and having $\mathcal{Q}_0, \mathcal{Q}_1, \ldots, \mathcal{Q}_{i-1}$ we pick $\mathcal{Q}_i$ as an element of $\mathcal{B}$ which does not appear in the countable subset $\bigcup_{d \in \mathbb{N}, j<i} \mathcal{V}_d(\mathcal{Q}_j)$. By Lemma 6.5.5 subobjects of critical conic objects are shifted critical conic objects hence Step 7 follows.

Combining Step 7 with Lemma 6.5.8 completes the proof. □

Remark 6.5.10. If A is of generic type A for which σ has infinite order then Lemma 6.5.9 may be proved alternatively by observing that for any conic object $\mathcal{Q} = \pi(A/vA)$ containing a shifted conic object $\mathcal{Q}' = \pi(A/v'A)$ we have

$$\operatorname{div}(\overline{v'}) = (\sigma^a p) + (\sigma^b q) + (\sigma^c r) + (\sigma^{-a-b-c} r) \qquad \text{for some } a, b, c \in \mathbb{Z}$$

where we have written $\operatorname{div}(\overline{v}) = (p)+(q)+(r)+(s)$ for the divisor of zeroes of $\overline{v} \in D_2$. This observation is proved by using similar methods as in [1], see also [3, Theorem 3.2]. Thus if A is of generic type A we do not need the hypothesis k is uncountable in Lemma 6.5.9.

6.6 Restriction of line bundles to the divisor C

In this section A is an elliptic cubic Artin-Schelter algebra and $X = \operatorname{Proj} A$, to which §1.9.4 we associate the geometric data $(C, \sigma, \mathcal{O}_C(1))$, the homogeneous coordinate ring $D = B(C, \sigma, \mathcal{O}_C(1))$ and the map of noncommutative schemes $u : C \to X$.

The following result is proved as Proposition 2.3.2.

Proposition 6.6.1. *Let A be an elliptic cubic Artin-Schelter regular algebra.*

1. *If $\mathcal{M}$ is a vector bundle in X then $L_j u^*\mathcal{M} = 0$ for $j > 0$ and $u^*\mathcal{M}$ is a vector bundle on C.*
2. *Assume σ has infinite order and let $\mathcal{M} \in D^b(\mathrm{coh}(X))$ for which $Lu^*\mathcal{M}$ is a vector bundle on C. Then $\mathcal{M}$ is a vector bundle on X.*

Although some of the following results may be generalized, for the rest of this Section 6.6 we will assume A is of generic type A (see Example 1.9.3). Thus E is a smooth divisor of bidegree $(2,2)$ on $\mathbb{P}^1 \times \mathbb{P}^1$. Since E has arithmetic genus 1 it is a smooth elliptic curve [41]. Fixing a group law on E the automorphism σ is a translation by some element $\xi \in C$ i.e. $\sigma p = p + \xi$ for all $p \in E$. We write o for the origin of the group law of C. For $p, q, r \in C$ we have $p + q = r$ if and only if the divisors $(p) + (q)$ and $(r) + (o)$ on C are linearly equivalent [41, Chapter IV §4].

Since E is reduced the geometric data $(E, \sigma, \mathcal{O}_E(1))$ and $(C, \sigma, \mathcal{O}_C(1))$ coincide and $g = h \in A_4$, $D = B = B(E, \sigma, \mathcal{O}_E(1))$.

The functor u^* induces a group homomorphism

$$u^* : K_0(X) \to K_0(C) : [\mathcal{M}] \mapsto \sum_j (-1)^j [L_j u^*\mathcal{M}] = [u^*\mathcal{M}] - [L_1 u^*\mathcal{M}]$$

Recall the basis $\mathcal{B}' = \{[\mathcal{O}], [\mathcal{S}], [\mathcal{Q}], [\mathcal{P}]\}$ for $K_0(X)$ from §6.2.2. The image of $\mathcal{B}'$ under u^* is computed in the following analogue of Lemma 2.3.3.

Lemma 6.6.2. *Assume A is a cubic Artin-Schelter algebra of generic type A. Then*

$$u^*[\mathcal{O}] = [\mathcal{O}_C]$$
$$u^*[\mathcal{S}] = [\mathcal{O}_p] + [\mathcal{O}_q] \qquad p, q \in C \text{ arbitrary but collinear}$$
$$u^*[\mathcal{Q}] = [\mathcal{O}_p] + [\mathcal{O}_q] + [\mathcal{O}_r] + [\mathcal{O}_s] \qquad p, q, r, s \in C \text{ arbitrary but } p + q + r + s = 2(p' + q' - \xi) \text{ for some collinear } p', q' \in C$$
$$u^*[\mathcal{P}] = [\mathcal{O}_p] - [\mathcal{O}_{\sigma^{-4}p}] \qquad p \in C \text{ arbitrary}$$

Proof. Since A is h-torsion free we have $L_1 u^*\mathcal{O} = 0$ hence $u^*[\mathcal{O}] = [u^*\mathcal{O}] = [\mathcal{O}_C]$. Second, write $S = A/aA$ for some $a \in A_1$ and $\mathcal{S} = \pi S$. Application of $- \otimes_A D$ on the exact sequence $0 \to A(-1) \xrightarrow{a\cdot} A \to S \to 0$ gives

$$0 \to \mathrm{Tor}_1^A(S, D) \to D(-1) \xrightarrow{\overline{a}\cdot} D \to S \otimes_A D \to 0$$

Since D is a domain the middle map is injective. Hence $\mathrm{Tor}_1^A(S, D) = 0$ (thus S is h-torsionfree) and therefore $L_1 u^*\mathcal{S} = 0$. Thus $u^*[\mathcal{S}] = [u^*\mathcal{S}]$. From [8] it follows that $u^*\mathcal{S} = \mathcal{O}_L$ where L is the scheme-theoretic intersection of C and the line $\{a = 0\} \times \mathbb{P}^1$. Since L consists of two points p, q we obtain $[\mathcal{O}_L] = [\mathcal{O}_p] + [\mathcal{O}_q]$. By definition p and q are collinear points. This proves the second equality.

Third, same reasoning as above yields $u^*\mathcal{Q}$ is a finite dimensional $\mathcal{O}_C$-module which corresponds to a divisor of degree four on C. Thus we may write $[u^*\mathcal{Q}] = [\mathcal{O}_p]+$

$[\mathcal{O}_q]+[\mathcal{O}_r]+[\mathcal{O}_s]$ for some $p,q,r,s \in C$. It is easy to see that $\mathcal{O}_C(-2) = \sigma^*\mathcal{O}_C(-1)\otimes_C \mathcal{O}_C(-1)$. Since $\mathcal{O}_C(-1) = \mathcal{O}_C(-L)$ (from the previous part) we find that its pullback via σ is equal to $\sigma^*\mathcal{O}_C(-1) = \mathcal{O}_C(-\sigma^{-1}L)$ hence $\mathcal{O}_C(-2) = \mathcal{O}_C(-L-\sigma^{-1}L)$. This means the divisor of $u^*\mathcal{Q}$ is linearly equivalent to $L+\sigma^{-1}L$, which means they have the same sum under the group law of C.

Finally we prove the fourth equation. Put $\mathcal{P} = \mathcal{N}_p$. Now $N_p \otimes_A D \cong N_p/N_p h = N_p$ thus $u^*\mathcal{N}_p = \widetilde{N_p} = \mathcal{O}_p$. Applying $N_p \otimes_A -$ to the short exact sequence of A-bimodules $0 \to A(-4) \xrightarrow{\cdot h} A \to D \to 0$ we get the exact sequence

$$0 \to \mathrm{Tor}_1^A(N_p, D) \to N_p(-4) \xrightarrow{\cdot \overline{h}} N_p \to N_p \otimes_A D \to 0$$

As $\overline{h} = 0$ we find $\mathrm{Tor}_1^A(N_p,D) = N_p(-4) = \left(N_{\sigma^{-4}p}\right)_{\geq 4}$. Thus $L_1 u^*\mathcal{N}_p = \mathrm{Tor}_1^A(N_p,D)^{\sim} = \left(N_{\sigma^{-4}p}\right)^{\sim} = \mathcal{O}_{\sigma^{-4}p}$. This ends the proof of the lemma. □

Let $\mathcal{M} \in \mathrm{coh}(X)$ and write $[\mathcal{M}] = r[\mathcal{O}] + a[\mathcal{S}] + b[\mathcal{Q}] + c[\mathcal{P}]$. By the previous lemma

$$\operatorname{rank} u^*[\mathcal{M}] = r = \operatorname{rank}\mathcal{M} \quad \text{and} \quad \deg u^*[\mathcal{M}] = 2a+4b \tag{6.16}$$

We deduce

Proposition 6.6.3. *Let A be a cubic Artin-Schelter algebra of generic type A.*

1. *If $\mathcal{I}$ is a line bundle on X then $u^*\mathcal{I}$ is a line bundle on C, and $\mathcal{I}$ is normalized if and only if* $\deg u^*\mathcal{I} = 0$.
2. *If $\mathcal{I}$ is a normalized line bundle on X with invariants (n_e, n_o) then*

$$c_1(u^*\mathcal{I}) = \mathcal{O}_C((o) - (2(n_e+n_o)\xi))$$

Proof. The first statement is immediate from the definition of a normalized line bundle on X. The second part results from a straightforward computation. □

Corollary 6.6.4. *Let A be a cubic Artin-Schelter regular algebra of generic type A and assume σ has infinite order. Then the category*

$$\mathcal{R}(X) = \coprod_{(n_e,n_o)\in N} \mathcal{R}_{(n_e,n_o)}(X) = \{\textit{normalized line bundles on } X\}$$

is equivalent to the full subcategory of $\mathrm{coh}(X)$ *with objects*

$$\{\mathcal{M} \in \mathrm{coh}(X) \mid u^*\mathcal{M} \textit{ is a line bundle on } C \textit{ of degree zero}\}.$$

Proof. Due to Proposition 6.6.3 it is sufficient to prove that if $\mathcal{M} \in \mathrm{coh}(X)$ for which $u^*\mathcal{M} \in \mathrm{coh}(C)$ is a line bundle of degree zero, then $\mathcal{M}$ is a normalized line bundle on X. Pick $M \in \mathrm{grmod}(A)$ for which $\pi M = \mathcal{M}$. We may assume M contains no subobject in tors(A). By Proposition 6.6.1 and (6.16) it suffices to prove $Lu^*\mathcal{M} = u^*\mathcal{M}$ i.e. $L_1u^*\mathcal{M} = 0$.

It is sufficient to prove M is torsion free, since it then follows that M is h-torsion free whence $L_1u^*\mathcal{M} = \ker(M(-4) \xrightarrow{\cdot h} M)^{\sim} = 0$. So let us assume by contradiction M is not torsion free. Let $T \subset M$ the maximal torsion submodule of M. Thus $0 \neq M/T$ is torsion free. Applying u^* to $0 \to \pi T \to \mathcal{M} \to \pi(M/T) \to 0$ then gives the exact sequence $0 \to u^*\pi T \to u^*\mathcal{M} \to u^*\pi(M/T) \to 0$ on C. Since $u^*\mathcal{M}$ is a line bundle on C, it is pure hence either $u^*\pi T$ is a line bundle or $u^*\pi T = 0$.

If $u^*\pi T$ would be a line bundle then $u^*\pi(M/T) = (M/T \otimes_A D)^{\sim}$ has rank zero. Thus $M/T \otimes_A D \in \operatorname{grmod}(D)$ has GK-dimension ≤ 1. But then $\operatorname{GKdim} M/T \leq 2$, a contradiction with the fact that $M/T \in \operatorname{grmod}(A)$ is non-zero and torsion free. Thus $u^*\pi T = 0$ i.e. $(T/hT)^{\sim} = 0$. This means $\pi(T/hT) = 0$ hence $T/hT \in \operatorname{tors}(A)$. By Lemma 1.9.19 we deduce $T \in \operatorname{tors}(A)$ thus $T = 0$ since M contains no subobjects in tors(A). This ends the proof. □

Remark 6.6.5. Some of the results above may be generalized to other elliptic cubic Artin-Schelter algebras. For example, if we consider the situation where $A = H_c$ is the enveloping algebra then one obtains the similar results

- If $\mathcal{I}$ is a line bundle on X then $u^*\mathcal{I}$ is a line bundle on $C = \Delta$ (the diagonal on $\mathbb{P}^1 \times \mathbb{P}^1$) and $L_1u^*\mathcal{I} = 0$. In addition $\mathcal{I}$ is normalized if and only if $u^*\mathcal{I}$ has degree zero, i.e. if and only if $u^*\mathcal{I} \cong \mathcal{O}_\Delta$ (since $\operatorname{Pic}(\Delta) \cong \mathbb{Z}$).

- The category $\mathcal{R}(X) = \coprod_{(n_e,n_o)} \mathcal{R}_{(n_e,n_o)}(X)$ is equivalent to the full subcategory of coh(X) with objects $\{\mathcal{M} \in \operatorname{coh}(X) \mid u^*\mathcal{M} \cong \mathcal{O}_\Delta\}$.

6.7 From line bundles to quiver representations

Throughout Section 6.7, A will be a cubic Artin-Schelter algebra. From §6.7.3 onwards we will furthermore assume A is elliptic (and often restrict to the case where σ has infinite order). We recycle the notations of Section 1.9.4 and write $u : C \to X$ for the map of noncommutative schemes as defined in Section 6.6.

6.7.1 Generalized Beilinson equivalence

We set $\mathcal{E} = \mathcal{O}(3) \oplus \mathcal{O}(2) \oplus \mathcal{O}(1) \oplus \mathcal{O}$ and $U = \operatorname{Hom}_X(\mathcal{E}, \mathcal{E}) = \bigoplus_{i,j=0}^{3} \operatorname{Hom}_X(\mathcal{O}(i), \mathcal{O}(j))$. The functor $\operatorname{Hom}_X(\mathcal{E}, -)$ from coh(X) to the category mod(U) of right U-modules extends to an equivalence $\operatorname{RHom}_X(\mathcal{E}, -)$ of bounded derived categories [18]

$$\mathrm{D}^b(\operatorname{coh}(X)) \underset{-\overset{\mathbf{L}}{\otimes}_U \mathcal{E}}{\overset{\operatorname{RHom}_X(\mathcal{E},-)}{\rightleftarrows}} \mathrm{D}^b(\operatorname{mod}(U)) \qquad (6.17)$$

where the inverse functor is given by $-\overset{\mathbf{L}}{\otimes}_U \mathcal{E}$. For the classical case $X = \mathbb{P}^n$ such an equivalence was found by Beilinson [15]. We refer to (6.17) as *generalized Beilinson equivalence*. For a non-negative integer i this equivalence restricts to an equivalence [12] between the full subcategories $\mathcal{X}_i = \{\mathcal{M} \in \mathrm{coh}(X) \mid \mathrm{Ext}^j_X(\mathcal{E}, \mathcal{M}) = 0 \text{ for } j \neq i\}$ and $\mathcal{Y}_i = \{M \in \mathrm{mod}(U) \mid \mathrm{Tor}^U_j(M, \mathcal{E}) = 0 \text{ for } j \neq i\}$. The inverse equivalences are given by $\mathrm{Ext}^i_X(\mathcal{E}, -)$ and $\mathrm{Tor}^U_i(-, \mathcal{E})$.

It is easy to see that $U \cong k\Gamma/(R)$ where $k\Gamma$ is the path algebra of the quiver Γ

$$-3 \underset{Y_{-3}}{\overset{X_{-3}}{\rightrightarrows}} -2 \underset{Y_{-2}}{\overset{X_{-2}}{\rightrightarrows}} -1 \underset{Y_{-1}}{\overset{X_{-1}}{\rightrightarrows}} 0 \tag{6.18}$$

with relations R reflecting the relations of A. If we write the relations of A as (1.18) then the relations R are given by

$$\begin{pmatrix} X_{-1} & Y_{-1} \end{pmatrix} \cdot M^t_A(X_{-2}, Y_{-2}, X_{-3}, Y_{-3}) = 0 \tag{6.19}$$

where $M^t_A(X_{-2}, Y_{-2}, X_{-3}, Y_{-3})$ is obtained from the matrix M^t_A by replacing x^2, xy, yx and y^2 by $X_{-2}X_{-3}, Y_{-2}X_{-3}, X_{-2}Y_{-3}$ and $Y_{-2}Y_{-3}$.

As agreed in Section 1.3 we write $\mathrm{Mod}(\Gamma)$ for the category of representations of the quiver Γ, where representations are assumed to satisfy the relations. For $M \in \mathrm{Mod}(\Gamma)$ we write M_i for the k-linear space located at vertex i of Γ and $M(X_i)$, $M(Y_i)$ for the linear maps corresponding to arrows X_i, Y_i of Γ ($i = -3, \dots, 0$). As usual we denote S_i for the simple representation corresponding to i. Since the category $\mathrm{Mod}(\Gamma)$ of representations of Γ is equivalent to the category of right $k\Gamma/(R)$-modules we deduce $\mathrm{Mod}(\Gamma) \cong \mathrm{Mod}(U)$. From now on we write $\mathrm{Mod}(\Gamma)$ instead of $\mathrm{Mod}(U)$. One verifies that the matrix representation of the Euler form $\chi : K_0(\Gamma) \times K_0(\Gamma) \to \mathbb{Z}$ with respect to the basis $\{S_{-3}, S_{-2}, S_{-1}, S_0\}$ of $K_0(\Gamma)$ is given by

$$\begin{pmatrix} 1 & -2 & 0 & 2 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{6.20}$$

6.7.2 Point, line and conic representations

For further use we determine the representations of Γ corresponding to point, line and conic objects on X. The following lemmas are proved in the same spirit as Lemmas 2.4.1 and 2.4.2.

Lemma 6.7.1. *Let $p \in C$ and put $(\alpha_i, \beta_i) = pr_1(\sigma^i p) \in \mathbb{P}^1$.*

1. *$H^j(X, \mathcal{N}_p(m)) = 0$ for all integers m and $j > 0$. In particular $\mathcal{N}_p \in \mathcal{X}_0$.*
2. *$\dim_k (\omega\mathcal{N}_p)_m = 1$ for all m and $(\omega\mathcal{N}_p)_{\geq m}$ is a shifted point module for all integers m. In particular $(\omega\mathcal{N}_p)_{\geq 0} = N_p$.*

3. $H^0(X, \mathcal{N}_p(m)) = (\omega\mathcal{N}_p)_m$ *for all integers* m.
4. *Write* $\mathrm{RHom}_X(\mathcal{E}, \mathcal{N}_p) = p$. *Then* $\underline{\dim} p = (1,1,1,1)$ *and* $p \in \mathrm{mod}(\Gamma)$ *corresponds to the representation*

$$k \underset{\beta_{-3}}{\overset{\alpha_{-3}}{\rightrightarrows}} k \underset{\beta_{-2}}{\overset{\alpha_{-2}}{\rightrightarrows}} k \underset{\beta_{-1}}{\overset{\alpha_{-1}}{\rightrightarrows}} k$$

Lemma 6.7.2. *Let* $n \geq 1$ *be an integer,* $w \in A_n$ *and put* $\mathcal{W} = \pi(A/wA)$.

1. $H^1(X, \mathcal{W}(m)) \cong (A/Aw)'_{-m-2}$ *for* $m \leq -1$
2. $H^j(X, \mathcal{W}(m)) = 0$ *for* $m \leq -1$ *and* $j \neq 1$. *In particular* $\mathcal{W}(-1) \in \mathcal{X}_1$.
3. *If* $\eta \in A_1$ *then the induced linear map* $H^1(X, \mathcal{W}(m)) \xrightarrow{\cdot\eta} H^1(X, \mathcal{W}(m+1))$ *corresponds to* $(\eta\cdot)'$ *on* $(A/Aw)'$.
4. *Write* $\mathrm{RHom}_X(\mathcal{E}, \mathcal{W}(-1)) = W[-1]$. *Then*

$$\underline{\dim} W = \begin{cases} (2,1,1,0) & \text{if } n = 1 \\ (3,2,1,0) & \text{if } n = 2 \\ (4,2,1,0) & \text{if } n > 2 \end{cases}$$

and $W \in \mathrm{mod}(\Gamma)$ *corresponds to the representation*

$$(A/Aw)'_2 \underset{(y\cdot)'}{\overset{(x\cdot)'}{\rightrightarrows}} (A/Aw)'_1 \underset{(y\cdot)'}{\overset{(x\cdot)'}{\rightrightarrows}} k \rightrightarrows 0 \qquad (6.21)$$

6.7.3 First description of $\mathcal{R}_{(n_e,n_o)}(X)$

From now on we assume in Section 6.7 A is an elliptic cubic Artin-Schelter algebra. As in (6.9) we put $N = \{(n_e, n_o) \in \mathbb{N}^2 \mid n_e - (n_e - n_o)^2 \geq 0\}$. Recall from §6.2.3 the set $R(A)$ of reflexive rank one graded right A-modules considered up to isomorphism and shift is in natural bijection with the isoclasses in the category $\coprod_{(n_e,n_o)\in N} \mathcal{R}_{(n_e,n_o)}(X)$ where $\mathcal{R}_{(n_e,n_o)}(X)$ is the full subcategory of $\mathrm{coh}(X)$ consisting of the normalized line bundles on X with invariants (n_e, n_o).

Let $\mathcal{I}$ be an object of $\mathcal{R}_{(n_e,n_o)}(X)$, considered as a complex in $\mathrm{D}^b(\mathrm{coh}(X))$ of degree zero. Theorem 6.2.11 implies $\mathcal{I} \in \mathcal{X}_1$. Thus the image of this complex is concentrated in degree one i.e. $\mathrm{RHom}_X(\mathcal{E}, \mathcal{I}) = M[-1]$ where $M = \mathrm{Ext}^1_X(\mathcal{E}, \mathcal{I})$ is a representation of Δ. By functoriality, multiplication by $x, y \in A$ induces linear maps $M(X_{-i}), M(Y_{-i}) : H^1(X, \mathcal{I}(-i)) \to H^1(X, \mathcal{I}(-i+1))$ hence M is given by the following representation of Γ

$$H^1(X,\mathcal{I}(-3)) \underset{M(Y_{-3})}{\overset{M(X_{-3})}{\rightrightarrows}} H^1(X,\mathcal{I}(-2)) \underset{M(Y_{-2})}{\overset{M(X_{-2})}{\rightrightarrows}} H^1(X,\mathcal{I}(-1)) \underset{M(Y_{-1})}{\overset{M(X_{-1})}{\rightrightarrows}} H^1(X,\mathcal{I})$$

We denote $\mathcal{C}_{(n_e,n_o)}(\Gamma)$ for the image of $\mathcal{R}_{(n_e,n_o)}(X)$ under the equivalence $\mathcal{X}_1 \cong \mathcal{Y}_1$. In an analogue way as Theorem 2.4.3 we obtain

Theorem 6.7.3. *Let A be an elliptic cubic Artin-Schelter algebra where σ has infinite order. Let $(n_e, n_o) \in N \setminus \{(0,0)\}$. Then there is an equivalence of categories*

$$\mathcal{R}_{(n_e,n_o)}(X) \underset{\operatorname{Tor}_1^{\Gamma}(-,\mathcal{E})}{\overset{\operatorname{Ext}^1_X(\mathcal{E},-)}{\rightleftarrows}} \mathcal{C}_{(n_e,n_o)}(\Gamma)$$

where

$$\mathcal{C}_{(n_e,n_o)}(\Gamma) = \{M \in \operatorname{mod}(\Gamma) \mid \underline{\dim} M = (n_o, n_e, n_o, n_e - 1) \text{ and } \operatorname{Hom}_\Gamma(M,p) = 0, \operatorname{Hom}_\Gamma(p,M) = 0 \text{ for all } p \in C\}. \quad (6.22)$$

Proof. Similar as the proof of Theorem 2.4.3. □

6.7.4 Line bundles on X with invariants $(1,0)$ and $(1,1)$

We may now parameterize the line bundles on X for some low invariants.

Corollary 6.7.4. *Let A be an elliptic cubic Artin-Schelter algebra where σ has infinite order.*

1. *The category $\mathcal{C}(1,0)$ consists of one object namely the simple object S_{-2}*

$$0 \;\underset{0}{\overset{0}{\rightrightarrows}}\; k \;\underset{0}{\overset{0}{\rightrightarrows}}\; 0 \;\underset{0}{\overset{0}{\rightrightarrows}}\; 0$$

2. *The representations in $\mathcal{C}(1,1)$ are the representations of Γ of the form*

$$k \;\underset{\beta}{\overset{\alpha}{\rightrightarrows}}\; k \;\underset{\beta'}{\overset{\alpha'}{\rightrightarrows}}\; k \;\underset{0}{\overset{0}{\rightrightarrows}}\; 0 \quad (6.23)$$

where $((\alpha,\beta),(\alpha',\beta')) \in (\mathbb{P}^1 \times \mathbb{P}^1) - C$.

Proof. The first statement being trivial by dimension arguments, we turn to the second part. Let $F \in \mathcal{C}(1,1)$. Then F is given by (6.23) for some scalars $\alpha, \beta, \alpha', \beta' \in k$. We will first show that $((\alpha,\beta),(\alpha',\beta')) \in \mathbb{P}^1 \times \mathbb{P}^1$ i.e. $(\alpha,\beta) \neq (0,0)$ and $(\alpha',\beta') \neq (0,0)$.

If α, β were both zero then for any $p \in C$ there is a non-zero morphism in $\operatorname{Hom}_\Gamma(p,M)$, given by (writing $\operatorname{pr}_1 \sigma^i p = (\alpha_i, \beta_i)$)

$$\begin{array}{ccccccc}
k & \underset{\beta_{-3}}{\overset{\alpha_{-3}}{\rightrightarrows}} & k & \underset{\beta_{-2}}{\overset{\alpha_{-2}}{\rightrightarrows}} & k & \underset{\beta_{-1}}{\overset{\alpha_{-1}}{\rightrightarrows}} & k \\
{\scriptstyle \mathrm{id}}\downarrow & & \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle 0} \\
k & \underset{0}{\overset{0}{\rightrightarrows}} & k & \underset{\beta'}{\overset{\alpha'}{\rightrightarrows}} & k & \underset{0}{\overset{0}{\rightrightarrows}} & k
\end{array}$$

Thus $(\alpha, \beta) \in \mathbb{P}^1$. Further, assume by contradiction that $(\alpha', \beta') = (0, 0)$. There is a point $p \in C$ such that $\mathrm{pr}_1\, \sigma^{-3} p = (\alpha, \beta)$. This follows from the fact that for any point $p_1 \in \mathbb{P}^1$ there is a point $p_2 \in \mathbb{P}^1$ such that $(p_1, p_2) \in C$. We then find a non-zero morphism in $\mathrm{Hom}_\Gamma(p, F)$ given by (writing $\mathrm{pr}_1\, \sigma^i p = (\alpha_i, \beta_i)$)

$$\begin{array}{ccccccc}
k & \overset{\alpha}{\underset{\beta}{\rightrightarrows}} & k & \overset{\alpha_{-2}}{\underset{\beta_{-2}}{\rightrightarrows}} & k & \overset{\alpha_{-1}}{\underset{\beta_{-1}}{\rightrightarrows}} & k \\
{\scriptstyle \mathrm{id}}\downarrow & & {\scriptstyle \mathrm{id}}\downarrow & & {\scriptstyle 0}\downarrow & & {\scriptstyle 0}\downarrow \\
k & \overset{\alpha}{\underset{\beta}{\rightrightarrows}} & k & \overset{0}{\underset{0}{\rightrightarrows}} & k & \overset{0}{\underset{0}{\rightrightarrows}} & k
\end{array}$$

yielding the desired contradiction. Thus we have shown that $((\alpha, \beta), (\alpha', \beta')) \in \mathbb{P}^1 \times \mathbb{P}^1$. Furthermore the condition $\mathrm{Hom}_\Gamma(p, F) = 0$ for all $p \in C$ implies $((\alpha, \beta), (\alpha', \beta')) \notin C$. Indeed, if $((\alpha, \beta), (\alpha', \beta')) = q$ were a point in C then $\tau = (\mathrm{id}, \mathrm{id}, \mathrm{id}, 0)$ is a non-zero morphism in $\mathrm{Hom}_\Gamma(p, F)$ where $p = \sigma^3 q$ given by the commutative diagram (write $\mathrm{pr}_1\, \sigma^{-1} p = (\alpha'', \beta'')$)

$$\begin{array}{ccccccc}
k & \overset{\alpha}{\underset{\beta}{\rightrightarrows}} & k & \overset{\alpha'}{\underset{\beta'}{\rightrightarrows}} & k & \overset{\alpha''}{\underset{\beta''}{\rightrightarrows}} & k \\
{\scriptstyle \mathrm{id}}\downarrow & & {\scriptstyle \mathrm{id}}\downarrow & & {\scriptstyle \mathrm{id}}\downarrow & & {\scriptstyle 0}\downarrow \\
k & \overset{\alpha}{\underset{\beta}{\rightrightarrows}} & k & \overset{\alpha'}{\underset{\beta'}{\rightrightarrows}} & k & \overset{0}{\underset{0}{\rightrightarrows}} & k
\end{array}$$

We conclude that $((\alpha, \beta), (\alpha', \beta')) \in (\mathbb{P}^1 \times \mathbb{P}^1) - C$.

Conversely let F as in (6.23) with $((\alpha, \beta), (\alpha', \beta')) \in (\mathbb{P}^1 \times \mathbb{P}^1) - C$. Then by consideration of the appropriate commutative diagrams we deduce $\mathrm{Hom}_\Gamma(p, F) = 0 = \mathrm{Hom}_\Gamma(F, p) = 0$ for all $p \in C$. □

6.7.5 Description of $\mathcal{R}_{(n_e, n_o)}(X)$ for the enveloping algebra

In this section we let A be the enveloping algebra H_c. Thus $C = E_{\mathrm{red}}$ is the diagonal Δ on $\mathbb{P}^1 \times \mathbb{P}^1$. Recall from §1.9.4 that the restriction σ_Δ is the identity. Our proof of the next lemma is in the same spirit as the proof of [16, Theorem 4.5(i)] for the homogenized Weyl algebra.

Lemma 6.7.5. *Let $\mathcal{I} \in \mathcal{R}_{(n_e, n_o)}(X)$ for some $(n_e, n_o) \in N \setminus \{(0, 0)\}$. Consider for any integer m the linear map $M(Z_m)$ induced by multiplication by $z = xy - yx$*

$$M(Z_m) : H^1(X, \mathcal{I}(-m)) \to H^1(X, \mathcal{I}(-m + 2))$$

Then $M(Z_m)$ is surjective for $m < 4$ and injective for $m > 2$.

Proof. Let m be any integer and put $\mathcal{Q} = \pi(A/zA) = \pi D$. Then $u^* \mathcal{Q} = \mathcal{O}_\Delta$. Applying $\mathcal{H}om_X(-, \mathcal{I})$ to $0 \to \mathcal{O}_X(m - 2) \xrightarrow{z} \mathcal{O}_X(m) \to \mathcal{Q}(m) \to 0$ yields

$$\mathrm{Ext}^1_X(\mathcal{Q}(m), \mathcal{I}) \to H^1(X, \mathcal{I}(-m)) \xrightarrow{M(Z_m)} H^1(X, \mathcal{I}(-m + 2)) \to \mathrm{Ext}^2_X(\mathcal{Q}(m), \mathcal{I}).$$

Furthermore Theorem 1.10.5 (Serre duality) implies

$$\mathrm{Ext}^1_X(\mathcal{Q}(m),\mathcal{I}) \cong \mathrm{Ext}^1_X(\mathcal{I},\mathcal{Q}(m-4))',\ \mathrm{Ext}^2_X(\mathcal{Q}(m),\mathcal{I}) \cong \mathrm{Hom}_X(\mathcal{I},\mathcal{Q}(m-4))'.$$

On the other hand since $\mathrm{RHom}_X(\mathcal{I},u_*\mathcal{O}_\Delta(m-4)) \cong \mathrm{RHom}_\Gamma(Lu^*\mathcal{I},\mathcal{O}_\Delta(m-4))$ by (6.17) and $Lu^*\mathcal{I} = \mathcal{O}_\Delta$ (see Remark 6.6.5) we derive

$$\mathrm{Hom}_X(\mathcal{I},\mathcal{Q}(m-4)) = H^0(\Delta,\mathcal{O}_\Delta(m-4)) = D_{m-4} = 0 \text{ for } m < 4$$

and by Serre duality on Δ

$$\mathrm{Ext}^1_X(\mathcal{I},\mathcal{Q}(m-4)) = H^1(\Delta,\mathcal{O}_\Delta(m-4)) \cong D'_{-m+2} = 0 \text{ for } m > 2$$

which completes the proof. □

Theorem 6.7.6. *Let $A = H_c$ be the enveloping algebra. Let $(n_e,n_o) \in N \setminus \{(0,0)\}$. Define for any $M \in \mathrm{mod}(\Gamma)$ the linear maps*

$$\begin{aligned} M(Z_{-3}) &= M(Y_{-2})M(X_{-3}) - M(X_{-2})M(Y_{-3}) \\ M(Z_{-2}) &= M(Y_{-1})M(X_{-2}) - M(X_{-1})M(Y_{-2}) \end{aligned}$$

There is an equivalence of categories

$$\mathcal{R}_{(n_e,n_o)}(X) \underset{\mathrm{Tor}_1^\Gamma(-,\mathcal{E})}{\overset{\mathrm{Ext}^1_X(\mathcal{E},-)}{\rightleftarrows}} \mathcal{C}_{(n_e,n_o)}(\Gamma)$$

where

$$\mathcal{C}_{(n_e,n_o)}(\Gamma) = \{M \in \mathrm{mod}(\Gamma) \mid \underline{\dim} M = (n_o,n_e,n_o,n_e-1) \text{ and } M(Z_{-3}) \text{ isomorphism and } M(Z_{-2}) \text{ surjective}\}. \quad (6.24)$$

Proof. Due to Theorem 6.7.3 it will be sufficient to prove that the descriptions (6.22) (6.24) coincide. One inclusion follows from directly from Lemma 6.7.5, so let us assume $M \in \mathrm{mod}(\Gamma)$ for which $M(Z_{-3})$ is an isomorphism and $M(Z_{-2})$ is surjective. Let $p = ((\alpha:\beta),(\alpha:\beta)) \in \Delta$ and write $p \in \mathrm{mod}(\Gamma)$ for the corresponding representation of the quiver Γ. Let $\tau = (\tau_{-3},\tau_{-2},\tau_{-1},\tau_0) \in \mathrm{Hom}_\Gamma(p,M)$ be any morphism. Thus we have a commutative diagram in $\mathrm{mod}(k)$

$$\begin{array}{ccccccc} k & \overset{\alpha}{\underset{\beta}{\rightrightarrows}} & k & \overset{\alpha}{\underset{\beta}{\rightrightarrows}} & k & \overset{\alpha}{\underset{\beta}{\rightrightarrows}} & k \\ \downarrow{\scriptstyle \tau_{-3}} & & \downarrow{\scriptstyle \tau_{-2}} & & \downarrow{\scriptstyle \tau_{-1}} & & \downarrow{\scriptstyle \tau_0} \\ M_{-3} & \overset{M(X_{-3})}{\underset{M(Y_{-3})}{\rightrightarrows}} & M_{-2} & \overset{M(X_{-2})}{\underset{M(Y_{-2})}{\rightrightarrows}} & M_{-1} & \overset{M(X_{-1})}{\underset{M(Y_{-1})}{\rightrightarrows}} & M_0 \end{array}$$

We claim $\tau_{-3} = 0$. Assume by contradiction this is not the case. Since $M(Z_{-3})$ is an isomorphism we surely have $v = M(Z_{-3})\tau_{-3}(1) \neq 0$. On the other hand, by the commutativity of the above diagram

$$M(Z_{-3})\tau_{-3}(1) = (M(Y_{-2})M(X_{-3}) - M(X_{-2})M(Y_{-3}))\,\tau_{-3}(1) = \tau_{-1}(\alpha\beta - \beta\alpha) = 0$$

leading to the desired contradiction. Thus $\tau_{-3} = 0$. It follows that $\tau = 0$.

