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Local equivalence and conjugacy of families of vector fields and diffeomorphisms

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Koen NEIRYNCK

Promotor : Prof. dr. P. Bonckaert
Copromotor : Dr. Vincent Naudot

LIMBURGS UNIVERSITAIR CENTRUM
PARTNER IN DE UNIVERSITEIT LIMBURG

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Dum Deus calculat, fit mundus. (G.W. Leibniz)

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Preface

In this thesis we work with C^∞ or analytic families of vector fields or diffeomorphisms. We are interested in local equivalences and conjugacies between such families and families in a "simple" form, sometimes called a *normal form*. Traditionally this normal form is chosen to be linear, but in some cases this choice prevents us from obtaining an analytic equivalence or conjugacy. So in such cases we will allow the presence of non-linear terms in the normal form.

A lot of work has already been done for individual vector fields and diffeomorphisms. It turns out that the eigenvalues of the linear part of the vector field, resp. diffeomorphism at the singular, resp. fixed point are determining whether the vector field or diffeomorphism is equivalent with or conjugate to its linear part. If the eigenvalues form a hyperbolic non-resonant set then there are celebrated results from Poincaré and Siegel telling us when an analytic conjugacy with the linear part can be obtained. If the eigenvalues form a hyperbolic resonant set then sometimes it is possible to obtain a finitely smooth conjugacy. In the non-hyperbolic case it becomes much more difficult to obtain smooth equivalences and conjugacies.

As in this thesis we are working with families of vector fields or diffeomorphisms, we will encounter the same problems concerning hyperbolicity and resonance as in the case of individual systems. An additional problem can be caused by the parameters that are in play in a family. As the parameter perturbs the eigenvalues, it can cause resonances which are absent for the unperturbed system. This phenomenon also has its impact on the smoothness of the equivalence or conjugacy.

This thesis is structured as follows.

In Chapter 1 we introduce the most important objects used in this thesis: vector fields, flows, fixed points, singular points, conjugacies and equivalences. We give a brief introduction on analytic functions in several variables. After this we give a profound discussion on normal forms. We start with the formal and smooth normal form for individual systems. After this we give some generalities on families of vector fields and diffeomorphisms. The section is finished with the normal form for families of hyperbolic vector fields and diffeomorphisms and the normal form for a deformation of a planar singularity of center type. The chapter ends with a short discussion on transition maps of planar vector fields. This way we have a natural introduction for the Ecalle–Roussarie compensator and the Melnikov functions.

In Chapter 2 the aim is to give an explicit construction for equivalences and conjugacies between nearly-resonant planar saddles and their linear parts. We start by proving a lower bound on the degree of the resonant terms that appear as the parameter varies. This bound will be crucial to obtain C^1 results. After this we discuss the explicit form for a C^1 equivalence between nearly-resonant planar saddles and their linear parts. The Ecalle–Roussarie compensator will play a prominent role in this part. We then move on to the C^1 conjugacy between nearly-resonant planar saddles and their linear parts. This time a second Ecalle–Roussarie compensator will appear in the computations. Introducing two new variables we prove that this conjugacy is C^∞ with respect to the two original and the two new variables. These new variables will be inspired by the Ecalle–Roussarie compensator. Next the conjugacies between nearly-resonant planar saddle diffeomorphisms and their linear parts are studied. To conclude we try to repeat the calculations that were made in the saddle case for a deformation of planar singularity of center type.

In Chapter 3 we consider the Poincaré map of a deformation of a planar singularity of center type. Traditionally one studies this map by means of an asymptotic expansion with respect to a one-dimensional parameter. The coefficients of this expansion, the so-called *Melnikov functions*, are then expressed by line and area integrals. We avoid the use of these integrals as they can be very difficult to calculate and use a multi-valued normal form. To give a good description of this multi-valued normal form we have to introduce some auxiliary functions. Once the multi-valued normal form has been properly described we apply it to two types of examples. The first type of examples, called Hopf–Takens models, are used to compare our results with the results obtained with the traditional techniques. The second type of example is the Hamiltonian triangle. Although this system is a degenerate center, our technique based on multi-valued normal forms can give an asymptotic expansion of the Melnikov functions.

In Chapter 4 we study local analytic models for analytic hyperbolic families of vector fields or diffeomorphisms. We first study the saddle situation. Here it is not possible to extend Siegel's Theorem to a family of systems, so we try to obtain a normal form consisting of the linear part plus high order terms which have a high degree in as well the stable as the unstable degree. To achieve this result we need to exclude the existence of small divisors in these systems. Once this result is achieved, we can prove the existence of an analytic conjugacy between the original family and the normal form we previously described. We also prove that if the original system admits a family of symmetries then the conjugacy will commute with these symmetries and the normal form admits the same symmetry. We conclude with an extension of Poincaré's Theorem to a family of systems. The techniques used to treat the saddle case will make it possible to give an elegant proof of Poincaré's Theorem for a family of vector fields or diffeomorphisms.

Contents

Chapter 1

Prerequisites and technicalities

In this chapter we want to recall some important notions on the local study of families of vector fields and diffeomorphisms. We classify these families depending on the properties of their singular points or fixed points, this is done in Section 1.1. It is important to make the remark that all singular points or fixed points will be assumed to be *isolated*, i.e. for each of these points P there exists an open neighbourhood U of P such that $U \setminus \{P\}$ does not contain any singular point or fixed point. In the neighbourhood of a singular point or a fixed point one likes to choose local coordinates such that the vector field or diffeomorphism obtains a "simple" form sometimes called a *normal form*. Techniques to obtain these normal forms are discussed in Section 1.3. In order to investigate in which case the local change of coordinates can be chosen to be analytic, we need some basic results on analytic functions in several variables. These results will be presented in Section 1.2. Special attention will be made to the transition maps of planar vector fields. They will be a key-tool in understanding the structure of equivalences and conjugacies between planar vector fields or diffeomorphisms on $\mathbb{R}^2$. This is why Section 1.4 has claimed a separate part in this chapter. We will always assume that the vector fields and diffeomorphisms are defined on an open subset of $\mathbb{R}^n$ (although in some situations we will extend the system to $\mathbb{C}^n$), $n \geq 2$.

1.1 Conjugacy and topological classification

1.1.1 Conjugacies and equivalences

Definition 1.1 *Two diffeomorphisms f and g on $\mathbb{R}^n$ are* **conjugate** *if there exists a homeomorphism h on $\mathbb{R}^n$ such that*

$$h^*(f) := h^{-1} \circ f \circ h = g. \tag{1.1}$$

The map $h^*(f)$ is called the **pull-back** *of f under h.*

If we want to define a conjugacy between two vector fields we need to introduce the flow of a vector field.

Definition 1.2 *If X is an autonomous vector field on an open subset of $\mathbb{R}^n$ then the map $\Phi : U \times I \subset \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n : (x_0, t) \mapsto \Phi(x_0, t)$ where U is an open subset of $\mathbb{R}^n$ and I is an open interval containing 0 as an inner point, is the* **flow** *of X if for any fixed $x_0 \in U$ the following two conditions hold:*

1. $\frac{\partial}{\partial t}\Phi(x_0, t) = X \circ \Phi(x_0, t)$, $\forall t \in I$ *and* $\forall x_0 \in U$
2. $\Phi(x_0, 0) = x_0$.

To prevent the notations from becoming cumbersome we will denote the flow by $\Phi_t(x_0)$.

Definition 1.3 *Two vector fields X and Y on an open subset of $\mathbb{R}^n$ are* **conjugate** *if their flows are conjugate, i.e. there exists a homeomorphism h such that*

$$h^*\Phi_t = \Psi_t, \forall t \in I \tag{1.2}$$

where I is an interval containing 0 as an inner point and where Φ_t is the flow of X and Ψ_t is the flow of Y.

In the case the map h is differentiable we can replace (1.2) by an expression containing the actual vector fields instead of their flows.

Proposition 1.1 *If two vector fields X and Y on an open subset of $\mathbb{R}^n$ are conjugate by a diffeomorphism h then we have that*

$$h^*X := (Dh)^{-1} \cdot X \circ h = Y. \tag{1.3}$$

The vector field h^*X is called the **pull-back** *of X under h.*

In the case of vector fields one can weaken the notion of conjugacy to equivalence. The essential difference between the two notions is that when two vector fields are equivalent to each other, they do not have to respect the time in which one "runs through" the orbits. To be more precise we have the following definition.

Definition 1.4 *Two vector fields X (with flow Φ_t) and Y (with flow Ψ_t) on an open subset of $\mathbb{R}^n$ are* **equivalent** *if there exists a homeomorphism h taking orbits of Φ_t onto those of Ψ_t and preserving their orientation.*

1.1.2 Classification based on the linear part

From now on we will assume that the vector fields and diffeomorphisms we are working with are at least of class C^2, i.e. they are at least two times continuously differentiable. We now introduce the following "special points" which we want to study throughout this thesis.

Definition 1.5 *A point x_0 is a* **fixed point** *of a diffeomorphism $f : \mathbb{R}^n \to \mathbb{R}^n$ iff $f(x_0) = x_0$.*

Definition 1.6 *A point x_0 is a* **singular point** *of a vector field X on $\mathbb{R}^n$ iff $\Phi(x_0, t) \equiv x_0$ where $\Phi(x_0, t)$ is the flow of X (in other words x_0 is a fixed point of the flow of X starting in x_0) or equivalently $X(x_0) = 0$.*

We now want to classify fixed points and singular points by means of the eigenvalues of the linear parts of the diffeomorphism or vector field taken in those points. We start with the singular points of a vector field.

Singular points

Let x_0 be a singular point of the vector field $X = \sum_{j=1}^{n} X_j(x) \frac{\partial}{\partial x_j}$, then the linear part of X in x_0 is the $n \times n$ matrix $DX(x_0)$. Denote the set of eigenvalues of $DX(x_0)$ with $\mathrm{Spec}(DX(x_0))$. If all elements of $\mathrm{Spec}(DX(x_0))$ have real part different from zero, then we say that $\mathrm{Spec}(DX(x_0))$ is a **hyperbolic set**. In this case we say that x_0 is a **singularity of hyperbolic type** or shortly a **hyperbolic singularity**. When working in the neighbourhood of a hyperbolic singularity, we will say that X is a **hyperbolic vector field**. This class of vector fields takes a special position within the set of vector fields because of the following result.

Theorem 1.2 (Hartman-Grobman Theorem) *Let x_0 be a hyperbolic singular point of the vector field X on an open subset of $\mathbb{R}^n$ and Y the linear vector field given by $Y = \sum_{j=1}^{n} DX_j(x_0) \cdot x \frac{\partial}{\partial x_j}$, then X and Y are C^0 conjugate in a neighbourhood of x_0.*

Proof: We refer the reader to [BK94, Gro59, Har60b]. □

If the conditions of Theorem 1.2 are fulfilled, then we say that X can be *linearised* in a C^0 way. For this reason we say that Theorem 1.2 gives a *topological* classification of such singular points.

If $\mathrm{Spec}(DX(x_0))$ forms a hyperbolic set and not all of the elements have real parts with the same sign, then we say that the singularity x_0 is of **saddle type** or shortly a **saddle**.

Outside the class of hyperbolic singularities, there is one type of singularities we will encounter in this thesis. In the case that all elements of $\mathrm{Spec}(DX(x_0))$ are pairs of complex conjugate imaginary numbers, we will say that x_0 is a **singularity of center type** or shortly a **center**. As the elements of $\mathrm{Spec}(DX(x_0))$ appear in pairs, it is obvious that centers can only exist in vector fields where n is even.

Fixed points

Let x_0 be a fixed point of the diffeomorphism $f : \mathbb{R}^n \to \mathbb{R}^n$, then the linear part of f in x_0 is the $n \times n$ matrix $Df(x_0)$. Denote the set of eigenvalues of $Df(x_0)$ with $\mathrm{Spec}(Df(x_0))$. As f is a diffeomorphism, all elements of $\mathrm{Spec}(Df(x_0))$ will be non-zero. If all elements of $\mathrm{Spec}(Df(x_0))$ have modulus different from one, then we say that $\mathrm{Spec}(Df(x_0))$ is a **multiplicatively hyperbolic set**. In this case we say that x_0 is a **fixed point of hyperbolic type** or shortly a **hyperbolic fixed point**. When working in the neighbourhood of a hyperbolic fixed point, we will say that f is a **hyperbolic diffeomorphism**. The class of hyperbolic diffeomorphisms takes a special position within the set of diffeomorphisms because of the following result.

Theorem 1.3 (Hartman-Grobman Theorem) *Let x_0 be a hyperbolic fixed point of the diffeomorphism $f : U \subset \mathbb{R}^n \to \mathbb{R}^n$ (where U is an open subset of $\mathbb{R}^n$) and g the linear map defined as $g : \mathbb{R}^n \to \mathbb{R}^n : x \mapsto Df(x_0) \cdot x$, then f and g are C^0 conjugate in a neighbourhood of x_0.*

Proof: We refer the reader to [BK94, Gro59, Har60b]. □

If the conditions of Theorem 1.3 are fulfilled, then we say that f can be *linearised* in a C^0 way. For this reason we say that Theorem 1.3 gives a *topological* classification of such fixed points.

If $\mathrm{Spec}(Df(x_0))$ forms a multiplicatively hyperbolic set and there are elements with modulus strictly greater than 1 and there are elements with modulus strictly smaller than 1, then we say that the fixed point x_0 is of **saddle type** or shortly a **saddle**.

1.2 Analytic functions in several variables

In this section we want to introduce some important notions and results on analytic functions in several variables. For more information and proofs we refer to [Hör73, KK83].

Although most of the results hold for analytic functions on an open subset of $\mathbb{C}^n$, we will restrict ourselves to analytic functions defined on a so-called **poly-disk**:

$$\mathbb{D}(a,R) := B(a_1,R_1) \times \cdots \times B(a_n,R_n)$$

where $a = (a_1, \cdots, a_n) \in \mathbb{C}^n$ and $R = (R_1, \cdots, R_n) \in (\mathbb{R}^+ \setminus \{0\})^n$.

Definition 1.7 • *A function $f : \mathbb{D}(a,R) \to \mathbb{C}$ is called* **partially analytic** *if, for each fixed $(z_1^0, \cdots, z_n^0) \in \mathbb{D}(a,R)$, and each $j = 1, \cdots, n$, the function of one variable determined by the assignment*

$$z_j \mapsto f(z_1^0, \cdots, z_{j-1}^0, z_j, z_{j+1}^0, \cdots, z_n^0)$$

is analytic on $B(a_j, R_j)$.

- *A continuous partially analytic function on a poly-disk is called* **analytic***.*

In [Hör73] it is proved that every partially analytic function is continuous, hence every partially analytic function is always analytic. We want to identify such an analytic function with a power series expansion. Therefore we use the following result.

Definition 1.8 *A series $\sum_{m\in\mathbb{N}^n} a_m(z)$* **converges normally** *on a poly-disk $\mathbb{D}(a,R)$ if*

$$\sum_{m\in\mathbb{N}^n} \sup_{z\in K} |a_m(z)|$$

converges on every compact set $K \subset \mathbb{D}(a,R)$.

Theorem 1.4 ([Hör73]) *If f is analytic on the poly-disk $\mathbb{D}(a,R)$, we have*

$$f(z) = \sum_{m\in\mathbb{N}^n} \frac{\partial^{|m|} f}{\partial z^m}(a) \frac{(z-a)^m}{m!}, \ z \in \mathbb{D}(a,R),$$

with normal convergence.