By similar arguments we may show for $\tau = (\tau_{-3}, \tau_{-2}, \tau_{-1}, \tau_0) \in \mathrm{Hom}_\Gamma(M, p)$ we have $\tau_0 = 0$, which implies $\tau = 0$. □

6.7.6 Restriction to a full subquiver

Let A be an elliptic cubic Artin-Schelter algebra. Although the description of $\mathcal{C}_{(n_e,n_o)}(\Gamma)$ in Theorem 6.7.3 is quite elementary, it is not easy to handle. Similar as in Chapter 2 and [52] for quadratic Artin-Schelter algebras we show representations in $\mathcal{C}_{(n_e,n_o)}(\Gamma)$ are completely determined by the four leftmost maps.

Let Γ^0 be the full subquiver of Γ consisting of the vertices $-3, -2, -1$ in (6.18). Let $\mathrm{Res} : \mathrm{Mod}(\Gamma) \to \mathrm{Mod}(\Gamma^0)$ be the obvious restriction functor. Res has a left adjoint which we denote by Ind. Note $\mathrm{Res} \circ \mathrm{Ind} = \mathrm{id}$. If $M \in \mathrm{Mod}(\Gamma)$ we will denote $\mathrm{Res}\, M$ by M^0.

In general, two objects A and B of an abelian category $\mathcal{C}$ are called *perpendicular*, denoted by $A \perp B$, if $\mathrm{Hom}_{\mathcal{C}}(A, B) = \mathrm{Ext}^1_{\mathcal{C}}(A, B) = 0$. For an object $B \in \mathcal{C}_f$ we define ${}^\perp B$ as the full subcategory of $\mathcal{C}_f$ which objects are

$${}^\perp B = \{A \in \mathcal{C}_f \mid A \perp B\}.$$

Repeating the arguments from the proof of Lemma 2.4.6 we have $M = \mathrm{Ind}\,\mathrm{Res}\, M$ for $M \in \mathrm{mod}(\Gamma)$ if and only if $M \perp S_0$. This means the functors Res and Ind define inverse equivalences [12]

$$\mathrm{mod}(\Gamma) \supset {}^\perp S_0 \underset{\mathrm{Ind}}{\overset{\mathrm{Res}}{\rightleftarrows}} \mathrm{mod}(\Gamma^0) \tag{6.25}$$

Lemma 6.7.7. *Let $(n_e, n_o) \in N \setminus \{(0,0)\}$. Then $\mathcal{C}_{(n_e,n_o)}(\Gamma) \subset {}^\perp S_0$.*

Proof. Similar as the proof of Lemma 2.4.7. See also [52]. □

Lemma 6.7.8. *Let $p \in C$ and $\mathcal{Q}$ be a conic object on X. Write $p = \mathrm{Hom}_X(\mathcal{E}, \mathcal{N}_p)$ and $Q = \mathrm{Ext}^1_X(\mathcal{E}, \mathcal{Q}(-1))$. Then $p \perp S_0$ and $Q \perp S_0$.*

Proof. That $p, Q \in \mathrm{mod}(\Gamma)$ follows from Lemmas 6.7.1 and 6.7.2. By (6.17) we have $\mathrm{Ext}^i_\Gamma(p, S_0) \cong \mathrm{Ext}^i_X(\mathcal{N}_p, \mathcal{O}) = 0$ for $i = 0, 1$. Similarly $\mathrm{Ext}^i_\Gamma(Q, S_0) = \mathrm{Ext}^{i-1}_\Gamma(Q[-1], S_0) \cong \mathrm{Ext}^{i-1}_X(\mathcal{Q}, \mathcal{O}) = 0$ for $i = 0, 1$. This proves what we want. □

6.7.7 Stable representations

Our next objective is to show that the representations in $\mathcal{C}_{(n_e,n_o)}(X)$ restricted to Γ^0 are stable. We will use the generalities on (semi)stable quiver representations from Section 1.3.

The following lemmas are elementary.

Lemma 6.7.9. *Let $p \in C$. Then $\operatorname{Res} p \in \operatorname{mod}(\Gamma^0)$ is θ-stable for $\theta = (-1, 0, 1)$.*

Proof. We have $(\underline{\dim} \operatorname{Res} p) \cdot \theta = (1, 1, 1) \cdot (-1, 0, 1) = 0$, so what remains to verify is that $(\underline{\dim} N) \cdot \theta > 0$ for all non-trivial subreprestations $N \subset \operatorname{Res} p$. For such $N \subset \operatorname{Res} p$ we have a commutative diagram (writing $pr_1 \sigma^i p = (\alpha_i, \beta_i)$)

$$\begin{array}{lccccc}
N: & N_{-3} & \overset{\gamma}{\underset{\delta}{\rightrightarrows}} & N_{-2} & \overset{\gamma'}{\underset{\delta'}{\rightrightarrows}} & N_{-1} \\
 & \downarrow{\scriptstyle \iota_{-3}} & & \downarrow{\scriptstyle \iota_{-2}} & & \downarrow{\scriptstyle \iota_{-1}} \\
\operatorname{Res} p: & k & \overset{\alpha_{-3}}{\underset{\beta_{-3}}{\rightrightarrows}} & k & \overset{\alpha_{-2}}{\underset{\beta_{-2}}{\rightrightarrows}} & k
\end{array}$$

where $\gamma, \delta, \gamma', \delta' \in k$ and $\iota_{-3}, \iota_{-2}, \iota_{-1}$ are injective maps. We claim that $\iota_{-3} = 0$. Indeed, if $\iota_{-3} \neq 0$ then it is easy to see from $(\alpha_{-3}, \beta_{-3}) \neq 0$ and $(\alpha_{-2}, \beta_{-2}) \neq 0$ that this implies $\iota_{-2} \neq 0$ and $\iota_{-1} \neq 0$. But then $N = \operatorname{Res} p$, contradiction. Hence $\iota_{-3} = 0$. Since ι_{-3} is injective we must have $N_{-3} = 0$ and consequently $\gamma = \delta = 0$. Similarly we have $\iota_{-2} \neq 0 \Rightarrow \iota_{-1} \neq 0$. Thus either $\underline{\dim} N = (0, 1, 1)$ or $\underline{\dim} N = (0, 0, 1)$. Note that both cases are possible:

$$\begin{array}{lccccc}
N: & 0 & \overset{0}{\underset{0}{\rightrightarrows}} & k & \overset{\alpha_{-2}}{\underset{\beta_{-2}}{\rightrightarrows}} & k \\
 & \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \mathrm{id}} \\
\operatorname{Res} p: & k & \overset{\alpha_{-3}}{\underset{\beta_{-3}}{\rightrightarrows}} & k & \overset{\alpha_{-2}}{\underset{\beta_{-2}}{\rightrightarrows}} & k
\end{array}$$

and

$$\begin{array}{lccccc}
N: & 0 & \overset{0}{\underset{0}{\rightrightarrows}} & 0 & \overset{0}{\underset{0}{\rightrightarrows}} & k \\
 & \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle \mathrm{id}} \\
\operatorname{Res} p: & k & \overset{\alpha_{-3}}{\underset{\beta_{-3}}{\rightrightarrows}} & k & \overset{\alpha_{-2}}{\underset{\beta_{-2}}{\rightrightarrows}} & k
\end{array}$$

Since for both cases we have $\theta \cdot \underline{\dim} N = 1 > 0$ we conclude that $\operatorname{Res} p$ is θ-stable for $\theta = (-1, 0, 1)$. ☐

Lemma 6.7.10. *Assume $0 \neq F, G \in \operatorname{mod}(\Gamma^0)$ are θ-semistable for $\theta = (-1, 0, 1)$.*

1. *If G is θ-stable then every non-zero map in $\operatorname{Hom}_{\Gamma^0}(F, G)$ is surjective.*
2. *If F is θ-stable then every non-zero map in $\operatorname{Hom}_{\Gamma^0}(F, G)$ is injective.*

Proof. Left to the reader. See also the proof of Lemma 2.4.15. □

Proposition 6.7.11. *Let $\mathcal{Q} = \pi(A/vA)$ be a conic object on X where $v = \alpha x^2 + \beta xy + \gamma yx + \delta y^2 \in A_2$ and write $Q = \operatorname{Ext}^1_X(\mathcal{E}, \mathcal{Q}(-1)) \in \operatorname{mod}(\Gamma)$. Let $(n_e, n_o) \in N \setminus \{(0,0)\}$, $\mathcal{I} \in \mathcal{R}_{(n_e,n_o)}(X)$ and write $M = \operatorname{Ext}^1_X(\mathcal{E}, \mathcal{I}) \in \operatorname{mod}(\Gamma)$. Then the following are equivalent:*

1. $M^0 \perp Q^0$
2. $\operatorname{Hom}_{\Gamma^0}(M^0, Q^0) = 0$
3. $\operatorname{Hom}_X(\mathcal{I}, \mathcal{Q}(-1)) = 0$
4. $\mathcal{I} \perp \mathcal{Q}(-1)$
5. *The following linear map is an isomorphism*

$$\begin{aligned} f =& \alpha M^0(X_{-2})M^0(X_{-3}) + \beta M^0(Y_{-2})M^0(X_{-3}) \\ &+ \gamma M^0(X_{-2})M^0(Y_{-3}) + \delta M^0(Y_{-2})M^0(Y_{-3}) : M_{-3} \to M_{-1} \end{aligned} \tag{6.26}$$

Proof. By definition (1) implies (2) and its converse is seen by $\chi(M^0, Q^0) = 0$. The equivalence (2) ⇔ (3) follows from (6.17) as

$$\begin{aligned} \operatorname{Hom}_{\Gamma^0}(M^0, Q^0) &= \operatorname{Hom}_\Gamma(\operatorname{Ind} M^0, Q) = \operatorname{Hom}_\Gamma(M, Q) = H^0(\operatorname{RHom}_\Gamma(M, Q)) \\ &\cong H^0(\operatorname{RHom}_X(\mathcal{I}, \mathcal{Q}(-1))) = \operatorname{Hom}_X(\mathcal{I}, \mathcal{Q}(-1)). \end{aligned}$$

To prove (3) ⇒ (4), as $\mathcal{I}$ is a normalized line bundle with invariants we may write $[\mathcal{I}] = [\mathcal{O}] - 2(n_e - n_o)[\mathcal{S}] + (n_e - n_o)[\mathcal{Q}] - n_o[\mathcal{P}]$. Furthermore (6.7) yields $[\mathcal{Q}(-1)] = [\mathcal{Q}] - [\mathcal{P}]$ and using Proposition 6.2.6 one computes $\chi(\mathcal{I}, \mathcal{Q}(-1)) = 0$. Since Serre duality gives $\operatorname{Ext}^2_X(\mathcal{I}, \mathcal{Q}(-1)) \cong \operatorname{Hom}_X(\mathcal{Q}(3), \mathcal{I})' = 0$ we conclude $\mathcal{I} \perp \mathcal{Q}(-1)$ if and only if $\operatorname{Hom}_X(\mathcal{I}, \mathcal{Q}(-1)) = 0$.

Finally we prove the equivalence between (4) and (5). Applying $\operatorname{Hom}_X(-, \mathcal{I})$ to $0 \to \mathcal{O}(1) \to \mathcal{O}(3) \to \mathcal{Q}(3) \to 0$ gives a long exact sequence of k-vector spaces

$$0 \to \operatorname{Ext}^1_X(\mathcal{Q}(3), \mathcal{I}) \to M_{-3} \xrightarrow{f} M_{-1} \to \operatorname{Ext}^2_X(\mathcal{Q}(3), \mathcal{I}) \to 0$$

where we have used Theorem 6.2.11. As the middle map f is given by (6.26) we deduce f is an isomorphism if and only if $\operatorname{Ext}^1_X(\mathcal{Q}(3), \mathcal{I}) = 0 = \operatorname{Ext}^2_X(\mathcal{Q}(3), \mathcal{I})$. Invoking Serre duality on X the latter is equivalent with $\mathcal{I} \perp \mathcal{Q}(-1)$. □

Remark 6.7.12. In case $A = H_c$ is the enveloping algebra we recover the property $M^0(Z_{-2})$ being an isomorphism (Theorem 6.7.6), as for the conic object $\mathcal{Q} = \pi(H_c/zH_c)$

$$\operatorname{RHom}_X(\mathcal{I}, \mathcal{Q}(-1)) = \operatorname{RHom}_X(\mathcal{I}, u_*\mathcal{O}_\Delta(-1)) \cong \operatorname{RHom}_\Delta(Lu^*\mathcal{I}, \mathcal{O}_\Delta(-1))$$

and since $Lu^*\mathcal{I} = \mathcal{O}_\Delta$ we obtain $\operatorname{Hom}_X(\mathcal{I}, \mathcal{Q}(-1)) \cong \operatorname{Hom}_\Delta(\mathcal{O}_\Delta, \mathcal{O}_\Delta(-1)) = 0$. As a consequence the restriction of the representations in $\mathcal{C}_{(n_e,n_o)}(\Gamma)$ to Γ^0 are θ-semistable for some $\theta \in \mathbb{Z}^3$. Since $\chi(-, \underline{\dim} Q^0) = - \cdot (-1, 0, 1)$ we may take $\theta = (-1, 0, 1)$.

Inspired by the previous remark one might try to find, for all elliptic cubic Artin-Schelter algebras A, a conic object $\mathcal{Q}$ on X for which $\mathrm{Hom}_X(\mathcal{I}, \mathcal{Q}(-1))$ is zero for all $\mathcal{I} \in \mathcal{R}_{(n_e,n_o)}(X)$. We did not manage to find such a conic object independent of $\mathcal{I}$. However, we are able to prove that for a fixed normalized line bundle $\mathcal{I}$ on X there is at least one conic object $\mathcal{Q}$ (which depends on $\mathcal{I}$) for which $\mathrm{Hom}_X(\mathcal{I}, \mathcal{Q}(-1)) = 0$. We will then show how this leads to a proof that the representations in $\mathcal{C}_{(n_e,n_o)}(X)$ restricted to Γ^0 are stable.

Proposition 6.7.13. *Assume k is uncountable and σ has infinite order. Let $(n_e, n_o) \in N$ such that $(n_e - 1, n_o - 1) \in N$. Let $\mathcal{I} \in \mathcal{R}_{(n_e,n_o)}(X)$. Then the set of conic objects $\mathcal{Q}$ for which $\mathrm{Hom}_X(\mathcal{I}, \mathcal{Q}(-1)) \neq 0$ is a curve of degree n_o in $\mathbb{P}(A_2)$. In particular this set is non-empty.*

Proof. By Proposition 6.7.11 we have $\mathrm{Hom}_X(\mathcal{I}, \mathcal{Q}(-1)) \neq 0$ if and only if $\det f = 0$. This is a homogeneous equation in $(\alpha, \beta, \gamma, \delta)$ of degree n_o and we have to show it is not identically zero, i.e. we have to show there is at least one $\mathcal{Q}$ for which $\mathrm{Hom}_X(\mathcal{I}, \mathcal{Q}(-1)) = 0$. This follows from Lemma 6.7.14 and Lemma 6.5.9 below. □

Lemma 6.7.14. *Assume k is uncountable and σ has infinite order. Let $(n_e, n_o) \in N$ and $l \geq 0$ as in* (6.11)*. Let $\mathcal{I} \in \mathcal{R}_{(n_e,n_o)}(X)$. Let $p, p' \in C$ such that $p \neq \sigma^m p'$ for all integers m. Modulo zero-dimensional objects, there exist at most l different critical conic objects $\mathcal{Q}$ on X for which $\mathrm{Hom}_X(\mathcal{I}, \mathcal{Q}(-1)) \neq 0$ and $\mathrm{Hom}_X(\mathcal{Q}, \mathcal{N}_p) \neq 0$, $\mathrm{Hom}_X(\mathcal{Q}, \mathcal{N}_{p'}) \neq 0$.*

Proof. That $p \neq \sigma^m p'$ for all integers m assures $\mathrm{Hom}_X(\mathcal{N}_p, \mathcal{N}_{p'}(m)) = 0$ and $\mathrm{Hom}_X(\mathcal{N}_{p'}(m), \mathcal{N}_p) = 0$ for all integers m, which we will use throughout this proof.

We prove the statement by induction on l. First let $l = 0$ and assume by contradiction there is a non-zero map $f : \mathcal{I} \to \mathcal{Q}(-1)$. Let $\mathcal{I}'(-2)$ be the kernel of f. By Lemma 6.5.5 the image of f is a shifted conic object $\mathcal{Q}'(-d)$ where $d \geq 1$. Using (6.7) one computes $[\mathcal{I}'] = [\mathcal{O}] - 2(n_e - n_o)[\mathcal{S}] + (n_e - n_o)[\mathcal{Q}] - (n_o - d)[\mathcal{P}]$. It follows that $\mathcal{I}'$ is a normalized line bundle on X with invariants $(n_e - d, n_o - d)$. Since $(n_e - d, n_o - d) \notin N$ this yields a contradiction with Proposition 6.3.2.

Let $l > 0$. Let $(\mathcal{Q}_i)_{i=1,\ldots,m}$ be different critical conic objects (modulo zero-dimensional objects) satisfying $\mathrm{Hom}_X(\mathcal{I}, \mathcal{Q}_i(-1)) \neq 0$ and $\mathrm{Hom}_X(\mathcal{Q}_i, \mathcal{N}_p) \neq 0$, $\mathrm{Hom}_X(\mathcal{Q}_i, \mathcal{N}_{p'}) \neq 0$. We will show $m \leq l$. If $m = 0$ then we are done. So assume $m > 0$. Let $\mathcal{Q}_i'(-1)$ be the kernel of a non-trivial map $\mathcal{Q}_i \to \mathcal{N}_p$. By Lemma 6.5.5 $\mathcal{Q}_i'$ is a critical conic object, and we have an exact sequence

$$0 \to \mathcal{Q}_i'(-1) \to \mathcal{Q}_i \to \mathcal{N}_p \to 0.$$

Applying $\mathrm{Hom}_X(-, \mathcal{N}_{p'})$ we find $\mathrm{Hom}_X(\mathcal{Q}_i'(-2), \mathcal{N}_{p'}(-1)) \neq 0$. Lemma 6.5.1(2) implies such a map factors through $\mathcal{Q}_i(-1)$. Let $\mathcal{Q}_i''(-3)$ be the kernel of a non-trivial map $\mathcal{Q}_i'(-2) \to \mathcal{N}_{p'}(-1)$. Again by Lemma 6.5.5 $\mathcal{Q}_i''$ is a critical conic object, and

$$0 \to \mathcal{Q}_i''(-3) \to \mathcal{Q}_i'(-2) \xrightarrow{\pi} \mathcal{N}_{p'}(-1) \to 0. \tag{6.27}$$

Applying $\operatorname{Hom}_X(-,\mathcal{N}_p(-2))$ yields $\operatorname{Hom}_X(\mathcal{Q}_i'',\mathcal{N}_p(1)) \neq 0$. Furthermore, as (6.27) is non-split, Serre duality (Theorem 1.10.5) implies $0 \neq \operatorname{Ext}^1_X(\mathcal{N}_{p'}(-1),\mathcal{Q}_i''(-3)) \cong \operatorname{Ext}^1_X(\mathcal{Q}_i'',\mathcal{N}_{p'}(-2))'$. By $\chi(\mathcal{Q}_i'',\mathcal{N}_{p'}(-2)) = 0$ and again by Serre duality $\operatorname{Ext}^2_X(\mathcal{Q}_i'',\mathcal{N}_{p'}(-2)) \cong \operatorname{Hom}_X(\mathcal{N}_{p'},\mathcal{Q}_i''(-1))' = 0$ we deduce $\operatorname{Hom}_X(\mathcal{Q}_i'',\mathcal{N}_{p'}(-2)) \neq 0$.

Let $\mathcal{I}'(-2)$ be the kernel of a non-trivial map $\iota : \mathcal{I} \to \mathcal{Q}_1(-1)$. As in first part of the proof one may show that $\mathcal{I}'$ is a normalized line bundle on X with invariants $(n_e - d, n_o - d)$ for some $d \geq 1$.

Since $\mathcal{N}_p(-1) = u_*\mathcal{O}_{p'}$ for some point $p' \in C$ it follows by adjointness $\dim_k \operatorname{Hom}_X(\mathcal{I},\mathcal{N}_p(-1)) = \dim_k \operatorname{Hom}_C(u^*\mathcal{I},\mathcal{O}_{p'}) = 1$. Hence for all i the composition $a_i : \mathcal{I} \to \mathcal{Q}_i(-1) \to \mathcal{N}_p(-1)$ is a scalar multiple of a_1. Thus for all i the map $a_i \circ \iota$ is a scalar multiple of $a_1 \circ \iota = 0$. Hence the composition $\mathcal{I}'(-2) \to \mathcal{I} \to \mathcal{Q}_i(-1)$ maps $\mathcal{I}'(-2)$ to $\mathcal{Q}_i'(-2)$.

As pointed out above the map π in (6.27) factors through $\mathcal{Q}_i(-1)$. Thus the composition $\mathcal{I}'(-2) \to \mathcal{Q}_i'(-2) \to \mathcal{N}_{p'}(-1)$ is the same as the composition $b_i : \mathcal{I}'(-2) \to \mathcal{I} \to \mathcal{Q}_i(-1) \to \mathcal{N}_{p'}(-1)$. Same reasoning as above shows $b_i = 0$ for all i hence the composition $\mathcal{I}'(-2) \to \mathcal{Q}_i'(-2)$ maps $\mathcal{I}'(-2)$ to $\mathcal{Q}_i''(-3)$.

We claim this map must be non-zero for $i > 1$. If not then there is a non-trivial map $\mathcal{I}/\mathcal{I}'(-2) \to \mathcal{Q}_i(-1)$ and since $\mathcal{I}/\mathcal{I}'(-2)$ is also a subobject of $\mathcal{Q}_1(-1)$ it follows that $\mathcal{Q}_1$ and $\mathcal{Q}_i$ have a common subobject. By Lemma 6.5.8 this contradicts the assumption $\mathcal{Q}_1$ and $\mathcal{Q}_i$ being different modulo zero dimensional objects.

Hence $\operatorname{Hom}_X(\mathcal{I}',\mathcal{Q}_i''(-1)) \neq 0$ for $i = 2,\ldots,m$. Since the $\mathcal{Q}_i''$ are still different modulo zero dimensional objects and $\operatorname{Hom}_X(\mathcal{Q}_i'',\mathcal{N}_p(1)) \neq 0$, $\operatorname{Hom}_X(\mathcal{Q}_i'',\mathcal{N}_{p'}(-2)) \neq 0$ we obtain by induction hypothesis $m-1 \leq l-d \leq l-1$ and hence $m \leq l$. □

The following lemma is similar to Lemma 2.4.16.

Lemma 6.7.15. *Put $\theta = (-1,0,1)$. Let $V \in \operatorname{mod}(\Gamma^0)$ and assume the forms $-\cdot\theta$ and $\chi(-,\underline{\dim}V)$ are proportional. Let $F \in \operatorname{mod}(\Gamma^0)$ for which $F \perp V$. Then*

1. *If $F' \subset F$ such that $\underline{\dim}F' \cdot \theta = 0$ then $F' \perp V$ and $F/F' \perp V$*
2. $\operatorname{Hom}_{\Gamma^0}(F,\operatorname{Res} p) = \operatorname{Hom}_{\Gamma^0}(\operatorname{Res} p, F) = 0$ *for all $p \in C$ for which* $\operatorname{Res} p$ *is not perpendicular to V.*

Proof. Using the Euler form (6.20) on Γ it is easy to see $\underline{\dim}V = (3l,2l,l)$ for some $l \in \mathbb{N}$. Now proceed as in the proof of Lemma 2.4.16. □

Lemma 6.7.16. *Assume σ has infinite order. Let $N \in \operatorname{mod}(\Gamma^0)$ with dimension vector (n_o,n_e,n_o) and assume $n_e \neq 0$. If $\operatorname{Hom}_{\Gamma^0}(N,\operatorname{Res} p) = \operatorname{Hom}_{\Gamma^0}(\operatorname{Res} p,N) = 0$ for all $p \in C$ then $\dim_k(\operatorname{Ind} N)_0 \leq n_e - 1$.*

Proof. This is the same as the proof of Lemma 2.4.17. □

We now come to the main result of this section.

Theorem 6.7.17. *Assume k is uncountable. Let A be an elliptic cubic Artin-Schelter algebra for which σ has infinite order. Let $(n_e, n_o) \in N \setminus \{(0,0)\}$. If $M \in \mathcal{C}_{(n_e,n_o)}(\Gamma)$ then M^0 is θ-stable for $\theta = (-1,0,1)$.*

Proof. Let $\mathcal{I} \in \mathcal{R}_{(n_e,n_o)}(X)$ such that $M[-1] = \mathrm{RHom}_X(\mathcal{E}, \mathcal{I})$ and write $F = M^0$. By Propositions 6.7.11 and 6.7.13 there exists a conic object $\mathcal{Q}$ on X for which $F \perp Q^0$, where $Q[-1] = \mathrm{RHom}_X(\mathcal{E}, \mathcal{Q}(-1))$. This shows F is θ-semistable for $\theta = (-1,0,1)$. Hence there is a representation $F' \subsetneq F$ such that F/F' is θ-stable. We will prove F' is necessarily zero, from which the theorem will follow.

Assume by contradiction $F' \neq 0$. Since F/F' is θ-stable we have $\theta \cdot \underline{\dim} F/F' = 0$ thus we may put $\underline{\dim} F/F' = (n_o - m_o, n_e - m_e, n_o - m_o)$ for some $m_o \leq n_o$, $m_e \leq n_e$ for which $(m_e, m_o) \neq (n_e, n_o)$. Note F' is θ-semistable. We now claim that for all $p \in C$ we have

$$\begin{aligned} &\mathrm{Hom}_{\Gamma^0}(F', \mathrm{Res}\, p) = \mathrm{Hom}_{\Gamma^0}(\mathrm{Res}\, p, F') = 0, \\ &\mathrm{Hom}_{\Gamma^0}(F/F', \mathrm{Res}\, p) = \mathrm{Hom}_{\Gamma^0}(\mathrm{Res}\, p, F/F') = 0. \end{aligned} \tag{6.28}$$

To show (6.28), let $p \in C$. Lemmas 6.5.9 and 6.7.14 imply there exists a conic object $\mathcal{Q}'$ on X for which $\mathrm{Hom}_X(\mathcal{Q}', \mathcal{N}_p(-3)) \neq 0$ and $\mathrm{Hom}_X(\mathcal{I}, \mathcal{Q}'(-1)) = 0$. Writing $Q'[-1] = \mathrm{RHom}_X(\mathcal{E}, \mathcal{Q}'(-1))$ and using Theorem 1.10.5 yields $\mathrm{Ext}^1_\Gamma(p, Q') \neq 0$ and $F \perp Q'$ by Proposition 6.7.11. In particular $\mathrm{Res}\, p$ is not perpendicular to $\mathrm{Res}\, Q'$. The claim (6.28) now follows from Lemma 6.7.15.

Combining (6.28) with Lemma 6.7.16 we find $\dim_k(\mathrm{Ind}\, F/F')_0 \leq n_e - m_e - 1$ and $\dim_k(\mathrm{Ind}\, F')_0 \leq m_e - 1$. Application of the right exact functor Ind on $0 \to F' \to F \to F/F' \to 0$ gives a long exact sequence in $\mathrm{mod}(\Gamma)$

$$\ldots \to \mathrm{Ind}\, F' \to \mathrm{Ind}\, F \to \mathrm{Ind}\, F/F' \to 0$$

and counting dimensions yields

$$\begin{aligned} n_e - 1 = \dim_k(\mathrm{Ind}\, F)_0 &\leq \dim_k(\mathrm{Ind}\, F')_0 + \dim_k(\mathrm{Ind}\, F/F')_0 \\ &\leq (m_e - 1) + (n_e - m_e - 1) = n_e - 2 \end{aligned}$$

which is absurd. Thus $F' = 0$ hence F is θ-stable, finishing the proof. □

Remark 6.7.18. In case A is of generic type A and σ has infinite order we do not need the hypothesis k is uncountable in Theorem 6.7.17. This is because we may prove Proposition 6.7.13 without the additional hypothesis on k, using the proof of Lemma 6.7.14 and Remark 6.5.10.

Remark 6.7.19. In case $A = H_c$ is the enveloping algebra we may choose $\mathcal{Q} = \pi(H_c/zH_c)$ in the proof of Theorem 6.7.17. See also Remark 6.7.12. Again we do not need the hypothesis k is uncountable.

6.7.8 Second description of $\mathcal{R}_{(n_e,n_o)}(X)$

For $(n_e, n_o) \in N \setminus \{(0,0)\}$ we denote $\mathcal{D}_{(n_e,n_o)}(\Gamma^0)$ for the image of $\mathcal{C}_{(n_e,n_o)}(\Gamma)$ under the equivalence (6.25).

Theorem 6.7.20. *Assume k is uncountable. Let A be an elliptic cubic Artin-Schelter algebra for which σ has infinite order. Let $\theta = (-1,0,1)$ and $(n_e,n_o) \in N \setminus \{(0,0),(1,1)\}$. Then there is an equivalence of categories*

$$\mathcal{C}_{(n_e,n_o)}(\Gamma) \underset{\text{Ind}}{\overset{\text{Res}}{\rightleftarrows}} \mathcal{D}_{(n_e,n_o)}(\Gamma^0)$$

where

$$\mathcal{D}_{(n_e,n_o)}(\Gamma^0) = \{F \in \text{mod}(\Gamma^0) \mid \underline{\dim} F = (n_o, n_e, n_o), F \textit{ is } \theta\textit{-stable}, \\ \dim_k(\text{Ind}\, F)_0 \geq n_e - 1\}.$$

Proof. Below we often use the equivalence $\mathcal{C}_{(n_e,n_0)}(\Gamma) \cong \mathcal{R}_{(n_e,n_o)}(X)$ from Theorem 6.7.3. We break the proof into five steps.

Step 1. $\text{Res}(\mathcal{C}_{(n_e,n_o)}(\Gamma)) \subset \mathcal{D}_{(n_e,n_o)}(\Gamma^0)$. This follows from Theorem 6.7.17 and Lemma 6.7.16.

Step 2. If $F \in \mathcal{D}_{(n_e,n_o)}(\Gamma^0)$ then $\text{Hom}_{\Gamma^0}(F, \text{Res}\, p) = \text{Hom}_{\Gamma^0}(\text{Res}\, p, F) = 0$ for all $p \in C$. Indeed, by Lemma 6.7.9 and Lemma 6.7.10 any non-zero morphism would yield an isomorphism $F \cong \text{Res}\, p$, contradicting the assumption $(n_e, n_o) \neq (1,1)$.

Step 3. If $F \in \mathcal{D}_{(n_e,n_o)}(\Gamma^0)$ then $\text{Hom}_\Gamma(\text{Ind}\, F, p) = \text{Hom}_\Gamma(p, \text{Ind}\, F) = 0$ for all $p \in E$. This follows from $0 = \text{Hom}_{\Gamma^0}(F, \text{Res}\, p) = \text{Hom}_\Gamma(\text{Ind}\, F, p)$ and

$$0 = \text{Hom}_{\Gamma^0}(\text{Res}\, p, F) = \text{Hom}_{\Gamma^0}(\text{Res}\, p, \text{Res}\,\text{Ind}\, F) = \text{Hom}_\Gamma(\text{Ind}\,\text{Res}\, p, \text{Ind}\, F)$$

where we have used Step 2 and $\text{Ind}\,\text{Res}\, p = p$ by Lemma 6.7.8.

Step 4. $\text{Ind}(\mathcal{D}_{(n_e,n_o)}(\Gamma^0)) \subset \mathcal{C}_{(n_e,n_o)}(\Gamma)$. Let $F \in \mathcal{D}_{(n_e,n_o)}(\Gamma^0)$. Combining Step 2 with Lemma 6.7.16 gives $\underline{\dim}\, \text{Ind}\, F = (n_o, n_e, n_o, n_e - 1)$. Now Step 3 shows $\text{Ind}\, F \in \mathcal{C}_{(n_e,n_o)}(\Gamma)$.

Step 5. Ind and Res are inverses to each other. To prove this we only need to show $\text{Ind}\,\text{Res}\, F = F$ for $F \in \mathcal{C}_{(n_e,n_o)}(\Gamma)$. This follows from Lemma 6.7.7. □

For $(n_e, n_o) \in N \setminus \{(0,0),(1,1)\}$ let $\alpha = (n_o, n_e, n_o)$ and put

$$\begin{aligned} \widetilde{D}_{(n_e,n_o)} &= \{F \in \text{Rep}_\alpha(\Gamma^0) \mid F \in \mathcal{D}_{(n_e,n_o)}(\Gamma^0)\} \\ &= \{F \in \text{Rep}_\alpha(\Gamma^0) \mid F \text{ is } \theta\text{-stable}, \ \dim_k(\text{Ind}\, F)_0 \geq n_e - 1\}. \end{aligned} \tag{6.29}$$

As $\widetilde{D}_{(n_e,n_o)}$ is a closed subset of the dense open subset of $\operatorname{Rep}_\alpha(\Gamma^0)$ consisting of all θ-stable representations we obtain that $\widetilde{D}_{(n_e,n_o)}$ is locally closed.

Denote $\mathrm{Gl}_\alpha(k) = \mathrm{Gl}_{n_e}(k) \times \mathrm{Gl}_{n_o}(k) \times \mathrm{Gl}_{n_e}(k)$. Put $D_{(n_e,n_o)} = \widetilde{D}_{(n_e,n_o)} /\!\!/ \mathrm{Gl}_\alpha(k)$. The next theorem provides the first part of Theorem 1 from the introduction. Our proof is, up to some minor computations, completely analogous to the proof of Theorem 2.4.21. For convenience of the reader we have included the proof.

Theorem 6.7.21. *Assume k is uncountable. Let A be an elliptic cubic Artin-Schelter algebra for which σ has infinite order. Then for $(n_e, n_o) \in N$ there exists a smooth locally closed variety $D_{(n_e,n_o)}$ of dimension $2\left(n_e - (n_e - n_o)^2\right)$ such that the isomorphism classes in $\mathcal{D}_{(n_e,n_o)}(\Gamma^0)$ (and hence in $\mathcal{R}_{(n_e,n_o)}(X)$) are in natural bijection with the points in $D_{(n_e,n_o)}$.*

Proof. For $(n_e, n_o) = (0, 0)$ or $(1, 1)$ we refer to Corollaries 6.2.12, 6.7.4 to see that we may take a point for $D_{(0,0)}$ and $(\mathbb{P}^1 \times \mathbb{P}^1) - C$ for $D_{(1,1)}$. So we may assume $(n_e, n_o) \in N \setminus \{(0, 0), (1, 1)\}$ throughout this proof.

Since all representations in $\widetilde{D}_{(n_e,n_o)}$ are stable, all $\mathrm{Gl}(\alpha)$-orbits on $\widetilde{D}_{(n_e,n_o)}$ are closed and so $D_{(n_e,n_o)}$ is really the orbit space for the $\mathrm{Gl}(\alpha)$ action on $\widetilde{D}_{(n_e,n_o)}$. This proves that the isomorphism classes in $\mathcal{D}_{(n_e,n_o)}(\Gamma)$ are in natural bijection with the points in $D_{(n_e,n_o)}$.

To prove $D_{(n_e,n_o)}$ is smooth it suffices to prove $\widetilde{D}_{(n_e,n_o)}$ is smooth [54]. We first estimate the dimension of $\widetilde{D}_{(n_e,n_o)}$. Write the equations of A in the usual form (1.18). For $n_e \times n_o$-matrices X, Y and $n_o \times n_e$-matrices X', Y' let $M_A^t(X', Y', X, Y)$ be obtained from M_A^t by replacing x^2, xy, yx, y^2 by $X'X$, $Y'X$, $X'Y$, $Y'Y$ (thus $M_A^t(X', Y', X, Y)$ is a $2n_o \times 2n_o$-matrix). Then $\widetilde{D}(n_e, n_o)$ has the following alternative description:

$$\widetilde{D}_{(n_e,n_o)} = \{(X, Y, X', Y') \in M_{n_e\times n_o}(k)^2 \times M_{n_o\times n_e}(k)^2 \mid (X, Y, X', Y') \text{ is } \theta\text{-stable} \\ \text{and } \operatorname{rank} M_A(X', Y', X, Y) \leq 2n_o - (n_e - 1)\}.$$

By Proposition 6.3.2, $\widetilde{D}_{(n_e,n_o)}$ is non-empty. The (X, Y, X', Y') for which the associated representation is stable are a dense open subset of $M_{n_e\times n_o}(k)^2 \times M_{n_o\times n_e}(k)^2$ and hence they represent a quasi-variety of dimension $4n_en_o$. Imposing $M_A(X', Y', X, Y)$ should have corank $\geq n_e - 1$ represents $(2n_o - (2n_o - (n_e - 1)))^2 = (n_e - 1)^2$ independent conditions. So the irreducible components of $\widetilde{D}_{(n_e,n_o)}$ have dimension $\geq 4n_en_o - (n_e - 1)^2$. Define $\widetilde{C}_{(n_e,n_o)}$ by

$$\{G \in \operatorname{Rep}(\Gamma, \widetilde{\alpha}) \mid G \cong \operatorname{Ind}\operatorname{Res} G, \operatorname{Res} G \in \widetilde{D}_{(n_e,n_o)}\}$$

where $\widetilde{\alpha} = (n_o, n_e, n_o, n_e - 1)$ (as usual we assume the points of $\operatorname{Rep}(\Gamma, \widetilde{\alpha})$ to satisfy the relation imposed on Γ). To extend $F \in \widetilde{D}_{(n_e,n_o)}$ to a point in $\widetilde{C}_{(n_e,n_o)}$ we need to choose a basis in $(\operatorname{Ind} F)_0$. Thus $\widetilde{C}_{(n_e,n_o)}$ is a principal $\mathrm{Gl}_{n_e-1}(k)$ fiber bundle over $\widetilde{D}_{(n_e,n_o)}$. In particular $\widetilde{C}_{(n_e,n_o)}$ is smooth if and only $\widetilde{D}_{(n_e,n_o)}$ is smooth and the irreducible components of $\widetilde{C}_{(n_e,n_o)}$ have dimension $\geq 4n_en_o - (n_e-1)^2 + (n_e-1)^2 = 4n_en_o$. By the description of $\mathcal{C}_{(n_e,n_o)}$ in Theorem 6.7.3 it follows that $\widetilde{C}_{(n_e,n_o)}$ is an open subset of $\operatorname{Rep}(\Gamma,\widetilde{\alpha})$.

Let $x \in \widetilde{C}_{(n_e,n_o)}$. The stabilizer of x consists of scalars thus if we put $G = \operatorname{Gl}(\widetilde{\alpha})/k^*$ then we have inclusions $\operatorname{Lie}(G) \subset T_x(\widetilde{C}_{(n_e,n_o)}) = T_x(\operatorname{Rep}(\Gamma,\widetilde{\alpha}))$. Next there is a natural inclusion $T_x(\operatorname{Rep}(\Gamma,\widetilde{\alpha}))/\operatorname{Lie}(G) \hookrightarrow \operatorname{Ext}^1_\Gamma(x,x)$. Now x corresponds to some normalized line bundle $\mathcal{I}$ on X and we have $\operatorname{Ext}^1_\Gamma(x,x) = \operatorname{Ext}^1_X(\mathcal{I},\mathcal{I})$. Lemma 6.2.13 implies $\dim_k \operatorname{Ext}^1_X(\mathcal{I},\mathcal{I}) = 2(n_e - (n_e-n_o)^2)$. Hence we obtain $4n_en_o \leq \dim T_x(\widetilde{C}_{(n_e,n_o)}) \leq \dim_k \operatorname{Ext}^1_\Gamma(x,x) + \dim G$ and the right-hand is equal to $2(n_e-(n_e-n_o)^2) + (n_o^2+n_e^2+n_o^2+(n_e-1)^2-1) = 4n_en_o$. Thus $\dim T_x(\widetilde{C}_n) = 4n_en_o$ is constant and hence $\widetilde{C}_{(n_e,n_o)}$ is smooth. We also obtain $\dim \widetilde{D}_{(n_e,n_o)} = 4n_en_o - (n_e-1)^2$. The dimension of $D(n_e,n_o)$ is equal to

$$\begin{aligned}\dim \widetilde{D}_{(n_e,n_o)} - \dim \operatorname{Gl}(\alpha) + 1 &= 4n_en_o - (n_e-1)^2 - (n_o^2+n_e^2+n_o^2) + 1 \\ &= 2(n_e - (n_e-n_o)^2).\end{aligned}$$

This finishes the proof. ☐

6.7.9 Descriptions of the varieties $D_{(n_e,n_o)}$ for the enveloping algebra

In case of the enveloping algebra we may further simplify the description of $D_{(n_e,n_o)}$.

Theorem 6.7.22. *Let $A = H_c$ be the enveloping algebra. Let $(n_e,n_o) \in N$. The isomorphism classes in $R_{(n_e,n_o)}(A)$ are in natural bijection with the points in the smooth affine variety $D_{(n_e,n_o)}$ of dimension $2(n_e-(n_e-n_o)^2)$ where*

$$\begin{aligned}D_{(n_e,n_o)} = \{(X,Y,X',Y') \in M_{n_e\times n_o}(k)^2 \times M_{n_o\times n_e}(k)^2 \mid Y'X - X'Y \textit{ isomorphism},\\ \operatorname{rank}\begin{pmatrix} Y'Y & X'Y-2Y'X \\ Y'X-2X'Y & X'X\end{pmatrix} \leq 2n_o - (n_e-1)\}/\operatorname{Gl}_\alpha(k)\end{aligned}$$

Proof. For $(n_e,n_o) = (0,0)$ or $(1,1)$ we refer to Corollaries 6.2.12, 6.7.4 to see that $D_{(n_e,n_o)}$ has the description as in the statement of the current theorem. So we may assume $(n_e,n_o) \in N \setminus \{(0,0),(1,1)\}$ throughout this proof.

Consider the conic object $\mathcal{Q} = \pi(A/zA)$ on X where $z = xy - yx$. Write $Q = \operatorname{Ext}^1_X(\mathcal{E},\mathcal{Q}(-1)) \in \operatorname{mod}(\Gamma)$ (Lemma 6.7.2) and put $V = \operatorname{Res} Q$.

It is sufficient to show $\mathcal{D}_{(n_e,n_o)}(\Gamma^0)$ has the alternative description

$$\begin{aligned}\mathcal{D}'_{(n_e,n_o)}(\Gamma^0) := \{F \in \operatorname{mod}(\Gamma^0) \mid \underline{\dim} F = (n_o,n_e,n_o),\\ F \perp V, \dim_k(\operatorname{Ind} F)_0 \geq n_e - 1\}.\end{aligned}$$

Indeed, if $\mathcal{D}_{(n_e,n_o)}(\Gamma^0) = \mathcal{D}'_{(n_e,n_o)}(\Gamma^0)$ we then have

$$\begin{aligned}\widetilde{D}_{(n_e,n_o)} &= \{F \in \operatorname{Rep}_\alpha(\Gamma^0) \mid F \in \mathcal{D}_{(n_e,n_o)}(\Gamma^0)\}\\ &= \{F \in \operatorname{Rep}_\alpha(\Gamma^0) \mid \phi_V(F) \neq 0, \dim_k(\operatorname{Ind} F)_0 \geq n_e - 1\}\end{aligned}$$

from which is clear that $\widetilde{D}_{(n_e,n_o)}$ is a closed subset of $\{\phi_V \neq 0\}$ so in particular $\widetilde{D}_{(n_e,n_o)}$ is affine. This means $D_{(n_e,n_o)} = \widetilde{D}_{(n_e,n_o)} / \operatorname{Gl}(\alpha)$ is an affine variety. Theorem 6.7.21 further implies $D_{(n_e,n_o)}$ is smooth of dimension $2(n_e - (n_e - n_o)^2)$ which points are in natural bijection with the isomorphism classes in $\mathcal{R}_{(n_e,n_o)}(X)$ whence in $R_{(n_e,n_o)}(A)$ by §6.2.3. Moreover, as in the proof of Theorem 6.7.21, $\widetilde{D}_{(n_e,n_o)}$ has the alternative description

$$\begin{aligned}\widetilde{D}_{(n_e,n_o)} = \{F = ((X,Y),(X',Y')) \in \operatorname{Rep}_{(n_o,n_e,n_o)}(\Gamma^0) \mid F \perp V\\ \text{and } \operatorname{rank} M_A(X',Y',X,Y) \leq 2n_o - (n_e-1)\}.\end{aligned}$$

As shown in Proposition 6.7.11 the condition $(X,Y,X',Y') \perp V$ is equivalent with saying $Y'X - X'Y$ is an isomorphism. Explicitely writing down M_A by (1.18), (1.8) yields the desired description of $D_{(n_e,n_o)}$.

So to prove the current theorem it remains to prove $\mathcal{D}_{(n_e,n_o)}(\Gamma^0) = \mathcal{D}'_{(n_e,n_o)}(\Gamma^0)$. We will do this by showing that the functors Res and Ind define inverse equivalences between $\mathcal{C}_{(n_e,n_o)}(\Gamma)$ and $\mathcal{D}'_{(n_e,n_o)}(\Gamma^0)$.

Step 1. $\operatorname{Res}(\mathcal{C}_{(n_e,n_o)}(\Gamma)) \subset \mathcal{D}'_{(n_e,n_o)}(\Gamma^0)$. Let $M \in \mathcal{C}_{(n_e,n_o)}(\Gamma)$. That $\operatorname{Res} M \perp V$ follows from Proposition 6.7.11 and Remark 6.7.12. Further, since $\operatorname{Ind}\operatorname{Res} M = M$ by Lemma 6.7.7 we find $\dim_k(\operatorname{Ind}\operatorname{Res} M)_0 = n_e - 1$ by Theorem 6.7.3.

Step 2. If $p \in C$ then $\operatorname{Res} p \in \operatorname{mod}(\Gamma^0)$ is not perpendicular to V. Indeed, by the equivalences (6.25) and (6.17) one finds $\operatorname{RHom}_{\Gamma^0}(\operatorname{Res} p, V) = \operatorname{RHom}_\Gamma(p, Q) = \operatorname{RHom}_X(\mathcal{N}_p, \mathcal{Q}[1])$. Thus $\operatorname{Ext}^1_{\Gamma_0}(\operatorname{Res} p, V) = \operatorname{Ext}^2_X(\mathcal{N}_p, \mathcal{Q})$. Further, Serre duality (Theorem 1.10.5) yields $\operatorname{Ext}^2_X(\mathcal{N}_p, \mathcal{Q}) \cong \operatorname{Hom}_X(\mathcal{Q}, \mathcal{N}_{\sigma^4 p})' = \operatorname{Hom}_X(\mathcal{Q}, \mathcal{N}_p)'$, which is non-zero by Lemma 6.5.1(1). This proves Step 2.

Step 3. $\operatorname{Ind}(\mathcal{D}'_{(n_e,n_o)}(\Gamma^0)) \subset \mathcal{C}_{(n_e,n_o)}(\Gamma)$. Let $F \in \mathcal{D}'_{(n_e,n_o)}(\Gamma^0)$. Combining Step 2 with Lemmas 6.7.24, 6.7.15 and 6.7.16 we obtain $\dim_k(\operatorname{Ind} F)_0 = n_e - 1$. It remains to show $\operatorname{Hom}_\Gamma(\operatorname{Ind} F, p) = \operatorname{Hom}_\Gamma(p, \operatorname{Ind} F) = 0$ for $p \in C$. By Lemma 6.7.9 we have $p = \operatorname{Ind}\operatorname{Res} p$. Thus $\operatorname{Hom}_\Gamma(\operatorname{Ind} F, p) = \operatorname{Hom}_{\Gamma^0}(F, \operatorname{Res} p) = 0$ and similarly

$$\operatorname{Hom}_\Gamma(p, \operatorname{Ind} F) = \operatorname{Hom}_{\Gamma^0}(\operatorname{Res} p, \operatorname{Res}\operatorname{Ind} F) = \operatorname{Hom}_{\Gamma^0}(\operatorname{Res} p, F) = 0$$

where we have used Lemma 6.7.15 again.

Step 4. Ind and Res are inverses to each other. This follows from Lemma 6.7.7. □

We further simplify the description of $D_{(n_e,n_o)}$ as

Theorem 6.7.23. *Let $A = H_c$ be the enveloping algebra. Let $(n_e, n_o) \in N$. The isomorphism classes in $R_{(n_e,n_o)}(A)$ are in natural bijection with the points in the smooth affine variety $D_{(n_e,n_o)}$ of dimension $2(n_e - (n_e - n_o)^2)$ where*

$$D_{(n_e,n_o)} = \{(\mathbb{X}, \mathbb{Y}, \mathbb{X}', \mathbb{Y}') \in M_{n_e \times n_o}(k)^2 \times M_{n_o \times n_e}(k)^2 \mid \mathbb{Y}'\mathbb{X} - \mathbb{X}'\mathbb{Y} = \mathbb{I} \text{ and } \operatorname{rank}(\mathbb{Y}\mathbb{X}' - \mathbb{X}\mathbb{Y}' - \mathbb{I}) \leq 1\} / \operatorname{Gl}_{n_e}(k) \times \operatorname{Gl}_{n_o}(k)$$

Proof. For $(n_e, n_o) = (0, 0)$ or $(1, 1)$ we refer to Corollaries 6.2.12, 6.7.4 to see that $D_{(n_e,n_o)}$ has the description as in the statement of theorem. So we assume $(n_e, n_o) \in N \setminus \{(0, 0), (1, 1)\}$ throughout this proof.

Similarly as in Theorem 6.7.6 we define for any $F \in \operatorname{mod}(\Gamma^0)$ the linear map

$$F(Z_{-3}) = F(Y_{-2})F(X_{-3}) - F(X_{-2})F(Y_{-3})$$

In order to prove the current theorem, it is sufficient to show $\mathcal{D}_{(n_e,n_o)}(\Gamma^0)$ has the alternative description

$$\mathcal{D}''_{(n_e,n_o)}(\Gamma^0) := \{F \in \operatorname{mod}(\Gamma^0) \mid \underline{\dim} F = (n_o, n_e, n_o), F(Z_{-3}) \text{ isomorphism}, \operatorname{rank}(F(Y_{-3})F(Z_{-3})^{-1}F(X_{-2}) - F(X_{-3})F(Z_{-3})^{-1}F(Y_{-2}) - \mathrm{id}) \leq 1\}.$$

Indeed, if $\mathcal{D}_{(n_e,n_o)}(\Gamma^0) = \mathcal{D}''_{(n_e,n_o)}(\Gamma^0)$ then

$$\begin{aligned}\widetilde{D}_{(n_e,n_o)} &= \{F \in \operatorname{Rep}_\alpha(\Gamma^0) \mid F \in \mathcal{D}_{(n_e,n_o)}(\Gamma^0)\} \\ &= \{(X, Y, X', Y') \in \operatorname{Rep}_\alpha(\Gamma^0) \mid Z := Y'X - X'Y \text{ isomorphism}, \\ &\qquad \operatorname{rank}(YZ^{-1}X' - XZ^{-1}Y' - \mathrm{id}) \leq 1\}\end{aligned}$$

and by $D_{(n_e,n_o)} = \widetilde{D}_{(n_e,n_o)} / \operatorname{Gl}_\alpha(k)$ the statement of the current theorem will follow.

What remains to prove is $\mathcal{D}_{(n_e,n_o)}(\Gamma^0) = \mathcal{D}''_{(n_e,n_o)}(\Gamma^0)$. We will do this by showing that the functors Res and Ind define inverse equivalences between $\mathcal{C}_{(n_e,n_o)}(\Gamma)$ and $\mathcal{D}''_{(n_e,n_o)}(\Gamma^0)$. This is done in the following three steps.

Step 1. $\operatorname{Res}(\mathcal{C}_{(n_e,n_o)}(\Gamma)) \subset \mathcal{D}''_{(n_e,n_o)}(\Gamma^0)$. Let $M \in \mathcal{C}_{(n_e,n_o)}(\Gamma)$ and put $F = \operatorname{Res} M$. For convenience we denote $X = M(X_{-3})$, $X' = M(X_{-2})$, $X'' = M(X_{-1})$ (similarly for Y). Theorem 6.7.6 already implies $Z = Y'X - X'Y$ is an isomorphism and $Z' = Y''X' - X''Y'$ is surjective. Thus to show Step 1, what remains to prove is $\operatorname{rank}(YZ^{-1}X' - XZ^{-1}Y' - \mathrm{id}) \leq 1$. From (6.19) we deduce

$$\begin{aligned} Y''Y'X - 2Y''X'Y + X''Y'Y = 0 \\ X''X'Y - 2X''Y'X + Y''X'X = 0 \end{aligned} \tag{6.30}$$

and these equations may be written as $X''Z = Z'X$, $Y''Z = Z'Y$. Since Z is an isomorphism we find $X'' = Z'XZ^{-1}$ and $Y'' = Z'YZ^{-1}$. Substitution yields

$$Z' = Y''X' - X''Y' = Z'(YZ^{-1}X' - XZ^{-1}Y')$$

thus $Z'(YZ^{-1}X' - XZ^{-1}Y' - \mathrm{id}) = 0$. As Z' is surjective, it has a one dimensional kernel, completing proof of Step 1.

Step 2. $\operatorname{Ind}(\mathcal{D}''_{(n_e,n_o)}(\Gamma^0)) \subset \mathcal{C}_{(n_e,n_o)}(\Gamma)$. To prove so, let $F \in \mathcal{D}''_{(n_e,n_o)}(\Gamma^0)$. We will construct a representation M of Γ for which $M \in \mathcal{C}_{(n_e,n_o)}(\Gamma)$ and $\operatorname{Res} M = F$. For then, $F \in \mathcal{D}_{(n_e,n_o)}(\Gamma^0)$ by Theorem 6.7.20 hence $\operatorname{Ind} F = M \in \mathcal{C}_{(n_e,n_o)}(\Gamma)$.

For simplicity we denote $X = F(X_{-3})$, $X' = F(X_{-2})$ (similarly for Y). Put $Z = Y'X - X'Y$. Let Z' denote the projection $F_{-2} \to F_{-2}/\operatorname{im}(YZ^{-1}X' - XZ^{-1}Y' - \mathrm{id})$. Define the linear maps $X'' = Z'XZ^{-1}$, $Y'' = Z'YZ^{-1}$. We now define M as

$$F_{-3} \underset{Y}{\overset{X}{\rightrightarrows}} F_{-2} \underset{Y'}{\overset{X'}{\rightrightarrows}} F_{-1} \underset{Y''}{\overset{X''}{\rightrightarrows}} \operatorname{im} Z'$$

In fact $\dim_k M_0 = n_e - 1$, as otherwise $\mathrm{id} = YZ^{-1}X' - XZ^{-1}Y'$ and by taking traces we find $n_e = \operatorname{Tr}(YZ^{-1}X' - XZ^{-1}Y') = \operatorname{Tr}(-Z^{-1}(Y'X - X'Y)) = \operatorname{Tr}(-Z^{-1}Z) = -n_e$ whence $n_e = 0$. By definition (6.9) of N this leads to $n_o = 0$, contradiction the assumption $(n_e, n_o) \neq (0, 0)$.

To prove $M \in \operatorname{mod}(\Gamma)$ we need to check the relations (6.30). This is easy to do. One also checks $Z' = Y''X' - X''Y'$. Now Theorem 6.7.6 implies $M \in \mathcal{C}_{(n_e,n_o)}(\Gamma)$.

By the construction of M we have $\operatorname{Res} M = F$. This proves Step 2.

Step 3. Ind and Res are inverses to each other. This follows from Lemma 6.7.7. □

6.7.10 Description of the varieties $D_{(n_e,n_o)}$ for generic type A

In Theorem 6.7.22 we have simplified the description of the varieties $D_{(n_e,n_o)}$ for the enveloping algebra. That such a simplification is possible is due to the fact that there exists a conic object $\mathcal{Q}$ on X for which $M^0 \perp Q^0$ for all $M \in \mathcal{C}_{(n_e,n_o)}(\Gamma)$.

It is therefore natural to ask if, for general cubic Artin-Schelter algebras A, there exists a conic object $\mathcal{Q}$ on X for which $M^0 \perp Q^0$ for all $M \in \mathcal{C}_{(n_e,n_o)}(\Gamma)$ where $\mathcal{Q}$ is independent of M. This is unknown (and probably unlikely). However there is another interpretation. In case of the enveloping algebra we relied on fact $u^*\mathcal{M} \cong \mathcal{O}_\Delta$ for all normalized line bundles $\mathcal{M}$ on X. For generic A we have

Lemma 6.7.24. *Let A be a cubic Artin-Schelter algebra of generic type A for which σ has infinite order. There exists $V \in \operatorname{mod}(\Gamma^0)$ with $\underline{\dim} V = (6, 4, 2)$ for which*

1. *for all $M \in \mathcal{C}_{(n_e,n_o)}(\Gamma)$ we have $M^0 \perp V$, and*
2. *if $p \in C$ then* $\operatorname{Res} p$ *is not perpendicular to V.*

Proof. Obtained by repeating the arguments in Lemma 2.4.22 where, in the current setting, we pick a degree zero line bundle $\mathcal{U}$ on C which is not of the form $\mathcal{O}((o) - (2(n_e + n_o)\xi))$ for $n_e, n_o \in \mathbb{N}$, see Proposition 6.6.3. □

Theorem 6.7.25. *Let A be a cubic Artin-Schelter algebra of generic type A for which σ has infinite order. Let $V \in \operatorname{mod}(\Gamma^0)$ be as in Lemma 6.7.24. Let $(n_e, n_o) \in N$.*

The isomorphism classes in $R(n_e, n_o)(A)$ *are in natural bijection with the points in the smooth affine variety* $D_{(n_e,n_o)}$ *of dimension* $2(n_e - (n_e - n_o)^2)$ *where*

$$D_{(n_e,n_o)} = \{F = (X, Y, X', Y') \in M_{n_e \times n_o}(k)^2 \times M_{n_o \times n_e}(k)^2 \mid F \perp V,$$
$$\operatorname{rank} \begin{pmatrix} aY'Y + cX'X & bX'Y + aY'X \\ bY'X + aX'Y & aX'X + cY'Y \end{pmatrix} \leq 2n_o - (n_e - 1)\}/\operatorname{Gl}_\alpha(k)$$

Proof. Analogous to the proof of Theorem 6.7.22 where one uses Lemma 6.7.24. □

6.8 Filtrations of line bundles and proof of Theorem 12

Let A be an elliptic cubic Artin-Schelter algebra for which σ has infinite order. The following analogue of Lemma 2.4.25 shows how to reduce the invariants of a line bundle.

Lemma 6.8.1. *Assume k is uncountable and σ has infinite order. Let $(n_e, n_o) \in N$ such that $(n_e - 1, n_o - 1) \in N$. Let $\mathcal{I} \in \mathcal{R}_{(n_e,n_o)}(X)$. Then there exists a conic object $\mathcal{Q}$ on X for which $\operatorname{Ext}^1_X(\mathcal{Q}(1), \mathcal{I}(-2)) \neq 0$. If $\mathcal{J} = \pi J$ is the middle term of a corresponding non-trivial extension and $\mathcal{J}^{**} = \pi J^{**}$ then $\mathcal{J}^{**} \in \mathcal{R}_{(m_e,m_o)}(X)$ with $m_e < n_e$, $m_o < n_o$. Furthermore $\mathcal{J}^{**}/\mathcal{I}(-2)$ is a shifted conic object on X.*

Proof. We have $\operatorname{Ext}^1_X(\mathcal{Q}(1), \mathcal{I}(-2)) \cong \operatorname{Ext}^1_X(\mathcal{I}(-2), \mathcal{Q}(-3))' = \operatorname{Ext}^1_X(\mathcal{I}, \mathcal{Q}(-1))'$ and $\operatorname{Ext}^2_X(\mathcal{I}, \mathcal{Q}(-1)) \cong \operatorname{Hom}_X(\mathcal{Q}(-1), \mathcal{I}(-4))' = 0$ by Theorem 1.10.5 (Serre duality). Thus $\chi(\mathcal{I}, \mathcal{Q}(-1)) = 0$ shows $\dim_k \operatorname{Hom}_X(\mathcal{I}, \mathcal{Q}(-1)) = \dim_k \operatorname{Ext}^1_X(\mathcal{I}, \mathcal{Q}(-1))$. Hence it follows from Proposition 6.7.13 there exist a conic object $\mathcal{Q}$ for which $\operatorname{Ext}^1_X(\mathcal{Q}(1), \mathcal{I}(-2)) \neq 0$.

Let $\mathcal{J} = \pi J$ be the middle term of a non-trivial extension of $\mathcal{I}(-2)$ by $\mathcal{Q}(1)$. It is easy to see $\mathcal{J}$. A computation yields $[\mathcal{I}(-2)] = [\mathcal{O}] - 2(n_e - n_o)[\mathcal{S}] + (n_e - n_o - 1)[\mathcal{Q}] - n_o[\mathcal{P}]$ hence $[\mathcal{J}] = [\mathcal{O}] - 2(n_e - n_o)[\mathcal{S}] + (n_e - n_o)[\mathcal{Q}] - (n_o - 1)[\mathcal{P}]$. Thus $\mathcal{J}$ is normalized with invariants $(n_e - 1, n_o - 1)$.

By Theorem 1.9.8 GKdim $J^{**}/J \leq 1$. As σ has infinite order, any zero dimensional object on X admits a filtration by shifted point objects [8]. Hence $[\mathcal{J}^{**}/\mathcal{J}] = c[\mathcal{P}]$ for some $c \geq 0$ and therefore $[\mathcal{J}^{**}] = [\mathcal{O}] - 2(n_e - n_o)[\mathcal{S}] + (n_e - n_o)[\mathcal{Q}] - (n_o - c - 1)[\mathcal{P}]$. Thus $\mathcal{J}^{**} \in \mathcal{R}_{(m_e,m_o)}(X)$ where $m_o = n_o - c - 1 < n_o$ and $m_e = n_e - c - 1$. Let $\mathcal{N} = \mathcal{J}^{**}/\mathcal{I}(-2)$. Then $\mathcal{N}$ is pure and furthermore we have $[\mathcal{N}] = [\mathcal{Q}] + (c + 1)[\mathcal{P}]$. Thus $e(\mathcal{N}) = 1$. Moreover $\mathcal{Q} \subset \mathcal{N}$ and $\mathcal{N}/\mathcal{Q}$ is zero dimensional. By Lemma 6.5.6 $\mathcal{N}$ is a shifted conic object on X. □

Theorem 6.8.2. *Assume k is uncountable. Let A be an elliptic cubic Artin-Schelter algebra and assume σ has infinite order. Let $(n_e, n_o) \in N$ and l as in (6.11). Let $\mathcal{I} \in \mathcal{R}_{(n_e,n_o)}(X)$. Then there exists an integer m, $0 \leq m \leq l$ together with a filtration of line bundles $\mathcal{I}_0 \supset \mathcal{I}_1 \supset \cdots \supset \mathcal{I}_m = \mathcal{I}(-2l)$ on X with the property that the $\mathcal{I}_i/\mathcal{I}_{i+1}$ are shifted conic objects on X and $\mathcal{I}_0$ has invariants $(m_e, m_o) = (n_e - l, n_o - l)$.*

Proof. By Lemma 6.8.1, (6.11) and downwards induction on l. □

We may now prove Theorem 12 from the introduction.

Proof of Theorem 12. The first part of Theorem 12 is due to Theorem 6.8.2 and the equivalence $R(A) = \coprod_{(n_e,n_o)\in N} \mathcal{R}_{(n_e,n_o)}$ from §6.2.3. Proposition 6.3.3 implies $I_0 = \omega\mathcal{I}_0$ having a minimal resolution of the form (6.3). By Proposition 6.3.3 and Remark 6.3.4, I_0 and $\mathcal{I}_0$ are up to isomorphism uniquely determined by (m_e, m_o), and therefore by (n_e, n_o). This finishes the proof. □

Remark 6.8.3. By Remarks 6.7.18 and 6.7.19 we do not need the hypothesis k is uncountable in Theorem 6.8.2 (hence Theorem 12) in case $A = H_c$ is the enveloping algebra or A is of generic type A and σ has infinite order. Furthermore it follows from Proposition 6.4.1 that Theorem 6.8.2 is (trivially) true in case A is a linear cubic Artin-Schelter algebra, again without the hypothesis k is uncountable.