This definition implies that $\sum_{m\in\mathbb{N}^n} a_m(z)$ exists and is independent of the order of summation and that the sum is analytic if all a_m are analytic. This is an immediate consequence of

Proposition 1.5 ([Hör73]) *If $(u_k)_{k\in\mathbb{N}}$ is a sequence of analytic functions on a poly-disk $\mathbb{D}(a,R)$ and $u_k \to u$ when $k \to +\infty$, uniformly on all compact subsets of $\mathbb{D}(a,R)$, then u is analytic on $\mathbb{D}(a,R)$.*

Remark 1.1 *In [Hör73] Proposition 1.5 is proved for open neighbourhoods of a instead of poly-disks.*

In what follows we want to work with functions that are analytic in a variable $z \in \mathbb{C}^p$ and a parameter $\varepsilon \in \mathbb{C}^q$. Equipping $\mathbb{C}^{p+q}$ with the maximum-norm, the cartesian product of a poly-disk in $\mathbb{C}^p$ with a poly-disk in $\mathbb{C}^q$ is a poly-disk in $\mathbb{C}^{p+q}$. Introducing $\mathbf{e} := (1, 1, \cdots, 1) \in \mathbb{C}^n$ (with $n \geq 1$) this choice of norm gives us that $B(a, R) = \mathbb{D}(a, R\mathbf{e}) \subset \mathbb{C}^n$. So by virtue of Theorem 1.4 and the definition of normal convergence, we have for each analytic function $f(z, \varepsilon)$ on $\mathbb{D}(a, R\mathbf{e}) \times \mathbb{D}(b, r)$ in $\mathbb{C}^p \times \mathbb{C}^q$ that

$$f(z, \varepsilon) = \sum_{m \in \mathbb{N}^p} f_m(\varepsilon)(z - a)^m,$$

for each $z \in \mathbb{D}(a, R)$ and $\varepsilon \in \mathbb{D}(b, r)$, with normal convergence and each $f_m(\varepsilon)$ is analytic in $\mathbb{D}(b, r)$.

Conversely if the series

$$\sum_{m \in \mathbb{N}^p} f_m(\varepsilon)(z - a)^m$$

converges normally on $\mathbb{D}(a, R) \times \mathbb{D}(b, r)$ and each $f_m(\varepsilon)$ is analytic on $\mathbb{D}(b, r)$, then by virtue of Proposition 1.5 the function defined by the sum

$$f(z, \varepsilon) := \sum_{m \in \mathbb{N}^p} f_m(\varepsilon)(z - a)^m$$

is analytic on $\mathbb{D}(a, R) \times \mathbb{D}(b, r)$ as it is clear that $f_m(\varepsilon)(z - a)^m$ is analytic on $\mathbb{D}(a, R) \times \mathbb{D}(b, r)$ and we have normal convergence.

From this, it is easy to see that the following result holds.

Proposition 1.6 *Let $f_m(\varepsilon)$ be an analytic function on $\mathbb{D}(b, r)$ for each $m \in \mathbb{N}^p$ and $g(z) = \sum_{m \in \mathbb{N}^p} g_m(z - a)^m$ is an analytic function on $\mathbb{D}(a, R)$ such that*

$$|f_m(\varepsilon)| \leq g_m, \forall m \in \mathbb{N}^p,$$

then the function $f : \mathbb{C}^{p+q} \to \mathbb{C}$ with

$$f(z, \varepsilon) := \sum_{m \in \mathbb{N}^p} f_m(\varepsilon)(z - a)^m$$

is analytic on $\mathbb{D}(a, R) \times \mathbb{D}(b, r)$.

The coefficients g_m in Proposition 1.6 need to be real and positive.

1.3 Formal and smooth normal forms

In this section it is our aim to give a comprehensive introduction to the theory of normal forms. First of all we discuss normal forms for a single vector field or diffeomorphism. In both cases we investigate the formal normal form calculation, the influence of resonance on the formal normal form and some results on smooth normal forms. Secondly we give the definition of a family of vector fields or diffeomorphisms together with some important results we need in the rest of this section. Thirdly we will discuss normal forms for a family of hyperbolic vector fields or diffeormorphisms. Again we investigate the normal form calculation, the influence of resonance on the formal normal form and some results on smooth normal forms. The latter subject will require some techniques that will be discussed in a more detailed manner. We end this section with a discussion on the formal normal form of a family of planar vector fields arising from a perturbed vector field of center type.

1.3.1 Normal form of a vector field

First of all we discuss the *formal normal form.* In this case we work on the k-jet of the vector field where $k \in \mathbb{N} \cup \{\infty\}$. In the second part we will discuss normal forms of analytic and C^∞ vector fields.

Formal normal form

Consider a vector field X on $\mathbb{C}^n$ with a singular point x_0. First of all we will perform a translation to obtain the singular point at the origin. So we apply the following change of coordinates

$$\tau_{x_0} : x \mapsto x - x_0$$

and obtain a vector field $\tilde{X}$ on $\mathbb{C}^n$ with a singular point at the origin. This way $\tilde{X}$ can be written as

$$\tilde{X} : \dot{x} = Ax + f(x),\ x \in \mathbb{C}^n,$$

where the complex $n \times n$ matrix A is the linear part at the origin of $\tilde{X}$, i.e. $A = D\tilde{X}(0)$, and we have for $f : \mathbb{C}^n \to \mathbb{C}^n$ that $f(x) = \mathcal{O}(|x|^2)$ for $x \to 0$. By virtue of the Jordan Normal Form Theorem from Linear Algebra, we have that by taking the eigenvectors and the generalised eigenvectors of A as basis we can transform A into its Jordan Normal Form $\hat{A}$. Applying this change of basis on $\tilde{X}$, we obtain the vector field

$$\hat{X} : \dot{x} = \hat{A}x + \hat{f}(x),\ x \in \mathbb{C}^n,$$

where we have for $\hat{f} : \mathbb{C}^n \to \mathbb{C}^n$ that $\hat{f}(x) = \mathcal{O}(|x|^2)$ for x close to the origin.

Having established this result, for the rest of this subsection we will assume we have a vector field

$$X : \dot{x} = Ax + f(x),\ x \in \mathbb{C}^n, \tag{1.4}$$

where A is a complex $n \times n$ matrix in Jordan Normal Form and for $f : \mathbb{C}^n \to \mathbb{C}^n$ we have that $f(x) = \mathcal{O}(|x|^2)$ for x close to the origin.

It is very natural to ask oneself if the function f appearing in (1.4) can't be put in a "simple" form just as we put the linear part into its Jordan Normal Form. Therefore we will use a sequence of near identity changes of coordinates of the form

$$h(y) = y + h^k(y),\ y \in U_k, \tag{1.5}$$

where $h^k : \mathbb{C}^n \to \mathbb{C}^n$ is a homogeneous polynomial of order $k \geq 2$ and U_k is a neighbourhood of the origin in $\mathbb{C}^n$. Applying the change of coordinates given by (1.5) to (1.4) means we want to calculate the pull-back of (1.4) under h. Using the Taylor expansion of f at the origin up to order k (where $k \geq 2$)

$$f(x) = f^2(x) + f^3(x) + \cdots + f^k(x) + \mathcal{O}(|x|^{k+1}),$$

we obtain

$$\begin{aligned} h^*X : \dot{y} &= Ay + f^2(y) + \cdots + f^{k-1}(y) \\ &+ \big(f^k(y) - [Dh^k(y)Ay - Ah^k(y)]\big) + \mathcal{O}(|y|^{k+1}), \end{aligned} \tag{1.6}$$

where $y \in U_k$. To simplify the term $f^k(y)$ we will have to choose a suitable $h^k(y)$ before making transformation (1.5). Let $\mathcal{H}_n^k$ denote the vector space of homogeneous polynomials of order k in n variables with values in $\mathbb{C}^n$, then we can define for each $k \geq 2$ the linear operator $L_A^k : \mathcal{H}_n^k \to \mathcal{H}_n^k$ by

$$(L_A^k h^k)(y) = Dh^k(y)Ay - Ah^k(y),\ h^k \in \mathcal{H}_n^k. \tag{1.7}$$

Then (1.6) can be rewritten as

$$h^*X : \dot{y} = f^2(y) + \cdots + f^{k-1}(y) + \big(f^k(y) - L_A^k h^k(y)\big) + \mathcal{O}(|y|^{k+1}). \tag{1.8}$$

As L_A^k is a linear operator, we know that $L_A^k(\mathcal{H}_n^k) =: \mathcal{R}^k$ is a linear subspace of $\mathcal{H}_n^k$. Let $\mathcal{C}^k$ be any complementary subspace to $\mathcal{R}^k$ in $\mathcal{H}_n^k$, then we have

$$\mathcal{H}_n^k = \mathcal{R}^k \oplus \mathcal{C}^k,\ k \geq 2. \tag{1.9}$$

Using the splitting given by (1.9) one proves the following result.

Theorem 1.7 *Let $X : \mathbb{C}^n \to \mathbb{C}^n$ be a vector field with $X(0) = 0$ and $DX(0) = A$. Let the decomposition (1.9) be given for $k = 2, \cdots, r$. Then there exists a sequence of near identity transformations $x = y + h^k(y)$, $y \in U_k$, where $h^k \in \mathcal{H}_n^k$ and U_k is a neighbourhood of the origin, $U_{k+1} \subseteq U_k$, $k = 2, \cdots, r$, such that (1.4) is transformed into*

$$\dot{y} = Ay + g^2(y) + \cdots + g^r(y) + \mathcal{O}(|y|^{r+1}),\ y \in U_r, \tag{1.10}$$

where $g^k \in \mathcal{C}^k$ for $k = 2, \cdots, r-1$.

PROOF: We refer the reader to [CLW94]. □

In this thesis we will always work with vector fields with a linear part that can be diagonalised over $\mathbb{C}$, in that case Theorem 1.7 can be improved. Before stating and proving this result we need to introduce the following definitions.

Definition 1.9 *Let* $\mathrm{Spec}(A) = \{\lambda_1, \cdots, \lambda_n\}$ *where* A *is defined by (1.4).*

- *The eigenvalues of* A *are* **resonant** *if there exist a* $m \in \mathbb{N}^n$ *and* $k \in \{1, \cdots, n\}$ *with*

$$|m| := \sum_{j=1}^{n} m_j \geq 2$$

and

$$\langle \Lambda, m \rangle - \lambda_k = 0, \tag{1.11}$$

where $\Lambda := (\lambda_1, \cdots, \lambda_n)$ *and* $\langle \Lambda, m \rangle := \sum_{j=1}^{n} \lambda_j m_j$.

- *Let* $(x_1, \cdots, x_n)$ *be coordinates with respect to the standard basis* $\{e_1, \cdots, e_n\}$ *of* $\mathbb{C}^n$ *in which the matrix* A *is in Jordan Normal Form where the diagonal elements are given by* Λ, *then a monomial* $x^m e_j = \prod_{1 \leq k \leq n} x_k^{m_k} e_j$ *is called* **resonant** *if (1.11) holds for the given* m *and* j.

We will refer to (1.11) as the *resonance equation* on the eigenvalues of A.

We now come to the following result.

Theorem 1.8 *If - under the conditions of Theorem 1.7 - we have that* $A = \mathrm{diag}(\lambda_1, \cdots, \lambda_n)$, *then a normal form up to order* $r \geq 2$ *can be chosen so that its nonlinear part consists only of resonant monomials up to order* r.

PROOF: Even though the proof can found in [CLW94] we give it anyway because it shows how resonance influences the formal normal form.

A direct calculation shows that for any monomial $x^m e_j$ with $|m| \geq 2$ and $1 \leq j \leq n$,

$$L_A^k(x^m e_j) = (\langle \Lambda, m \rangle - \lambda_j)\, x^m e_j. \tag{1.12}$$

Hence $\ker(L_A^k)$ is obviously a complementary subspace to $L_A^k(\mathcal{H}_n^k)$ and $\ker(L_A^k)$ is spanned by all resonant monomials of order k for each $k \geq 2$. Then the desired result follows from Theorem 1.7. □

Smooth linearisation of a vector field

The next question that arises is if it is possible to eliminate all non-linear terms and obtaining a linear vector field. From (1.12) it is clear why a resonant monomial cannot be eliminated by a polynomial change of variables. So resonant vector fields are in general not formally linearisable. When the eigenvalues of the linear part at the singularity are not resonant there are known cases in which an analytic vector field is not analytically linearisable. The phenomenon appearing there is known as the problem of the *small divisors*: although $\langle \Lambda, m \rangle - \lambda_j \neq 0$ for any j and m, we may have that $\langle \Lambda, m \rangle - \lambda_j$ becomes very small when $|m|$ is large.

If one is interested in an analytic conjugacy between a vector field and its linear part, one has the well-known results from Poincaré and Siegel which we will discuss after introducing some definitions.

Definition 1.10 *Let S be a subset of $\mathbb{C}^n$, then the* **convex hull** $\text{Conv}(S)$ *is the intersection of all convex subsets of $\mathbb{C}^n$ containing S. If $S = \{s_1, \cdots, s_N\}$ is a finite subset of $\mathbb{C}^n$, then*

$$\text{Conv}(S) = \left\{ \sum_{j=1}^{N} r_j s_j \,\middle|\, r_j \geq 0 \text{ and } \sum_{j=0}^{N} r_j = 1 \right\}.$$

Definition 1.11 *If the convex hull of* $\text{Spec}(A)$ *in the complex plane does not contain the origin of $\mathbb{C}$, then* $\text{Spec}(A)$ *is said to be in the* **Poincaré domain***; otherwise we say that* $\text{Spec}(A)$ *is in the* **Siegel domain**.

So now we state the two theorems and one should note how restrictive the demands are to obtain an analytic linearisation. This permits us to conclude that this type of linearisation is rather exceptional.

Theorem 1.9 (Poincaré's Theorem) *Let $A = \text{diag}(\lambda_1, \cdots, \lambda_n)$. If* $\text{Spec}(A)$ *is in the Poincaré domain and the resonance equations for A have no solutions for $|m| \geq 2$ and $1 \leq j \leq n$, then there exists an analytic change of variables $x = y + \xi(y)$, $y \in U$, where $\xi(y) = \mathcal{O}(|y|^2)$ as $y \to 0$ and U is a neighbourhood of the origin in $\mathbb{C}^n$, which transforms the analytic vector field*

$$\dot{x} = Ax + f(x), \;\; x \in \mathbb{C}^n,$$

with $f(x) = \mathcal{O}(|x|^2)$ if $x \to 0$, into the linear system

$$\dot{y} = Ay, \;\; y \in \mathbb{C}^n.$$

Proof: We refer the reader to [Poi79, CLW94]. □

Remark 1.2 *We will obtain Theorem 1.9 as a corollary of Theorem 4.17.*

Theorem 1.10 (Siegel's Theorem) *Let* $\mathrm{Spec}(A) = \{\lambda_1, \cdots, \lambda_n\}$. *If there exist* $C_0 > 0$ *and* $\mu > 0$ *such that for any* $m \in \mathbb{N}^n$ *with* $|m| \geq 2$

$$|\langle \Lambda, m\rangle - \lambda_j| \geq \frac{C_0}{|m|^\mu}, \ 1 \leq j \leq n, \tag{1.13}$$

then the analytic vector field

$$\dot{x} = Ax + f(x), \ x \in \mathbb{C}^n,$$

with $f(x) = \mathcal{O}(|x|^2)$ *if* $x \to 0$, *can be transformed into the linear system*

$$\dot{y} = Ay, \ y \in \mathbb{C}^n$$

by an analytic transformation.

Proof: We refer the reader to [Sie52, CLW94]. □

The condition (1.13) was weakened by Bruno to the so-called "Condition ω" and "Condition A", where Condition ω poses a restriction on the eigenvalues of A and Condition A poses a restriction on the formal normal form of the vector field. This way Bruno obtained a generalisation of Theorem 1.10. For more details we refer to [Brun71].

In the case one works with a real C^∞ vector field, one has the following result from Sternberg.

Theorem 1.11 (Sternberg's Theorem) *Let*

$$X : \dot{x} = Ax + f(x), \ x \in \mathbb{R}^n$$

be a C^∞ *vector field with a singularity at the origin and with* $f(x) = \mathcal{O}(|x|^2)$ *for* $x \to 0$. *If the eigenvalues of* A *are non-resonant, then there exists a* C^∞ *conjugacy between* X *and the linear system*

$$Y : \dot{y} = Ay.$$

Proof: We refer the reader to [Ste58, Ste59]. □

1.3.2 Normal form of a diffeomorphism

This subsection holds very similar results to the previous one. So we start with a discussion on the formal normal form of a diffeomorphism and after that we will discuss some results on smooth normal forms of diffeomorphisms.

Formal normal form

The calculation of the formal normal form of a diffeomorphism $f : U \subset \mathbb{C}^n \to \mathbb{C}^n$ is quite similar to the calculations we did in the case of a vector field in the previous subsection. That is why will skip most of the details here.

So assume we have a diffeomorphism $f : U \subset \mathbb{C}^n \to \mathbb{C}^n$ with a fixed point x_0 and where U is an open subset of $\mathbb{C}^n$, then after a translation and a suitable change of variables (given by the Jordan Normal Form Theorem) we may assume that we have a diffeomorphism

$$f : \mathbb{C}^n \to \mathbb{C}^n : x \mapsto Ax + F(x),\ x \in \mathbb{C}^n, \tag{1.14}$$

where A is a complex $n \times n$ matrix in Jordan Normal Form and for $F : \mathbb{C}^n \to \mathbb{C}^n$ we have that $F(x) = \mathcal{O}(|x|^2)$ for $x \to 0$.

Using a sequence of near identity change of coordinates we now calculate a formal normal form up to order k ($k \geq 2$). In the case of a diffeomorphism one encounters the linear operator $L_A^k : \mathcal{H}_n^k \to \mathcal{H}_n^k$ defined as:

$$L_A^k h(x) = h(Ax) - Ah(x),\ h \in \mathcal{H}_n^k. \tag{1.15}$$

Therefore it is easy to see that Theorem 1.7 can be imitated for diffeomorphisms. As throughout this thesis we will always work with diffeomorphisms with a linear part at the fixed point that can be diagonalised over $\mathbb{C}$, we like to have a variant of Theorem 1.8. This is possible if we use the following definition.

Definition 1.12 *Let* $\mathrm{Spec}(A) = \{\lambda_1, \cdots, \lambda_n\}$ *where* A *is defined by (1.14).*

- *The eigenvalues of* A *are* **multiplicatively resonant** *if there exists a* $m \in \mathbb{N}^n$ *and* $k \in \{1, \cdots, n\}$ *with*

$$|m| := \sum_{j=1}^{n} m_j \geq 2$$

and

$$\Lambda^m - \lambda_k = 0, \tag{1.16}$$

where $\Lambda := (\lambda_1, \cdots, \lambda_n)$ *and* $\Lambda^m = \prod_{1 \leq j \leq n} \lambda_j^{m_j}$.

- *Let* $(x_1, \cdots, x_n)$ *be coordinates with respect to the standard basis* $\{e_1, \cdots, e_n\}$ *of* $\mathbb{C}^n$ *in which the matrix* A *is in Jordan Normal Form where the diagonal elements are given by* Λ*, then a monomial* $x^m e_k$ *is called* **multiplicatively resonant** *if (1.16) holds for given* m *and* k*.*

We will also refer to (1.16) as the *resonance equation* on the eigenvalues of A. From the context it will always be clear whether we are talking about the resonance equation on the eigenvalues of the linear part of a vector field or the resonance equation on the eigenvalues on the linear part of a diffeomorphism.

Smooth linearisation of a diffeomorphism

Again we ask ourselves if it is possible to eliminate all non-linear terms in order to obtain a linear diffeomorphism. Applying the operator L_A^k defined by (1.15) to a monomial of the form $x^m e_j$ gives us

$$L_A^k(x^m e_j) = (\Lambda^m - \lambda_j)\, x^m e_j. \tag{1.17}$$

In the multiplicatively resonant case it is obvious that the resonant terms cannot be eliminated, but when $\Lambda^m - \lambda_j$ becomes very small for large values of $|m|$ we encounter again the *small divisor* problem.

If one is interested in an analytic conjugacy between a vector field and its linear part, there are variants on the results from Poincaré and Siegel.

Theorem 1.12 *Let* $A = \mathrm{diag}(\lambda_1, \cdots, \lambda_n)$*. If* $0 < |\lambda_j| < 1$ *for all* $j = 1, \cdots, n$ *and the resonance equations for* A *have no solutions for* $|m| \geq 2$ *and* $1 \leq j \leq n$*, then there exists an analytic change of variables* $x = y + \xi(y)$*,* $y \in U$*, where* $\xi(y) = \mathcal{O}(|y|^2)$ *as* $y \to 0$ *and* U *is a neighbourhood of the origin in* $\mathbb{C}^n$*, which transforms the analytic diffeomorphism*

$$f : \mathbb{C}^n \to \mathbb{C}^n : x \mapsto Ax + F(x),$$

with $F(x) = \mathcal{O}(|x|^2)$ *if* $x \to 0$*, into the linear diffeomorphism*

$$g : \mathbb{C}^n \to \mathbb{C}^n : x \mapsto Ax.$$

PROOF: We refer the reader to [Mey75]. □

Remark 1.3 *We will obtain Theorem 1.12 as a corollary of Theorem 4.19.*

Theorem 1.13 *Let* $\mathrm{Spec}(A) = \{\lambda_1, \cdots, \lambda_n\}$*. If there exist* $C_0 > 0$ *and* $\mu > 0$ *such that for any* $m \in \mathbb{N}^n$ *with* $|m| \geq 2$

$$|\Lambda^m - \lambda_j| \geq \frac{C_0}{|m|^\mu}, \;\; 1 \leq j \leq n, \tag{1.18}$$

then the analytic diffeomorphism

$$f : \mathbb{C}^n \to \mathbb{C}^n : x \mapsto Ax + F(x)$$

with $F(x) = \mathcal{O}(|x|^2)$ *if* $x \to 0$*, can be transformed into the linear diffeomorphism*

$$g : \mathbb{C}^n \to \mathbb{C}^n : x \mapsto Ax$$

by an analytic transformation.

PROOF: We refer the reader to [Zeh77]. □

If one works with a C^∞ diffeomorphism on $\mathbb{R}^n$ then one has the following result from Sternberg.

Theorem 1.14 (Sternberg's Theorem) *Let*

$$f : \mathbb{R}^n \to \mathbb{R}^n : x \mapsto Ax + F(x)$$

be a C^∞ diffeomorphism with a fixed point at the origin and with $F(x) = \mathcal{O}(|x|^2)$ for $x \to 0$. If the eigenvalues of A are multiplicatively non-resonant, then there exists a C^∞ conjugacy between f and the linear diffeomorphism

$$g : \mathbb{R}^n \to \mathbb{R}^n : x \mapsto Ax.$$

Proof: We refer the reader to [Ste58, Ste59]. □

1.3.3 Family of vector fields or diffeomorphisms

In this thesis we want to work with more than one vector field or diffeomorphism at the time. This can be done if one works with (local) families.

Definition 1.13

- *A* **(local)** C^k **family of vector fields** *($k \in \mathbb{N} \cup \{\infty, \omega\}$) is the germ at $(x, \varepsilon) = (0, 0)$ of the vector field defined by a system of equations*

$$\begin{cases} \dot{x} &= f(x,\varepsilon) \\ \dot{\varepsilon} &= 0 \end{cases},$$

where $(x, \varepsilon) \in (\mathbb{R}^{n+p}, (0, 0))$ and f is a C^k function of (x, ε).

- *A* **(local)** C^k **family of diffeomorphisms** *($k \in \mathbb{N} \cup \{\infty, \omega\}$) is the germ at $(x, \varepsilon) = (0, 0)$ of the C^k map*

$$(x, \varepsilon) \mapsto (f(x, \varepsilon), \varepsilon)$$

where $(x, \varepsilon) \in (\mathbb{R}^{n+p}, (0, 0)$.

When working with (local) families of vector fields, one usually omits the equation $\dot{\varepsilon} = 0$ from the notation.

Sometimes one constructs a (local) family of vector fields or diffeomorphisms by perturbing a given vector field or diffeomorphism. The (local) family obtained this way is called a *(local) deformation*.

Definition 1.14

- *A* **(local) deformation of a vector field** $\dot{x} = g(x)$ *is a local family of vector fields $\dot{x} = f(x, \varepsilon)$, $(x, \varepsilon) \in (\mathbb{R}^{n+p}, (0, 0))$ where $f(x, 0) = g(x)$ for all $x \in (\mathbb{R}^n, 0)$.*

- *A* **(local) deformation of a diffeomorphism** $f : \mathbb{R}^n \to \mathbb{R}^n$ *is a local family of diffeomorphisms*

$$(x, \varepsilon) \mapsto (g(x, \varepsilon), \varepsilon),$$

where $(x, \varepsilon) \in (\mathbb{R}^{n+p}, (0, 0))$ and $g(x, 0) = f(x)$ for all $x \in (\mathbb{R}^n, 0)$.

Conjugacies (or equivalences) between (local) families or (local) deformations of vector fields or diffeomorphisms are defined as in Subsection 1.1.1 with the difference that now we will have a (local) family of conjugacies (or equivalences).

To end this subsection we want to give two results which we will need to obtain useful normal forms for families of vector fields or diffeomorphisms.

Proposition 1.15 *Let A_ε be a real $n \times n$ matrix where the coefficients are C^∞, resp. analytic functions of a parameter $\varepsilon \in B(0,r)$, resp. $\varepsilon \in \mathbb{D}(0,R)$ such that the eigenvalues of A_0 have multiplicity 1, then there exists a $\tilde{r} \in \mathbb{R}^+ \setminus \{0\}$, resp. $\tilde{r} \in (\mathbb{R}^+ \setminus \{0\})^p$ such that the eigenvalues of A_ε are C^∞, resp. analytic functions of ε for all $\varepsilon \in B(0,\tilde{r})$, resp. $\varepsilon \in \mathbb{D}(0,\tilde{r})$.*

Proof: First we treat the analytic case. We consider the following function

$$f : \mathbb{C} \times \mathbb{C}^p : (\lambda, \varepsilon) \mapsto \det(A_\varepsilon - \lambda I_n),$$

where I_n is the n-dimensional unity matrix, then f is an analytic function of (λ, ε) as A_ε is analytic in ε. The eigenvalues $\nu_1, \cdots, \nu_n$ of A_0 are the solutions of the equation

$$f(\lambda, 0) = 0$$

and as all eigenvalues have multiplicity 1 for $\varepsilon = 0$ we have

$$\begin{aligned} f(\nu_j, 0) &= 0 \\ \frac{\partial f}{\partial \lambda}(\nu_j, 0) &\neq 0, \end{aligned}$$

for $j = 1, \cdots, n$. Now the Implicit Function Theorem permits us to conclude that, for each $j = 1, \cdots, n$ there exist an analytic function λ_j and a $\tilde{r}^{(j)} \in (\mathbb{R}^+ \setminus \{0\})^p$,

$$\lambda_j : \mathbb{D}(0, \tilde{r}^{(j)}) \to \mathbb{C} : \varepsilon \mapsto \lambda_j(\varepsilon)$$

such that $f(\lambda_j(\varepsilon), \varepsilon) = 0$ and $\lambda_j(0) = \nu_j$. Taking

$$\tilde{r}_k := \min_{1 \le j \le n} \tilde{r}_k^{(j)}$$

for each $k = 1, \cdots, p$, we obtain the $\tilde{r}$ we are looking for.

We now come to the C^∞ case. We consider the function

$$f : \mathbb{C} \times \mathbb{R}^p \to \mathbb{C} : (\lambda, \varepsilon) \mapsto \det(A_\varepsilon - \lambda I_n).$$

We now identify $\mathbb{C}$ with $\mathbb{R}^2$, this gives us the function

$$\tilde{f} : \mathbb{R}^2 \times \mathbb{R}^p \to \mathbb{R}^2 : (a, b, \varepsilon) \mapsto \left(\Re(f(a + ib, \varepsilon)), \Im(f(a + ib, \varepsilon))\right).$$

One observes that $\tilde{f}$ is C^∞ in (a, b, ε). Now the proof is analogous to the analytic case so we conclude that for each $j = 1, \cdots, n$ there exists C^∞ functions a_j, b_j and a $\tilde{r}^{(j)} > 0$,

$$\begin{aligned} a_j &: B(0, \tilde{r}^{(j)}) \to \mathbb{R}^2 : \varepsilon \mapsto a_j(\varepsilon), \\ b_j &: B(0, \tilde{r}^{(j)}) \to \mathbb{R}^2 : \varepsilon \mapsto b_j(\varepsilon), \end{aligned}$$

such that $\tilde{f}(a_j(\varepsilon), b_j(\varepsilon), \varepsilon) = 0$ and $a_j(0) + ib_j(0) = \nu_j$. Taking $\lambda_j := a_j + ib_j$ and $\tilde{r} = \min_{1 \le j \le n} \tilde{r}^{(j)}$, we have that $f(\lambda_j(\varepsilon), \varepsilon) = 0$, $\lambda_j(0) = \nu_j$ and we obtain the $\tilde{r}$ we are looking for. □

We have just established that the eigenvalues of A_ε are of the same class of differentiability as A_ε. We will now prove that the same result holds for the eigenvectors associated with the eigenvalues of A_ε.

Proposition 1.16 *Under the same conditions of Proposition 1.15 we have that the eigenvectors associated with the eigenvalues of A_ε are C^∞, resp. analytic functions of ε for $\varepsilon \in B(0, \tilde{r})$, resp. $\varepsilon \in \mathbb{D}(0, \tilde{r})$.*

Proof: As each eigenvalue has multiplicity 1, the system

$$A_\varepsilon v - \lambda_j(\varepsilon) v = 0$$

is row equivalent with

$$\left(\begin{array}{cccccc|c} 1 & 0 & 0 & \cdots & 0 & \hat{a}_1(\varepsilon) & 0 \\ 0 & 1 & 0 & \cdots & 0 & \hat{a}_2(\varepsilon) & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & \hat{a}_{n-1}(\varepsilon) & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 \end{array} \right)$$

where $\hat{a}_j(\varepsilon)$ is a rational function in the coefficients of A_ε hence it is a C^∞, resp. analytic function of ε. This means that the eigenvector belonging to the jth eigenvalue is given by

$$v^{(j)} = \rho(-\hat{a}_1(\varepsilon), -\hat{a}_2(\varepsilon), \cdots, -\hat{a}_{n-1}(\varepsilon), 1)$$

for $\rho \in \mathbb{R} \setminus \{0\}$. This permits us to conclude that all eigenvectors are C^∞, resp. analytic functions of ε. □

1.3.4 Normal form of a (local) family of hyperbolic vector fields

As in Subsection 1.3.1 we start with a discussion on the formal normal form of a (local) family of hyperbolic vector fields. After this we will discuss the smooth normal form of a local family of hyperbolic vector fields. Besides the main results, we will also discuss some of the techniques that are being employed to treat this kind of problem.