6.9 Invariant ring of the first Weyl algebra and proof of Theorem 14

In this final section we show how application of the previous results for the enveloping algebra H_c may be used to classify the right ideals of an invariant ring of the first Weyl algebra.

Let $A = H_c$ denote the enveloping algebra and write $z = xy - yx \in A_2$. In the notations of §1.9.4 the canonical normalizing element g is given by $z^2 \in A_4$ and $h = z$ is central. Consider the graded algebra $\Lambda = A[h^{-1}]$, the localisation of A at the powers of $h = z$, and its subalgebra Λ_0 of elements of degree zero. It is shown in [8, Theorem 8.20] that $\Lambda_0 = A_1^{\langle\varphi\rangle}$, the algebra of invariants of the first Weyl algebra $A_1 = k < x, y > /(xy - yx - 1)$ under the automorphism φ defined by $\varphi(x) = -x$, $\varphi(y) = -y$.

For any positive integer l, let V_l be the k-linear space spanned by the set $\{x^i y^j \mid i + j \text{ even and } i + j \leq 2l\}$. Then $k = V_0 \subset V_1 \subset \dots$ endow $A_1^{\langle\varphi\rangle}$ with a positive filtration.

The associated Rees ring $\mathrm{Rees}(A_1^{\langle\varphi\rangle}) = \bigoplus_{l\in\mathbb{N}} V_l$ is identified with the subring $\bigoplus_{l\in\mathbb{N}} V_l t^l$ of the ring of Laurent polynomials $A_1^{\langle\varphi\rangle}[t, t^{-1}]$, and identifying $t = h$ we see the Rees ring of $A_1^{\langle\varphi\rangle}$ is isomorphic to $A^{(2)}$, the 2-Veronese of A. The associated graded algebra $\mathrm{gr}(A_1^{\langle\varphi\rangle}) = \bigoplus_{l\in\mathbb{N}} V_l/V_{l-1}$ is isomorphic to $A^{(2)}/hA^{(2)} = k[x, y]^{(2)}$, the 2-Veronese of the commutative polynomial ring $k[x, y]$.

Similarly, for a filtered $A_1^{\langle\varphi\rangle}$-module M we write $\mathrm{Rees}(M) = \bigoplus_{l\in\mathbb{N}} M_l$, which is isomorphic to an object in $\mathrm{GrMod}(A^{(2)})$ and $\mathrm{gr}(M) = \bigoplus_{l\in\mathbb{N}} M_l/M_{l-1}$ for the associated graded module, identified with an object of $\mathrm{Mod}(k[x, y]^{(2)})$.

Write $\mathrm{Filt}(A_1^{\langle\varphi\rangle})$ for the category which objects are the filtered right $A_1^{\langle\varphi\rangle}$-modules and morphisms the $A_1^{\langle\varphi\rangle}$-morphisms $f : M \to N$ which are strict i.e. $N_n \cap \mathrm{im}(f) =$

$f(M_n)$ for all n. Write $\mathrm{GrMod}(A^{(2)})_h$ for the full subcategory of $\mathrm{GrMod}(A^{(2)})$ consisting of the h-torsion free modules. The exact functor

$$\mathrm{Rees}(-) : \mathrm{Filt}(A_1^{\langle\varphi\rangle}) \to \mathrm{GrMod}(A^{(2)})_h$$

is an equivalence and $(\mathrm{Rees}(M)[h^{-1}])_0 \cong M$ for all $M \in \mathrm{Filt}(A_1^{\langle\varphi\rangle})$.

Let $R(A_1^{\langle\varphi\rangle})$ denote the set of isomorphism classes of right $A_1^{\langle\varphi\rangle}$-ideals. Note

$$R(A_1^{\langle\varphi\rangle}) = \{M \in \mathrm{mod}(A_1^{\langle\varphi\rangle}) \mid M \text{ torsion free of rank one}\}/\text{ iso}$$

Performing a similar treatment as in [16, §4] yields

Proposition 6.9.1. *The set* $R(A_1^{\langle\varphi\rangle})$ *is in natural bijection with the isomorphism classes in the full subcategory of* $\mathrm{coh}(\Delta)$ *with objects*

$$\{\mathcal{M} \in \mathrm{coh}(X) \mid u^*\mathcal{M} \cong \mathcal{O}_\Delta\} \tag{6.31}$$

Proof. We will make use of the following commutative diagram

$$\begin{array}{ccccccc} \mathrm{filt}(A_1^{\langle\varphi\rangle}) & \xrightarrow{\mathrm{Rees}} & \mathrm{grmod}(A^{(2)}) & \xrightarrow{\pi} & \mathrm{tails}(A^{(2)}) & \xrightarrow{\cong} & \mathrm{tails}(A) \\ \cong\downarrow & & & & & & \downarrow u^* \\ \mathrm{filt}(A_1^{\langle\varphi\rangle}) & \xrightarrow{\mathrm{gr}} & \mathrm{grmod}(k[x,y]^{(2)}) & \xrightarrow{\pi} & \mathrm{tails}(k[x,y]^{(2)}) & \xrightarrow{\cong} & \mathrm{tails}(k[x,y]) \end{array} \tag{6.32}$$

Let $\mathcal{M} \in \mathrm{coh}(X)$ for which $u^*\mathcal{M} \cong \mathcal{O}_\Delta$. It follows from Remark 6.6.5 and §6.2.3 that $\mathcal{M} = \pi M$ for some $M \in \mathrm{grmod}(A)$ which is torsion free of rank one (and therefore critical). Thus $M[h^{-1}]_0$ is a critical rank one object i.e. $M[h^{-1}]_0 \in R(A_1^{\langle\varphi\rangle})$. To show this correspondence $\mathcal{M} \mapsto M[h^{-1}]_0$ is bijective it suffices to give its inverse.

Let $M \in \mathrm{mod}(A_1^{\langle\varphi\rangle})$ be torsion free of rank one and let us fix, temporarily, an embedding of M as an ideal of $A_1^{\langle\varphi\rangle}$. The filtration V_l on $A_1^{\langle\varphi\rangle}$ induces a filtration $M_l = M \cap V_l$ on M. Let us still write M for the associated object in $\mathrm{filt}(A_1^{\langle\varphi\rangle})$.

Arguing on the bottom half of (6.32) it follows that $\mathrm{gr}(M) \subset k[x,y]^{(2)}$ is a homogeneous ideal and writing $\overline{\mathcal{M}} \in \mathrm{tails}(k[x,y]) = \mathrm{coh}(\Delta)$ for the image of $\pi\,\mathrm{gr}(M)$ we deduce $\overline{\mathcal{M}} \subset \mathcal{O}_\Delta$. As the quotient $\mathcal{O}_\Delta/\overline{\mathcal{M}}$ has rank zero it is finite dimensional i.e. of the form $\mathcal{O}_D$ for some divisor D on Δ of degree $d = \deg D \geq 0$. Thus $\overline{\mathcal{M}} \cong \mathcal{O}_\Delta(-D)$. Redefine the filtration on M by $M_l^0 := M_{l+d}$ and write M^0 for the associated object in $\mathrm{filt}(A_1^{\langle\varphi\rangle})$. Repeating the arguments we now find $\overline{\mathcal{M}^0} \cong \mathcal{O}_\Delta$. By the commutativity of the diagram (6.32), M^0 now corresponds to an object $\mathcal{M}^0 \in \mathrm{tails}(A)$ for which $u^*\mathcal{M}^0 \cong \mathcal{O}_\Delta$. This finishes the proof. □

We end with the

Proof of Theorem 14. By Theorem 6.7.23, Proposition 6.9.1 and Remark 6.6.5. □

Appendix A

Serre duality for graded rings

This section is taken from [27, Appendix A], where we prove that (a generalization of) Bondal-Kapranov-Serre duality holds for graded rings. For the convenience of the reader we restate some definitions so that this appendix can be read independently of the rest of this work.

Let $\mathcal{A}$ be a k-linear Ext finite triangulated category. By this we mean that for all $\mathcal{M}, \mathcal{N} \in \mathcal{A}$ we have $\sum_n \dim_k \operatorname{Hom}_{\mathcal{A}}(\mathcal{M}, \mathcal{N}[n]) < \infty$. The category $\mathcal{A}$ is said to satisfy Bondal-Kapranov-Serre (BKS) duality if there is an autoequivalence $F : \mathcal{A} \to \mathcal{A}$ together with for all $A, B \in \mathcal{A}$ natural isomorphisms

$$\operatorname{Hom}_{\mathcal{A}}(A, B) \to \operatorname{Hom}_{\mathcal{A}}(B, FA)'$$

where $(-)'$ denotes the k-dual.

Let $\mathcal{C}$ be an abelian category. An object O in $D^b(\mathcal{C})$ is said to have finite projective (injective) dimension if $\operatorname{Ext}^i_{\mathcal{C}}(O, \mathcal{C}) = 0$ ($\operatorname{Ext}^i_{\mathcal{C}}(\mathcal{C}, O) = 0$) for $|i| > u$ for some $u \geq 0$. The minimal such u we call the projective (injective) dimension of O.

In this appendix we assume that A is a connected graded noetherian ring over a k. By $(-)'$ we denote the functor on graded vectorspaces which sends M to $\oplus_n M'_{-n}$. If we use notations which refer to the left structure of A then we adorn them with a superscript "$\circ$".

We make the following additional assumptions on A:

1. A satisfies χ and the functor τ has finite cohomogical dimension.

2. A satisfies χ° and the functor τ° has finite cohomogical dimension.

These conditions imply that A has a *balanced dualizing complex* [85] given by $R = R\tau(A)' = R\tau^\circ(A)'$ [77, 85]. Below we freely use the properties of such dualizing complexes.

We let $D(A)$ be the derived category of graded right A-modules. $D^b_f(A)$ will be the full subcategory of objects in $D^b(A)$ with finitely generated homology. The category

$D^b_f(\mathrm{Tails}(A))$ is the full subcategory of $D^b(\mathrm{Tails}(A))$ consisting of complexes with homology in $\mathrm{tails}(A)$.

We let $D^b_f(A)_{\mathrm{fpd}}$ ($D^b_f(A)_{\mathrm{fid}}$) be the full category of $D^b_f(A)$ consisting of objects of finite projective (injective) dimension. The categories $D^b_f(\mathrm{Tails}(A))_{\mathrm{fpd}}$, $D^b_f(\mathrm{Tails}(A))_{\mathrm{fid}}$ are defined in a similar way. The fact that τ has finite cohomological dimension implies $\pi A(n) \in D^b_f(\mathrm{Tails}(A))_{\mathrm{fpd}}$.

We will denote the functors $R\underline{\mathrm{Hom}}_A(-,R)$ and $R\underline{\mathrm{Hom}}_{A^\circ}(-,R)$ by D. Since they define a duality between $D^b_f(A)$ and $D^b_f(A^\circ)$ it is clear that they define a duality between $D^b_f(A)_{\mathrm{fid}}$ and $D^b_f(A^\circ)_{\mathrm{fpd}}$ and between $D^b_f(A)_{\mathrm{fpd}}$ and $D^b_f(A^\circ)_{\mathrm{fid}}$.

It is also clear that these functors induce a duality between $D^b_f(\mathrm{Tails}(A))$ and $D^b_f(\mathrm{Tails}(A^{\mathrm{opp}}))$. We denote these induced functors also by D. Again they define a duality between $D^b_f(\mathrm{Tails}(A))_{\mathrm{fid}}$ and $D^b_f(\mathrm{Tails}(A^\circ))_{\mathrm{fpd}}$ and between $D^b_f(\mathrm{Tails}(A))_{\mathrm{fpd}}$ and $D^b_f(\mathrm{Tails}(A^\circ))_{\mathrm{fid}}$. Recall the following:

Lemma A.1. *Let $\mathcal{P} \in D^b_f(\mathrm{Tails}(A))_{\mathrm{fpd}}$. Then there exists an object $P \in D^b_f(A)_{\mathrm{fpd}}$ such that $\mathcal{P}$ is a direct summand of πP.*

Proof. This can be deduced from general results about compact objects in triangulated categories. For simplicity we give a direct proof based on a trick which we learned from Maxim Kontsevich. Take M arbitrary such that $\pi M = \mathcal{P}$.

Take a quasi-isomorphism $Q \to M$ where Q is a right bounded complex of finitely generated projective modules. This yields a triangle:

$$(\pi Z)[a] \to \sigma_{\geq -a}\pi Q \to \mathcal{P}$$

where $Z = \ker(Q_{-a} \to Q_{-a+1})$. This triangle corresponds to an element of $\mathrm{Ext}^{a+1}(\mathcal{P}, \pi Z)$ which must be zero for large a. Hence $\sigma_{\geq a}\pi Q = \mathcal{P} \oplus (\pi Z)[a]$. This proves the lemma. □

We recall the following fact.

Proposition A.2. *The functors $- \overset{\mathbf{L}}{\otimes}_A R$ and $R\underline{\mathrm{Hom}}_A(R,-)$ induce inverse equivalences between $D^b_f(A)_{\mathrm{fpd}}$ and $D^b_f(A)_{\mathrm{fid}}$.*

Proof. If $P \in D^b_f(A)_{\mathrm{fpd}}$ then it is quasi-isomorphic to a bounded complex of finitely generated projective A-modules. For such such a complex it is clear that $P \otimes_A R$ has finite injective dimension. There is a canonical map $P \to R\underline{\mathrm{Hom}}(R, P \otimes_A R)$ which is an isomorphism for $P = A$. By induction over triangles one shows that it is an isomorphism for all P.

Conversely assume $I \in D^b_f(A)_{\mathrm{fid}}$. Then by duality $R\underline{\mathrm{Hom}}(R,I) = R\underline{\mathrm{Hom}}(DI, A)$. By the above discussion $DI \in D^b_f(A^\circ)_{\mathrm{fpd}}$. Hence $R\underline{\mathrm{Hom}}_A(DI,A) \in D^b_f(A)_{\mathrm{fpd}}$. We also find $R\underline{\mathrm{Hom}}_A(DI,A) \otimes_A R = R\underline{\mathrm{Hom}}_A(DI,R) = I$. □

The functor $-\otimes_A R$ induces a functor $D^-(\mathrm{Tails}(A)) \to D^-(\mathrm{Tails}(A))$ which we denote by $-\otimes \mathcal{R}$. Similarly the functor $\mathrm{RHom}_A(R,-)$ induces a functor $D^+(\mathrm{Tails}(A)) \to D^+(\mathrm{Tails}(A))$ which we denote by $\mathrm{R}\mathcal{H}om(\mathcal{R},-)$.

Proposition A.3. *The functors* $-\otimes\mathcal{R}$ *and* $\mathrm{R}\mathcal{H}om(\mathcal{R},-)$ *induces inverse equivalences between* $D^b_f(\mathrm{Tails}(A))_{\mathrm{fpd}}$ *and* $D^b_f(\mathrm{Tails}(A))_{\mathrm{fid}}$.

Proof. If $\mathcal{P}\in D^b_f(\mathrm{Tails}(A))_{\mathrm{fpd}}$ then by Lemma A.1 $\mathcal{P}$ is direct summand of some πP with $P\in D^b_f(A)_{\mathrm{fpd}}$. Using the proof of the previous proposition this easily implies that $\mathcal{P}\otimes\mathcal{R}\in D^b_f(\mathrm{Tails}(A))_{\mathrm{fid}}$ and $\mathrm{R}\mathcal{H}om(\mathcal{R},\mathcal{P}\otimes\mathcal{R})=\mathcal{P}$ (essentially because we may reduce to $\mathcal{P}=\pi A(n)$ for some n).

Conversely assume $\mathcal{I}=\pi I\in D^b_f(\mathrm{Tails}(A))_{\mathrm{fid}}$. Then $\mathrm{R}\mathcal{H}om(\mathcal{R},\mathcal{I})=\pi R\underline{\mathrm{Hom}}(R,I)=\pi R\underline{\mathrm{Hom}}(DI,A)$. We have by definition $\pi DI=D\pi I$, and hence $\pi DI\in D^b_f(A)_{\mathrm{fpd}}$. Then it follows from Lemma A.1 that πDI is a direct summand of some πQ with $Q\in D^b_f(Q)_{\mathrm{fpd}}$. We easily deduce from this that $\pi R\underline{\mathrm{Hom}}(DI,A)$ is a direct summand of $\pi R\underline{\mathrm{Hom}}(Q,A)$ and hence $\mathrm{R}\mathcal{H}om(\mathcal{R},\mathcal{I})=\pi R\underline{\mathrm{Hom}}(DI,A)\in D^b_f(\mathrm{Tails}(A))_{\mathrm{fpd}}$. The proof now continous as the proof of Proposition A.2. □

Theorem A.4. *(Serre duality) For all* $\mathcal{M}\in D^b_f(\mathrm{Tails}(A))_{\mathrm{fpd}}$, $\mathcal{N}\in D^b_f(\mathrm{Tails}(A))$ *there are natural isomorphisms*

$$\mathrm{Hom}(\mathcal{M},\mathcal{N})\cong\mathrm{Hom}(\mathcal{N},F\mathcal{M})'$$

where

$$F\mathcal{M}=(\mathcal{M}\overset{\mathbf{L}}{\otimes}\mathcal{R})[-1] \tag{A.1}$$

Furthermore F *defines an equivalence between* $D^b_f(\mathrm{Tails}(A))_{\mathrm{fpd}}$ *and* $D^b_f(\mathrm{Tails}(A))_{\mathrm{fid}}$.

Proof. As in [86] our proof of Serre duality is based on the local duality formula [77, 85]. The formulation of local duality in [77] used the functor $R\tau$ but the same proof works for the functor RQ where $Q=\omega\circ\pi$. Furthermore it is possible to throw an extra perfect complex into the bargain. If we do this we obtain canonical isomorphisms

$$\underline{\mathrm{Hom}}_A(N,P\otimes_A(RQA)')\cong\underline{\mathrm{Hom}}_A(P,RQN)' \tag{A.2}$$

for $N\in D(A)$ and $P\in D^b_f(A)_{\mathrm{fpd}}$. By adjointness $\underline{\mathrm{Hom}}_A(P,RQN)_0=\mathrm{Hom}_{\mathrm{Tails}(A)}(\pi P,\pi N)$. In addition, if we apply (A.2) with N finite dimensional then we find $\underline{\mathrm{Hom}}_A(N,P\otimes_A(RQA)')=0$. Thus using Lemma A.1 we obtain for $N\in D^b_f(A)$: $\underline{\mathrm{Hom}}_A(N,P\otimes_A(RQA)')_0=\mathrm{Hom}_{\mathrm{Tails}(A)}(\pi N,\pi(P\otimes_A(RQA)')$. Now the standard triangle for local cohomology yields $RQA=\mathrm{cone}(R\tau A\to A)$ and thus $(RQA)'=\mathrm{cone}(A'\to R)[-1]$. Using the fact that A' is torsion we easily obtain from this: $\pi(P\otimes_A(RQA)')=F(\pi P)$ where F is defined as in the statement of the theorem. So now we have shown

$$\mathrm{Hom}_{\mathrm{Tails}(A)}(\pi N,F(\pi P))\cong\mathrm{Hom}_{\mathrm{Tails}(A)}(\pi P,\pi N)' \tag{A.3}$$

Now we obtain from Lemma A.1 that $\mathcal{M}$ is a direct summand of a complex πP with $P\in D^b_f(A)_{\mathrm{fpd}}$. Thus (A.3) is true for $\mathcal{M}$ and this finishes of the the first part of the theorem. The last part is Proposition A.3. □

Corollary A.5. *If* $\mathrm{Tails}(A)$ *has finite global dimension then* $D^b_f(\mathrm{Tails}(A))$ *satisfies BKS-duality.*

Appendix B

Upper semi-continuity for noncommutative Proj

This section appeared in [28], where we discuss some results which are definitely at least implicit in [11] but for which we have been unable to find a convenient reference. The methods are quite routine. We refer to [11, 41] for more details.

Below R will be a noetherian commutative ring and $A = R + A_1 + A_2 + \cdots$ is a noetherian connected graded R-algebra.

Lemma B.1. *Let $M \in \operatorname{grmod}(A)$ be flat over R and $n \in \mathbb{Z}$. Then the function*

$$\operatorname{Spec} R \to \mathbb{Z} : x \mapsto \dim_k \operatorname{Tor}_i^{A_{k(x)}}(M_{k(x)}, k(x))_n$$

is upper semi-continuous.

Proof. Because of flatness we have $\operatorname{Tor}_i^{A_{k(x)}}(M_{k(x)}, k(x)) = \operatorname{Tor}_i^A(M, k(x))$. Let $F^{\cdot} \to M \to 0$ be a graded resolution of M consisting of free A-modules of finite rank. Then $\operatorname{Tor}_i^A(M, k(x))_n$ is the homology of $(F^{\cdot})_n \otimes_A k(x)$. Since $(F^{\cdot})_n$ is a complex of free R-modules, the result follows in the usual way. □

Now we write $X = \operatorname{Proj} A$ and we use the associated notations as in §1.4 - §1.6. In addition we will assume that A satisfies the following conditions.

1. A satisfies χ [10].
2. $\Gamma(X, -)$ has finite cohomological dimension.

Under these hypotheses we prove

Proposition B.2. *Let $\mathcal{G} \in \operatorname{coh}(X)$ be flat over R and let $\mathcal{F} \in \operatorname{coh}(X)$ be arbitrary. Then there is a complex $L^{\cdot}$ of finitely generated projective R-modules such that for any $M \in \operatorname{Mod}(R)$ and for any $i \geq 0$ we have*

$$\operatorname{Ext}^i(\mathcal{F}, \mathcal{G} \otimes_R M) = H^i(L^{\cdot} \otimes_R M)$$

Proof.

Step 1. We first claim that there is an N such that for $n \geq N$ one has that $\Gamma(X, \mathcal{G}(n))$ is a projective R-module, $\Gamma(X, \mathcal{G}(n) \otimes_R M) = \Gamma(X, \mathcal{G}(n)) \otimes_R M$ and $R^i\Gamma(X, \mathcal{G}(n) \otimes_R M) = 0$ for $i > 0$ and all M. We start with the last part of this claim. We select N is such a way that $R^i\Gamma(X, \mathcal{G}(n)) = 0$ for $i > 0$ and $n \geq N$. Using the fact that $\Gamma(X, -)$ has finite cohomological dimension and degree shifting in M we deduce that indeed $R^i\Gamma(X, \mathcal{G}(n) \otimes_R M) = 0$ for $i > 0$ and all M. Thus $\Gamma(X, \mathcal{G}(n) \otimes_R -)$ is an exact functor. Applying this functor to a projective presentation of M yields $\Gamma(X, \mathcal{G}(n) \otimes_R M) = \Gamma(X, \mathcal{G}(n)) \otimes_R M$. Since $\Gamma(X, \mathcal{G}(n) \otimes_R -)$ is left exact and $\Gamma(X, \mathcal{G}(n)) \otimes -$ is right exact this implies that $\Gamma(X, \mathcal{G}(n))$ is flat. Finally since A satisfies χ and R is noetherian $\Gamma(X, \mathcal{G}(n))$ is finitely presented and hence projective.

Step 2. Now let N be as in the previous step and take a resolution $\mathcal{P}^{\cdot} \to \mathcal{F} \to 0$ where the $\mathcal{P}_i$ are finite direct sums of objects $\mathcal{O}(-n)$ with $n \geq N$.

Then $\operatorname{Ext}^i(\mathcal{F}, \mathcal{G} \otimes_R M)$ is the homology of

$$\operatorname{Hom}(\mathcal{P}^{\cdot}, \mathcal{G} \otimes_R M)) = \operatorname{Hom}(\mathcal{P}^{\cdot}, \mathcal{G}) \otimes_R M$$

where the equality follows from Step 1. We put $L^{\cdot} = \operatorname{Hom}(\mathcal{P}^{\cdot}, \mathcal{G})$ which is term wise projective, also by Step 1. This finishes the proof. □

For a point $x \in \operatorname{Spec} R$ we denote the base change functor $- \otimes_R k(x)$ by $(-)_x$. We also put $X_x = \operatorname{Proj} A_x$.

Corollary B.3. *If $\mathcal{G}$ is as in the previous proposition then the function*

$$\operatorname{Spec} R \to \mathbb{N} : x \mapsto \dim_{k(x)} R\Gamma^i(X_x, \mathcal{G}_x)$$

is upper semi-continuous.

Proof. By [11, Lemma C6.6] we have $R\Gamma^i(X_x, \mathcal{G}_x) = R\Gamma^i(X, \mathcal{G} \otimes_R k(x))$. This implies

$$R\Gamma^i(X_x, \mathcal{G}_x) = H^i(L^{\cdot} \otimes_R k(x)) \tag{B.1}$$

The fact that the dimension of the right hand side of (B.1) is upper semi-continuous is an elementary fact from linear algebra. □

Corollary B.4. *Assume that $\mathcal{G}$ is as in the previous proposition and assume that R is a domain. Assume furthermore that the function*

$$\operatorname{Spec} R \to \mathbb{N} : x \mapsto \dim_{k(x)} R\Gamma^i(X_x, \mathcal{G}_x)$$

is constant. Then $R\Gamma^i(X, \mathcal{G})$ is projective over R and in addition for any $M \in \operatorname{Mod}(R)$ the natural map

$$R\Gamma^i(X, \mathcal{G}) \otimes_R M \to R\Gamma^i(X, \mathcal{G} \otimes_R M)$$

is an isomorphism for all $x \in \operatorname{Spec} R$.

Proof. This is proved as [41, Corollary 12.9]. □

Appendix C

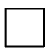
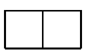
Hilbert series of ideals with small invariants

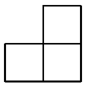
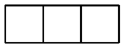
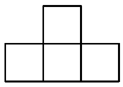
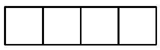
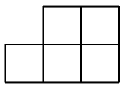
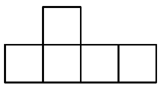
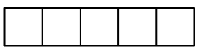
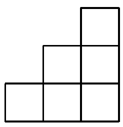
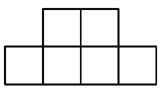
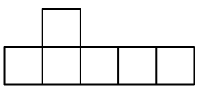
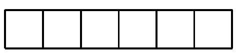
C.1 Quadratic Artin-Schelter algebras

Let A be a quadratic Artin-Schelter algebra, and let I be a normalized rank one torsion free graded right A-module of projective dimension one with invariant n. According to Theorem 4 the Hilbert series of I has the form

$$h_I(t) = \frac{1}{(1-t)^3} - \frac{s_I(t)}{1-t}$$

where $s_I(t)$ is a Castelnuovo polynomial of weight n. For the cases $n \leq 6$ we list the possible Hilbert series for I, the corresponding Castelnuovo polynomial, the dimension of the stratum (given by $\dim_k \operatorname{Ext}^1_A(I,I)$) and the possible minimal resolutions of I.

$n=0$	$h_I(t) = 1 + 3t + 6t^2 + 10t^3 + 15t^4 + 21t^5 + \dots$ $s_I(t) = 0$ $\dim_k \operatorname{Ext}^1_A(I,I) = 0$ $0 \to A \to I \to 0$
$n=1$ 	$h_I(t) = 2t + 5t^2 + 9t^3 + 14t^4 + 20t^5 + 27t^6 + \dots$ $s_I(t) = 1$ $\dim_k \operatorname{Ext}^1_A(I,I) = 2$ $0 \to A(-2) \to A(-1)^2 \to I \to 0$
$n=2$ 	$h_I(t) = t + 4t^2 + 8t^3 + 13t^4 + 19t^5 + 26t^6 + \dots$ $s_I(t) = 1 + t$ $\dim_k \operatorname{Ext}^1_A(I,I) = 4$ $0 \to A(-3) \to A(-1) \oplus A(-2) \to I \to 0$

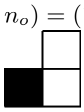
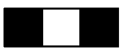
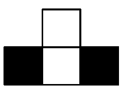
$n = 3$ 	$h_I(t) = 3t^2 + 7t^3 + 12t^4 + 18t^5 + 25t^6 + \dots$ $s_I(t) = 1 + 2t$ $\dim_k \operatorname{Ext}^1_A(I, I) = 6$ $0 \to A(-3)^2 \to A(-2)^3 \to I \to 0$
	$h_I(t) = t + 3t^2 + 7t^3 + 12t^4 + 18t^5 + 25t^6 + \dots$ $s_I(t) = 1 + t + t^2$ $\dim_k \operatorname{Ext}^1_A(I, I) = 5$ $0 \to A(-4) \to A(-1) \oplus A(-3) \to I \to 0$
$n = 4$ 	$h_I(t) = 2t^2 + 6t^3 + 11t^4 + 17t^5 + 24t^6 + \dots$ $s_I(t) = 1 + 2t + t^2$ $\dim_k \operatorname{Ext}^1_A(I, I) = 8$ $0 \to A(-4) \to A(-2)^2 \to I \to 0$ $0 \to A(-3) \oplus A(-4) \to A(-2)^2 \oplus A(-3) \to I \to 0$
	$h_I(t) = t + 3t^2 + 6t^3 + 11t^4 + 17t^5 + 24t^6 + \dots$ $s_I(t) = 1 + t + t^2 + t^3$ $\dim_k \operatorname{Ext}^1_A(I, I) = 6$ $0 \to A(-5) \to A(-1) \oplus A(-4) \to I \to 0$
$n = 5$ 	$h_I(t) = t^2 + 5t^3 + 10t^4 + 16t^5 + 23t^6 + \dots$ $s_I(t) = 1 + 2t + 2t^2$ $\dim_k \operatorname{Ext}^1_A(I, I) = 10$ $0 \to A(-4)^2 \to A(-2) \oplus A(-3)^2 \to I \to 0$
	$h_I(t) = 2t^2 + 5t^3 + 10t^4 + 16t^5 + 23t^6 + \dots$ $s_I(t) = 1 + 2t + t^2 + t^3$ $\dim_k \operatorname{Ext}^1_A(I, I) = 8$ $0 \to A(-3) \oplus A(-5) \to A(-2)^2 \oplus A(-4) \to I \to 0$
	$h_I(t) = t + 3t^2 + 6t^3 + 10t^4 + 16t^5 + 23t^6 + \dots$ $s_I(t) = 1 + t + t^2 + t^3 + t^4$ $\dim_k \operatorname{Ext}^1_A(I, I) = 7$ $0 \to A(-6) \to A(-1) \oplus A(-5) \to I \to 0$
$n = 6$ 	$h_I(t) = 4t^3 + 9t^4 + 15t^5 + 22t^6 + 30t^7 + \dots$ $s_I(t) = 1 + 2t + 3t^2$ $\dim_k \operatorname{Ext}^1_A(I, I) = 12$ $0 \to A(-4)^3 \to A(-3)^4 \to I \to 0$
	$h_I(t) = t^2 + 4t^3 + 9t^4 + 15t^5 + 22t^6 + \dots$ $s_I(t) = 1 + 2t + 2t^2 + t^3$ $\dim_k \operatorname{Ext}^1_A(I, I) = 11$ $0 \to A(-5) \to A(-2) \oplus A(-3) \to I \to 0$ $0 \to A(-4) \oplus A(-5) \to A(-2) \oplus A(-3) \oplus A(-4) \to I \to 0$
	$h_I(t) = 2t^2 + 5t^3 + 9t^4 + 15t^5 + 22t^6 + \dots$ $s_I(t) = 1 + 2t + t^2 + t^3 + t^4$ $\dim_k \operatorname{Ext}^1_A(I, I) = 9$ $0 \to A(-3) \oplus A(-6) \to A(-2)^2 \oplus A(-5) \to I \to 0$
	$h_I(t) = t + 3t^2 + 6t^3 + 10t^4 + 15t^5 + 22t^6 + \dots$ $s_I(t) = 1 + t + t^2 + t^3 + t^4 + t^5$ $\dim_k \operatorname{Ext}^1_A(I, I) = 8$ $0 \to A(-7) \to A(-1) \oplus A(-6) \to I \to 0$

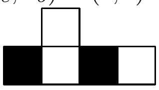
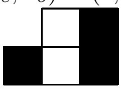
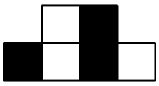
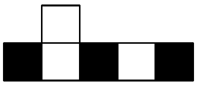
C.2 Cubic Artin-Schelter algebras

Let A be a cubic Artin-Schelter algebra, and let I be a normalized rank one torsion free graded right A-module of projective dimension one with invariants (n_e, n_o). According to Theorem 13 the Hilbert series of I has the form

$$h_I(t) = \frac{1}{(1-t)^2(1-t^2)} - \frac{s_I(t)}{1-t^2}$$

where $s_I(t)$ is a Castelnuovo polynomial of even weight n_e and odd weight n_o. For the cases $n_e \leq 3$, $n_o \leq 3$ we list the possible Hilbert series for I, the corresponding Castelnuovo polynomial, $\dim_k \operatorname{Ext}^1_A(I,I)$ and the possible minimal resolutions of I.

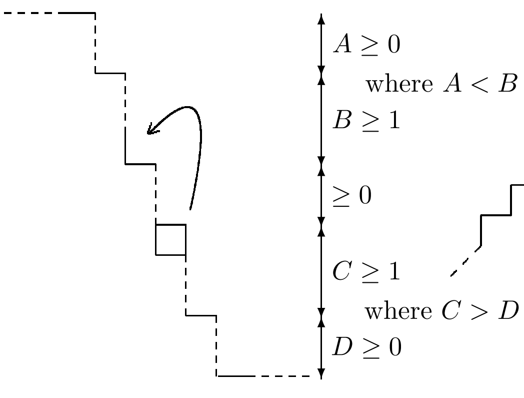
$(n_e, n_o) = (0,0)$	$h_I(t) = 1 + 2t + 4t^2 + 6t^3 + 9t^4 + 12t^5 + \ldots$ $s_I(t) = 0$ $\dim_k \operatorname{Ext}^1_A(I,I) = 0$ $0 \to A \to I \to 0$
$(n_e, n_o) = (1,0)$ 	$h_I(t) = 2t + 3t^2 + 6t^3 + 8t^4 + 12t^5 + 15t^6 + \ldots$ $s_I(t) = 1$ $\dim_k \operatorname{Ext}^1_A(I,I) = 0$ $0 \to A(-2) \to A(-1)^2 \to I \to 0$
$(n_e, n_o) = (1,1)$ 	$h_I(t) = t + 3t^2 + 5t^3 + 8t^4 + 11t^5 + 15t^6 + \ldots$ $s_I(t) = 1 + t$ $\dim_k \operatorname{Ext}^1_A(I,I) = 2$ $0 \to A(-3) \to A(-1) \oplus A(-2) \to I \to 0$
$(n_e, n_o) = (1,2)$ 	$h_I(t) = 3t^2 + 4t^3 + 8t^4 + 10t^5 + 15t^6 + \ldots$ $s_I(t) = 1 + 2t$ $\dim_k \operatorname{Ext}^1_A(I,I) = 0$ $0 \to A(-3)^2 \to A(-2)^3 \to I \to 0$
$(n_e, n_o) = (2,1)$ 	$h_I(t) = t + 2t^2 + 5t^3 + 7t^4 + 11t^5 + 14t^6 + \ldots$ $s_I(t) = 1 + t + t^2$ $\dim_k \operatorname{Ext}^1_A(I,I) = 2$ $0 \to A(-4) \to A(-1) \oplus A(-3) \to I \to 0$
$(n_e, n_o) = (2,2)$ 	$h_I(t) = 2t^2 + 4t^3 + 7t^4 + 10t^5 + 14t^6 + \ldots$ $s_I(t) = 1 + 2t + t^2$ $\dim_k \operatorname{Ext}^1_A(I,I) = 4$ $0 \to A(-4) \to A(-2)^2 \to I \to 0$ $0 \to A(-3) \oplus A(-4) \to A(-2)^2 \oplus A(-3) \to I \to 0$
	$h_I(t) = t + 2t^2 + 4t^3 + 7t^4 + 10t^5 + 14t^6 + \ldots$ $s_I(t) = 1 + t + t^2 + t^3$ $\dim_k \operatorname{Ext}^1_A(I,I) = 3$ $0 \to A(-5) \to A(-1) \oplus A(-4) \to I \to 0$

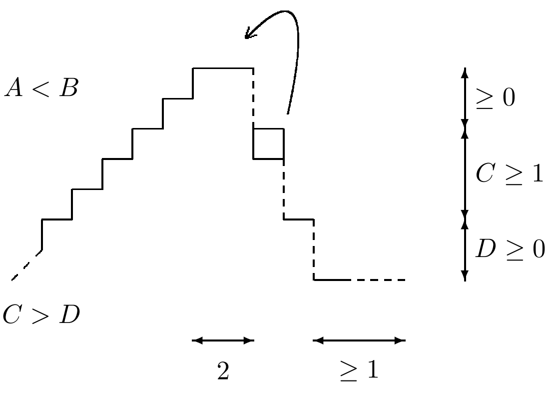
$(n_e, n_o) = (2,3)$ 	$h_I(t) = 2t^2 + 3t^3 + 7t^4 + 9t^5 + 14t^6 + \dots$ $s_I(t) = 1 + 2t + t^2 + t^3$ $\dim_k \operatorname{Ext}^1_A(I,I) = 2$ $0 \to A(-3) \oplus A(-5) \to A(-2)^2 \oplus A(-4) \to I \to 0$
$(n_e, n_o) = (3,2)$ 	$h_I(t) = t^2 + 4t^3 + 6t^4 + 10t^5 + 13t^6 + \dots$ $s_I(t) = 1 + 2t + 2t^2$ $\dim_k \operatorname{Ext}^1_A(I,I) = 4$ $0 \to A(-4)^2 \to A(-2) \oplus A(-3)^2 \to I \to 0$
	$h_I(t) = t + 2t^2 + 4t^3 + 6t^4 + 10t^5 + 13t^6 + \dots$ $s_I(t) = 1 + t + t^2 + t^3 + t^4$ $\dim_k \operatorname{Ext}^1_A(I,I) = 3$ $0 \to A(-6) \to A(-1) \oplus A(-5) \to I \to 0$
$(n_e, n_o) = (3,3)$ 	$h_I(t) = t^2 + 3t^3 + 6t^4 + 9t^5 + 13t^6 + \dots$ $s_I(t) = 1 + 2t + 2t^2 + t^3$ $\dim_k \operatorname{Ext}^1_A(I,I) = 6$ $0 \to A(-5) \to A(-2) \oplus A(-3) \to I \to 0$ $0 \to A(-4) \oplus A(-5) \to A(-2) \oplus A(-3) \oplus A(-4) \to I \to 0$
	$h_I(t) = 2t^2 + 3t^3 + 6t^4 + 9t^5 + 13t^6 + \dots$ $s_I(t) = 1 + 2t + t^2 + t^3 + t^4$ $\dim_k \operatorname{Ext}^1_A(I,I) = 4$ $0 \to A(-3) \oplus A(-6) \to A(-2)^2 \oplus A(-5) \to I \to 0$
	$h_I(t) = t + 2t^2 + 4t^3 + 6t^4 + 9t^5 + 13t^6 + \dots$ $s_I(t) = 1 + t + t^2 + t^3 + t^4 + t^5$ $\dim_k \operatorname{Ext}^1_A(I,I) = 4$ $0 \to A(-7) \to A(-1) \oplus A(-6) \to I \to 0$

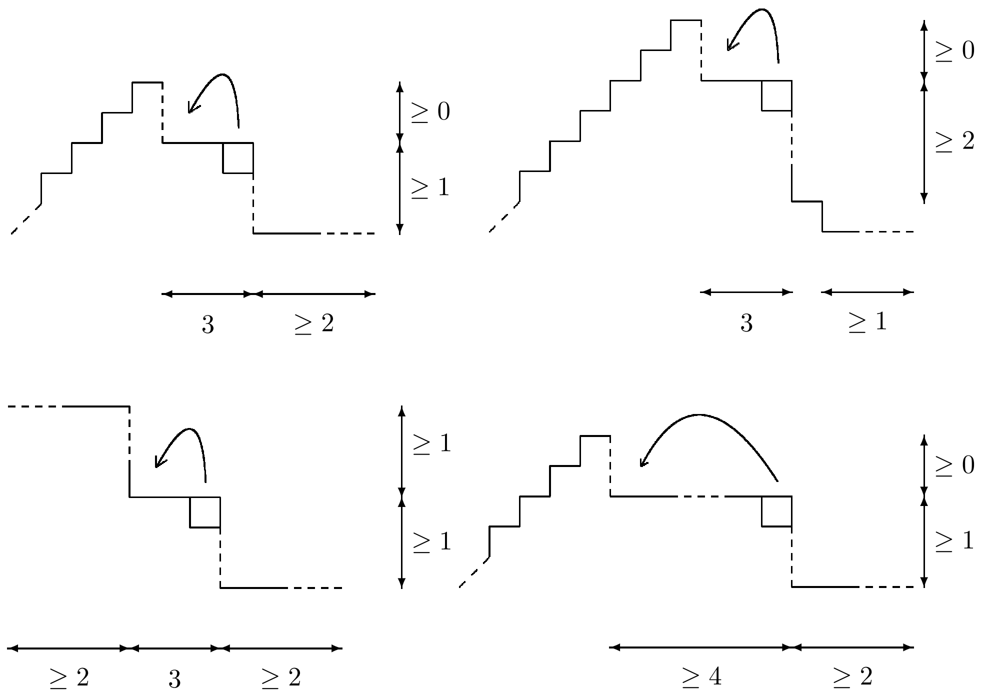
Appendix D

A visual criterion for incidence problems of length zero

In this appendix we provide a visual criterion for the three conditions in Theorem 9 of Chapter 5. The reader may easily check these using Condition C, Proposition 5.3.1 and (5.7) in Chapter 5. We let (φ, ψ) be a pair of Hilbert series of degree n and length zero. Then $H_\varphi \subset \overline{H_\psi}$ if and only if the Castelnuovo diagram s_φ of φ has one of the following six forms, where the diagram s_ψ is obtained by moving the upper square as indicated.






≥ 0
≥ 1
3
≥ 2
≥ 0
≥ 2
3
≥ 1
≥ 1
≥ 1
≥ 2
3
≥ 2
≥ 0
≥ 1
≥ 4
≥ 2

Appendix E

Maple programs

In this appendix we provide some procedures, implemented in Maple™, which allows us to compute various data concerning Chapter 3 and Chapter 5. These procedures will be illustrated with examples after we have briefly recalled a some earlier definitions and results from Chapters 3, Chapter 5. In this way the reader may save some time in turning pages.

Some of the procedures in this chapter are based on programs in [38, Annexe C].

E.1 Procedures for Chapter 3: Examples

In this part we illustrate procedures for Chapter 3. The procedures themselves will be listed in Section E.3. For convenience we recall some definitions and results.

In the sequel we identify a function $f : \mathbb{Z} \to \mathbb{C}$ with its generating function $f(t) = \sum_n f(n)t^n$. We refer to $f(t)$ as a polynomial or a series depending on whether the support of f is finite or not.

Let n be a positive integer. A Castelnouvo polynomial is a polynomial in $s(t) \in \mathbb{Z}[t]$ of the form

$$s(t) = 1 + 2t + 3t^2 + \cdots + ut^{u-1} + s_u t^u + \cdots + s_v t^v$$

for some integers $0 \leq u \leq v$, where $s_u \geq s_{u+1} \geq \cdots \geq s_v \geq 0$. The integers u and $\sum_i s_i$ are called the height and the weight of $s(t)$.

Let A be a quadratic Artin-Schelter algebra. Due to Theorem 4 the equation

$$h(t) = \frac{1}{(1-t)^3} - \frac{s(t)}{1-t}$$

gives a bijection between Castelnuovo polynomials $s(t)$ of weight n and the Hilbert series $h(t)$ of torsion free graded right A-modules I of rank one and projective dimension one, which are normalized of invariant n. These objects I are parameterized by the scheme $\mathrm{Hilb}_n(\mathbb{P}^2_{\mathrm{q}})$. There is a (weak) stratification into smooth, non-empty connected locally closed sets

$$\mathrm{Hilb}_n(\mathbb{P}^2_q) = \bigcup_h \mathrm{Hilb}_h(\mathbb{P}^2_q)$$

where the union runs over the (finite set) of admissible Hilbert series of weight n and where the points in $\mathrm{Hilb}_h(\mathbb{P}^2_q)$ represents the points in $\mathrm{Hilb}_n(\mathbb{P}^2_q)$ corresponding to objects with Hilbert series h. The appearing Hilbert series $h(t)$ are called admissible Hilbert series of weight n. Similarly we will refer to a polynomial of the form $q(t) = 1 - s(t)(1-t)^2$ as an *admissible characteristic polynomial of weight* n. Finitely supported sequences $(a_i), (b_i)$ of integers are called *admissible Betti numbers of weight* n if $\sum_i (a_i - b_i)t^i$ is an admissible characteristic polynomial of weight n. This is equivalent with saying that $(a_i), (b_i)$ occur as the Betti numbers of a torsion free graded right A-module I of projective dimension one and rank one which is normalized of invariant n, i.e. a minimal resolution of I has the form

$$0 \to \oplus_i A(-i)^{b_i} \to \oplus_i A(-i)^{a_i} \to I \to 0$$

Note that counting the number of admissible Betti numbers of weight n is the same as counting the possible minimal resolutions.

We now come to the discussion of the procedures of Section E.3.

- "Castelnuovo" computes all Castelnuovo polynomials with given weight n.
  Input: positive integer n ($n < 60$ is recommended).
  Output: list L for which each member $L[i]$ is a list of coefficients of a corresponding Castelnuovo polynomial of weight n. In the sequel we refer to such a list of coefficients as a *Castelnuovo list*. L contains all Castelnuovo lists of weight n.
  Example: all Castelnuovo lists of weight 9.

```
> Castelnuovo(9);
```

$$[[1,1,1,1,1,1,1,1,1],[1,2,1,1,1,1,1,1],[1,2,2,1,1,1,1],$$
$$[1,2,2,2,1,1],[1,2,2,2,2],[1,2,3,1,1,1],[1,2,3,2,1],[1,2,3,3]]$$

- "validCastelnuovo" determines if a list of numbers is a Castelnuovo list.
  Input: list of numbers.
  Output: true if input is a Castelnuovo list, false otherwise.
  Example: the list $[1,2,3,4,5,5,5,5,2,2,3,1]$ is not a Castelnuovo list.

```
> validCastelnuovo([1,2,3,4,5,5,5,5,2,2,3,1]);
```

false

- "weight" computes the weight of a Castelnuovo polynomial.
  Input: Castelnuovo list (if this list is not a Castelnuovo list a warning is given).
  Output: the weight of the Castelnuovo list.
  Example: the weight of the Castelnuovo list $[1,2,3,4,5,5,5,4,2,2,2,1]$.

```
> weight([1,2,3,4,5,5,5,4,2,2,2,1]);
```

36

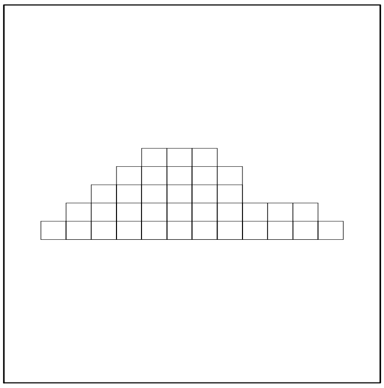
- "height" computes the height of a Castelnuovo polynomial.
  Input: Castelnuovo list (if this list is not a Castelnuovo list a warning is given).
  Output: the height of the Castelnuovo list.
  Example: the height of the Castelnuovo list $[1, 2, 3, 4, 5, 5, 5, 4, 2, 2, 2, 1]$.

  ```
  > height([1,2,3,4,5,5,5,4,2,2,2,1]);
  ```

  5

- "diagram" plots the Castelnuovo diagram of a given Castelnouvo polynomial.
  Input: Castelnuovo list (if this list is not a Castelnuovo list a warning is given).
  Output: plot of the corresponding Castelnuovo diagram.
  Example: the Castelnuovo diagram of $[1, 2, 3, 4, 5, 5, 5, 4, 2, 2, 2, 1]$.

  ```
  > diagram([1,2,3,4,5,5,5,4,2,2,2,1]);
  ```



- "savediagram" saves a Castluovo diagram as an eps-file.
  Input: sequence filename.eps,L where L is a Castelnuovo list and "filename" is the name you want to give to the eps-file.
  Output: the file filename.eps has been saved on the computer.
  Example: save diagram of $[1, 2, 3, 4, 5, 5, 5, 4, 2, 2, 2, 1]$ as drawing.eps.[1]

  ```
  > savediagram("drawing.eps",[1,2,3,4,5,5,5,4,2,2,2,1]);
  ```

- "Hilbertideal" computes the initial terms of the Hilbert series corresponding to a given Castelnuovo polynomial of weight n.
  Input: Castelnuovo list (if this list is not a Castelnuovo list a warning is given).
  Output: list L of the initial terms of the Hilbert series h corresponding to the input Castelnuovo list. The initial term of L is the coefficient of t^0 in $h(t)$. If the difference between the two final terms of L is m then the next term is

[1]To present the Castelnuovo diagram as shown above, the author included drawing.eps by use of the LaTeX command \parbox[c]{1cm}{\includegraphics[height=5cm,width=5cm]{drawing.eps}}.

determined by adding $m+1$ to the final term. Repeating this proces allows the user to compute higher terms.
Example: Hilbert series corresponding to $[1,2,3,4,5,5,3,2,1,1,1]$.

```
> Hilbertideal([1,2,3,4,5,5,3,2,1,1,1]);
```

$$[0,0,0,0,0,1,5,11,19,28,38,50,63,77,92,108,125]$$

- "allHilbertideal" computes the initial terms of all weight n admissible Hilbert series.
Input: positive integer n.
Output: complete list for which each member is a list of the initial terms of an admissible Hilbert series of weight n.
Example: the admissible Hilbert series of weight 3.

```
> allHilbertideal(3);
```

$$[[0,1,3,7,12,18,25,33,42],[0,0,3,7,12,18,25,33]]$$

- "characteristic" computes the characteristic polynomial of weight n corresponding to a given Castelnuovo polynomial of weight n.
Input: Castelnuovo list (if this list is not a Castelnuovo list a warning is given).
Output: list L of all coefficients of the characteristic polynomial $q(t)$ corresponding to the input Castelnuovo list. The initial term of L is the coefficient of t^0 in $q(t)$.
Example: the admissible characteristic polynomial of $[1,2,3,4,5,5,3,2,1,1,1]$.

```
> characteristic([1,2,3,4,5,5,3,2,1,1,1]);
```

$$[0,0,0,0,0,1,2,-1,0,-1,0,1,-1]$$

- "allcharacteristic" computes all weight n admissible characteristic polynomials.
Input: positive integer n.
Output: complete list for which each member is a list of coefficients of an admissible weight n characteristic polynomial.
Example: the admissible characteristic polynomials of weight 3.

```
> allcharacteristic(3);
```

$$[[0,1,0,1,-1],[0,0,3,-2]]$$

- "validcharacteristic" determines if a given list appears as the coefficients of an admissible characteristic polynomial.
Input: list of numbers.
Output: sequence True,L if the input list agrees with the coefficients of $t^d q(t)$ for some integer d and some admissible characteristic polynomial $q(t)$. In that case L is the associated Castelnuovo polynomial. False otherwise.
Example: $[0,0,0,0,0,1,2,1,0,0,-3]$ is a list of terms of an admissible characteristic polynomial (up to shifting).

```
> validcharacteristic([0,0,0,0,0,1,2,1,0,0,-3]);
```

$$true, [1,2,3,4,5,6,7,8,9,10,11,11,9,6,3]$$

- “Betti” computes all admissible Betti numbers $(a_i)_i, (b_i)_i$ corresponding to a given Castelnuovo polynomial.
  Input: Castelnuovo list (if this list is not a Castelnuovo list a warning is given).
  Output: a list of the form $[[\ldots, [i, v \ldots v'], \ldots], [\ldots, [i, u \ldots u'], \ldots]]$. The sequences $(a_i)_i, (b_i)_i$ with support the appearing i and for which $u \leq a_i \leq u'$, $v \leq b_i \leq v'$ where $a_i - b_i = u - v$ are exactly the admissible Betti numbers corresponding to the input list.
  Example: admissible Betti numbers corresponding to $[1,2,3,4,5,5,3,2,1,1,1]$.

  ```
  > Betti([1,2,3,4,5,5,3,2,1,1,1]);
  ```

  $$[[[4,0],[6,2],[7,0\ldots1],[8,0\ldots1],[11,1]],[[7,1\ldots2],[8,0\ldots1],[9,1],[12,1]]]$$

- “numberminimalresolutions” computes the number of admissible Betti numbers corresponding to the given Castelnuovo polynomial.
  Input: Castelnuovo list (if this list is not a Castelnuovo list a warning is given).
  Output: the number of admissible Betti numbers corresponding to input.
  Example: number of admissible Betti numbers for $[1,2,3,4,5,5,3,2,1,1,1]$.

  ```
  > numberminimalresolutions([1,2,3,4,5,5,3,2,1,1,1]);
  ```

  $$4$$

- “numberallminimalresolutions” computes the number of admissible Betti numbers of weight n.
  Input: positive integer n.
  Output: the number of admissible Betti numbers of weight n.
  Example: number of admissible Betti numbers of weight 20.

  ```
  > numberallminimalresolutions(20);
  ```

  $$104$$

- “dimension” computes the dimension of the stratum $\mathrm{Hilb}_h(\mathbb{P}^2_{\mathrm{q}})$ in $\mathrm{Hilb}_n(\mathbb{P}^2_{\mathrm{q}})$ corresponding to a given Hilbert series.
  Input: Castelnuovo list (if this list is not a Castelnuovo list a warning is given).
  Output: the dimension of the stratum corresponding to the Hilbert series determined by the input Castelnuovo list.
  Example: dimension of the stratum corresponding to $[1,2,3,4,5,5,3,2,1,1,1]$.

  ```
  > dimension([1,2,3,4,5,5,3,2,1,1,1]);
  ```

  $$42$$

- “alldimension” computes the dimensions of all strata in $\mathrm{Hilb}_n(\mathbb{P}^2_{\mathrm{q}})$.
  Input: positive integer n.
  Output: list of dimensions of all strata in $\mathrm{Hilb}_n(\mathbb{P}^2_{\mathrm{q}})$.
  Example: the dimensions of the strata in $\mathrm{Hilb}_n(\mathbb{P}^2_{\mathrm{q}})$ for $n = 12$.

  ```
  > alldimension(12);
  ```

$$[14, 15, 16, 16, 16, 17, 17, 18, 17, 18, 19, 20, 21, 20, 24]$$

E.2 Procedures for Chapter 5: Examples

In this part we illustrate procedures for Chapter 5, listed in Section E.4. We first recall some definitions and results from Chapter 5. For more details we refer to Chapters 3 and 5.

Let $A = k[x, y, z]$ and write $\mathrm{Hilb}_n(\mathbb{P}^2)$ for the Hilbert scheme parameterizing zero-dimensional subschemes of length n in $\mathbb{P}^2$. Associated to X is an ideal $\mathcal{I}_X \subset \mathcal{O}_{\mathbb{P}^2}$ and a graded ideal $I_X = \oplus_n H^0(\mathbb{P}^2, \mathcal{I}_X(n)) \subset A$. The Hilbert function h_X of X is the Hilbert function of the graded ring $A(X) = A/I_X$. A function $\varphi : \mathbb{Z} \to \mathbb{N}$ is of the form h_X for some $X \in \mathrm{Hilb}_n(\mathbb{P}^2)$ if and only if $h(m) = 0$ for $m < 0$ and $h(m) - h(m-1)$ is a Castelnuovo function of weight n.

We refer to a series φ for which $\varphi = h_X$ for some $X \in \mathrm{Hilb}_n(\mathbb{P}^2)$ as a *Hilbert function of degree* n. The set of all Hilbert functions of degree n (or equivalently the set of all Castelnuovo functions of weight n) will be denoted by Γ_n. For $\varphi, \psi \in \Gamma_n$ we have that $\psi(t) - \varphi(t)$ is a polynomial, and we write $\varphi \le \psi$ if its coefficients are non-negative. In this way $\le$ becomes a partial ordering on Γ_n and we call the associated directed graph the *Hilbert graph*, also denoted by Γ_n.

Let $X \in \mathrm{Hilb}_n(\mathbb{P}^2)$. The graded ideal I_X associated to X admits a minimal free resolution of the form

$$0 \to \oplus_i A(-i)^{b_i} \to \oplus_i A(-i)^{a_i} \to I_X \to 0$$

where $(a_i), (b_i)$ are sequences of non-negative integers which have finite support, called the *graded Betti numbers* of I_X (and X). They are related to the Hilbert series of X as

$$h_X(t) = h_A(t)\left(1 - \sum_i (a_i - b_i)t^i\right) = \frac{1 - \sum_i (a_i - b_i)t^i}{(1-t)^3}$$

So the Betti numbers determine the Hilbert series of X. For generic X (in a stratum H_ψ) the converse is true since in that case a_i and b_i are not both non-zero. We will call such $(a_i)_i$, $(b_i)_i$ *generic Betti numbers.*

The *tangent function* t_φ of a Hilbert function $\varphi \in \Gamma_n$ is defined as the Hilbert function of $\mathcal{I}_X \otimes_{\mathbb{P}^2} \mathcal{T}_{\mathbb{P}^2}$, where $X \in H_\varphi$ is generic.

For any Hilbert function ψ of degree n one defines a smooth connected subscheme $H_\psi = \{X \in \mathrm{Hilb}_n(\mathbb{P}^2) \mid h_X = \psi\}$ of $\mathrm{Hilb}_n(\mathbb{P}^2)$. The family $\{H_\psi\}_{\psi \in \Gamma_n}$ forms a stratification of $\mathrm{Hilb}_n(\mathbb{P}^2)$ in the sense that

$$\overline{H_\psi} \subset \bigcup_{\varphi \le \psi} H_\varphi$$

In general it is still an open problem to find necessary and sufficient conditions for the existence of an inclusion $H_\varphi \subset \overline{H_\psi}$. This problem is sometimes referred to as the incidence problem.

It follows that

$$H_\varphi \subset \overline{H_\psi} \Rightarrow \begin{cases} \varphi \leq \psi & \text{cohomological condition} \\ \dim H_\varphi < \dim H_\psi & \text{dimension condition} \\ t_\varphi \geq t_\psi & \text{tangent condition} \end{cases} \tag{E.1}$$

A pair of Hilbert functions (φ, ψ) of degree n has *length zero* if $\varphi < \psi$ and there are no Hilbert functions τ of degree n such that $\varphi < \tau < \psi$. The main result of Chapter 5 is that the converse of the implication (E.1) is true if (φ, ψ) has length zero, see Theorem 9. In [38] Guerimand proved this result under the additional hypothesis (φ, ψ) has type zero (see §5.1 for the definition of type zero).

We now discuss of the procedures of Section E.4.

- "Hilbertsubscheme" computes the Hilbert function corresponding to a given Castelnuovo polynomial.
  Input: Castelnuovo list (if this list is not a Castelnuovo list a warning is given).
  Output: list L of the (initial) values of the Hilbert function φ corresponding to input. The i-th term $L[i]$ of L is equal to the value $\varphi(i-1)$, and if $L[m]$ is the final term of L then $L[m]$ is the degree of φ and $\varphi(j) = \varphi(m-1) = L[m]$ for all $j \geq m$.
  Example: Hilbert function corresponding to $[1, 2, 3, 4, 5, 5, 3, 2, 1, 1, 1]$.

```
> Hilbertsubscheme([1,2,3,4,5,5,3,2,1,1,1]);
```

$$[1, 3, 6, 10, 15, 20, 23, 25, 26, 27, 28, 28, 28, 28]$$

- "allHilbertsubscheme" computes all Hilbert functions of given degree n.
  Input: positive integer n.
  Output: complete list L for which each member $L[i]$ are the (initial) values of a Hilbert function φ of degree n.
  Example: all Hilbert functions of degree 3.

```
> allHilbertsubscheme(3);
```

$$[[1, 2, 3, 3, 3, 3], [1, 3, 3, 3, 3]]$$

- "genericBetticoefficients" computes, given i, the numbers a_i, b_i of the generic Betti-numbers $(a_i)_i, (b_i)_i$ associated to a given Castelnuovo polynomial.
  Input: Castelnuovo list (if this list is not a Castelnuovo list a warning is given).
  Output: $[[a_i], [b_i]]$ where the integers a_i, b_i are the i-th generic Betti numbers.
  Example: the 8-th generic Betti numbers of $[1, 2, 3, 4, 5, 5, 3, 2, 1, 1, 1]$.

```
> genericBetticoefficients([1,2,3,4,5,5,3,2,1,1,1],8);
```

$$[[0], [0]]$$

- "genericBetti" computes the generic Betti numbers $(a_i), (b_i)$ associated to a given Castelnuovo polynomial.

Input: Castelnuovo list (if this list is not a Castelnuovo list a warning is given).
Output: $[[..., [i, b_i], ...], [..., [i, a_i], ...]]$ where $(a_i)_i$, $(b_i)_i$ are the generic Betti numbers corresponding to the input Castelnuovo list.
Example: generic Betti numbers of $[1, 2, 3, 4, 5, 5, 3, 2, 1, 1, 1]$.

```
> genericBetti([1,2,3,4,5,5,3,2,1,1,1]);
```

$$[[[7, 1], [9, 1], [12, 1]], [[5, 1], [6, 2], [11, 1]]]$$

- "allgenericBetti" computes all generic Betti numbers $(a_i)_i, (b_i)_i$ associated to the set Γ_n of all Hilbert functions of given degree n.
  Input: positive integer n.
  Output: complete list of all generic Betti numbers $(a_i)_i, (b_i)_i$.
  Example: all generic Betti numbers of all Hilbert functions of degree 3.

```
> allgenericBetti(3);
```

$$[[[[4, 1]], [[1, 1], [3, 1]]], [[[3, 2]], [[2, 3]]]]$$

- "tangentcoefficient" computes a specific coefficient of the tangent function associated to a given Castelnuovo polynomial.
  Input: L, d where L is a Castelnuovo list and d an integer.
  Output: the value $t_\varphi(d)$ where t_φ is the tangent function of the Hilbert function φ associated to the given Castelnuovo list.
  Example: the coefficient $t_\varphi(3)$ where t_φ is the tangent function associated to $[1, 2, 3, 2, 1, 1]$.

```
> tangentcoefficient([1,2,3,2,1,1],3);
```

$$16$$

- "tangent" computes the values of the tangent function associated to a given Castelnuovo polynomial.
  Input: Castelnuovo list (if this list is not a Castelnuovo list a warning is given).
  Output: list L of the initial values of the tangent function t_φ of the Hilbert series φ associated to the input Castelnuovo list. The i-th term $L[i]$ of L is the value $t_\varphi(i-2)$. Further, $t_\varphi(j) = 0$ for $j \leq -2$ and if $L[m]$ are the two final terms of L then $t_\varphi(j) = 2t_\varphi(j-1) - t_\varphi(j-2) + 2$ for $j > m-2$.
  Example: (initial) values of the tangent function t_φ of the Hilbert series φ associated to $[1, 2, 3, 2, 1, 1]$.

```
> tangent([1,2,3,2,1,1]);
```

$$[0, 0, 0, 7, 16, 28, 43, 60, 79, 100, 123, 148, 175]$$

- "cohomologicalcondition" determines if a pair of Hilbert series (φ, ψ) of the same degree satisfies the cohomological condition $\varphi \leq \psi$.
  Input: Castelnuovo lists L, M of the same weight (if at least one of them is not

a Castelnuovo list or if they do not have the same weight a warning is given).
Output: true if the pair of Hilbert series (φ, ψ) corresponding to the input L, M satisfies the cohomological condition, false otherwise.
Example: cohomological condition for the pair of Hilbert series corresponding to the Castelnuovo lists $[1, 2, 3, 3, 2, 2, 2, 1]$ and $[1, 2, 3, 4, 2, 2, 1, 1]$.

```
> K:=[1,2,3,3,2,2,2,1]; L:=[1,2,3,4,2,2,1,1];
  cohomologicalcondition(K,L);
  Hilbertsubscheme(K); Hilbertsubscheme(L);
```

$$K := [1, 2, 3, 3, 2, 2, 2, 1]$$
$$L := [1, 2, 3, 4, 2, 2, 1, 1]$$
$$\mathit{true}$$
$$[1, 3, 6, 9, 11, 13, 15, 16, 16, 16, 16]$$
$$[1, 3, 6, 10, 12, 14, 15, 16, 16, 16, 16]$$

- "dimensioncondition" determines if a pair of Hilbert series (φ, ψ) of the same degree satisfies the dimension condition $\dim H_\varphi < \dim H_\psi$.
  Input: Castelnuovo lists L, M of the same weight (if at least one of them is not a Castelnuovo list or if they do not have the same weight a warning is given).
  Output: true if the pair of Hilbert series (φ, ψ) corresponding to the input L, M satisfies the dimension condition, false otherwise.
  Example: dimension condition for the pair of Hilbert series with corresponding Castelnuovo lists $[1, 2, 3, 4, 4, 1, 1, 1]$ and $[1, 2, 3, 4, 4, 2, 1]$.

```
> K:=[1,2,3,4,4,1,1,1]; L:=[1,2,3,4,4,2,1];
  dimensioncondition(K,L); dimension(K); dimension(L);
```

$$K := [1, 2, 3, 4, 4, 1, 1, 1]$$
$$L := [1, 2, 3, 4, 4, 2, 1]$$
$$\mathit{true}$$
$$28$$
$$29$$

- "tangentcondition" determines if a pair of Hilbert series (φ, ψ) of the same degree satisfies the tangent condition $t_\varphi \geq t_\psi$.
  Input: Castelnuovo lists L, M of the same weight (if at least one of them is not a Castelnuovo list or if they do not have the same weight a warning is given).
  Output: true if the pair of Hilbert series corresponding to the input L, M satisfies the tangent condition, false otherwise.
  Example: tangent condition for the pair of Hilbert series with corresponding Castelnuovo lists $[1, 2, 2, 2, 2, 1]$ and $[1, 2, 3, 2, 1, 1]$.

```
> K:=[1,2,2,2,2,1]; L:=[1,2,3,2,1,1];
  tangentcondition(K,L); tangent(K); tangent(L);
```

$$K := [1, 2, 2, 2, 2, 1]$$
$$L := [1, 2, 3, 2, 1, 1]$$

$$false$$
$$[0, 0, 3, 8, 15, 28, 43, 60, 79, 100, 123, 148, 175]$$
$$[0, 0, 0, 7, 16, 28, 43, 60, 79, 100, 123, 148, 175]$$

- "lengthzero" determines if a pair of Hilbert series (φ, ψ) of the same degree n has length zero, i.e. $\varphi < \psi$ and there are no Hilbert series τ of degree n such that $\varphi < \tau < \psi$.
  Input: Castelnuovo lists L, M of the same weight (if at least one of them is not a Castelnuovo list or if they do not have the same weight a warning is given).
  Output: true if the pair of Hilbert series (φ, ψ) corresponding to input has length zero, false otherwise.
  Example: length zero condition for the pair of Hilbert series with corresponding Castelnuovo lists $[1, 2, 3, 2, 2, 2, 2, 1]$ and $[1, 2, 3, 3, 2, 2, 2]$.

```
> K:=[1,2,3,2,2,2,2,1]; L:=[1,2,3,3,2,2,2]; lengthzero(K,L);
```

$$K := [1, 2, 3, 2, 2, 2, 2, 1]$$
$$L := [1, 2, 3, 3, 2, 2, 2]$$
$$false$$

- "typezero" determines if a pair of Hilbert series (φ, ψ) of the same degree has type zero.
  Input: pair of Castelnuovo lists L, M of the same weight (if at least one of them is not a Castelnuovo list or if they do not has the same weight a warning is given).
  Output: true if the pair of Hilbert series (φ, ψ) corresponding to input has type zero, false otherwise.
  Example: the type zero condition for the pair of Hilbert series with corresponding Castelnuovo lists $[1, 2, 3, 4, 4, 1, 1, 1]$ and $[1, 2, 3, 4, 4, 2, 1]$.

```
> K:=[1,2,3,4,4,1,1,1]; L:=[1,2,3,4,4,2,1]; typezero(K,L);
```

$$K := [1, 2, 3, 4, 4, 1, 1, 1]$$
$$L := [1, 2, 3, 4, 4, 2, 1]$$
$$true$$

- "alltypezero" determines all pairs of Hilbert series (φ, ψ) of given degree which have type zero.
  Input: positive integer n (less than 25 is recommended).
  Output: list of all pairs $[i, j]$ such that the pair of Hilbert series associated to the Castelnuovo lists $G[i]$,$G[j]$ has type zero. Here G stands for the output of the procedure Castelnuovo(n).
  Example: all pairs of Castelnuovo lists of weight 15 such that the corresponding Hilbert series have length zero.

```
> alltypezero(15);G:=Castelnuovo(15):G[17];G[19];
```

$$[[12,13],[17,19],[18,19]]$$
$$[1,2,3,3,2,2,2]$$
$$[1,2,3,3,3,2,1]$$

- "incidencelengthzero" determines if for a pair of Hilbert series (φ,ψ) of the same degree and length zero satisfies $H_\varphi \subset \overline{H_\psi}$. In other words, this procedure solves the incidence problem for (φ,ψ) of length zero.
  Input: pair of Castelnuovo lists L, M of the same weight and of length zero (if at least one of them is not a Castelnuovo list or if they do not have the same weight or if they are not of length zero a warning is given).
  Output: true if $H_\varphi \subset \overline{H_\psi}$ where (φ,ψ) is the pair of Hilbert series associated to input. False otherwise.
  Example: the incidence problem for (φ,ψ) associated to the Castelnuovo lists $[1,2,3,4,4,1,1,1]$ and $[1,2,3,4,4,2,1]$.

  ```
  > K:=[1,2,3,4,4,1,1,1]; L:=[1,2,3,4,4,2,1];
    incidencelengthzero(K,L);
  ```

  $$K := [1,2,3,4,4,1,1,1]$$
  $$L := [1,2,3,4,4,2,1]$$
  $$\mathit{true}$$

- "equivalenceBC" checks if the conditions B and C in Chapter 5 are equivalent for a particular invariant n.
  Input: a positive integer n.
  Output: true if the conditions B and C are equivalent for n, false otherwise.
  Example: check if the conditions B and C are equivalent for $n = 10$.

  ```
  > equivalenceBC(10);
  ```

  $$\mathit{true},\ [\ ]$$

  Remark E.2.1. Using this program we have checked that the conditions B and C in Chapter 5 are equivalent for $1 \leq n \leq 70$. The computer needed approximately 827000 seconds to do this.