Formal normal form

We consider a C^∞ or analytic family of real hyperbolic vector fields $X_\varepsilon : \dot{x} = f(x, \varepsilon)$ and work on its k-jet with respect to x ($k \in \mathbb{N} \cup \{\infty\}$).

From the Implicit Function Theorem we know that there exists a hyperbolic singular point $x_0(\varepsilon)$ for X_ε for any value of ε, i.e. $X_\varepsilon \xi_\varepsilon(x) = 0$ where $\xi_\varepsilon : x \mapsto x_0(\varepsilon)$. Just as we did for a single vector field, we can make sure that this singular point becomes the origin using the translation

$$\tau_{x_0(\varepsilon)} : x \mapsto x - x_0(\varepsilon).$$

This way we obtain a family of vector fields

$$\tilde{X}_\varepsilon : \dot{x} = A_\varepsilon x + \tilde{f}(x, \varepsilon),$$

where A_ε is a real $n \times n$ matrix with coefficients that are analytic or C^∞ functions of ε (depending on the smoothness of X_ε) and for $\tilde{f}$ we have that $\tilde{f}(x, \varepsilon) = \mathcal{O}(|x|^2)$ for $x \to 0$.

The next step now would be to apply the Jordan Normal Form Theorem. We assume that all eigenvalues of A_0 (or equivalently the eigenvalues of $DX_0(x_0)$) have multiplicity 1.

By virtue of Proposition 1.15 and Proposition 1.16 we can apply the Jordan Normal Form Theorem without loss of the differentiability of the system. So after a suitable linear change of variables, we can assume that the family of vector fields takes the form

$$\hat{X}_\varepsilon : \dot{x} = \hat{A}_\varepsilon x + \hat{f}_\varepsilon(x), \tag{1.19}$$

where $\hat{f}_\varepsilon(x) = \mathcal{O}(|x|^2)$ is a C^∞, resp. analytic function of (x, ε) (on a poly-disk $\mathbb{D}(0, R)$ in the analytic case) and $\hat{A}_\varepsilon$ is in Jordan Normal Form.

From this point on, one can repeat the techniques with near identity transformations - described in Subsection 1.3.1 - to transform (4.1) into a normal form up to order $r \geq 2$ where the nonlinear terms are resonant monomials. We must take in account that the resonance equation will also contain the parameter ε, i.e.

$$\langle \Lambda_\varepsilon, m \rangle - \lambda_j(\varepsilon) = 0. \tag{1.20}$$

It is possible to have a solution of (1.20) for only one specific value of ε, anyhow we will still consider this to be a solution of the resonance equation as we consider the whole family and not one of the vector fields of the family. This situation can lead to a gigantic number of resonant monomials, that is why in these cases ε is restricted in a "sufficiently" small neighbourhood of the origin (in general an open ball or a poly-disk).

From now on we will omit the $\hat{}$ notation and assume that the family of vector fields

$$X_\varepsilon : \dot{x} = A_\varepsilon x + f_\varepsilon(x) \tag{1.21}$$

is in a formal normal form up to order $r \geq 2$ and the formal expansion of f_ε up to order r contains only resonant monomials and $f_\varepsilon(x) = \mathcal{O}(|x|^2)$ for $x \to 0$.

Smooth normal form, decomposition of flat functions and the homotopic method

We want to give an important result on smooth normal forms for local deformations of a hyperbolic vector field. The results we present are stated and proved in [IY91] and are generalisations of known results on smooth normal forms for a single hyperbolic vector field.

Theorem 1.17 ([IY91]) *Given a C^∞ local deformation (1.21) of a C^∞ hyperbolic vector field where the eigenvalues of A_0 form a non-resonant set, then for all $k \in \mathbb{N}$ we have that (1.21) is C^k conjugate to*

$$\dot{x} = A_\varepsilon x$$

on a k-dependent neighbourhood of the origin in $\mathbb{R}^{n+p}$.

We like to point out that Theorem 1.17 does not claim that (1.21) is C^∞ conjugate to its linear part. As the eigenvalues are perturbed by a parameter ε, high order resonances can appear and these resonances cannot be removed in a C^∞ way.

In the resonant case a result similar to Theorem 1.17 can be proved. Before stating this result we need to introduce the following definition.

Definition 1.15 *Let* $\mathrm{Spec}(A) = \{\lambda_1, \cdots, \lambda_n\}$ *where A is a complex $n \times n$ matrix. The eigenvalues of A are* **strongly one-resonant** *if all the resonance equations on the eigenvalues follow from a single equality*

$$\langle \Lambda, m \rangle = 0, \tag{1.22}$$

where $m \in \mathbb{N}^n$ with $|m| \geq 1$.

Remark 1.4 *If $n = 2$ in the previous definition, then strongly one-resonance is the same as resonance.*

So if (1.21) is a strongly one-resonant system, then all resonance equations are given by

$$\langle \Lambda, m + e_j \rangle = \lambda_j,$$

where $e_j = (0, \cdots, 0, 1, 0, \cdots, 0)$ with a 1 on the jth place and where $j = 1, \cdots, n$.

Theorem 1.18 ([IY91]) *Given a C^∞ local deformation (1.21) of a C^∞ hyperbolic vector field where the eigenvalues of A_0 form a strongly one-resonant set, then (1.21) is finitely smooth conjugate to*

$$\dot{x}_j = x_j g_j(u(x), \varepsilon)), \ 1 \leq j \leq n, \tag{1.23}$$

where $g(u,\varepsilon)$ is a vector polynomial of the scalar variable $u \in \mathbb{R}$ whose coefficients depend finitely smooth on ε and $x_j u(x)$ is the resonant monomial of the jth resonance equation. More precisely, given any natural number N, there is a polynomial $\tilde{g}_N(u,\varepsilon)$ with the aforementioned properties such that (1.21) is C^N conjugate to (1.23) with $g = \tilde{g}_N$ in some N-dependent neighbourhood of the origin in $\mathbb{R}^{n+p}$.

Not only the actual result of Theorem 1.17 and Theorem 1.18 are interesting, also the techniques used to obtain these results are of great importance. Techniques like globalising a vector field, decomposing flat functions and applying the homotopic method to eliminate flat functions are very important. As we will need the latter two techniques in Chapter 2, we want to give more details on them here.

A vector function w that is N-flat at the singular point, i.e. w has a zero N-jet at the singular point, is not easily annihilated by the homotopic method. The following result shows that such a N-flat function can be decomposed as the sum of two terms each of which is N-flat at all points of one of the subspaces in the decomposition $\mathbb{R}^n = \mathbb{R}^{n+} \oplus \mathbb{R}^{n-}$.

Lemma 1.19 ([Bon97, BD84]) *Assume that a C^∞ vector function $w : \mathbb{R}^n \times (\mathbb{R}^p, 0) \to \mathbb{R}^n$ has a compact support and a zero N-jet at $x = 0$ ($N \leq \infty$). Let $\mathbb{R}^n = \mathbb{R}^{n-} \oplus \mathbb{R}^{n+}$ be an arbitrary decomposition of $\mathbb{R}^n$ into a direct sum of subspaces (without loss of generality, the subspaces can be regarded as coordinate planes). There is then a decomposition*

$$w = w_- + w_+ \tag{1.24}$$

such that the function w_- has a zero N-jet at all points of $\mathbb{R}^{n-}$ and the function w_+ has a zero jet of the same order at $x \in \mathbb{R}^{n+}$, and the supports of both functions remain compact.

Now we come to the homotopic method. Actually this technique transforms the problem of conjugacy between vector fields into the solubility problem of an equation.

Lemma 1.20 *Let $X_\varepsilon : \dot{x} = F(x,\varepsilon)$ and $Y_\varepsilon : \dot{x} = F(x,\varepsilon) + w(x,\varepsilon)$ be two local families of vector fields in $\mathbb{R}^{n+p}$. We assume that there is a C^k smooth vector field $Z_{\varepsilon,\tau}$ depending on an additional parameter $\tau \in [0,1]$ and defined by a differential equation $\dot{x} = h(x,\varepsilon,\tau)$ such that the commutation relation*

$$[F + \tau w, h] = w \tag{1.25}$$

is satisfied, where $h(0,\tau,\varepsilon) \equiv 0$. The families X_ε and Y_ε are then C^k conjugate.

PROOF: We refer the reader to [IY91]. □

Equation (1.25) is called the *homological equation*. We now come to the result that gives us that this equation always has a solution if we restrict to hyperbolic systems. We refer to [IY91] for a proof that a local family of hyperbolic vector fields fulfills all criterions that are posed by Lemma 1.21.

Lemma 1.21 *Let $X_\mu : \dot{x} = v(x, \mu)$ be a local family of vector fields on $\mathbb{R}^n$ depending smoothly on a finite-dimensional parameter $\mu \in B \subseteq \mathbb{R}^m$. We assume that there is a submanifold $M \subseteq \mathbb{R}^n$ that is invariant under all the fields X_μ and globally exponentially stable. This means that there is a constant $\lambda > 0$ such that the action of the phase flow operator Φ_t of X_μ satisfies the following estimate uniformly for all $\mu \in B$:*

$$\operatorname{dist}(\Phi_t(x), M) \leq Ce^{-\lambda t} \operatorname{dist}(x, M)$$

for all positive $t > 0$. Furthermore, we assume that all the trajectories of the fields X_μ can be extended without bounds and the divergence $\operatorname{div}(X_\mu)$ is uniformly bounded.

Let us consider the homological equation

$$[v, h] = w \tag{1.26}$$

for an unknown field $Z_\mu : \dot{x} = h(x, \mu)$ depending on a parameter μ, whose right-hand side is a compactly supported C^∞ local family of vector fields on $\mathbb{R}^n$.

Under the above hypotheses, given any natural number $k < \infty$ we can find a finite N such that the homological equation whose right-hand side w is N-flat at all points of M has a C^k-smooth solution whose k-jet on M is also zero (the smoothness is with respect to the phase variables x as well as the parameters μ).

Proof: We refer the reader to [IY91]. □

Lemma 1.21 can be generalised to $N = \infty$. For a proof we refer to [IY91].

Proposition 1.22 *Under the conditions of Lemma 1.21 the homological equation (1.26) whose right-hand side is ∞-flat at all points of M has a C^∞-smooth solution that is ∞-flat on M.*

1.3.5 Normal form of a (local) family of hyperbolic diffeomorphisms

As in Subsection 1.3.2 we start with a brief discussion on the formal normal form of a (local) family of hyperbolic diffeomorphisms. After this we discuss the smooth normal form of a local family of hyperbolic diffeomorphisms. Beside the main results, we also discuss some of the techniques that are being employed to prove these results.

Formal normal form

Consider a C^∞ or analytic family of diffeomorphisms $f_\varepsilon : U \subset \mathbb{R}^n \to \mathbb{R}^n$ with a hyperbolic fixed point $x_0(\varepsilon)$. Combining the techniques and results from Subsection 1.3.2 and Subsection 1.3.4 we may assume that the family of diffeomorphisms

$$f_\varepsilon : \mathbb{R}^n \to \mathbb{R}^n : x \mapsto A_\varepsilon x + F_\varepsilon(x) \tag{1.27}$$

is in a formal normal form up to order $r \geq 2$, A_ε is in Jordan Normal Form and the formal expansion of F_ε up to order r contains only resonant monomials and $F_\varepsilon(x) = \mathcal{O}(|x|^2)$ for $x \to 0$.

Smooth normal form, decomposition of flat functions and the homotopic method

We want to give an important result on smooth normal forms for local deformations of a hyperbolic diffeomorphism. The results we present are stated and proved in [IY91] and are the "discrete versions" of the results presented in Subsection 1.3.4.

Theorem 1.23 ([IY91]) *Given a C^∞ local deformation (1.27) of a C^∞ hyperbolic diffeomorphism where the eigenvalues of A_0 form a multiplicatively non-resonant set, then for all $k \in \mathbb{N}$ we have that (1.27) is C^k conjugate to*

$$g_\varepsilon : \mathbb{R}^n \to \mathbb{R}^n : x \mapsto A_\varepsilon x$$

on a k-dependent neighbourhood of the origin in $\mathbb{R}^{n+p}$.

In the resonant case a result similar to Theorem 1.23 can be proved. Before stating this result we need to introduce the following definition.

Definition 1.16 *Let* $\mathrm{Spec}(A) = \{\lambda_1, \cdots, \lambda_n\}$ *where A is defined by (1.14). The eigenvalues of A are* **multiplicatively strongly one-resonant** *if all the resonance equations on the eigenvalues follow from a single equality*

$$\Lambda^m = 1, \tag{1.28}$$

where $m \in \mathbb{N}^n$ with $|m| \geq 1$.

Remark 1.5 *If $n = 2$ in the previous definition, then multiplicatively strongly one-resonant is the same as multiplicatively resonant.*

So if (1.27) is a multiplicatively strongly one-resonant diffeomorphism, then all resonance equations are given by

$$\Lambda^{m+e_j} = \lambda_j,$$

where $e_j = (0, \cdots, 0, 1, 0, \cdots, 0)$ with a 1 on the jth place and where $j = 1, \cdots, n$.

Theorem 1.24 ([IY91]) *Given a C^∞ local deformation (1.27) of a C^∞ hyperbolic vector field where the eigenvalues of A_0 form a multiplicatively strongly one-resonant set, then (1.27) is finitely smooth conjugate to*

$$g_\varepsilon : \mathbb{R}^n \to \mathbb{R}^n : x \mapsto x_j g_j(u(x), \varepsilon)),\ 1 \leq j \leq n, \tag{1.29}$$

where $g(u, \varepsilon)$ is a vector polynomial of the scalar variable $u \in \mathbb{R}$ whose coefficients depend finitely smooth on ε and $x_j u(x)$ is the resonant monomial of the jth resonance equation. More precisely, given any natural number N, there is a polynomial $\tilde{g}_N(u, \varepsilon)$ with the aforementioned properties such that (1.27) is C^N conjugate to (1.29) with $g = \tilde{g}_N$ in some N-dependent neighbourhood of the origin in $\mathbb{R}^{n+p}$.

We briefly recall the results on the homotopic method applied to a family of diffeomorphisms.

Lemma 1.25 *Let f_ε and $f_\varepsilon + w_\varepsilon$ be two smooth diffeomorphisms of $\mathbb{R}^n$ depending on parameters $\varepsilon \in (\mathbb{R}^p, 0)$. We assume that there is a C^k-smooth diffeomorphism $h_{\varepsilon,\tau}$ on $\mathbb{R}^n$ depending on the parameters ε and also on an additional parameter $\tau \in [0, 1]$ such that*

$$\frac{\partial}{\partial x}(f_\varepsilon + \tau w_\varepsilon) \cdot h_{\varepsilon,\tau} - h_{\varepsilon,\tau} \circ (f_\varepsilon + \tau w_{\varepsilon,\tau}) = -\varphi, h_{\varepsilon,\tau}(0) = 0, \tag{1.30}$$

holds for all $\varepsilon \in (\mathbb{R}^p, 0)$, $\tau \in [0, 1]$. The diffeomorphisms f_ε and $f_\varepsilon + w_\varepsilon$ are then C^k-smoothly conjugate.