E.3 Procedures for Chapter 3

In this section we define the procedures which were used in Section E.1.

```
> restart;with(linalg):with(plots);with(combinat,partition):
  plotoptions(noborders,portrait,'height=100,width=100'):
  interface(plotdevice=inline):
```

```
> Castelnuovo:=proc(n)
  local i,j,k,P,Castel;
  i:=1; Castel:=[];
  if n=0 then
     Castel:=[[]]
  fi;
  while n-i*(i+1)/2 >= 0 do
     P:=partition(n-i*(i+1)/2,min(n-i*(i+1)/2,i));
     Castel:=[op(Castel),seq([seq(j,j=1..i),
     seq(P[k][nops(P[k])-m+1],m=1..nops(P[k]))],k=1..nops(P))];
     i:=i+1
  od;
  RETURN(Castel)
  end:
> validCastelnuovo:=proc(L)
  local i,j,k,n,test,sigma,s;
  test:=false;
  if nops(L)=0 or (nops(L)=1 and L[1]=1) then
     test:=true;
  else
     n:=add(L[k],k=1..nops(L));i:=1;
     while L[i]=i and i < nops(L) do
           i:=i+1
     od;
     sigma:=i-1;s:=sigma;
     if i = nops(L) and L[i]=i then
        test:=true
     else
        test:=member([seq(L[nops(L)-j+1],j=1..nops(L)-s)],
        partition(n-s*(s+1)/2,min(s,n-s*(s+1)/2)));
     fi;
  fi;
  RETURN(test);
  end:
> weight:=proc(L)
  local i,wt;
  global validcastelnuovo;
  if validCastelnuovo(L) then
     wt:=add(L[i],i=1..nops(L));RETURN(wt)
  else
     RETURN("the input is not a valid Castelnuovo list")
  fi;
  end:
```

```
> height:=proc(L)
  local i,ht;
  global validcastelnuovo;
  if validCastelnuovo(L) then
     ht:=0;
     for i from 1 to nops(L) do
        ht:=max(ht,L[i])
     od;
     RETURN(ht)
  else
     RETURN("the input is not a valid Castelnuovo list")
  fi;
  end:
```

```
> diagram:=proc(L)
  local i,j,M,S,xmax,ymax;
  global weight,height,validCastelnuovo;
  if validCastelnuovo(L) then
     M:=[seq(seq([j,i],i=1..L[j]),j=1..nops(L))];
     S:={seq([[M[j][1]-1,0],[M[j][1],0],[M[j][1],M[j][2]],
     [M[j][1]-1,M[j][2]],[M[j][1]-1,0]],j=1..nops(M))};
     xmax:=nops(L);ymax:=height(L);
     print(plot(S,x=0..xmax,0..ymax,xtickmarks=0,ytickmarks=0,
     color=BLACK,axes=none,scaling=constrained));
  else
     RETURN("the input is not a valid Castelnuovo list")
  fi;
  end:
```

```
> savediagram:=proc(filename,L)
  global validCastelnuovo,diagram;
  if validCastelnuovo(L) then
     plotsetup(ps,plotoutput=filename,plotoptions=`portrait,
     noborder`);interface(plotdevice=ps);
     try
        print(diagram(L))
     finally
        interface(plotdevice=inline);NULL;
     end try;
  else
     RETURN("the input is not a valid Castelnuovo list")
  fi;
  end:
```

```
> Hilbertideal:=proc(L)
  local i,j,k,n,Hilb;
  global validCastelnuovo;
  if validCastelnuovo(L) then
     n:=add(L[i],i=1..nops(L));
     Hilb:=[seq(k*(k+1)/2-add(L[j],j=1..k),k=1..nops(L)),
     seq(m*(m+1)/2-n,m=nops(L)+1...nops(L)+6)];
     RETURN(Hilb);
  else
     RETURN("the input was not a valid Castelnuovo list")
  fi;
  end:

> allHilbertideal:=proc(n)
  local i,j,k,L,Hilb;
  global Castelnuovo,Hilbertideal;
  L:=Castelnuovo(n);Hilb:=[seq(Hilbertideal(L[i]),i=1..nops(L))];
  RETURN(Hilb);
  end:

> characteristic:=proc(L)
  local i,M,charac;
  global validCastelnuovo;
  if nops(L)=0 then
     charac:=[1]
  else
     if validCastelnuovo(L) then
        M:=[L[1],seq(L[j]-L[j-1],j=2..nops(L)),-L[nops(L)]];
        charac:=[1-M[1],seq(-M[j]+M[j-1],j=2..nops(M)),M[nops(M)]];
        RETURN(charac);
     else
        RETURN("the input is not a valid Castelnuovo list")
     fi;
  fi;
  end:

> allcharacteristic:=proc(n)
  local i,L,charac;
  global Castelnuovo,characteristic;
  L:=Castelnuovo(n);
  charac:=[seq(characteristic(L[i]),i=1..nops(L))];
  RETURN(charac);
  end:
```

```
> validcharacteristic:=proc(Q)
  local i,j,k,s,V,W,X,Y,Z,test;
  if nops(Q)=0 then
     test:=false
  else
     test:=true;i:=1;
     while Q[i]=0 do
        i:=i+1;
     od;
     while add(Q[j],j=1..i)>0 and i<nops(Q) do
        i:=i+1
     od;
     if add(Q[i],i=1..nops(Q))<>1 or i<>nops(Q) then
        test:=false
     fi;
  fi;
  if test=true then
     s:=add((k-1)*Q[k],k=1..nops(Q));
     if s>0 then
        V:=[seq(Q[k],k=s+1..nops(Q))];
     else
        V:=[seq(0,k=1..-s),op(Q)]
     fi;
     W:=[1-V[1],seq(-V[i],i=2..nops(V))];
     X:=[seq(add(W[k],k=1..l),l=1..nops(W))];
     Y:=[seq(add(X[k],k=1..l),l=1..nops(X))];
     Z:=[];i:=1;
     while Y[i]<>0 and i<=nops(Y) do
        Z:=[op(Z),Y[i]];i:=i+1
     od;
     RETURN(test,Z)
  else
     RETURN(test)
  fi;
  end:
```

```
> Betti:=proc(L)
  local i,j,m,A,B,Q,sigma;
  global validCastelnuovo,height,characteristic;
  if validCastelnuovo(L) then
     Q:=characteristic(L);sigma:=height(L);A:=[];B:=[];
     if sigma=0 then
        A:=[[0,1]]
     else
        A:=[op(A),[sigma-1,Q[sigma]]];
        for i from sigma+2 to nops(Q) do
              m:=add(Q[j],j=1..i);
              if max(Q[i],0)=m-1 then
                 if Q[i]>0 then
                    A:=[op(A),[i-1,Q[i]]]
                 fi;
                 if Q[i]<0 then
                    B:=[op(B),[i-1,-Q[i]]]
                 fi;
              else
                 A:=[op(A),[i-1,max(Q[i],0)..m-1]];
                 B:=[op(B),[i-1,max(Q[i],0)-Q[i]..m-1-Q[i]]]
              fi;
        od;
        RETURN([A,B]);
     fi;
  else
     RETURN("the input is not a valid Castelnuovo list")
  fi;
  end:
```

```
> numberminimalresolutions:=proc(L)
  local i,K,T,sigma,s;
  global validCastelnuovo,height;
  if validCastelnuovo(L) then
     sigma:=height(L);s:=sigma;K:=[op(L),0];
     T:=product(1+min(K[i-1]-K[i],K[i-2]-K[i-1]),i=s+2..nops(K));
     RETURN(T)
  else
     RETURN("the input is not a valid Castelnuovo list")
  fi;
  end:
```

```
> numberallminimalresolutions:=proc(n)
  local i,G,T;
  global Castelnuovo;numberminimalresolutions;
  G:=Castelnuovo(n);T:=0;
  for i from 1 to nops(G) do
     T:=T+numberminimalresolutions(G[i])
  od;
  RETURN(T)
  end:
> dimension:=proc(L)
  local dim,i,j;
  global validCastelnuovo;
  if validCastelnuovo(L) then
     if nops(L)=0 then
        dim:=0
     else
        dim:=1+add(L[i],i=1..nops(L))+add(L[j]*L[j-1],
        j=2..nops(L))-add(L[j]*L[j-2],j=3..nops(L))
     fi;
     RETURN(dim);
  else
     RETURN("the input is not a valid Castelnuovo list")
  fi;
  end:
> alldimension:=proc(n)
  local i,L,dim;
  global Castelnuovo,dimension;
  L:=Castelnuovo(n);dim:=[seq(dimension(L[i]),i=1..nops(L))];
  RETURN(dim);
  end:
```

E.4 Additional procedures for Chapter 5

In this part we define procedures in Maple™ additional to the ones given in Section E.3.

```
> Hilbertsubscheme:=proc(L)
  local i,j,k,n,Hilb;
  global validCastelnuovo;
  if validCastelnuovo(L) then
     n:=add(L[i],i=1..nops(L));
     Hilb:=[seq(add(L[j],j=1..k),k=1..nops(L)),n,n,n];
     RETURN(Hilb)
  else
     RETURN("the input is not a valid Castelnuovo list")
  fi;
  end:
```

```
> allHilbertsubscheme:=proc(n)
  local i,j,k,L,Hilb;
  global Castelnuovo,Hilbertsubscheme;
  L:=Castelnuovo(n);
  Hilb:=[seq(Hilbertsubscheme(L[i]),i=1..nops(L))];RETURN(Hilb);
  end:
```

```
> genericBetticoefficients:=proc(L,i)
  local A,B,Q,sigma;
  global validCastelnuovo,characteristic;
  if validCastelnuovo(L) then
     Q:=characteristic(L);sigma:=height(L);
     if i<sigma or i>nops(Q)-1 then
        A:=[0];B:=[0];
     else
        if Q[i+1]>=0 then
           A:=[Q[i+1]];B:=[0]
        fi;
        if Q[i+1]<0 then
           A:=[0];B:=[-Q[i+1]]
        fi;
     fi;
     RETURN([A,B]);
  else
     RETURN("the input is not a valid Castelnuovo list")
  fi;
  end:
```

```
> genericBetti:=proc(L)
  local i,Q,A,B;
  global validCastelnuovo,characteristic;
  if validCastelnuovo(L) then
     Q:=characteristic(L);A:=[];B:=[];
     for i from 1 to nops(Q) do
        if Q[i]>0 then
           A:=[op(A),[i-1,Q[i]]]
        fi;
        if Q[i]<0 then
           B:=[op(B),[i-1,-Q[i]]]
        fi;
     od;
     RETURN([B,A]);
  else
     RETURN("the input is not a valid Castelnuovo list")
  fi;
  end:
```

```
> allgenericBetti:=proc(n)
  local i,L,Betti;
  global Castelnuovo,genericBetti;
  L:=Castelnuovo(n);Betti:=[seq(genericBetti(L[i]),i=1..nops(L))];
  RETURN(Betti);
  end:
```

```
> tangentcoefficient:=proc(L,d)
  local i,T,Q;
  global characteristic;
  if validCastelnuovo(L) then
     Q:=characteristic(L);
     T:=add(Q[i+1]*max((d-i+2),0)*(d-i+4),i=0..nops(Q)-1);
     if 1<=d+4 and d+4<=nops(Q) then
        T:=T+max(-Q[d+4],0)
     fi;
     RETURN(T);
  else
     RETURN("the input is not a valid Castelnuovo list")
  fi;
  end:
```

```
> tangent:=proc(L)
  local d,T;
  global tangentcoefficient,validCastelnuovo;
  if validCastelnuovo(L) then
     T:=[seq(tangentcoefficient(L,d),d=-1..nops(L)+5)];
     RETURN(T);
  else
     RETURN("the input is not a valid Castelnuovo list")
  fi;
  end:
```

```
> cohomologicalcondition:=proc(K,L)
  local g,h,i,m,G,H,test;
  global validCastelnuovo,Hilbertsubscheme,weight;
  test:=false;
  if validCastelnuovo(K) and validCastelnuovo(L) and
  weight(K)=weight(L) then
     G:=Hilbertsubscheme(K);H:=Hilbertsubscheme(L);
     m:=min(nops(K),nops(L));i:=1;
     while G[i]<=H[i] and i<=m do
        i:=i+1
     od;
     if i=m+1 then
        test:=true
     fi;
     RETURN(test);
  else
     RETURN("the input is not a pair of valid
     Castelnuovo functions of the same degree")
  fi;
  end:
```

```
> dimensioncondition:=proc(K,L)
  local test;
  global dimension,weight;
  test:=false;
  if validCastelnuovo(K) and validCastelnuovo(L) and
  weight(K)=weight(L) then
     if dimension(K)<dimension(L) then
        test:=true
     fi;
     RETURN(test)
  else
     RETURN("the input is not a pair of valid
     Castelnuovo functions of the same degree")
  fi;
  end:
```

```
> tangentcondition:=proc(K,L)
  local i,m,G,H,test;
  global validcastelnuovo,tangent,weight;
  test:=false;
  if validCastelnuovo(K) and validCastelnuovo(L) and
  weight(K)=weight(L) then
     G:=tangent(K);H:=tangent(L);m:=min(nops(K),nops(L));i:=1;
     while H[i]<=G[i] and i <= m do
        i:=i+1
     od;
     if i=m+1 then
        test:=true
     fi;
     RETURN(test);
  else
     RETURN("the input is not a pair of valid
     Castelnuovo functions of the same degree")
  fi;
  end:
```

```
> lengthzero:=proc(K,L)
  local i,G,V,test;
  global validCastelnuovo,Castelnuovo,
  weight,cohomologicalcondition;
  if validCastelnuovo(K) and validCastelnuovo(L) and
  weight(K)=weight(L) then
     test:=false;
     if cohomologicalcondition(K,L) then
        G:=Castelnuovo(weight(K));V:=[];
        for i from 1 to nops(G) do
           if cohomologicalcondition(K,G[i]) and
            cohomologicalcondition(G[i],L) then
               V:=[op(V),G[i]]
           fi;
        od;
        if nops(V)=2 then
           test:=true
        fi;
     fi;
     RETURN(test)
  else
     RETURN("the input is not a pair of valid
     Castelnuovo functions of the same degree")
  fi;
  end:
```

```
> typezero:=proc(K,L)
  local i,j,P,test;
  global validCastelnuovo,weight,height;
  test:=false;
  if validCastelnuovo(K) and validCastelnuovo(L) and
  weight(K)=weight(L) then
     P:=[op(K),0,0];
     for i from height(K) to nops(K)-4 do
        if P[i]=P[i+1] and P[i+1]>P[i+2] and P[i+2]=P[i+3] and
        P[i+3]=P[i+4] and P[i+4]>P[i+5] and P[i+5]=P[i+6] and
        (equal([seq(K[j],j=1..i+1),K[i+2]+1,K[i+3],K[i+4]-1,
        seq(K[j],j=i+5..nops(K))],L) or
        equal([seq(K[j],j=1..i+1),K[i+2]+1,K[i+3],K[i+4]-1,
        seq(K[j],j=i+5..nops(K))],[op(L),0])) then
           test:=true
        fi;
     od;
     RETURN(test);
  else
     RETURN("the input is not a pair of valid
     Castelnuovo functions of the same degree")
  fi;
  end:
```

```
> alltypezero:=proc(n)
  global Castelnuovo,lengthzero;
  local i,j,G,S;
  G:=Castelnuovo(n);S:=[];
  for i from 1 to nops(G) do
     for j from i+1 to nops(G) do
        if typezero(G[i],G[j]) then
           S:=[op(S),[i,j]]
        fi;
     od;
  od;
  RETURN(S)
  end:
```

```
> incidencelengthzero:=proc(K,L)
  local test;
  global validCastelnuovo,weight,lengthzero,
  dimensioncondition,tangentcondition;
  if validCastelnuovo(K) and validCastelnuovo(L) and
  weight(K)=weight(L) then
     if lengthzero(K,L) then
        test:=false;
        if dimensioncondition(K,L) and tangentcondition(K,L) then
           test:=true
        fi;
        RETURN(test);
     else
        RETURN("the input is not of length zero")
     fi;
  else
     RETURN("the input is not a pair of valid
     Castelnuovo functions of the same degree")
  fi;
  end:
```

```
> valid:=proc(S)
  local i,j,test;
  test:=false;
  i:=1;
  while S[i]=i and i<nops(S) do
     i:=i+1
  od;
  j:=i;
  while S[i-1]>=S[i] and i<nops(S) do
     i:=i+1
  od;
  if (i=nops(S) and S[i-1]>=S[i]) or j=nops(S) then
     test:=true
  fi;
  RETURN(test);
  end:
```

```
> setonesquare:=proc(n)
  global Castelnuovo;
  local i,j,k,r,s,t,G,H,S;
  G:=Castelnuovo(n);
  H:=[];
  for i from 1 to nops(G) do
     for j from height(G[i])+1 to nops(G[i])-1 do
        for k from j+1 to nops(G[i]) do
           S:=G[i]+[seq(0,r=1..j-1),1,seq(0,s=j+1..k-1),-1,
           seq(0,t=k+1..nops(G[i]))];
           if valid(S) then
              if S[nops(S)]=0 then
                 S:=[seq(S[r],r=1..nops(S)-1)]
              fi;
              H:=[op(H),[G[i],S]]
           fi;
        od;
     od;
  od;
  RETURN(H);
  end:
```

```
> setlengthzero:=proc(n)
  global setonesquare;
  local i,j,H,S;
  H:=setonesquare(n);S:=convert(H,set);
  for i from 1 to nops(H) do
     for j from i to nops(H) do
        if H[i][2]=H[j][1] and member([H[i][1],H[j][2]],S) then
           S:=S minus {[H[i][1],H[j][2]]}
        fi;
     od;
  od;
  H:=convert(S,list);RETURN(H);
  end:
```

```
> equivalenceBCdetail:=proc(K,L)
  global genericBetticoefficients,dimensioncondition,
  tangentcondition;
  local a,b,i,m,u,v,V,test,testBetti,testgeometric;
  test:=false;testBetti:=false;testgeometric:=false;
  m:=min(nops(K),nops(L));u:=0;
  V:=[seq(K[i]-L[i],i=1..m),seq(K[i],i=m+1..nops(K)),
  seq(L[i],i=m+1..nops(L))];
  while V[u+1]<>-1 do
     u:=u+1;
  od;
  v:=u;
  while V[v+2]<>1 do
     v:=v+1;
  od;
  a[u]:=genericBetticoefficients(K,u)[1][1];
  a[v+2]:=genericBetticoefficients(K,v+2)[1][1];
  b[u+1]:=genericBetticoefficients(K,u+1)[2][1];
  b[v+3]:=genericBetticoefficients(K,v+3)[2][1];
  if v=u and a[u]>0 and b[v+3]>0 then
     testBetti:=true
  fi;
  if v=u+1 and a[u]>0 and b[u+1]<=a[u] and a[u]<=b[u+1]+1
  and b[v+3]=a[v+2] then
     testBetti:=true
  fi;
  if v=u+1 and a[u]=b[u+1]+1 and b[v+3]=a[v+2]-1 and b[v+3]>0 then
     testBetti:=true
  fi;
  if v>=u+2 and a[u]=b[u+1]+1 and b[v+3]=a[v+2] then
     testBetti:=true
  fi;
  if dimensioncondition(K,L) and tangentcondition(K,L) then
     testgeometric:=true
  fi;
  if testBetti=testgeometric then
     test:=true
  fi;
  RETURN(test);
  end:
```

```
> equivalenceBC:=proc(n)
  local i,H,T,test;
  global setlenthzero,equivalenceBCdetail;
  T:=[];test:=true;H:=setlengthzero(n);
  for i from 1 to nops(H) do
     if not(equivalenceBCdetail(H[i][1],H[i][2])) then
        test:=false;T:=[op(T),[H[i][1],H[i][2]]]
     fi;
  od;
  RETURN(test,T);
  end:
```

Appendix F

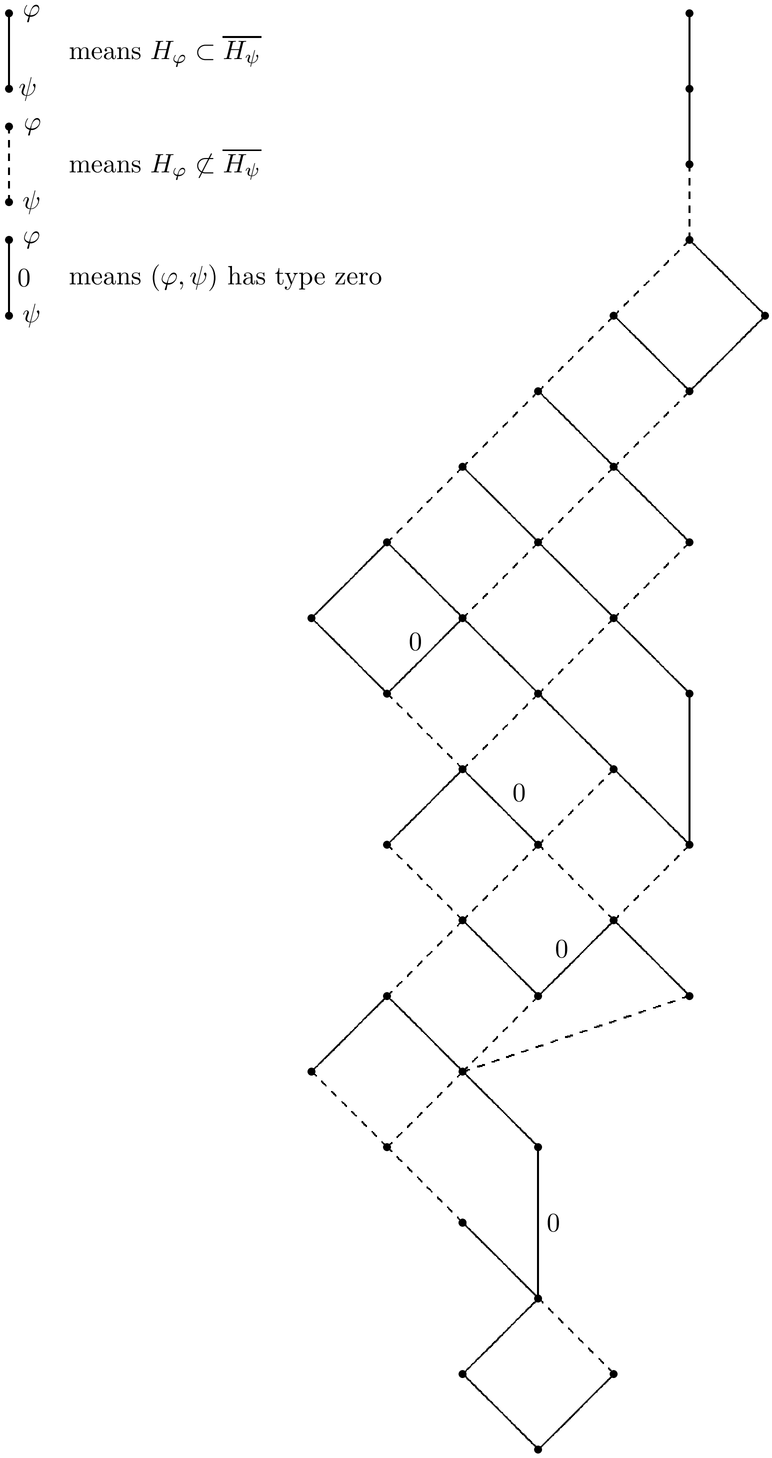
Hilbert graphs and combinatorics

In this appendix we examine Hilbert graphs (as defined in §5.1) more closely.

F.1 Hilbert graphs and incidence problems for low invariants

For low values of n the Hilbert graph Γ_n is rather trivial. However when n becomes bigger the number of Hilbert functions increase rapidly (see Remark 5.1.2) and so the Hilbert graphs become more complicated. As an illustration we have included the Hilbert graph for $n = 17$ where we used Theorem 9 to solve the incidence problems of length zero (the picture gives little information on incidence problems which are not of length zero). By convention the minimal Hilbert series is on top.

The reader will notice that the Hilbert graph contains pentagons. This shows that the Hilbert graph is not catenary and also contradicts [38, Lemme 2.1.2].


φ
ψ
means $H_\varphi \subset \overline{H_\psi}$
φ
ψ
means $H_\varphi \not\subset \overline{H_\psi}$
φ
0
ψ
means (φ, ψ) has type zero
0
0
0
0

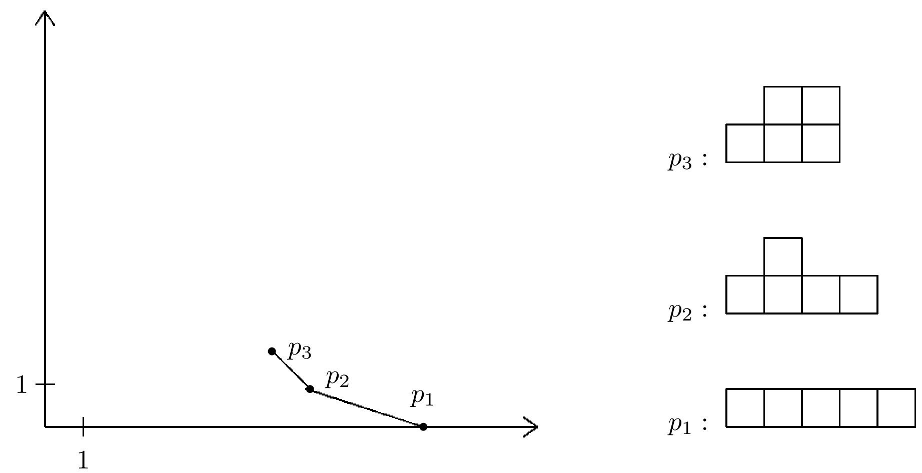
F.2 General Hilbert graphs

As n increases the reader will convince himself that it becomes difficult to visualize the Hilbert graph Γ_n. It is not even clear to us that Γ_n is in general a planar graph.

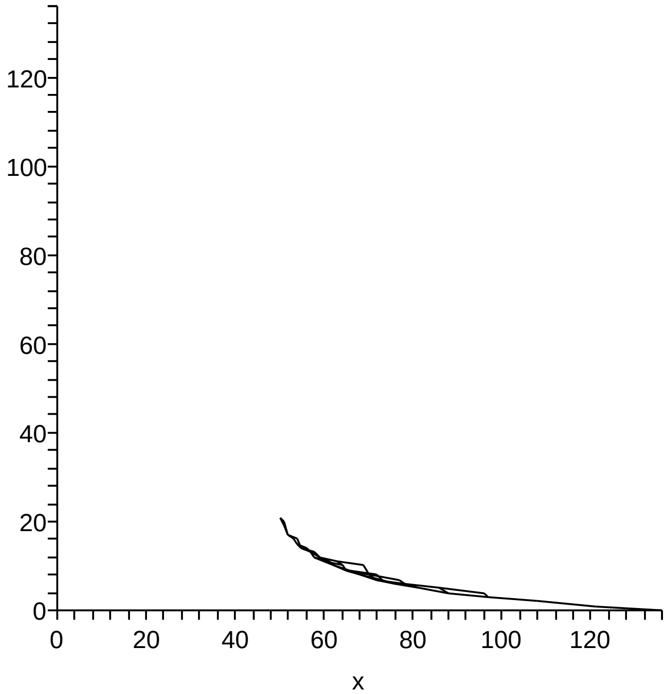
After a talk the author held at the University of Washington in the summer of 2004, Rekha Thomas proposed a way to draw Hilbert graps. We describe her idea.

Fix an integer $n > 0$. To the point $p(s_{\max}) = (n(n-1)/2, 0) \in \mathbb{R}^2$ we associate the Castelnuovo polynomial $s_{\max}(t) = 1 + t + t^2 + \cdots + t^{n-1}$. It is clear that any Castelnuovo diagram s of weight n can be obtained from $s_{\max}$ by making a number of unit squares "jump to the left" while, at each step, preserving the Castelnuovo property. Writing $v \in \mathbb{R}^2$ for the sum of all intermediate movement vectors of these unit squares, we define $p(s) = p(s_{\max}) + v \in \mathbb{R}^2$ for the coordinates corresponding to the Castelnuovo diagram s. The relation $\leq$ on the Hilbert graph Γ_n induces a directed graph on this configuration $\{p(s)\}_{s \in \Gamma_n}$ which we will denote by H_n.

Example F.2.1. For $n = 5$ the graph H_n is shown below

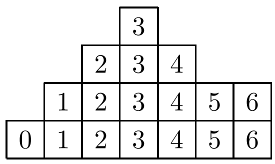


Example F.2.2. For $n = 17$ the graph H_n is given by

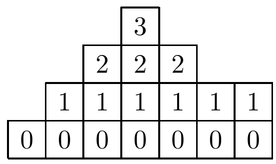


It is easy to give an explicit description of the coordinates of p_s in H_n: For a Castelnuovo diagram s of weight n we obtain the first (resp. second) coordinate of $p(s)$ by attaching to each unit square in the i-column (resp. i-th row) the integer $i-1$ and computing the sum over all i. Indeed, it follows that $p(s_{\max}) = (n(n-1)/2, 0)$. Further, moving one unit square in $s \in \Gamma_n$ from row α, column β to row α', column β' will decrease the first sum by $-(\alpha-1)+(\alpha'-1)$ and increase the second sum by $-(\beta-1)+(\beta'-1)$. Hence the vector of movement is given by $(\alpha'-\alpha, \beta'-\beta)$ which corresponds to the definition p_s.

Example F.2.3. Consider the Castelnuovo polynomial $s(t) = 1 + 2t + 3t^2 + 4t^3 + 3t^4 + 2t^5 + 2t^6$ of weight 17. The correponding point in H_{17} is given by $p(s) = (54, 15)$.