Proof: We refer the reader to [IY91]. □

Equation (1.30) is the discrete version of (1.25) and is therefore also called *homological.*

Lemma 1.26 *Let $f(x, \mu)$ be a family of diffeomorphisms of $\mathbb{R}^n$ depending on parameters $\mu \in B \subseteq \mathbb{R}^m$. Let there be a submanifold $M \subseteq \mathbb{R}^n$ invariant under all $f(\cdot, \mu)$ that is exponentially asymptotically stable in the sense that the following uniform estimate is satisfied:* $\forall t \in \mathbb{N}$

$$\operatorname{dist}(f^t(x, \mu), M) \leq c\lambda^t \operatorname{dist}(x, M), 0 < \lambda < 1.$$

Then, given any $k \in \mathbb{N}$, we can find a $N = N(k)$ such that the homological equation

$$\left(\frac{\partial f}{\partial x}\right) \cdot h - h \circ f = w \tag{1.31}$$

whose right-hand side w is compactly supported, smooth, and N-flat on M, has a solution in the class of C^k-smooth diffeomorphisms that is k-flat at points $x \in M$.

PROOF: We refer the reader to [IY91]. □

Lemma 1.26 can be generalised to $N = \infty$. For a proof we refer to [IY91].

Proposition 1.27 *Under the conditions of Lemma 1.26 the homological equation (1.31) whose right-hand side is ∞-flat at all points of M has a C^∞-smooth solution that is ∞-flat on M.*

1.3.6 Normal form of a local deformation of a planar singularity of center type

We start with a real planar vector field Y with a singularity (x_0, y_0) of center type, i.e. $DY_{(x_0,y_0)}$ has eigenvalues $\pm i\alpha$ with $\alpha > 0$. Let X_ε be a local C^∞ deformation of Y (so $X_0 \equiv Y$) such that $(x_\varepsilon, y_\varepsilon)$ is a singularity for X_ε. We note that in general $(x_\varepsilon, y_\varepsilon)$ will not be a singularity of center type for all vector fields of the local deformation, i.e. for $\varepsilon = 0$ we have a singularity of center type and for $\varepsilon \neq 0$ we have a singularity of hyperbolic type. This will become clear later on.

First we will perform the translation

$$\tau_{(x_\varepsilon,y_\varepsilon)} : \mathbb{R}^2 \to \mathbb{R}^2 (x, y) \mapsto (x - x_\varepsilon, y - y_\varepsilon)$$

to obtain a local C^∞ deformation $\tilde{X}_\varepsilon$ with a singularity at the origin which is of center type for $\varepsilon = 0$. We will assume that $\varepsilon \in (\mathbb{R}, 0)$. By virtue of Proposition 1.15 and Proposition 1.16 we can apply the Jordan Normal Form Theorem without loss of differentiability and by means of a linear transformation we obtain that $\tilde{X}_\varepsilon$ is conjugate to

$$\hat{X}_\varepsilon : \left\{ \begin{array}{rcl} \dot{x} &=& \beta(\varepsilon)x - \alpha(\varepsilon)y + \hat{f}(x, y; \varepsilon) \\ \dot{y} &=& \alpha(\varepsilon)x + \beta(\varepsilon)y + \hat{g}(x, y; \varepsilon) \end{array} \right. , \tag{1.32}$$

where the functions appearing in (1.32) are C^∞ and we have that $\alpha(0) \neq 0$, $\beta(0) = 0$, $\hat{f}(x, y; \varepsilon) = \mathcal{O}(|(x, y)|^2)$ and $\hat{g}(x, y; \varepsilon) = \mathcal{O}(|(x, y)|^2)$ for $(x, y) \to (0, 0)$. We will assume that $\frac{\partial \beta}{\partial \varepsilon}(0) \neq 0$.

Proposition 1.28 *There exist a C^∞ function $a : (\mathbb{R}, 0) \to (\mathbb{R}, 0)$ such that (1.32) can be rewritten as*

$$\check{X}_\delta : \left\{ \begin{array}{rcl} \dot{x} &=& \delta x - a(\delta)y + \check{f}(x, y; \delta) \\ \dot{y} &=& a(\delta)x + \delta y + \check{g}(x, y; \delta) \end{array} \right. . \tag{1.33}$$

PROOF: As $\alpha(0) \neq 0$, $\beta(0) = 0$ and $\frac{\partial \beta}{\partial \varepsilon}(0) \neq 0$, we have that the curves $\gamma_\pm$ defined by

$$\gamma_\pm : (\mathbb{R}, 0) \to (\mathbb{C}, 0) : \varepsilon \mapsto \beta(\varepsilon) \pm i\alpha(\varepsilon)$$

will intersect the imaginary axis transversally in the eigenvalues $\pm i\alpha(0)$, i.e. $\frac{\partial \gamma_\pm}{\partial \varepsilon}(0) \neq 0$. We know that α and β are C^∞ functions in ε so we have that

$$\begin{aligned}\alpha(\varepsilon) &= \alpha(0) + \mathcal{O}(|\varepsilon|)\\ \beta(\varepsilon) &= r\varepsilon + \mathcal{O}(|\varepsilon|^2)\\ &= r\varepsilon(1 + \mathcal{O}(|\varepsilon|))\end{aligned}$$

where $r = \frac{\partial \beta}{\partial \varepsilon}(0) \neq 0$.

We now consider the equation

$$\delta = \beta(\varepsilon) = r\varepsilon(1 + \mathcal{O}(|\varepsilon|))$$

for δ close to zero. As $r \neq 0$ we can apply the Inverse Function Theorem which gives us the existence of a C^∞ function b for which we have that $\varepsilon = b(\delta)$ for δ sufficiently close to zero and $b(\delta) = \tilde{r}\delta + \mathcal{O}(|\delta|^2)$ where $\tilde{r} \neq 0$.

We now have that

$$\begin{aligned}\alpha(\varepsilon) &= \alpha(b(\delta))\\ &= \alpha(0) + \mathcal{O}(|b(\delta)|)\\ &= \alpha(0) + \mathcal{O}(|\tilde{r}\delta + \mathcal{O}(|\delta|^2)|)\\ &= \alpha(0) + \mathcal{O}(|\delta|.|\tilde{r} + \mathcal{O}(|\delta|)|)\\ &= \alpha(0) + \mathcal{O}(|\delta|).\end{aligned}$$

Defining $a(\delta) := \alpha(b(\delta))$ we obtain that

$$\beta(\varepsilon) \pm i\alpha(\varepsilon) = \delta \pm i a(\delta)$$

for δ sufficiently close to zero. We introduce the following definitions

$$\begin{aligned}\check{f}(x, y; \delta) &= \hat{f}(x, y; \beta^{-1}(\delta))\\ \check{g}(x, y; \delta) &= \hat{g}(x, y; \beta^{-1}(\delta))\end{aligned}$$

for δ sufficiently close to zero. As $\frac{\partial \beta}{\partial \varepsilon}(0) \neq 0$ we know that β is invertible for ε close to zero. This means that (1.32) can be written as (1.33). □

Changing notations this means that we can work with the following C^∞ local deformation

$$X_\varepsilon : \left\{ \begin{aligned}\dot{x} &= \varepsilon x - \alpha(\varepsilon) y + f(x, y; \varepsilon)\\ \dot{y} &= \alpha(\varepsilon) x + \varepsilon y + g(x, y; \varepsilon)\end{aligned}\right. , \tag{1.34}$$

where $f(x, y; \varepsilon) = \mathcal{O}(|(x, y)|^2)$ and $g(x, y; \varepsilon) = \mathcal{O}(|(x, y)|^2)$ for $(x, y) \to (0, 0)$.

The linear part of X_ε is diagonal if we turn to complex coordinates given by the linear change of variables

$$\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

In this complex setting we can apply Theorem 1.8 with the adaptation that the coefficients of the transformations are C^∞ functions of ε instead of constants. This time we will apply Theorem 1.8 for all $k \in \mathbb{N}$. This leads to a sequence of C^∞ functions and a normal form containing a series of resonant monomials. As we know that the transformation needed to eliminate terms of degree ℓ does not effect any of the terms of degree smaller than ℓ this limit process will lead to a well-defined formal power series. By virtue of Borel's Theorem we know there exist C^∞ functions P, Q, R_1 and R_2 such that X_ε becomes

$$\begin{aligned}\mathcal{X}_\varepsilon(x,y) &= (\varepsilon x - \alpha(\varepsilon)y + xP(x^2+y^2;\varepsilon) - yQ(x^2+y^2;\varepsilon))\frac{\partial}{\partial x}\\ &+ (\alpha(\varepsilon)x + \varepsilon y + xQ(x^2+y^2;\varepsilon) + yP(x^2+y^2;\varepsilon))\frac{\partial}{\partial y}\\ &+ R_1(x,y;\varepsilon)\frac{\partial}{\partial x} + R_2(x,y;\varepsilon)\frac{\partial}{\partial y} \qquad (1.35)\end{aligned}$$

where

$$P(0;\varepsilon)=0, Q(0;\varepsilon)=0 \text{ and } \frac{\partial^{j+k}R_i}{\partial x^j y^k}(0,0;\varepsilon)=0, \forall j,k\in\mathbb{N}, i=1,2.$$

The functions R_1 and R_2 are called *infinitely flat* functions.

We introduce

$$\begin{aligned}\mathcal{X}^N_\varepsilon(x,y) &= (\varepsilon x - \alpha(\varepsilon)y + xP(x^2+y^2;\varepsilon) - yQ(x^2+y^2;\varepsilon))\frac{\partial}{\partial x} \qquad (1.36)\\ &+ (\alpha(\varepsilon)x + \varepsilon y + xQ(x^2+y^2;\varepsilon) + yP(x^2+y^2;\varepsilon))\frac{\partial}{\partial y}\end{aligned}$$

which is a C^∞ vector field without infinitely flat terms.

Putting

$$\begin{aligned}x &= \rho\cos\theta,\\ y &= \rho\sin\theta,\end{aligned}$$

a direct calculation shows that (1.36) is transformed into

$$\left\{\begin{aligned}\dot\rho &= \varepsilon\rho + \rho P(\rho^2;\varepsilon)\\ \dot\theta &= \alpha(\varepsilon) + Q(\rho^2;\varepsilon).\end{aligned}\right. \qquad (1.37)$$

As $\alpha(\varepsilon)\neq 0$ for all ε close to zero, the function $\alpha(\varepsilon)+Q(\rho^2;\varepsilon)$ doesn't change sign for sufficiently small ε.

Dividing this function, there exists a C^∞ function F such that $\mathcal{X}^N_\varepsilon$ - in polar coordinates - is equivalent with

$$\mathcal{X}^P_\varepsilon : \left\{\begin{aligned}\dot\rho &= \varepsilon\rho + \rho^3F(\rho^2;\varepsilon)\\ \dot\theta &= 1\end{aligned}\right. . \qquad (1.38)$$

Consider now $\mathcal{M} = \mathbb{R}^+ \times \mathbb{R}$ the universal cover of $\mathbb{R}^+ \times \mathbb{S}^1$ with covering

$$\mathbb{P} : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}^+ \times \mathbb{S}^1 : \ (\rho, \theta) \mapsto (\rho, \theta \bmod 2\pi)$$

Since $\mathcal{X}_\varepsilon^P$ is (obviously) periodic in θ, the vector field

$$\hat{\mathcal{X}}_\varepsilon = \left(\varepsilon\rho + \rho^3 F(\rho^2; \varepsilon)\right) \frac{\partial}{\partial\rho}$$

is such that $\mathbb{P}_*(\hat{\mathcal{X}}_\varepsilon) = \mathcal{X}_\varepsilon^P$. Now (1.38) provides the normal form we were looking for.

1.4 Transition maps of planar vector fields

In this section we want to introduce some results on transition maps of planar vector fields. It is by no means our intention to give a complete introduction to this topic as it is studied and applied in a large variety of problems. We confine ourselves to the case of a planar vector field as we will only apply results on transition maps in this case.

Consider a real planar vector field X with a singular point x_0 and with a flow $\Phi_t(x)$.

Definition 1.17 *Given a curve Σ with parameterisation $\sigma : I \to \mathbb{R}^2 : t \mapsto \sigma(t)$ where I is an interval, we say that Σ is* **transversal** *to the flow Φ_t of a planar vector field X if the tangent vector to Σ is not parallel to the tangent vector to the flow Φ_t at each point of Σ, i.e.*

$$\forall t \in I, \forall k \in \mathbb{R} : \sigma'(t) \neq kX(\sigma(t)).$$

So consider two curves Σ_1 and Σ_2 (in this context we will call them sections) transversal to the flow $\Phi_t(x)$ of the planar vector field X such that for all $x^{(1)} \in \Sigma_1$ there exists a $t_0 > 0$ such that $x^{(2)} := \Phi_{t_0}(x^{(1)}) \in \Sigma_2$. The map

$$\Sigma_1 \to \Sigma_2 : x^{(1)} \mapsto x^{(2)}$$

is called the **transition map** from Σ_1 to Σ_2.

In the following subsections we consider two types of vector fields in which we can define and study transition maps with special properties.

1.4.1 Dulac map near a saddle singularity of a planar vectorfield

Consider a local C^∞ deformation of a real planar vector field with a saddle singularity, then by virtue of the results form Subsection 1.3.4, this local deformation is given by

$$X_\varepsilon : \left\{ \begin{array}{rcl} \dot{x} & = & \lambda_1(\varepsilon)x + f_\varepsilon(x, y) \\ \dot{y} & = & \lambda_2(\varepsilon)y + g_\varepsilon(x, y), \end{array} \right. \tag{1.39}$$

where

$$\frac{\lambda_1(\varepsilon)}{\lambda_2(\varepsilon)} \in \mathbb{R}^- \setminus \{0\},$$

$f_\varepsilon(x,y)$ and $g_\varepsilon(x,y)$ are C^∞ functions with $f_\varepsilon(x,y) = \mathcal{O}(|(x,y)|^2)$ and $g_\varepsilon(x,y) = \mathcal{O}(|(x,y)|^2)$ for $(x,y) \to (0,0)$. For sufficiently small values of ε the origin will always be a singularity of saddle type for (1.39).

Denoting

$$r(\varepsilon) := \left|\frac{\lambda_1(\varepsilon)}{\lambda_2(\varepsilon)}\right|,$$

then (1.39) is C^∞ equivalent to

$$\tilde{X}_\varepsilon : \begin{cases} \dot{x} &= x \\ \dot{y} &= -r(\varepsilon)y + \tilde{g}_\varepsilon(x,y), \end{cases} \tag{1.40}$$

where $\tilde{g}_\varepsilon(x,y)$ is a C^∞ function with $\tilde{g}_\varepsilon(x,y) = \mathcal{O}(|(x,y)|^2)$ for $(x,y) \to (0,0)$.

If $r(0)$ is irrational, then by Theorem 1.17 we have that for each $k \in \mathbb{N}$ system (1.40) is C^k conjugate to

$$Y_\varepsilon : \begin{cases} \dot{x} &= x \\ \dot{y} &= -r(\varepsilon)y. \end{cases} \tag{1.41}$$

If $r(0) = \frac{p}{q}$ (with $\gcd(p,q) = 1$) is rational, then by Theorem 1.18 we have that for each $k \in \mathbb{N}$ there exists an integer $N(k)$ such that (1.40) is C^k conjugate to

$$Y_\varepsilon : \begin{cases} \dot{x} &= x \\ \dot{y} &= y\left(-r_0 + \sum_{j=0}^{N(k)} \alpha_{j+1}(\varepsilon)(x^p y^q)^j\right). \end{cases} \tag{1.42}$$

where $\alpha_1(\varepsilon) = r_0 - r(\varepsilon)$.

This way we can define the following transition map

$$D_\varepsilon : \Sigma_1 \to \Sigma_2 : P_1 \mapsto P_2,$$

where $\Sigma_1 = [0,1] \times \{1\}$, $\Sigma_2 = \{1\} \times [0,1]$.

The map D_ε is called the *Dulac map* and can be extended continously by $D_\varepsilon(0) \equiv 0$.

In the non-resonant case we use the normal form given by (1.41), then the Dulac map can be calculated explicitly:

$$D_\varepsilon(x) = x^{r(\varepsilon)},$$

as P_1 has coordinates $(x,1)$. In the resonant case we use the normal form given by (1.42), so the Dulac map cannot be calculated explicitly. Therefore we are interested in an asymptotic expansion of the Dulac map.

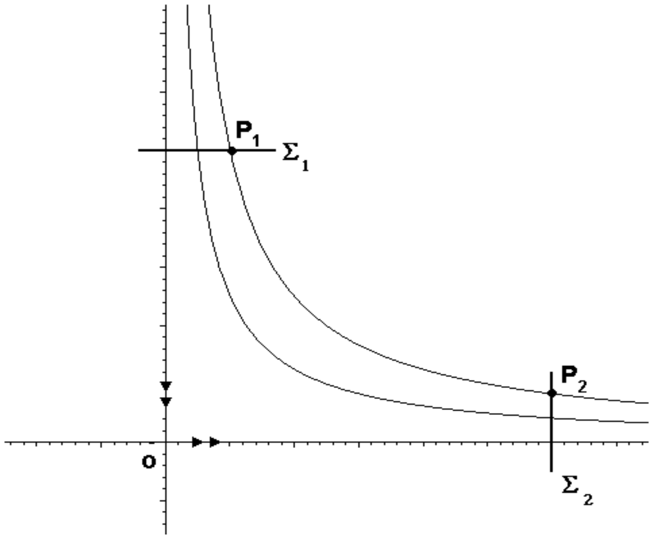

Figure 1.1: Dulac map near a saddle

Definition 1.18 *Given* $\alpha \in \mathbb{R}$, *the* **Ecalle–Roussarie compensator** *is defined as*

$$\omega(x,\alpha) := \begin{cases} \dfrac{x^{-\alpha}-1}{\alpha} & \textit{if} \quad \alpha \neq 0 \\ -\ln x & \textit{if} \quad \alpha = 0. \end{cases} \tag{1.43}$$

By virtue of this definition one obtains the following formal expansion of the Dulac map

$$\begin{aligned} D_\varepsilon(x) &= x^{r_0} + \alpha_1(\varepsilon)x^{r_0}\omega(x,\alpha_1(\varepsilon)) \\ &+ \sum_{\substack{1\le j\le i+1\le K(k)\\ 1\le i}} \alpha_{ij}(\varepsilon)x^{(iq+1)r_0}\omega(x,\alpha_1(\varepsilon))^j + \psi_k(x,\varepsilon), \end{aligned}$$

where $r_0 = \frac{p}{q}$ such that $\gcd(p,q) = 1$, $\alpha_{ij}(\varepsilon)$ are polynomials of $\alpha_1(\varepsilon)$, $\cdots$, $\alpha_{N(k)+1}(\varepsilon)$ and where $K(k)$ is a positive integer depending on k such that $\psi_k(x,\varepsilon)$ is a C^k function which is k-flat with respect to $x = 0$.

For a proof of this result and more information on and applications of the Dulac map, we refer - for instance - to [BRS96, Cau04, CR02, DMR96, DRR94, GR99, IY95, JM94, MMR94, Mou91, Rou86, Rou97].

Anyhow, it is clear that the Ecalle–Roussarie compensator is an important tool in order to give an explicit (asymptotic) description of the dynamics in the neighbourhood of a saddle singularity of planar vector field.

1.4.2 Poincaré map, Melnikov functions and Abelian integrals

Let $H : \mathbb{R}^2 \to \mathbb{R} : (x, y) \mapsto H(x, y)$ be a C^∞ Hamiltonian function such that the Hamiltonian vector field $X_H = \frac{\partial H}{\partial y}(x, y)\frac{\partial}{\partial x} - \frac{\partial H}{\partial x}(x, y)\frac{\partial}{\partial y}$ or

$$X_H : \begin{cases} \dot{x} & = & \frac{\partial H}{\partial y}(x, y) \\ \dot{y} & = & -\frac{\partial H}{\partial x}(x, y) \end{cases}$$

has a center type singularity at the origin.

Considering the parameter $\lambda = (\bar{\lambda}, \varepsilon)$, $\varepsilon \in [0, \varepsilon_0[$, $\bar{\lambda} \in (\mathbb{R}^k, 0)$, then by perturbation we obtain the following local deformation of X_H:

$$X_\lambda = X_H + \sum_{1\leq k\leq n} \varepsilon^k X_{\bar{\lambda}}^{(k)} + \mathcal{O}(\varepsilon^{n+1}) \tag{1.44}$$

where $\mathcal{O}(\varepsilon^{n+1})$ is a C^∞ vector field depending on (x, y, λ).

We take the dual 1-form of X_λ

$$\omega_\lambda = dH + \sum_{1\leq k\leq n} \varepsilon^k \nu_{\bar{\lambda}}^{(k)} + \mathcal{O}(\varepsilon^{n+1}) \tag{1.45}$$

with respect to $\Omega = dx \wedge dy$, the standard symplectic area-form on $\mathbb{R}^2$, where $\mathcal{O}(\varepsilon^{n+1})$ is a C^∞ 1-form depending on (x, y, λ).

We now have the following situation:

- γ_h is the level curve defined by $H(x, y) = h$,
- Σ is a transversal section to the level curves of the Hamiltonian H,
- Γ is the part of the orbit of X_λ starting at the point denoted by h on Σ and ending at the point of first return on Σ,
- τ is the segment on Σ between h and $\mathcal{P}_\lambda(h)$,
- Δ is the shaded region.

In practice one always tries to choose a segment on one of the axis as transversal section Σ.

We will now integrate ω_λ over Γ, by (1.45) we have

$$\int_\Gamma \omega_\lambda = \int_\Gamma dH + \int_\Gamma \sum_{1\leq j\leq n} \varepsilon^j \nu_{\bar{\lambda}}^{(j)} + \mathcal{O}(\varepsilon^{n+1}) \tag{1.46}$$

as ω_λ is the dual form of X_λ and Γ is an orbit of X_λ we have that $\int_\Gamma \omega_\lambda = 0$. We also have that $\int_\Gamma dH = \mathsf{P}_\lambda(h) - h$.

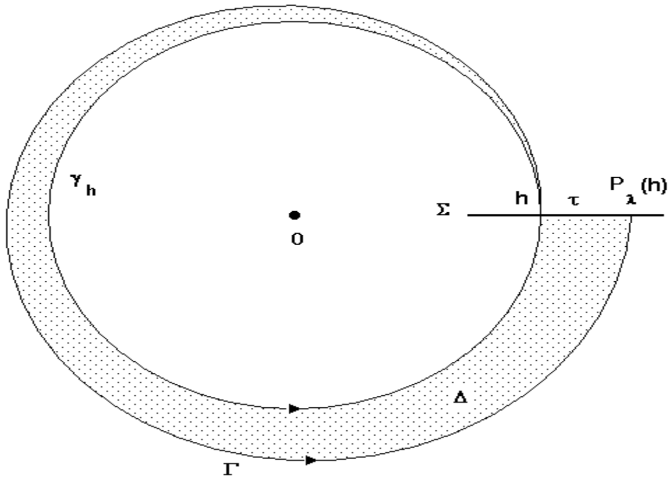

Figure 1.2: Poincaré map near a center singularity

The map $\mathsf{P}_\lambda : \Sigma \to \Sigma : h \mapsto \mathsf{P}_\lambda(h)$ is the **Poincaré map** of X_λ. The Poincaré map is - together with the Dulac map - a well-known example of a transition map. Note that here we work with only one transversal section, hence the Poincaré map detects the first time an orbit returns on a given section: for this reason the Poincaré map is also known as the *first return map*. The Poincaré map is very important in the study of limit cycles as the zeros of the Poincaré map correspond to limit cycles. We remark that in practice we take the Poincaré map $\mathsf{P}_\lambda : \Sigma_0 \to \Sigma$, where Σ_0 is a transversal $\Sigma_0 \subset \Sigma$ that is sufficiently small such that $\mathsf{P}_\lambda(h)$ exists for all values of h under consideration.

In general it is not possible to give an explicit form of the Poincaré map, therefore we are interested in a formal expansion of the Poincaré map. We will calculate an expansion with respect to the parameter ε:

$$\mathsf{P}_\lambda(h) = h + \sum_{1 \leq j \leq n} M_{\bar{\lambda}}^{(j)}(h)\varepsilon^j + \mathcal{O}(\varepsilon^{n+1}). \tag{1.47}$$

The function $M_{\bar{\lambda}}^{(j)}$ appearing in (1.47) is called the j**th order Melnikov function** for $j = 1, \cdots, n$. We now want to discuss how the first order Melnikov function can be expressed by Abelian integrals.

Consider $\int_\Delta d\nu_{\bar{\lambda}}^{(1)}$, by Stokes's Theorem we have

$$\int_\Delta d\nu_{\bar{\lambda}}^{(1)} = \int_{\partial\Delta} \nu_{\bar{\lambda}}^{(1)} = \int_\Gamma \nu_{\bar{\lambda}}^{(1)} + \int_\tau \nu_{\bar{\lambda}}^{(1)} - \int_{\gamma_h} \nu_{\bar{\lambda}}^{(1)} \tag{1.48}$$

where the minus sign appears by choice of orientation of the curve γ_h.

As for $\varepsilon = 0$ we have that $\mathsf{P}_\lambda(h) \equiv h$ and τ is the segment between h and $\mathsf{P}_\lambda(h)$ on the transversal section Σ we know that length of τ is $\mathcal{O}(\varepsilon)$. Hence $\int_\tau \nu_{\bar\lambda}^{(1)} = \mathcal{O}(\varepsilon)$. Using this result it is a straightforward application of calculus to prove that the area of Δ is $\mathcal{O}(\varepsilon)$ and $\int_\Delta d\nu_{\bar\lambda}^{(1)} = \mathcal{O}(\varepsilon)$.

So (1.48) is reduced to

$$\int_\Gamma \nu_{\bar\lambda}^{(1)} = \int_{\gamma_h} \nu_{\bar\lambda}^{(1)} + \mathcal{O}(\varepsilon). \tag{1.49}$$

Introducing

$$I_{\bar\lambda}(h) = \int_{\gamma_h} \nu_{\bar\lambda}, \tag{1.50}$$

then $I_{\bar\lambda}(h)$ is called an **Abelian integral**.

Using (1.46), (1.49) and (1.50) we have the following asymptotic expansion in ε for the Poincaré map:

$$\mathsf{P}_\lambda(h) = h - \varepsilon I_{\bar\lambda}(h) + \mathcal{O}(\varepsilon^2). \tag{1.51}$$

So the first Melnikov function can be written as

$$M_{\bar\lambda}^{(1)} = -I_{\bar\lambda}(h). \tag{1.52}$$

By (1.52) we obtain the natural link between the Poincaré map and Abelian integrals. For more information on the Poincaré map and on how to calculate higher order Melnikov functions we refer to e.g. [BF02, Cau04, CR02, Ili98, LLZ04, ZZ02].

Chapter 2

Nearly-resonant saddles

2.1 Introduction

In Subsection 1.4.1 we observed how the Ecalle–Roussarie compensator appears in the formal expansion of the Dulac map near a planar saddle singularity. So one can expect that these compensators also can be used in the calculations of equivalences or conjugacies between vector fields with saddle singularities. Even in the case of conjugacies between diffeomorphisms with saddle fixed points we can expect the involvement of Ecalle–Roussarie compensators if one considers the results in [BRS96].

In the previous chapter we briefly discussed the Hartman–Grobman Theorem. As saddles are hyperbolic systems, the Hartman–Grobman Theorem can be applied to them and gives a C^0 conjugacy between the vector field or diffeomorphism and its linear part. In [Har60a] it is proved that in the case of a planar vector field with a saddle singularity or a diffeomorphism on $\mathbb{R}^2$ with a saddle fixed point there exists a C^1 conjugacy between the vector field or diffeomorphism and its linear part. For saddles in higher dimensions this is no longer true, we refer to [Har60a, BK94] for a counterexample in 3 dimensions. Therefore we will restrict ourselves to the two-dimensional case.

Our approach will be different than in most proofs on equivalences and conjugacies between vector fields or diffeomorphisms. Usually the change of variables to do this is given as the solution of a functional equation or as the fixed point of some contraction. With our "explicit" approach we can give more information on these transformations. This way we know more than just the fact that the transformation is C^1. A similar approach in the higher dimensional case (resulting in a C^0 conjugacy) was done in [BNY03a, BNY03b].

If the eigenvalues of the saddle are non-resonant, resp. multiplicatively non-resonant then by Theorem 1.17, resp. Theorem 1.23 we know that the family of saddles is finitely smooth linearisable. So we suppose that the eigenvalues at the saddle singularity are close to resonance in the case of vector fields and that the eigenvalues at the saddle fixed points are close to multiplicatively resonance. We will specify this later on.

This chapter is structured as follows. In Section 2.2 we make clear what we mean with "nearly-resonance" and calculate equivalences between vector fields with saddle singularities. The topic of Section 2.2 was already discussed in [BN02]. In Section 2.3 we generalise the results of Section 2.2 to the case of conjugacies between vector fields with saddle singularities. In Section 2.4 we discuss the fact that the conjugacies are actually C^∞ functions if one introduces new variables. In Section 2.5 we calculate conjugacies between diffeomorphisms with saddle fixed points. Of course there will be similarities between the results from Section 2.3 and the results from Section 2.5. In Section 2.6 we make an attempt to establish a similar result on the conjugacies between vector fields with center singularities. Therefore we will need to introduce some complex functions which we will call *complex compensators*.