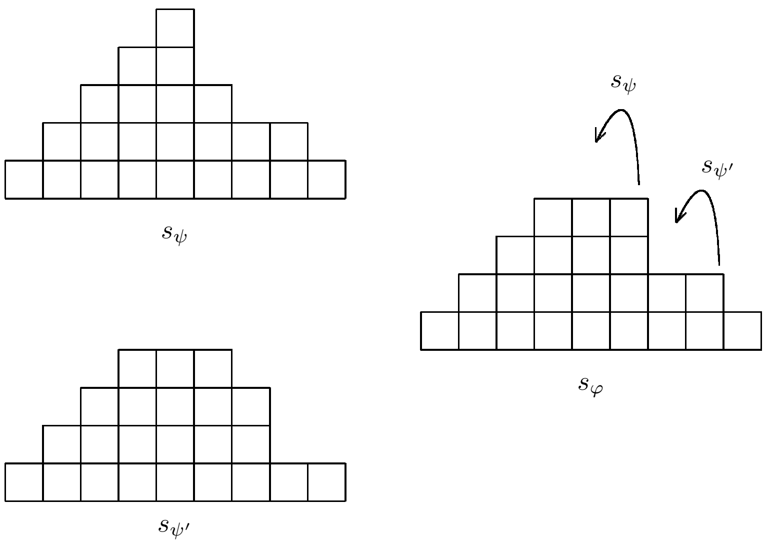
54



15

Unfortunately it may occur that for different Castelnuovo diagrams s, s' of weight n the associated vertices $p(s)$, $p(s')$ in H_n are equal. Moreover, writing φ, φ' for the corresponding Hilbert functions it may happen that incidence problems (φ, ψ) and (φ', ψ) of length zero have different solutions. Similar for incidence problems (φ, ψ) and (φ, ψ') of length zero where $p(s_\psi) = p(s_{\psi'})$. These phenomena are illustrated in the example below.

Example F.2.4. Let $n = 23$ and let s_φ, s_ψ and $s_{\psi'}$ be the Castelnuovo diagrams as indicated below. It follows that $p(s_\psi) = p(s_{\psi'})$. Also (φ, ψ) and (φ, ψ') have length zero. One verifies (for example by using the visual criterion given in Appendix D) that $H_\varphi \subset \overline{H_\psi}$ while $H_\varphi \not\subset \overline{H_{\psi'}}$.



Despite the fact that points of the graph H_n may represent multiple Hilbert series, H_n still give us an approximation of the Hilbert graph Γ_n. At moment of writing we do not understand the shape of H_n as n tends to infinity. However we do want to point out the link between the vertices of H_n and certain graphs associated to partitions of n, called partition graphs. We will do this below.

F.2.1 Partition graphs

Let us first recall some basic notions concerning partitions. We refer to [4] for more details.

A *partition* λ of a positive integer n is a finite sequence of positive integers

$\lambda_1, \lambda_2, \ldots, \lambda_r$ for which

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r \geq 0 \quad \text{and} \quad \sum_{i=1}^{r} \lambda_i = n$$

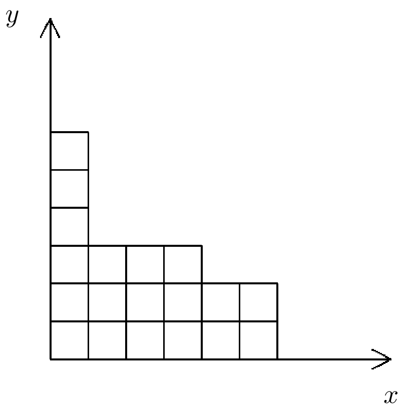
We will often put $\lambda_i = 0$ for $i < 1$ and $i > r$. The partition $(\lambda_1, \lambda_2, \ldots, \lambda_r)$ will be denoted by λ. We refer to the integers $\lambda_1, \ldots, \lambda_r$ as the *parts* of λ. In case all parts of λ are distinct we say λ is a *partition in distinct parts.* Write $\mathcal{P}_n$ for the set of all partitions of n and $\mathcal{D}_n \subset \mathcal{P}_n$ for the subset of all partitions in distinct parts.

It is standard to visualize a partition $\lambda \in \mathcal{P}_n$ using the graph of the staircase function

$$F(\lambda) : \mathbb{R} \to \mathbb{N} : x \mapsto \lambda'_{\lfloor x \rfloor}$$

where as usual $\lfloor x \rfloor$ stands for the greatest integer less or equal than $x \in \mathbb{R}$. We divide the area under this graph $F(\lambda)$ in unit cases. This graph is called the *Ferrers graph* of λ.

Example F.2.5. $\lambda = (6, 6, 4, 1, 1, 1)$ is a partition of length 6 and weight 19. The Ferrers graph of λ is presented by



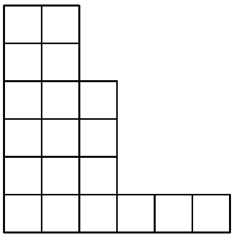
In the sequel we will omit the axes in these graphs.

If $\lambda \in \mathcal{P}_n$ is a partition we may define a new partition $\lambda' = (\lambda'_1, \lambda'_2, \ldots, \lambda'_{r'})$ by defining λ'_i as the number of parts of λ greater or equal than i (for $i \geq 1$)

$$\lambda'_i = \text{kard}\{j \mid \lambda_j \geq i\}$$

The partition λ' is called the *conjugate* of λ. Note that $\lambda' \in \mathcal{P}_n$ and the Ferrers graph of λ' is obtained by reflection of the graph of λ along the diagonal.

Example F.2.6. The conjugate of the partition $\lambda = (6, 6, 4, 1, 1, 1)$ from Example F.2.5 is $\lambda' = (6, 3, 3, 3, 2, 2)$, presented by

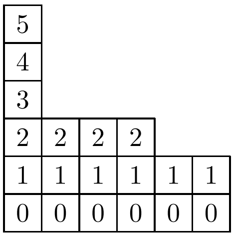


For any partition λ of n we define the number $n(\lambda)$ as

$$n(\lambda) = \sum_i (i-1)\,\lambda_i \in \mathbb{N}$$

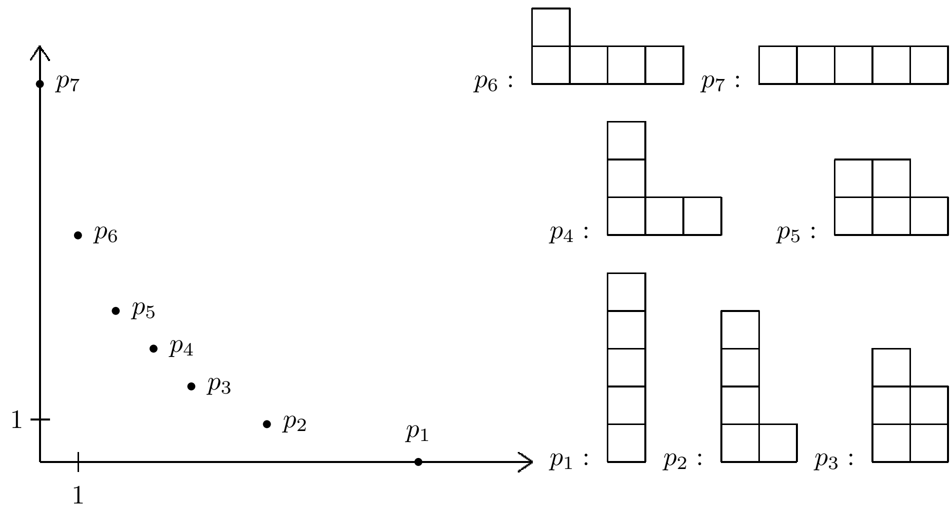
Now $n(\lambda)$ has the following interpretation on the Ferrers graph of λ: Attaching to each unit square in the i-th row the integer $i-1$, $n(\lambda)$ is the total sum over all i.

Example F.2.7. For the partition $\lambda = (6, 6, 4, 1, 1, 1)$ we have $n(\lambda) = 26$.

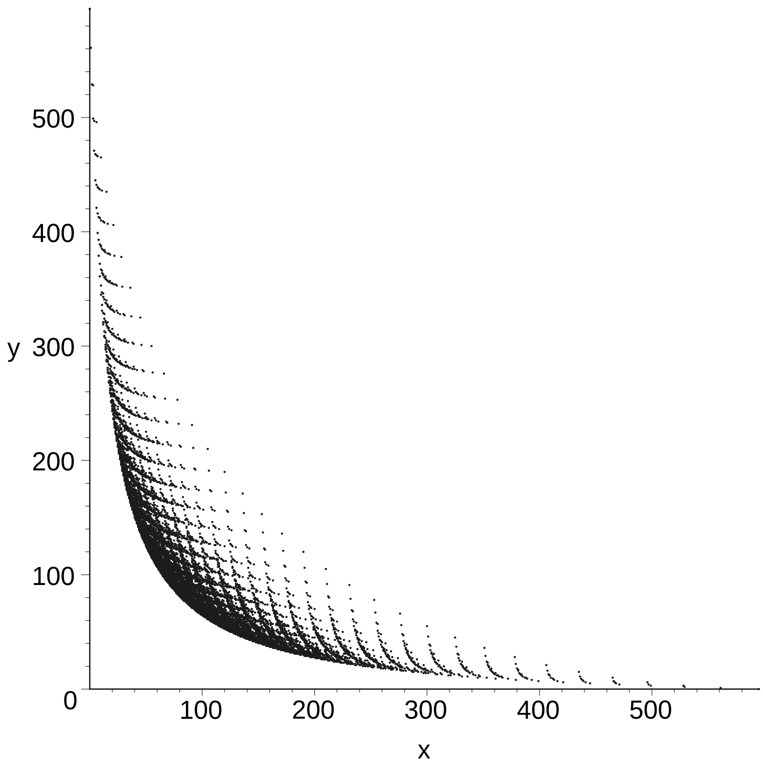


We refer to the set of points $\{(n(\lambda), n(\lambda')) \mid \lambda \in \mathcal{P}_n\} \subset \mathbb{R}^2$ as the *partition graph* P_n. Similarly its subgraph $\{(n(\lambda), n(\lambda')) \mid \lambda \in \mathcal{D}_n\} \subset \mathbb{R}^2$ is called the *distinct partition graph* D_n.

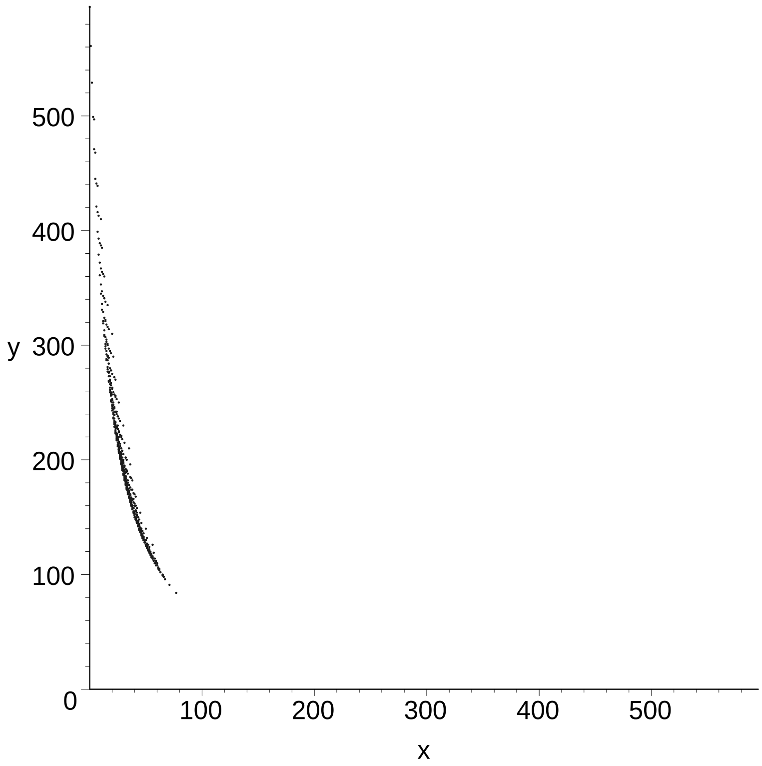
Example F.2.8. For $n = 5$ the partition graph P_n is shown below. The distinct partition graph are the points p_5, p_6, p_7.



Example F.2.9. For $n = 35$ the partition graph P_n is of the form



and the distinct partition graph D_n is given by



F.2.2 Twisted partitiongraphs

We now show how partition graphs are related to Hilbert graphs.

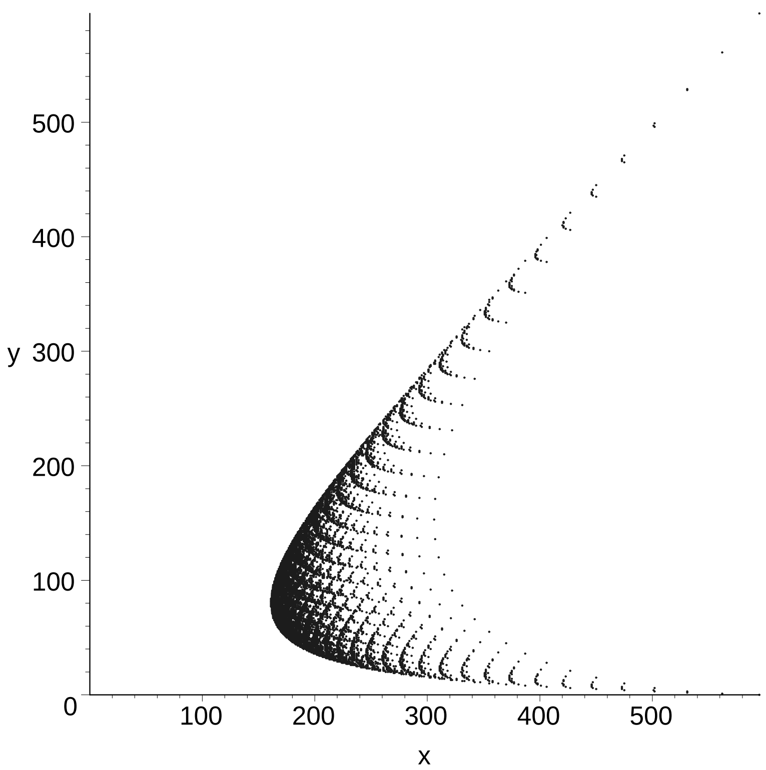
As observed in Remark 3.1.3, by shifting the rows in a Castelnuovo diagram in such a way they are left aligned, we obtain a partition in distinct graphs. In this way we obtain a bijective correspondence $\mathcal{T} : \mathcal{D}_n \to \Gamma_n$ between the set $\mathcal{D}_n$ of partitions of n in distinct parts and the set Γ_n of Castelnuovo diagrams of degree n. Moreover

Proposition F.2.10. *The map $T : \mathbb{R}^2 \to \mathbb{R}^2 : (x, y) \mapsto (x + y, x)$ restricts to a bijection between the twisted partition graph D_n and the vertex graph of H_n, where $T(n(\lambda), n(\lambda')) = p(\mathcal{T}(\lambda))$ for all $\lambda \in \mathcal{D}_n$.*

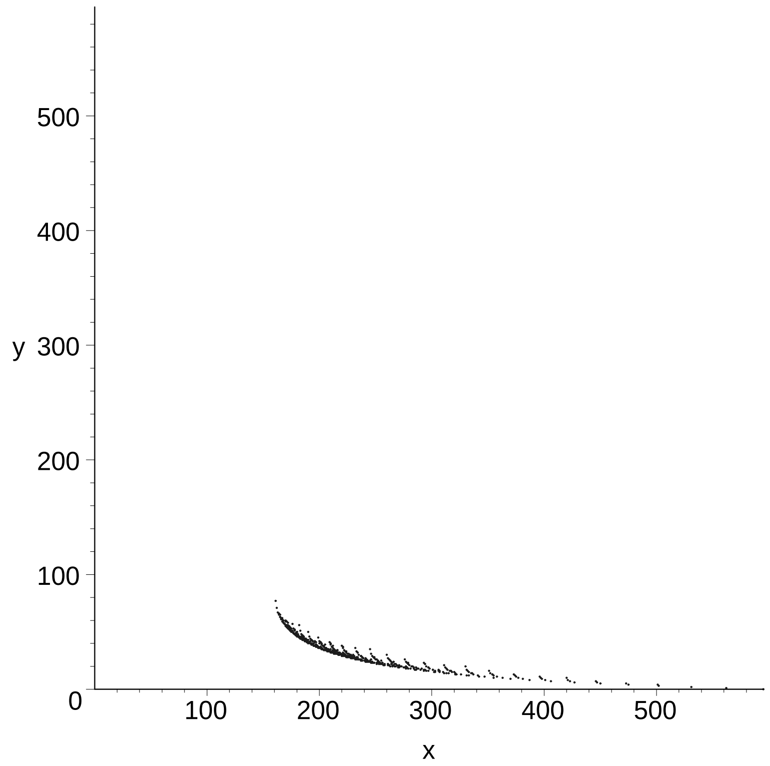
Proof. It is sufficient to prove that for any partition λ of n in distinct parts we have $p(\mathcal{T}(\lambda)) = (n(\lambda) + n(\lambda'), n(\lambda))$. Let $s = \mathcal{T}(\lambda)$ be the associated Castelnuovo diagram and write $p(s) = (u, v)$. By the explicit description of H_n we immediately obtain $v = n(\lambda)$. Further, $u - v$ now correspond to the associating the integer i to the unit squares in the $(i - 1)$-th (lower) diagonal of s and summing over all i. By shifting the rows in s in such a way they are left aligned we see $u - v = n(\lambda')$, proving the statement. □

We refer to the image of the partition graph P_n under T as the *twisted partition graph*, denoted by TP_n. Similarly, TD_n is called the *twisted distinct partition graph.* By the previous result $TD_n = H_n$ (omiting the edges).

Example F.2.11. For $n = 35$ the twisted partition graph TP_n is of the form



and the twisted distinct partition graph TD_n (and hence the vertex graph of H_n) is given by



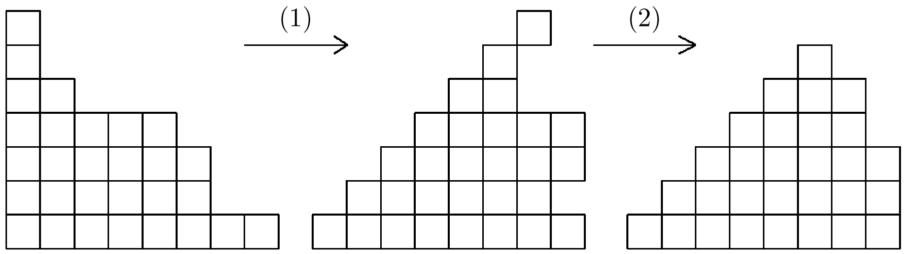
From the above examples one expects that TD_n is located in the lower part of the twisted partition graph TP_n. We prove

Proposition F.2.12. *Let $p = (x, y)$ represent a point in the twisted partition graph TP_n. Then there is a vertex $q \in H_n$ of the form $q = (x, y')$ where $y' \leq y$.*

Proof. Let $\lambda \in \mathcal{P}_n$ be a partition of n associated to $p \in TP_n$ (note λ does not need to be unique). To the Ferrers graph of λ we associate another graph by

1. shifting the i-th row $(i-1)$ units to the right, for all i, and
2. if necessary filling the "holes" by applying gravity

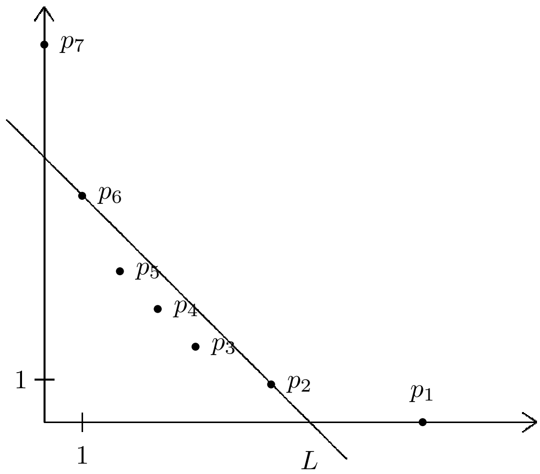
We illustrate this operation below by the example $\lambda = (8, 6, 6, 5, 2, 1, 1)$



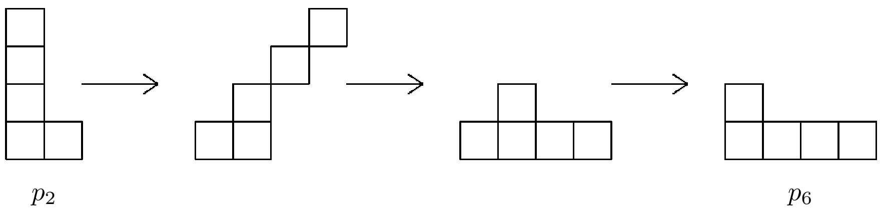
It follows that the end result is a Castelnuovo graph s of weight n. It is also easy to see $p(s) = (n(\lambda) + n(\lambda'), y') \in H_n$ for some $y' \leq n(\lambda)$, and $y' = n(\lambda)$ if and only if λ is a partition in distinct parts i.e. $p \in H_n$. This proves the statement. □

The interpretation on the partition graph is clear: For any vertical line L the lowest intersection point with TP_n belongs to $TD_n = H_n$. Using the function T this property is translated to partition graphs as follows: For any line L with slope -1 the leftmost intersection point with P_n belongs to D_n. Thus to each partition λ we may associate a partition in distinct parts.

Example F.2.13. Applied to Example F.2.1 and partition $\lambda = (2, 1, 1, 1)$ (correspoonding to p_2) we find the partition $\lambda' = (4, 1)$ in distinct parts (point p_6).



and the correspondence $p_2 \mapsto p_6$ is obtained by the procedure



Appendix G

An inequality on broken chessboards

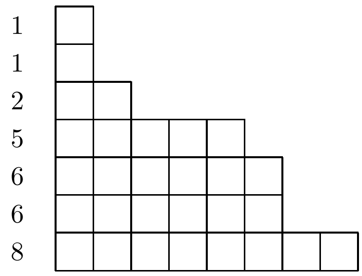
For any partition of a positive integer we consider the chess (or draughts) colouring of its associated Ferrers graph. Let b denote the total number of black unit squares, and w the number of white squares. In this appendix we characterise all pairs (b, w) which arise in this way. This simple combinatorical result was discovered by characterising Hilbert series of certain right modules over cubic three dimensional Artin-Schelter algebras in Chapter 6. However in this part we present a purely combinatorical proof.

The result is (at least partially) known in literature [75, Problem 10], but we found it interesting to present an alternative and perhaps more natural proof based on the notion of Castelnuovo polynomials.

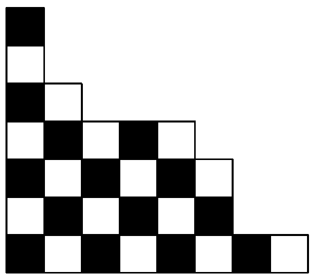
This appendix is joint work with N. Marconnet [31].

G.1 Introduction

A partition of a positive integer n is a finite nonincreasing sequence of positive integers $\lambda_1, \lambda_2, \ldots, \lambda_r$ such that $\sum_{i=1}^{r} \lambda_i = n$. We denote $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r)$. To each partition λ is associated its Ferrers graph: A pattern of unit squares with the i-th row (counting from $i = 0$) having la_{i+1} unit squares (see §G.2.1 for a more formal definition). As an example the Ferrers graph of the partition $\lambda = (8, 6, 6, 5, 2, 1, 1)$ of 29 is given by



For such a Ferrers graph we consider the chess (or draughts) colouring on it, with the convention that the unit square left below is black. For example the chess Ferrers graph of the partition $\lambda = (8, 6, 6, 5, 2, 1, 1)$ is given by



For a partition λ we write $b(\lambda)$ (resp. $w(\lambda)$) for the number of black (resp. white) squares in its chess Ferrers graph. Our main result in this appendix is

Theorem A. *Let* $(b, w) \in \mathbb{N}^2$. *Then there exists a partition* λ *such that* $(b(\lambda), w(\lambda)) = (b, w)$ *if and only if*

$$(b - w)^2 \leq b \tag{G.1}$$

Furthermore the same statement holds if we restrict ourselves to partitions in distinct parts.

If $b \neq 0$ then (G.1) may be written as

$$\left(1 - \frac{w}{b}\right)^2 \leq \frac{1}{b}$$

which measures how close the ratio w/b is to 1. As a byproduct of the proof of Theorem A presented in this note, the appearing $(b, w) \in \mathbb{N}^2$ are described in an explicit way.

Theorem B. *Let* $(b, w) \in \mathbb{N}^2$. *Then there exists a partition* λ *such that* $(b(\lambda), w(\lambda)) = (b, w)$ *if and only if there exist positive integers* $k, l \in \mathbb{N}$ *such that either*

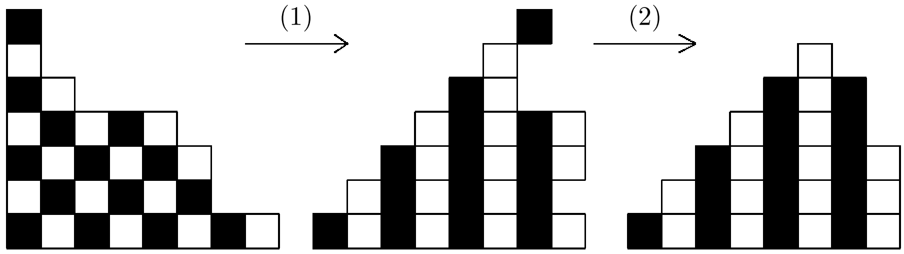
$$(b, w) = \big(k^2 + l, k(k + 1) + l\big) \text{ or } (b, w) = \big((k + 1)^2 + l, k(k + 1) + l\big)$$

Let us indicate how we prove Theorem A. To any chess Ferrers graph we associate another graph by

1. shifting the first row one place to the right, the second row two places to the right, etc. and afterwards

2. if necessary filling the "holes" by applying gravity

For example for the partition $\lambda = (8, 6, 6, 5, 2, 1, 1)$ we find

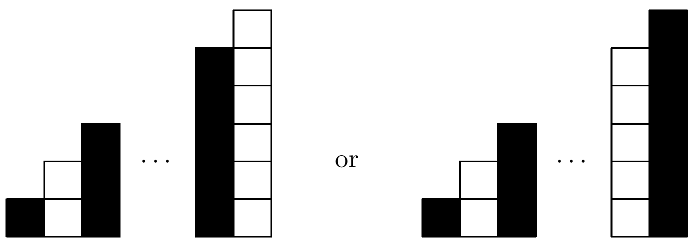


It is easy to see these obtained graphs are characterised by the the property that they consist of a finite number of unit squares and regarded from left to right they increase one square at a time untill at some point they are only allowed to be non increasing. The underlying uncoloured graphs are usually called Castelnuovo diagrams or graphs, see [26] or Chapter 3.

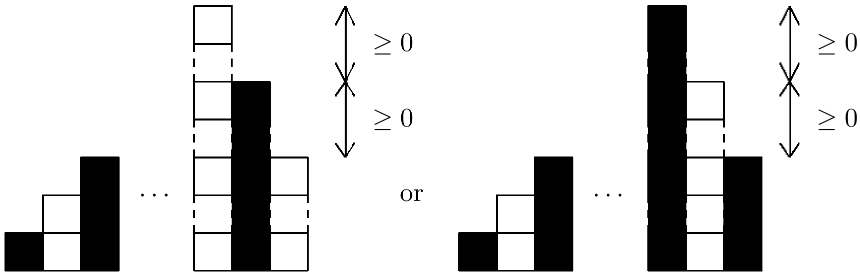
Next we consider the following action on the coloured Castelnuovo graph:

(3) delete one white and black unit square, both on top and on the at most right position as possible

We repeat (3) as many times as possible in such a way that after every removement the underlying uncoloured graph is a valid Castelnuovo graph. It is easy to see that the inequality (G.1) holds if it holds after applying (3). We then show that applying (3) a finite number of times we obtain a "maximal" diagram of the form



for which (G.1) is (trivially) true. This proves the condition (G.1) is necessary. To prove that (G.1) is sufficient we show there exists a (coloured) Castelnuovo graph of the form



case 1 case 2

where the sum of black (resp. white) unit squares is equal to b (resp. w). By reversing the above proces we find a partition λ for which $(b(\lambda), w(\lambda)) = (b, w)$. As a refinement, this partition has distinct parts.

Remark G.1.1. The inequality (G.1) in Theorem A was originally discovered in Lemma 6.2.13 while investigating the appearing Hilbert series of graded right ideals of cubic Artin-Schelter algebras. We refer to §6.3 for the relationship between these modules and Theorem A.

The rest of this appendix is organized as follows. In Section G.2 we have included some preliminaries on partitions and Castelnuovo functions where we develop their relation which we will need lateron. In G.2.3 the proof of Theorem A is given, and Section G.3 presents the proof of Theorem B, as a consequence of Section G.2.3. Finally in part G.4 we make the connection to [75, Problem 10].

G.2 Generalities

It this section we recall some basic notions. See [4] for an introduction to the theory of partitions.

G.2.1 Partitions and chess Ferrers graphs

A *partition* λ of a positive integer n is a finite sequence of positive integers $\lambda_1, \lambda_2, \ldots, \lambda_r$ for which

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r \geq 0 \quad \text{and} \quad \sum_{i=1}^{r} \lambda_i = n.$$

We will often not specify the integer n, and put $\lambda_i = 0$ for $i < 1$ and $i > r$. The partition $(\lambda_1, \lambda_2, \ldots, \lambda_r)$ will be denoted by λ and for convenience we assume that the appearing entries in λ are non-zero. Thus the empty sequence $\lambda = (\ )$ forms the only partition of zero. We refer to the integers $\lambda_1, \ldots, \lambda_r$ as the *parts* of λ. In case all parts of λ are distinct we say λ is a *partition in distinct parts*. The sum $n = \lambda_1 + \lambda_2 + \cdots + \lambda_r$ is called the *weight* of λ. Write $\mathcal{P}$ for the set of all partitions (of weight n where n runs through all positive integers). Similarly we let $\mathcal{D} \subset \mathcal{P}$ be the set of all partitions in distinct parts.

If $\lambda \in \mathcal{P}$ is a partition we may define a new partition $\lambda' = (\lambda'_1, \lambda'_2, \ldots, \lambda'_{r'})$ by defining λ'_i as the number of parts of λ greater or equal than i (for $i \geq 1$)

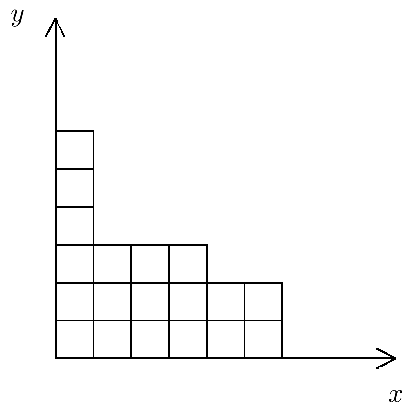
$$\lambda'_i = \#\{j \mid \lambda_j \geq i\}$$

The partition λ' is called the *conjugate* of λ. Note that weight λ = weight λ'. It is standard to visualize a partition $\lambda \in \mathcal{P}$ using the graph of the staircase function

$$F(\lambda) : \mathbb{R} \to \mathbb{N} : x \mapsto \lambda'_{\lfloor x \rfloor}$$

where $\lfloor x \rfloor$ stands for the greatest integer less or equal than $x \in \mathbb{R}$. We divide the area under this graph $F(\lambda)$ in unit cases. This graph is called the *Ferrers graph* of λ. Note that the number of unit squares in the diagram is equal to the weight of λ. We label the columns from left to right, and rows from down to up, starting by index number zero.

Example G.2.1. $\lambda = (6, 6, 4, 1, 1, 1)$ is a partition of length 6 and weight 19. Then its conjugate is given by $\lambda' = (6, 3, 3, 3, 2, 2)$ and the Ferrers graph of λ is presented by



In the sequel we will omit the axes in Ferrers graphs. For any partition $\lambda \in \mathcal{P}$ we colour the unit squares of the Ferrers graph $F(\lambda)$ of λ as follows. A unit square in row r and column c has colour black if $r + c$ is even, and colour white if $r + c$ is odd. The resulting coloured graph is called the *chess Ferrers graph* of λ. We let $b(\lambda)$ be the sum of all black unit squares, and $w(\lambda)$ the sum of all white unit squares. Obviously

$b(\lambda) + w(\lambda) = n$. More formally,

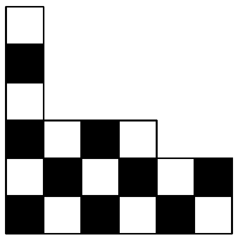
$$\begin{aligned} b(\lambda) &= \lceil \frac{\lambda_1}{2} \rceil + \lfloor \frac{\lambda_2}{2} \rfloor + \lceil \frac{\lambda_3}{2} \rceil + \lfloor \frac{\lambda_4}{2} \rfloor + \dots \\ &= \sum_j \lceil \frac{\lambda_{2j+1}}{2} \rceil + \sum_j \lfloor \frac{\lambda_{2j}}{2} \rfloor \end{aligned}$$

and

$$\begin{aligned} w(\lambda) &= \lfloor \frac{\lambda_1}{2} \rfloor + \lceil \frac{\lambda_2}{2} \rceil + \lfloor \frac{\lambda_3}{2} \rfloor + \lceil \frac{\lambda_4}{2} \rceil + \dots \\ &= \sum_j \lfloor \frac{\lambda_{2j+1}}{2} \rfloor + \sum_j \lceil \frac{\lambda_{2j}}{2} \rceil \end{aligned}$$

where $\lceil x \rceil$ is the notation for the least integer greater or equal than $x \in \mathbb{R}$.