2.2 Equivalences between saddle vector fields

2.2.1 Settings, nearly resonance and compensators

Settings and nearly resonance

We consider a local C^∞ deformation X_λ of a resonant planar saddle singularity. By virtue of the results in Subsection 1.3.4 we can assume that $X_\lambda(0) = 0$ and for the linear part we have that

$$DX_\lambda(0) = \begin{pmatrix} \alpha(\lambda) & 0 \\ 0 & \beta(\lambda) \end{pmatrix}$$

where $\beta(\lambda) < 0 < \alpha(\lambda)$. By virtue of Proposition 1.15 we know that the eigenvalues $\alpha(\lambda)$ and $\beta(\lambda)$ are C^∞ functions of λ. Proposition 1.15 and Proposition 1.16 assure us that DX_λ can be linearised by a C^∞ change of coordinates. Using a submersion $\lambda \mapsto (\alpha(\lambda), \beta(\lambda))$ we can view (α, β) as new parameters and omit λ in the notation. As we assume that the system is nearly resonant, there exists $p, q \in \mathbb{N}$ with $\gcd(p, q) = 1$ such that

$$\varepsilon = \frac{p}{q} + \frac{\beta}{\alpha}, \tag{2.1}$$

is close to zero, hence

$$\beta = \left(\varepsilon - \frac{p}{q}\right)\alpha. \tag{2.2}$$

Looking at (2.2) it is natural to consider (α, ε) as parameters instead of (α, β) on condition we take ε close to zero. Therefore we can assume that

$$DX_{\alpha,\varepsilon}(0) = \begin{pmatrix} \alpha & 0 \\ 0 & \left(\varepsilon - \frac{p}{q}\right)\alpha \end{pmatrix}. \tag{2.3}$$

From now on we consider α as a strictly positive constant, hence we obtain a local deformation X_ε depending on the one-dimensional parameter $\varepsilon \in (\mathbb{R}, 0)$.

For $\varepsilon = 0$ the only resonant monomials that will appear in the formal normal form of X_ε are of the form $x(x^p y^q)^k \frac{\partial}{\partial x}$ and $y(x^p y^q)^k \frac{\partial}{\partial y}$ for $k \in \mathbb{N} \setminus \{0\}$; this means that all other monomials are non-resonant and thus can be eliminated formally.

The following result will be useful in order to know which monomials are resonant.

Proposition 2.1 *There exists a constant $K > 0$ depending on p and q such that for all small $\varepsilon \neq 0$ the resonant monomials of X_ε are all of order $> \frac{K}{|\varepsilon|}$.*

Proof: The linear part of X_ε is given by $\alpha x \frac{\partial}{\partial x} + \left(\varepsilon - \frac{p}{q}\right) \alpha y \frac{\partial}{\partial y}$, so the resonance equations are

$$r_1 + \left(\varepsilon - \frac{p}{q}\right) r_2 \quad = \quad 1, \tag{2.4}$$

$$r_1 + \left(\varepsilon - \frac{p}{q}\right) r_2 \quad = \quad \varepsilon - \frac{p}{q}, \tag{2.5}$$

where $r_1, r_2 \in \mathbb{N}$ and

$$r_1 + r_2 \geq 2. \tag{2.6}$$

If $\varepsilon \notin \mathbb{Q}$, then there aren't any solutions for (2.4) and (2.5), so we take $\varepsilon \in \mathbb{Q}$ and ε close to zero. This gives us $\varepsilon = \frac{m}{n}$ where $n \in \mathbb{N} \setminus \{0\}$, $m \in \mathbb{Z}$ and $|m| < n$.

Now (2.4) becomes

$$r_1 + \left(\frac{mq - np}{nq}\right) r_2 = 1.$$

So necessarily we have $r_2 = kQ$ where $k \in \mathbb{N} \setminus \{0\}$ and $Q = \frac{nq}{\gamma}$ and $\gamma = \gcd(mq - np, nq)$. Substituting this in (2.4) we have

$$r_1 + kP = 1, \tag{2.7}$$

where $P = \frac{mq-np}{\gamma} \in \mathbb{Z}^- \setminus \{0\}$. So (2.7) gives us $r_1 = 1 - Pk$.

There is resonance iff $r_1 \geq 0$, $r_2 \geq 0$ and $r_1 + r_2 \geq 2$. The first two conditions are trivially fulfilled, the third condition is equivalent with

$$\begin{aligned} 1 + (Q - P)k \geq 2 \quad &\Leftrightarrow \quad (Q - P)k \geq 1 \\ &\Leftrightarrow \quad k \geq \frac{1}{Q - P} \\ &\Leftrightarrow \quad k \geq 1, \end{aligned}$$

where we used that $Q-P>1$ and $k\in\mathbb{N}$. So the solutions of (2.4) are

$$\left\{\left(1-k\frac{mq-np}{\gamma},k\frac{nq}{\gamma}\right)\middle|\,k\in\mathbb{N}\setminus\{0\}\right\}.$$

The solutions of (2.5) are found in the same way:

$$\left\{\left(-k\frac{mq-n)}{\gamma},1+k\frac{nq}{\gamma}\right)\middle|\,k\in\mathbb{N}\setminus\{0\}\right\}.$$

So the formal normal form of X_ε is

$$\left\{\begin{array}{rcl}\dot{x} & = & \alpha x+x\sum\limits_{k\geq 1}x^{\frac{np-mq}{\gamma}}y^{k\frac{nq}{\gamma}}\\ \dot{y} & = & \alpha\left(\varepsilon-\frac{p}{q}\right)y+y\sum\limits_{k\geq 1}x^{k\frac{np-mq}{\gamma}}y^{k\frac{nq}{\gamma}}.\end{array}\right. \tag{2.8}$$

We denote the lowest degree of a resonant monomial of (2.8) by

$$\Delta\left(1,-\frac{p}{q},\varepsilon\right)=1+\frac{np-mq+nq}{\gamma},$$

where $\varepsilon=\frac{m}{n}$.

First we consider the case where $q=1$.

Lemma 2.2

$$\Delta\left(1,-p,\varepsilon\right)\geq\frac{p+1}{|\varepsilon|}.$$

Proof: As $q=1$, we have that $\gamma=\gcd(m-np,n)=1$ because $\gcd(m,n)=1$. So $\Delta\left(1,-p,\varepsilon\right)=1+np-m+n$. Hence

$$\Delta\left(1,-p,\varepsilon\right)=1+np-m+n\geq\frac{p+1}{|\varepsilon|}=(p+1)\frac{n}{|m|}$$

iff

$$|m|m-((p+1)n+1)|m|+(p+1)n\leq 0. \tag{2.9}$$

It is clear that (2.9) describes a quadratic inequality in m which we solve separately in the case $m>0$ and $m<0$.

- Assume $m>0$. This means that $\varepsilon=\frac{m}{n}>0$. So (2.9) is reduced to

  $$m^2-((p+1)n+1)m+(p+1)n\leq 0. \tag{2.10}$$

  The solutions of this equation are $(p+1)n$ and 1. So (2.10) is fulfilled iff

  $$1\leq m\leq(p+1)n$$

  or equivalently

  $$\frac{1}{n}\leq\frac{m}{n}=\varepsilon\leq p+1.$$

  The latter inequalities are fulfilled as $0<\varepsilon<1$.

- Assume $m < 0$. In this case (2.9) reduces to

$$m^2 - ((p+1)n+1)m - (p+1)n \geq 0. \tag{2.11}$$

The equations of (2.11) are

$$m_\pm = \frac{((p+1)n+1) \pm \sqrt{((p+1)n+1)^2 + 4(p+1)n}}{2}.$$

Clearly $m_+ > 0$, so (2.11) is fulfilled if $m \leq m_-$. We now prove that $m_- \geq -1$, so by the fact that $m \in \mathbb{Z}^- \setminus \{0\}$ we can conclude that (2.11) is fulfilled.

$$\begin{aligned} m_- \geq -1 \quad &\Leftrightarrow \quad ((p+1)n+1) - \sqrt{((p+1)n+1)^2 + 4(p+1)n} \geq -2 \\ &\Leftrightarrow \quad ((p+1)n+1) + 2 \leq \sqrt{((p+1)n+1)^2 + 4(p+1)n} \\ &\Leftrightarrow \quad ((p+1)n+1)^2 + 4((p+1)n+1) + 4 \\ &\qquad \geq ((p+1)n+1)^2 + 4(p+1)n \\ &\Leftrightarrow \quad 8 \geq 0. \end{aligned}$$

As the latter inequality is obviously true, we can conclude the proof.

□

We use Lemma 2.2 in order to prove the crucial Lemma 2.3.

Lemma 2.3 *For sufficiently small ε we have that*

$$\Delta\left(1, -\frac{p}{q}, \varepsilon\right) \geq \frac{\frac{p}{q}}{q|\varepsilon|}.$$

PROOF: With ε sufficiently small, we mean that there exists $k \in \mathbb{N} \setminus \{0\}$ such that

$$-k < -\frac{p}{q} + \varepsilon < -k + 1.$$

If $q = 1$, then Lemma 2.2 gives us that

$$\Delta\left(1, -\frac{p}{q}, \varepsilon\right) \geq \frac{p+1}{|\varepsilon|},$$

hence

$$\Delta\left(1, -\frac{p}{q}, \varepsilon\right) \geq \frac{\frac{p}{1}}{1.|\varepsilon|}.$$

So we can assume that $q \geq 2$. Because $\frac{p}{q} \notin \mathbb{Z}$, we have

$$-k < -\frac{p}{q} < -k + 1.$$

Define the real number μ by

$$\mu = \frac{p}{q} - k,$$

then $\mu < 0$ and

$$-\frac{p}{q} + \varepsilon = -k - \mu + \varepsilon.$$

One observes that $\gamma = \gcd(q,n)^\rho$ where $\rho = 1$ or $\rho = 2$, so $\gamma \leq q^2$. To obtain this result we use the following arguments. We know that $\gamma = \gcd(qn, mq - np)$. So we write

$$\begin{aligned} p &= a_1 a_2 a_3, \\ q &= a_4 a_5 a_6, \\ m &= a_1 a_4 a_7, \\ n &= a_2 a_5 a_8, \end{aligned}$$

where

$$\gcd(a_i, a_j) = \begin{cases} 1 & \text{if} \quad i \neq j \\ a_i & \text{if} \quad i = j \end{cases}.$$

So we have that

$$\begin{aligned} qn &= a_2 a_4 a_5^2 a_6 a_8, \\ mq - pn &= a_1 a_5 (a_4^2 a_6 a_7 - a_2^2 a_3 a_8). \end{aligned}$$

Hence

$$\gcd(qn, mq - np) = \begin{cases} a_5^2 & \text{if} \qquad a_5 | \gcd\left(\frac{qn}{a_5}, \frac{mq-pn}{a_5}\right) \\ a_5 & \text{otherwise} \end{cases}.$$

We now have:

$$\begin{aligned} \Delta\left(1, -\frac{p}{q}, \varepsilon\right) &= \Delta\left(1, -k, \varepsilon - \mu\right) \\ &= 1 + Q(k + 1 - \varepsilon + \mu) \\ &= 1 + Q(k - \varepsilon) + Q(1 + \mu) \\ &\geq 1 + \frac{nq}{\gamma}(k - \varepsilon) \\ &\geq 1 + \frac{n}{q}(k - \varepsilon) \\ &\geq \frac{1 + n(k - \varepsilon)}{q} \\ &= \frac{1 + n(k - 1 + 1 - \varepsilon)}{q} \\ &= \frac{\Delta\left(1, k - 1, -\varepsilon\right)}{q}, \end{aligned}$$

hence by virtue of Lemma 2.2 we have

$$\Delta\left(1, -\frac{p}{q}, \varepsilon\right) \geq \frac{k}{q|\varepsilon|}.$$

As $k > \frac{p}{q}$ we have that

$$\Delta\left(1, -\frac{p}{q}, \varepsilon\right) \geq \frac{p}{q^2|\varepsilon|}.$$

□

We now finish the proof of Proposition 2.1. Choosing

$$K := \frac{p}{q^2},$$

the statement follows directly from Lemma 2.3. □

In the first stage we prefer not to eliminate the "near" resonant monomials of low degree. More precisely: we fix a small $\varepsilon_0 > 0$ and let N be the integer part of $\frac{K}{\varepsilon_0}$. Using the results from Subsection 1.3.4 there exists a C^∞ change of coordinates such that X_ε takes the following form:

$$X_\varepsilon : \left\{ \begin{array}{lcl} \dot{x} & = & \alpha x(1 + P_\varepsilon(x^p y^q)) + \mathcal{O}(|(x,y)|^{N+1}) \\ \dot{y} & = & \alpha\left(\varepsilon - \frac{p}{q}\right) y(1 + Q_\varepsilon(x^p y^q)) + \mathcal{O}(|(x,y)|^{N+1}) \end{array} \right. , \tag{2.12}$$

where P_ε and Q_ε are polynomials of degree at most $\frac{N}{p+q}$.

As the topic of this section is equivalences, we are allowed to divide X_ε by a strictly positive function. Using invariant manifolds we can and do assume that $\{x = 0\}$ and $\{y = 0\}$ are invariant. Hence we can start from a local family of the form

$$X_\varepsilon : \left\{ \begin{array}{lcl} \dot{x} & = & x(1 + P_\varepsilon(x^p y^q) + R_\varepsilon(x,y)) \\ \dot{y} & = & \left(\varepsilon - \frac{p}{q}\right) y \end{array} \right. \tag{2.13}$$

where $R_\varepsilon(x,y) = \mathcal{O}(|(x,y)|^{N+1})$. By virtue of Theorem 1.18 or [Bon97] we know that for a given integer $k > 0$ and for N large enough (depending on k, p, q), i.e. ε_0 sufficiently small, there exists a C^k change of variables eliminating R_ε.

Compensators

Before we start with the computations of the equivalences, we introduce the variable

$$\omega_1 := \omega(|x|^{-q}, -\varepsilon) = \frac{1 - |x|^{-q\varepsilon}}{\varepsilon} \tag{2.14}$$

where ω is the Ecalle–Roussarie compensator defined in (1.43). Of course ω_1 depends on q and ε but for brevity this dependence is surpressed in the notation. A direct calculation gives us

$$\frac{\partial \omega_1}{\partial x} = q\frac{1 - \omega_1 \varepsilon}{x} \tag{2.15}$$

and also

$$\lim_{\varepsilon\to 0}\omega_1 = q\ln|x|, \tag{2.16}$$

$$\omega_1(x_1x_2) = \omega_1(x_1)+\omega_1(x_2)-\varepsilon\omega_1(x_1)\omega_1(x_2) \tag{2.17}$$

In the changes of variables in the sequel we will use monomials of the form $(x^py^q)^N\omega_1^j$; we define a partial ordering $\prec$ on them by putting

$$(x^py^q)^n\omega_1^j \prec (x^py^q)^m\omega_1^k \tag{2.18}$$

iff

$$n<m \text{ or } (n=m \text{ and } j>k). \tag{2.19}$$

We will say that the monomial on the left-hand side of (2.18) is lower order than the right-hand side.

Proposition 2.4 *If* $(x^py^q)^n\omega_1^j \prec (x^py^q)^m\omega_1^k$ *then for* x,y,ε *sufficiently small one has*

$$\left|(x^py^q)^n\omega_1^j\right| \geq \left|(x^py^q)^m\omega_1^k\right|,$$

i.e. lower order terms are "more important". For all $\eta>0$ *one also has*

$$(x^py^q)^n\omega_1^j = \mathcal{O}(|x|^{np-\eta})\cdot\mathcal{O}(|y|^{nq}).$$

PROOF: We start with the proof of the first statement. By the definition of $\prec$ there are 2 possiblities.