Example G.2.2. Consider the partition $\lambda = (6, 6, 4, 1, 1, 1)$. Then $b(\lambda) = 9$ and $w(\lambda) = 10$. The chess Ferrers diagram F_λ of λ is given by



G.2.2 From partitions to Castelnuovo functions

In the sequel we identify a function $f : \mathbb{Z} \to \mathbb{C}$ with its generating function $f(t) = \sum_n f(n)t^n$. We refer to $f(t)$ as a polynomial or a series depending on whether the support of f is finite or not.

Recall from §3.1 that a Castelnuovo function is a finitely supported function $s : \mathbb{N} \to \mathbb{C}$ such that

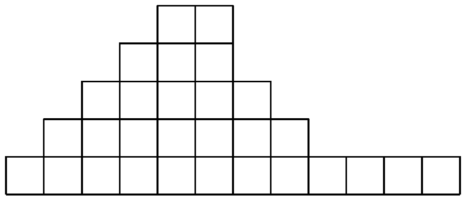
$$s(0) = 1, s(1) = 2, \dots, s(\sigma - 1) = \sigma \text{ and } s(\sigma - 1) \geq s(\sigma) \geq s(\sigma + 1) \geq \dots \geq 0 \quad \text{(G.2)}$$

for some integer $\sigma \geq 0$. We write $\mathcal{S}$ for the set of all Castelnuovo functions. It is convenient to visualize a Castelnuovo function $s \in \mathcal{S}$ using the graph of the staircase function

$$F(s) : \mathbb{R} \to \mathbb{N} : x \mapsto s(\lfloor x \rfloor)$$

and to divide the area under this graph in unit cases. We will call the result a *Castelnuovo graph* (or *Castelnuovo diagram*). The *weight* of a Castelnuovo function is the sum of its values, i.e. the number of unit squares in the graph.

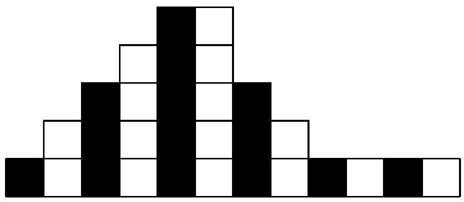
Example G.2.3. $s(t) = 1 + 2t + 3t^2 + 4t^3 + 5t^4 + 5t^5 + 3t^6 + 2t^7 + t^8 + t^9 + t^{10} + t^{11}$ is a Castelnuovo polynomial of weight 28. The corresponding Castelnuovo graph is



Given a Castelnuovo function s we colour the unit squares of its Castelnuovo graph $F(s)$ of s as follows: An unit square in column c has colour black if c is even, and colour white if c is odd. Again we agree the columns are indexed from left to right, and the most left column has index zero. The resulting coloured graph is called the *coloured Castelnuovo graph* of s. We let $b(s)$ be the sum of all black cases, and $w(s)$ the sum of all white cases. Obviously

$$b(s) = \sum_i s_{2i}, \quad w(s) = \sum_i s_{2i+1}$$

Example G.2.4. For the Castelnuovo polynomial $s(t) = 1 + 2t + 3t^2 + 4t^3 + 5t^4 + 5t^5 + 3t^6 + 2t^7 + t^8 + t^9 + t^{10} + t^{11}$ from Example G.2.3 we have $b(s) = 14$, $w(s) = 15$. The corresponding coloured Castelnuovo graph is given by



We next describe the relationship between partitions and Castelnuovo functions. For a partition $\lambda = (\lambda_0, \lambda_1, \ldots, \lambda_{l-l})$ we let $s_\lambda : \mathbb{N} \to \mathbb{N}$ be the function defined by

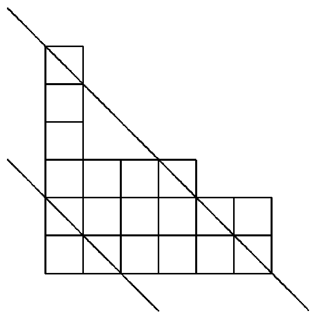
$$s_\lambda(m) = \mathrm{kard}\{j \in \mathbb{N} \mid j \leq m + 1 \text{ and } m + 2 - j \leq \lambda_j\}$$

It is easy to see that $s_\lambda(m)$ is exactly the sum of unit squares which meet the line $D_m : y = -x + m$ in the Ferrers graph of λ. This corresponds to the interpretation in the introduction.

Example G.2.5. Consider the partition $\lambda = (6, 6, 4, 1, 1, 1)$ from Example G.2.2. We compute

$$s_\lambda(t) = 1 + 2t + 3t^2 + 4t^3 + 4t^4 + 4t^5 + t^6$$

The interpretation for the associated Ferrers graph $F(\lambda)$ is illustrated for $s_\lambda(1)$ and $s_\lambda(5)$. The line D_1 meets two unit squares hence $s_\lambda(1) = 2$. Similarly $s_\lambda(5) = 4$.



$D_1 : y = -x + 1$ $D_5 : y = -x + 5$

The following is immediately clear.

Proposition G.2.6. *For any partition λ the function s_λ is a Castelnuovo function of the same weight. The correspondence $\lambda \mapsto s_\lambda$ is a surjective map from the set $\mathcal{P}$ of partitions to the set $\mathcal{S}$ of Castelnuovo functions. Furthermore $(b(\lambda), w(\lambda)) = (b(s_\lambda), w(s_\lambda))$.*

Remark G.2.7. As observed in [28, Remark 1.3] follows that the correspondence $\lambda \mapsto s_\lambda$ restricts to a bijective correspondence between the set $\mathcal{D}$ of partitions in distinct parts and the set $\mathcal{S}$ of Castelnuovo functions.

G.2.3 Proof of Theorem A

G.2.4 Proof that the condition in Theorem A is necessary

In this subsection we prove the condition (G.1) in Theorem A is necessary. Throughout §G.2.3 $\lambda \in \mathcal{P}$ is a partition and we denote $(b, w) = (b(\lambda), w(\lambda))$.

Consider the map

$$(-)^* : \mathbb{Z}[t] \to \mathbb{Z}[t]$$

$$f(t) \mapsto f^*(t) = \begin{cases} f(t) - t^{d-1} - t^d & \text{if } f(t) \neq 0 \text{ and } d = \deg f(t) > 0 \\ f(t) & \text{else} \end{cases}$$

Lemma G.2.8. *Assume $f(t) \neq 0$ is a Castelnuovo polynomial such that* $\deg f(t) > 0$. *If $f^*(t)$ is not a Castelnuovo polynomial then $f(t)$ is of the form*

$$f(t) = 1 + 2t + 3t^2 + \cdots + (u+1)t^u$$

for some integer $u > 0$.

Proof. Since $f(t)$ is a Castelnuovo polynomial we may write

$$f(t) = 1 + 2t + 3t^2 + \cdots + (u+1)t^u + f_{u+1}t^{u+1} + \cdots + f_{v-1}t^{v-1} + f_v t^v$$

for some integers $0 \leq u \leq v$ and such that $u+1 \geq f_{u+1} \geq \cdots \geq f_{v-1} \geq f_v > 0$. It is easy to see that in case $u < v$ then $f^*(t)$ is a Castelnuovo polynomial. Therefore, if $f^*(t)$ is not a Castelnuovo polynomial then this means $u = v$. This also implies $u > 0$, otherwise $f(t) = 1$ and $\deg f(t) = 0$. Ending the proof. □

Write $s = s_\lambda$ for the Castelnuovo function associated to λ. Proposition G.2.6 implies $(b, w) = (b(s), w(s))$. We put

$$s_0(t) = s(t), s_1(t) = s^*(t), s_2(t) = s^{**}(t), \ldots$$

Either s_k is a Castelnuovo function for all integers $k \in \mathbb{N}$, or not. We will treat these two cases separately.

Case 1. s_k is a Castelnuovo function for all integers $k \in \mathbb{N}$.

It is clear that $s_k = s_{k+1}$ implies $s_{k+1} = s_{k+2}$ for all integers $k \in \mathbb{N}$. Define

$$l = \max\{k \in \mathbb{N} \mid s_k \neq s_{k+1}\} + 1$$

Then $s_0 \neq s_1 \neq \cdots \neq s_{l-1} \neq s_l = s_{l+1} = s_{l+2} = \ldots$. By definition of the map $(-)^*$ and the fact that s_k is a Castelnuovo function we deduce either $s_l(t) = 1$ or $s_l(t) = 0$. Since for all $k \in \mathbb{N}$

$$(b, w) = (b(s), w(s)) = (b(s_k) + k, w(s_k) + k)$$

we either have $(b, w) = (l, l)$ or $(b, w) = (l + 1, l)$, for which (G.1) is easily checked.

Case 2. There exists an integer k such that s_k is *not* a Castelnuovo function.

Put

$$l = \max\{k \in \mathbb{N} \mid s_k \text{ is a Castelnuovo function}\}$$

This definition makes sense because $s = s_0$ is a Castelnuovo function. Lemma G.2.8 implies $s_l(t)$ is of the form

$$s_l(t) = 1 + 2t + 3t^2 + \cdots + (u + 1)t^u$$

for some integer $u > 0$. One easily computes

$$(b(s_l), w(s_l)) = \begin{cases} \left((u+2)^2/4, u(u+2)/4\right) & \text{if } u \text{ is even} \\ \left((u+1)^2/4, (u+1)(u+3)/4\right) & \text{if } u \text{ is odd} \end{cases} \tag{G.3}$$

and combining with $(b, w) = (b(s), w(s)) = (b(s_l) + l, w(s_l) + l)$ we find

$$\frac{1}{b} - \left(1 - \frac{w}{b}\right)^2 = \frac{l}{b} \geq 0$$

which completes the proof.

G.2.5 Proof that the condition in Theorem A is sufficient

Let $b, w \in \mathbb{N}$ be positive integers such that (G.1) holds. If $b = 0$ then it follows that $w = 0$, and it is clear that for the empty partition $\lambda = (\ )$ we have $(b, w) = (0, 0) = (b(\lambda), w(\lambda))$. Hence we may assume $b > 0$. Let

$$l = \max\{j \in \mathbb{N} \mid \sum_{i=0}^{j}(2i+1) \leq b \text{ and } \sum_{i=0}^{j} 2i \leq w\}.$$

It is clear that there exist positive integers $b', w' \in \mathbb{N}$ for which either Case 1 or Case 2 is true:

$$\text{Case 1: } \begin{cases} b = 1 + 3 + 5 + \cdots + (2l-1) + b' \\ w = 2 + 4 + 6 + \cdots + 2l + w' \end{cases} \quad \text{where } b' < 2l + 1$$

$$\text{Case 2: } \begin{cases} b = 1 + 3 + 5 + \cdots + (2l+1) + b' \\ w = 2 + 4 + 6 + \cdots + 2l + w' \end{cases} \quad \text{where } w' < 2l + 2$$

Lemma G.2.9. *Let $b, w \in \mathbb{N}$ such that* (G.1) *holds, i.e.*

$$(b - w)^2 \leq b$$

Consider the associated integers $l, b', w' \in \mathbb{N}$ as defined above. We have

1. *If Case 1 is true then $w' \leq b'$, and*
2. *if Case 2 is true then $b' \leq w'$.*

Proof. 1. First assume Case 1 is true. Then

$$\begin{cases} b = l^2 + b' \\ w = l(l+1) + w' \end{cases}$$

From the inequality (G.1) we find $0 \leq b - (b - w)^2$ hence

$$\begin{aligned} 0 &\leq (l^2 + b') - \left(l^2 + b' - l(l+1) - w'\right)^2 \\ &= l^2 + b' - (b' - w' - l)^2 \\ &= b' - (b' - w')^2 + 2(b' - w')l \end{aligned}$$

Assume by contradiction $w' > b'$ i.e. $b' - w' \leq -1$. Then we further deduce

$$\begin{aligned} 0 &\leq b' - (b' - w')^2 + 2(b' - w')l \\ &< b' - (b' - w')^2 - 2l \\ &\leq -(b' - w')^2 \end{aligned}$$

where we have used $b' \leq 2l$. We conclude $0 < -(b' - w')^2$, clearly a contradiction. Hence $w' \leq b'$.

2. Second, assume Case 2 is true. We now have

$$b = (l+1)^2 + b'$$
$$w = l(l+1) + w'$$

and $0 \le b - (b-w)^2$ leads to

$$\begin{aligned} 0 &\le \big((l+1)^2 + b'\big) - \big((l+1)^2 + b' - l(l+1) - w'\big)^2 \\ &= (l+1)^2 + b' - ((b'-w') + (l+1))^2 \\ &= b' - (b'-w')^2 - 2(b'-w')(l+1) \end{aligned}$$

Assume by contradiction $w' < b'$. This means $1 \le b' - w'$ and also $(b'-w') \le (b'-w')^2$. Invoking these inequalities we further deduce

$$\begin{aligned} 0 &\le b' - (b'-w')^2 - 2(b'-w')(l+1) \\ &\le b' - (b'-w') - 2(b'-w')(l+1) \\ &\le b' - (b'-w') - 2(l+1) \end{aligned}$$

and therefore

$$2l + 2 \le w'$$

which contradicts $w' < 2l + 2$. We conclude $w' \ge b'$, which proves the lemma.

□

We now put

$$s(t) = \begin{cases} 1 + 2t + 3t^2 + \cdots + (2l-1)t^{2l-2} + (2l)t^{2l-1} + b't^{2l} + w't^{2l+1} & \text{if Case 1} \\ 1 + 2t + 3t^2 + \cdots + (2l)t^{2l-1} + (2l+1)t^{2l} + w't^{2l+1} + b't^{2l+2} & \text{if Case 2} \end{cases}$$

As a consequence of Lemma G.2.9 we have that $s(t)$ is a Castelnuovo polynomial for which $(b(\lambda), w(\lambda)) = (b, w)$. By Proposition G.2.6 there exists a partition (in distinct parts) λ for which $(b(\lambda), w(\lambda)) = (b, w)$. This proves that the condition (G.1) in Theorem A is sufficient.

G.3 Proof of Theorem B

In this section we prove Theorem B. First let $\lambda \in \mathcal{P}$ be any partition. As shown in Section G.2.4 there exists integers k, l for which $(b(\lambda), w(\lambda))$ is either equal to

- (l, l), or
- $(l+1, l)$, or
- $\big((k+1)^2 + l, k(k+1) + l\big)$ (put $k = u/2$ in (G.3) if u is even), or

- $\big(k^2+l, k(k+1)+l\big)$ (put $k=(u+1)/2$ in (G.3) if u is odd).

Hence there exist positive integers $k, l \in \mathbb{N}$ such that either

$$(b,w)=\big((k+1)^2+l, k(k+1)+l\big)$$

or

$$(b,w)=\big(k^2+l, k(k+1)+l\big)$$

Conversely, let $k, l \in \mathbb{N}$. Putting

$$(b,w)=\big((k+1)^2+l, k(k+1)+l\big)$$

it is easy to verify $b-(b-w)^2=l$. Hence (G.1) holds. By Theorem A there exists a partition λ such that $(b(\lambda), w(\lambda))=(b,w)$. Similar treatment if we put $(b,w)=\big(k^2+l, k(k+1)+l\big)$. This ends the proof of Theorem B.

G.4 A reformulation

In this final part we make the connection with Problem 10 of [75]. For convenience for the reader we recall the question as it was stated in [75].

Problem 10. Let n be a positive integer. Let $a_1, a_2, \ldots, a_m$ be a partition of n. Represent this partition as a left-justified array of boxes, with a_1 boxes in the first row, a_2 in the second, and so on, and label the boxes with 1 and -1 in a chess-board pattern, starting with a 1 in the top-left corner. Let c be the sum of these labels. For instance, if $n=11$ and the partition is $4,3,3,1$ then $c=-1$, as one sees by summing the labels in the diagram:

1	−1	1	−1
−1	1	−1	
1	−1	1	
−1			

Prove that $n \geq c(2c-1)$, and determine when equality occurs.

Let us now indicate how we use Theorem A and Theorem B to solve Problem 10. Write $\lambda=(a_1, a_2, \ldots, a_m)$, and put $(n(\lambda), c(\lambda))=(n,c)$ and $(b,w)=(b(\lambda), w(\lambda))$. It is clear that $n=b+w$, $c=b-w$. Hence $b=(n+c)/2$, $w=(n-c)/2$ and it follows that $n+c$ and $n-c$ are even, i.e. n and c have the same parity (either n and c are both even, or they are both odd). Further inequality (G.1) is equivalent with

$$\begin{aligned}(b-w)^2 \leq b \Leftrightarrow \left(\frac{n+c}{2}-\frac{n-c}{2}\right) &\leq \frac{n+c}{2}\\ \Leftrightarrow 2c^2 &\leq n+c\\ \Leftrightarrow c(2c-1) &\leq n\end{aligned}$$

Hence Theorem A implies $c(2c-1) \leq n$. Conversely, given any $(n, c) \in \mathbb{N} \times \mathbb{Z}$ of the same parity for which $c(2c-1) \leq n$ holds, we see that by putting $b = (n+c)/2$, $w = (n-c)/2$ that (G.1) holds, hence Theorem A implies there exists a partition λ such that $(n(\lambda), c(\lambda)) = (n, c)$.

To see when equality in $c(2c-1) \leq n$ occurs, we may invoke Theorem B: The appearing integers b, w are of the form

$$(b, w) = \big((k+1)^2 + l, k(k+1) + l\big) \text{ or } (b, w) = \big(k^2 + l, k(k+1) + l\big)$$

for some $k, l \in \mathbb{N}$, and conversely for any (b, w) of this form there exists a partition λ for which $(b, w) = (b(\lambda), w(\lambda))$. By replacing $b = (n+c)/2$, $w = (n-c)/2$ we find

$$(n, c) = (2k^2 + k + 2l, -k) \text{ or } (n, c) = (2k^2 + 3k + 1 + 2l, k+1) \tag{G.4}$$

for some $k, l \in \mathbb{N}$, and conversely for any (n, c) of this form there exists a partition λ for which $(n, c) = (n(\lambda), c(\lambda))$. Hence for any $c \in \mathbb{Z}$ the appearing $n \in \mathbb{Z}$ for which (G.4) holds are

$$n = c(2c-1) + 2l, \quad l \in \mathbb{N}.$$

Note it follows that $n \in \mathbb{N}$. Hence equality in $c(2c-1) \leq n$ occurs if and only if $l = 0$. Using the resuls of Section G.2.4 we find $n = c(2c-1)$ if and only if the associated Castelnuovo function is of the "maximal" form from the introduction, i.e. the partition is of the form $\lambda = (m, m-1, \ldots, 2, 1)$ for some $m \in \mathbb{N}$. We have proved

Solution 10 (To Problem 10). Let $(n, c) \in \mathbb{N} \times \mathbb{Z}$. Then there exists a partition λ such that $(n(\lambda), c(\lambda)) = (n, c)$ if and only if

$$n, c \text{ have the same parity and } c(2c-1) \leq n$$

In this case, $n = c(2c-1) + 2l$ for some $l \in \mathbb{N}$. For any partition λ we have $c\,(2c-1) = n$ if and only if $\lambda = (m, m-1, \ldots, 2, 1)$ for some $m \in \mathbb{N}$.
Furthermore the same statement holds if we restrict ourselves to partitions in distinct parts.

Remark G.4.1. The reader will notice that the presented solution of Problem 10 is different from the one presented in [75, Problem 10]. Our version is somewhat longer, however the description is more detailed as we alse give the necessary conditions for (n, c) to correspond to a partition. As a consequence, for any partition λ the difference of n and $c(2c-1)$ is always even.

Bibliography

[1] K. Ajitabh, *Modules over elliptic algebras and quantum planes*, Proc. Lond. Math. Soc. **72** (1996), no. 3, 567–587.

[2] ______, *Esistence of critical modules of GK-dimension 2 over elliptic algebras*, Proc. American Math. Soc. **128** (2000), no. 10, 2843–2849.

[3] K. Ajitabh and M. Van den Bergh, *Presentation of critical modules of GK-dimension 2 over elliptic algebras*, Proc. American Math. Soc. **127** (1999), no. 6, 1633–1639.

[4] G. E. Andrews, *The theory of partitions*, Cambridge Mathematical Library, Cambridge University Press, Cambridge, 1998, Reprint of the 1976 original.

[5] M. Artin and W. Schelter, *Graded algebras of global dimension 3*, Adv. in Math. **66** (1987), 171–216.

[6] M. Artin, L. W. Small, and J. J. Zhang, *Generic flatness for strongly Noetherian algebras*, J. Algebra **221** (1999), no. 2, 579–610.

[7] M. Artin, J. Tate, and M. Van den Bergh, *Some algebras associated to automorphisms of elliptic curves*, The Grothendieck Festschrift, vol. 1, Birkhäuser, 1990, pp. 33–85.

[8] ______, *Modules over regular algebras of dimension 3*, Invent. Math. **106** (1991), 335–388.

[9] M. Artin and M. Van den Bergh, *Twisted homogeneous coordinate rings*, J. Algebra **133** (1990), 249–271.

[10] M. Artin and J. J. Zhang, *Noncommutative projective schemes*, Adv. in Math. **109** (1994), no. 2, 228–287.

[11] ______, *Abstract Hilbert schemes*, Algebr. Represent. Theory **4** (2001), no. 4, 305–394.

[12] D. Baer, *Tilting sheaves in representation theory of algebras*, Manuscripta Math. **60** (1988), no. 3, 323–347.

[13] V. Baranovsky, V. Ginzburg, and A. Kuznetsov, *Quiver varieties and a noncommutative* $\mathbb{P}^2$, Compositio Math. **134** (2002), no. 3, 283–318.

[14] K. Bauwens, *Lie superalgebras and noncommutative geometry*, Ph.D. thesis, Limburgs Universitair Centrum, 1996.

[15] A. A. Beilinson, *Coherent sheaves on* $\mathbb{P}^n$ *and problems of linear algebra*, Funct. Anal. Appl. **12** (1979), 214–216.

[16] Y. Berest and G. Wilson, *Ideal classes of the Weyl algebra and noncommutative projective geometry (with an appendix by Michel Van den Bergh)*, Int. Math. Res. Not. (2002), no. 26, 1347–1396.

[17] ______, *Automorphisms and ideals of the Weyl algebra*, Math. Ann. **318** (2000), no. 1, 127–147.

[18] A. I. Bondal, *Helices, representions of quivers and Koszul algebras*, Helices and Vector bundles, 75–95, London Math. Soc. Lecture Note Ser., **148**, Cambridge Univ. Press, Cambridge, (1990).

[19] A. I. Bondal and M. M. Kapranov, *Representable functors, Serre functors, and reconstructions*, Izv. Akad. Nauk SSSR Ser. Mat. **53** (1989), no. 6, 1183–1205, 1337.

[20] A. I. Bondal and A. E. Polishchuk, *Homological properties of associative algebras: the method of helices*, Russian Acad. Sci. Izv. Math **42** (1994), 219–260.

[21] J. Brun and A. Hirschowitz, *Le problème de Brill-Noether pour les idéaux de* $\mathbb{P}^2$, Ann. Sci. Ec. Norm. Super., IV, **20** (1987), 171–200.

[22] R. C. Cannings and M. P. Holland, *Right ideals of rings of differential operators*, J. Algebra **167** (1994), no. 1, 116–141.

[23] C. Ciliberto, A. V. Geramita and F. Orecchia, *Remarks on a theorem of Hilbert-Burch*, Bollettino U.M.I. (7) **2**-B (1988), 463–483.

[24] M. A. Coppo, *Familles maximales de systemes de points surabondants dans le plan projectif*, Math. Ann., **291** (1991), 725–735.

[25] M. A. Coppo and C. Walter, *Composante centrale du lieu de Brill-Noether de* $\mathrm{Hilb}^2(\mathbb{P}^2)$, Lect. Notes Pure Appl. Math., **200** (1998), 341–349.

[26] E. D. Davis, 0-*dimensional subschemes of* $\mathbb{P}^2$*: new applications of Castelnuovo's function*, An. Univ. Ferrara, **32** (1986), 93–107.

[27] K. De Naeghel and M. Van den Bergh, *Ideal classes of three dimensional Sklyanin algebras*, J. of Algebra **276** (2004) no. 2 515–551.

[28] ———, *Ideal classes of three dimensional Artin-Schelter regular algebras*, J. of Algebra **283** (2005) no. 1 399–429.

[29] ———, *On incidence between strata of the Hilbert scheme of points on* $\mathbb{P}^2$, submitted to the Math Zeitschrift.

[30] K. De Naeghel and N. Marconnet, *Ideals of three dimensional cubic Artin-Schelter regular algebras*, submitted to J. of Algebra.

[31] ———, *An inequality on broken chessboards*, preprint math.CO/0601094, 2006.

[32] P. Gabriel, *Auslander-Reiten sequences and representation-finite algebras*, Lecture Notes in Math. 831 (Springer, New-York, 1980) 1–71.

[33] A. V. Geramita, T. Harima and Yong Su Shin, *an alternative to the Hilbert function for the ideal of a finite set of points in* $\mathbb{P}^n$, Illinois Journal of Mathematics, **45** 1 (2001), 1–23.

[34] A. V. Geramita, P. Maroscia, and L. G. Roberts, *The Hilbert function of a reduced* k*-algebra*, J. London Math. Soc. (2) **28** (1983) no. 3 443–452.

[35] S. R. Ghorpade, *Hilbert functions of ladder determinantal varieties*, Discrete Math. **246** (2002), no. 1-3, 131–175, Formal power series and algebraic combinatorics (Barcelona, 1999).

[36] G. Gotzmann, *A stratification of the Hilbert scheme of points on the projective plane*, Math. Zeitschrift **199** (1988) no. 4 539–547.

[37] L. Gruson and C. Peskine, *Genre des courbes de l'espace projectif*, Lecture notes in Math., vol. 687, 31–59, Springer verlag, Berlin, 1978.

[38] F. Guerimand, *Sur l'incidence des strates de Brill-Noether du schéma de Hilbert des points du plan projectif*, PhD thesis, Université de Nice - Sophia Antipolis, 2002.

[39] J. Harris, *Algebraic geometry: a first course*, Springer-Verlag, 1992.

[40] R. Hartshorne, *Residues and duality*, Lecture notes in mathematics, vol. 20, Springer Verlag, Berlin, 1966.

[41] ———, *Algebraic geometry*, Springer-Verlag, 1977.

[42] W. Heisenberg, *Über den anschaulichen Inhalt der quantentheoretischen Kinematik und Mechanik*, Z. für Phys. **43**, 172–198, 1927.

[43] A. Hirschowitz, O. Rahavandrainy, and C. Walter, *Quelques strates de Brill-Noether du schéma de Hilbert de* $\mathbb{P}^2$, C.R. Acad. Sci. Paris, Sér. I Math. **319** (1994) no. 6 589–594.

[44] K. Hulek, *On the classification of stable rank-r vector bundles over the projective plane*, Vector bundles and differential equations (Proc. Conf., Nice, 1979), pp. 113144, Progr. Math., 7, Birkhauser, Boston, Mass., 1980.

[45] A. Kapustin, A. Kuznetsov, and D. Orlov, *Noncommutative instantons and twistor transform*, Comm. Math. Phys. **221** (2001), no. 2, 385–432.

[46] M. Kashiwara and P. Schapira, *Sheaves on manifolds*, Die Grundlehren der Mathematischen Wissenschaften, vol. 292, Springer Verlag, 1994.

[47] A. D. King, *Moduli of representations of finite-dimensional algebras*, Quart. J. Math. Oxford Ser. (2) **45** (1994), no. 180, 515–530.

[48] K. M. Kouakou, *Isomorphismes entre algèbres d'opérateurs différentiels sur les courbes algébriques affines*, Ph.D. thesis, Université Claude Bernard-Lyon 1, 1994.

[49] ———, *Codimension B-W d'un idéal à droite non nul de* $A_1(\mathbb{C})$, Bulletin de la SMF 133 (2005).

[50] G. R. Krause and T. H. Lenagan, *Growth of algebras and Gelfand-Kirillov dimension*, Research Notes in Mathematics, vol. 116, Pitman, Boston, 1985.

[51] L. Le Bruyn, *Moduli spaces for right ideals of the Weyl algebra*, J. Algebra **172** (1995), 32–48.

[52] ———, Noncommutative geometry@n, forgotten book (2000).

[53] W. Lowen, *Obstruction theory for objects in abelian and derived categories*, preprint math.KT/0407019.

[54] D. Luna, *Slices étales*, Bull. Soc. Math. France **33** (1973), 81–105.

[55] F. S. Macaulay, Some properties of enumerations in the theory of modular systems, Proc. London Math. Soc. **26**, (1927), 531–555.

[56] J. C. McConnell and J. C. Robson, *Noncommutative Noetherian rings*, Pure and Applied Mathematics (New York). A Wiley-Interscience Publication. John Wiley and Sons, Ltd., Chichester, 1987.

[57] I. Mori and S. P. Smith, *Bézout's theorem for non-commutative projective spaces*, J. Pure Appl. Algebra **157** (2001), no. 2-3, 279–299.

[58] H. Nakajima, *Lectures on Hilbert schemes of points on surfaces*, University Lecture Series **18**, Amer. Math. Soc., Providence, RI, 1999.

[59] C. Nastasescu and F. Van Oystaeyen, *Graded and filtered rings and modules*, Springer, Berlin, 1979.

[60] T. A. Nevins and J. T. Stafford, Sklyanin algebras and Hilbert schemes of points, preprint math.AG/0310045, 2003.

[61] A. V. Odesskii and B. L. Feigin, *Sklyanin algebras associated with an elliptic curve*, preprint Institute for Theoretical Physics, Kiev, 1989.

[62] ———, *Sklyanin's elliptic algebras*, Functional Anal. Appl. **23** (1989), no. 3, 207–214.

[63] O. Rahavandrainy, *Quelques composantes des strates de Brill-Noether de* $\mathrm{Hilb}^n(\mathbb{P}^2)$, C.R. Acad. Sci. Paris, **322** (1996), 455-460.

[64] A. Schofield, *Semi-invariants of quivers*, J. London Math. Soc. (2) **43** (1991), no. 3, 385–395.

[65] A. Schofield and M. Van den Bergh, *Semi-invariants of quivers for arbitrary dimension vectors*, Indag. Math. (N.S.) **12** (2001), 125–138.

[66] S. P. Smith, *Noncommutative algebraic geometry*, 2000.

[67] ———, *Subspaces of non-commutative spaces*, Trans. Amer. Math. Soc. **354** (2002), no. 6, 2131–2171 (electronic).

[68] S. T. Stafford, *Noncommutative projective geometry (ICM '2002 talk)*.

[69] J. T. Stafford and M. Van den Bergh, *Noncommutative curves and noncommutative surfaces*, Bull. Amer. Math. Soc. (N.S.) **38** (2001), no. 2, 171–216 (electronic).

[70] B. Stenström, *Rings of quotients*, Die Grundlehren der Mathematischen Wissenschaften in Einzeldarstellungen, vol. 217, Springer Verlag, Berlin, 1975.

[71] D. R. Stephenson, *Artin-Schelter regular algebras of global dimension three*, PhD thesis, University of Michigan, UMI, 1994.

[72] ———, *Artin-Schelter regular algebras of global dimension three*, J. Algebra **183** (1996), 55–73.

[73] ———, *Algebras associated to elliptic curves*, Trans. Amer. Math. Soc. **349** (1997), 2317–2340.

[74] D. R. Stephenson and J. J. Zhang, *Growth of graded Noetherian rings*, Proc. Amer. Math. Soc. **125** (1997), no. 6, 1593–1605.

[75] Sydney University Mathematical Society Problems Competition 2004, http://www.maths.usyd.edu.au/u/SUMS/sols2004.pdf

[76] M. Van Gastel and M. Van den Bergh, *Graded modules of Gelfand-Kirillov dimension one over three-dimensional Artin-Schelter regular algebras*, J. Algebra **196** (1997), no. 1, 251–282.

[77] M. Van den Bergh, *Existence theorems for dualizing complexes over non-commutative graded and filtered rings*, J. Algebra **195** (1997), no. 2, 662–679.

[78] ______, *Blowing up of non-commutative smooth surfaces*, Mem. Amer. Math. Soc. **154** (2001), no. 734, x+140.

[79] ______, *Non-commutative quadrics*, Preprint.

[80] F. Van Oystaeyen and L. Willaert, *Grothendieck topology, coherent sheaves and Serre's theorem for schematic algebras*, J. Pure Appl. Algebra **104** (1995), no. 1, 109–122.

[81] A. B. Verevkin, *On a noncommutative analogue of the category of coherent sheaves on a projective scheme*, Amer. Math. Soc. Trans. (2) **151** (1992).

[82] D. Voigt, *Induzierte Darstellungen in der Theorie der endlichen, algebraischen Gruppen*, Springer-Verlag, Berlin, 1977, Mit einer englischen Einführung, Lecture Notes in Mathematics, Vol. 592.

[83] C.A. Weibel, *An introduction to homological algebra*, Cambridge University Press, 1994.

[84] G. Wilson, *Collisions of Calogero-Moser particles and an adelic Grassmannian*, Invent. Math. **133** (1998), no. 1, 1–41, With an appendix by I. G. Macdonald.

[85] A. Yekutieli, *Dualizing complexes over noncommutative graded algebras*, J. Algebra **153** (1992), 41–84.

[86] A. Yekutieli and J. J. Zhang, *Serre duality for non-commutative projective schemes*, Proc. Amer. Math. Soc. (1997), no. 125, 697–707.

[87] J. J. Zhang, *Twisted graded algebras and equivalences of graded categories*, Proc. London Math. Soc., (2) 72 (1996), 281–311.