- $n<m$. In this case we know that $|y^{qn}|\geq|y^{qm}|$ as $|y|<1$. Also we have the following equivalent statements

$$\begin{aligned} |x|^{np}\left|\frac{1-|x|^{-q\varepsilon}}{\varepsilon}\right|^j &\geq |x|^{pm}\left|\frac{1-|x|^{-q\varepsilon}}{\varepsilon}\right|^k \\ &\Updownarrow \\ 1 &\geq |x|^{p(m-n)}\left|\frac{1-|x|^{-q\varepsilon}}{\varepsilon}\right|^{k-j} \\ &\Updownarrow \\ |\varepsilon|^{k-j} &\geq |x|^{p(m-n)}|1-|x|^{-q\varepsilon}|^{k-j}. \end{aligned} \tag{2.20}$$

  As $n<m$ we have that $m-n>0$ so for ε sufficiently close to zero we have

$$p(m-n)-q\varepsilon(k-j)>0.$$

  Therefore the right-hand side of (2.20) converges to zero if $x\to 0$, therefore (2.20) is fulfilled for x sufficiently close to zero.

- $n = m$ and $j > k$. In this case we have the following equivalent statements

$$\begin{aligned}
|\omega_1(x)|^j &\geq |\omega_1(x)|^k \\
&\Updownarrow \\
\left|\frac{1-|x|^{-q\varepsilon}}{\varepsilon}\right|^j &\geq \left|\frac{1-|x|^{-q\varepsilon}}{\varepsilon}\right|^k \\
&\Updownarrow \\
\left|\frac{1-|x|^{-q\varepsilon}}{\varepsilon}\right|^{j-k} &\geq 1 \\
&\Updownarrow \\
|1-|x|^{-q\varepsilon}|^{j-k} &\geq |\varepsilon|^{j-k}. \qquad (2.21)
\end{aligned}$$

  As the left-hand side of (2.21) converges to 1 (for $\varepsilon < 0$) or $+\infty$ (for $\varepsilon > 0$), we know that (2.21) always will be fulfilled.

We now come to the proof of the second statement. As for $\varepsilon = 0$ we have that $\omega_1(x) = q \ln x$, the statement is true as $\ln x = \mathcal{O}(x^{-\eta})$ for any small $\eta > 0$ if $x \downarrow 0$. By continuity with respect to ε, we can conclude that the second statement remains true for ε sufficiently close to the origin. □

2.2.2 Computation of the equivalences

By virtue of (2.13) we can start from

$$X_\varepsilon(x,y) = x\left(1 + f(\varepsilon)x^p y^q + g(\varepsilon, x^p y^q)(x^p y^q)^2\right)\frac{\partial}{\partial x} + \left(\varepsilon - \frac{p}{q}\right) y \frac{\partial}{\partial y}. \qquad (2.22)$$

We begin by eliminating the first non-linear term using ω_1.

Lemma 2.5 *The system given by (2.22) is locally conjugate to*

$$\left\{\begin{array}{rcl} \dot{\xi} &=& \xi\left(1 + \tilde{g}(\varepsilon, \xi^p y^q \omega_1(\xi))\xi^{2p} y^{2q}\right) \\ \dot{y} &=& \left(\varepsilon - \frac{p}{q}\right) y \end{array}\right. \qquad (2.23)$$

by means of a change of variables of the form

$$x = \xi + \alpha(\varepsilon)\xi^{p+1}y^q + \beta(\varepsilon)\xi^{p+1}y^q\omega_1(\xi) \qquad (2.24)$$

where $\tilde{g}$ has the same smoothness as g.

Proof: Differentiating (2.24) with respect to t gives us

$$\begin{aligned}
\dot{x} &= \dot{\xi}\left[1 + ((p+1)\alpha(\varepsilon) + q\beta(\varepsilon))\xi^p y^q + (p+1-q\varepsilon)\beta(\varepsilon)\xi^p y^q \omega_1(\xi)\right] \\
&+ (q\varepsilon - p)\alpha(\varepsilon)\xi^{p+1}y^q + (q\varepsilon - p)\beta(\varepsilon)\xi^{p+1}y^q\omega_1(\xi). \qquad (2.25)
\end{aligned}$$

Substituting (2.24) into (2.22) gives us

$$\begin{aligned}\dot{x} &= \xi\left[1+(\alpha(\varepsilon)+f(\varepsilon))\xi^p y^q + \beta(\varepsilon)\xi^p y^w \omega_1(\xi)\right. \\ &+ \left.\bar{g}(\varepsilon,\xi^p y^q,\xi^p y^q\omega_1(\xi))(\xi^p y^q)^2 + \bar{\bar{g}}(\varepsilon,\xi^p y^q,\xi^p y^q\omega_1(\xi))(\xi^p y^q)^2\omega_1(\xi)\right].\end{aligned} \tag{2.26}$$

Equating (2.25) and (2.26) we obtain

$$\begin{aligned}\dot{x} &= \xi\frac{1+((p+1-q\varepsilon)\alpha(\varepsilon)+f(\varepsilon))\xi^p y^q + (p+1-q\varepsilon)\beta(\varepsilon)\xi^p y^q\omega_1(\xi)}{1+((p+1)\alpha(\varepsilon)+q\beta(\varepsilon))\xi^p y^q + (p+1-q\varepsilon)\beta(\varepsilon)\xi^p y^q\omega_1(\xi)} \\ &+ \hat{g}(\varepsilon,\xi^p y^q,\xi^p y^q\omega_1(\xi)).\end{aligned} \tag{2.27}$$

We will obtain (2.23) iff

$$\beta(\varepsilon) = -\varepsilon\alpha(\varepsilon) + \frac{1}{q}f(\varepsilon). \tag{2.28}$$

We see that there is freedom to choose, for example $\alpha(\varepsilon)$. Let us take $\alpha(\varepsilon)\equiv 0$, then $\beta = \frac{f(\varepsilon)}{q}$. This means that our transformation is implicitly given by

$$x = \xi + \frac{f(\varepsilon)}{q}\xi^{p+1}y^q\omega_1(\xi). \tag{2.29}$$

By virtue of the Inverse Function Theorem the map $(\xi,y,\varepsilon)\mapsto(x,y,\varepsilon)$ is invertible. □

Remark 2.1 *The limiting behaviour of the change of variables in (2.24) is, for* $\varepsilon\to 0$,

$$x = \xi + f(0)\xi^{p+1}y^q\ln|\xi|$$

which is a well known formula for eliminating resonant monomials.

In a next step we develop $\tilde{g}$ in (2.23), which gives the equation

$$\begin{aligned}\dot{\xi} &= \xi + \tilde{g}(\varepsilon,0,0)\xi^{2p+1}y^{2q} + g_1(\varepsilon,\xi^p y^q\omega_1(\xi))\xi^{3p+1}y^{3q} \\ &+ g_2(\varepsilon,\xi^p y^q,\xi^p y^q\omega_1(\xi))\xi^{3p+1}y^{3q}\omega_1(\xi)).\end{aligned}$$

Let us indicate now how to continue the elimination process in general. We must take in account that the monomials that we want to eliminate may contain ω_1.

Lemma 2.6 *The system*

$$\begin{cases}\dot{x} = x\left(1+\sum_{j=0}^{N-2} G_j(\varepsilon,x^p y^q,x^p y^q\omega_1(x))x^{Np}y^{Nq}\omega_1(x)^j\right) \\ \dot{y} = \left(\varepsilon-\frac{p}{q}\right)y\end{cases} \tag{2.30}$$

with $p, q \in \mathbb{N} \setminus \{0\}$ *and* $N \geq 2$ *is locally conjugate to the system*

$$\begin{cases} \dot{\xi} &= \xi \left(1 + \sum_{j=0}^{N-1} \tilde{G}_j(\varepsilon, \xi^p y^q, \xi^p y^q \omega_1(\xi)) \xi^{(N+1)p} y^{(N+1)q} \omega_1(\xi)^j\right) \\ \dot{y} &= \left(\varepsilon - \frac{p}{q}\right) y \end{cases} \tag{2.31}$$

by means of a change of variables of the form

$$x = \xi \left(1 + \sum_{j=0}^{N} \alpha_j(\varepsilon) \xi^{Np} y^{Nq} \omega_1(\xi)^j\right) \tag{2.32}$$

and where the functions $\alpha_0, \cdots, \alpha_N$ *are recursively calculated by the formula*

$$\alpha_{j+1}(\varepsilon) = \frac{j-N}{j+1} \varepsilon \alpha_j(\varepsilon) + \frac{G_j(\varepsilon, 0, 0)}{(j+1)q}, \tag{2.33}$$

$0 \leq j \leq N-1$. *We have the freedom to choose* $\alpha_0(\varepsilon)$. *The functions* $\tilde{G}_j$ *have the same smoothness as the* G_j*'s.*

Proof: Differentiating (2.32) with respect to t gives us

$$\begin{aligned} \dot{x} &= \dot{\xi} \left[1 + \sum_{j=0}^{N} (Np + 1 - qj\varepsilon) \alpha_j(\varepsilon) \xi^{Np} y^{Nq} \omega_1(\xi)^j \right. \\ &+ \left. \sum_{j=0}^{N-1} q(j+1) \alpha_{j+1}(\varepsilon) \xi^{Np} y^{Nq} \omega_1(\xi)^j \right] \\ &+ N(q\varepsilon - p) \sum_{j=0}^{N} \alpha_j(\varepsilon) \xi^{Np+1} y^{Nq} \omega_1(\xi)^j. \end{aligned} \tag{2.34}$$

Substituting (2.32) into (2.30) we obtain

$$\begin{aligned} \dot{x} &= \xi \left[1 + \sum_{j=0}^{N} \alpha_j(\varepsilon) (x^p y^q)^N \omega_1(\xi)^j + \sum_{j=0}^{N-1} g_j(\varepsilon) (\xi^p y^q)^N \omega_1(\xi)^j\right] \\ &+ \sum_{j=0}^{N-1} \hat{G}_j(\varepsilon, \xi^p y^q, \xi^p y^q \omega_1(\xi)^j) \xi^{(N+1)p+1} y^{(N+1)q} \omega_1(\xi)^j. \end{aligned} \tag{2.35}$$

Equating (2.34) and (2.35) we find

$$\begin{aligned} \dot{\xi} &= \xi \frac{1 + \sum_{j=0}^{N} ((1 + Np - Nq\varepsilon) \alpha_j(\varepsilon) + G_j(\varepsilon, 0, 0)) \xi^{Np+1} y^{Nq} \omega_1(\xi)^j}{\mathcal{N}} \\ &+ \sum_{j=0}^{N-1} \bar{G}_j(\varepsilon, \xi^p y^q, \xi^p y^q \omega_1(\xi)^q) \xi^{(N+1)p+1} y^{(N+1)q} \omega_1(\xi)^j, \end{aligned}$$

where

$$\begin{aligned}\mathcal{N} &= 1+\sum_{j=0}^{N-1}((1+Np-jq\varepsilon)\alpha_j(\varepsilon)+q(j+1)\alpha_{j+1}(\varepsilon))\xi^{Np+1}y^{Nq}\omega_1(\xi)^j \\ &+ (Np+1-Nq\varepsilon)\alpha_N(\varepsilon)\xi^{Np}y^{Nq}\omega_1(\xi)^N.\end{aligned}$$

The $\bar{G}_j$'s have the same smoothness as the G_j's and $G_N(\varepsilon,0,0)\equiv 0$. We will obtain (2.31) iff

$$\alpha_{j+1}(\varepsilon)=\frac{j-N}{j+1}\varepsilon\alpha_j(\varepsilon)+\frac{G_j(\varepsilon,0,0)}{(j+1)q}, \tag{2.36}$$

for $0\le j\le N-1$. Choosing $\alpha_0(\varepsilon)\equiv 0$, we can use (2.36) as a recursive relation and as a result we find that the α_j's are as smooth as the G_j's are in ε, so

$$\lim_{\varepsilon\to 0}\alpha_j(\varepsilon)\in\mathbb{R},$$

$\forall j\in\{0,1,\cdots,N\}$. This means that our transformation is implicitly given by

$$x=\xi+\sum_{j=1}^{N}\alpha_j(\varepsilon)\xi^{Np+1}y^{Nq}\omega_1(\xi)^j \tag{2.37}$$

where the α_j's are defined by (2.36) with initital condition $\alpha_0(\varepsilon)\equiv 0$. □

Remark 2.2 *The limiting behaviour of the change of variables in (2.32) is given by the transformation*

$$x=\xi+\sum_{j=1}^{N-1}\frac{1}{j}G_{j-1}(0,0,0)q^{j-1}\xi^{Np+1}y^{Nq}(\ln|\xi|)^j.$$

This is in contrast to the polynomial change of variables discussed in Subsection 1.3.4 which would be divergent for $\varepsilon\to 0$ in the case that $\frac{n}{m}=\frac{p}{q}$.

2.2.3 Conclusions

We start with a system as in (2.22)

$$\begin{cases}\dot{x} &= x\left(1+f_0(\varepsilon)x^py^q+g_0(\varepsilon,x^py^q)(x^py^q)^2\right)\\ \dot{y} &= \left(\varepsilon-\frac{p}{q}\right)y\end{cases}$$

If we apply Lemma 2.5 then we obtain the system

$$\begin{cases}\dot{x}_1 &= x_1\Big(1+f_1(\varepsilon)x_1^{2p}y^{2q}+g_1(\varepsilon,x_1^py^q,x_1^py^q\omega_1(x_1))x_1^{3p}y^{3q}\\ &+ g_2(\varepsilon,x_1^py^q,x_1^py^q\omega_1(x_1))x_1^{3p}y^{3q}\omega_1(x_1)\Big)\\ \dot{y} &= \left(\varepsilon-\frac{p}{q}\right)y\end{cases}.$$

By multiple application of Lemma 2.6 we obtain, given a positive integer $N \in \mathbb{N} \setminus \{0\}$, a system where all nonlinear terms are of equal or lower order than $\xi^{Np+1}y^{Nq}$ (if we name the final variable ξ):

$$\begin{cases} \dot{\xi} &= \xi\left(1+\sum_{j=0}^{N-1} \tilde{G}_j(\varepsilon, \xi^p y^q, \xi^p y^q \omega_1(\xi))\xi^{(N+1)p}y^{(N+1)q}\omega_1(\xi)^j\right) \\ \dot{y} &= \left(\varepsilon - \frac{p}{q}\right) y \end{cases}. \tag{2.38}$$

From Proposition 2.4 it follows that the nonlinear term in (2.38) is of order $\mathcal{O}(\xi^{(N+1)p+1-\eta}) \cdot \mathcal{O}(y^{(N+1)q})$ for any small $\eta > 0$. We want to eliminate this term, be it in a non-explicit way, by application of the methods in [Bon97, IY91]. For that purpose we need to know that this nonlinear term is sufficiently smooth. From elementary methods of calculus we obtain:

Lemma 2.7 *Let $K, L \in \mathbb{N} \setminus \{0\}$. The function*

$$\begin{aligned} g(\varepsilon, x) &:= x^K \omega_1(x)^L \text{ for } \varepsilon \neq 0 \\ g(0, x) &:= x^K (q \ln |x|)^L \end{aligned}$$

is of class C^{K-1} in the x variable for ε sufficiently small; moreover for all $k \in \{0, 1, \cdots, K-1\}$ one has

$$\lim_{\varepsilon \to 0} \frac{\partial^k}{\partial x^k} g(\varepsilon, x) = \frac{\partial^k}{\partial x^k} g(0, x).$$

Moreover, if the functions $\tilde{G}_j$ in the right-hand side of (2.38) are of class $C^{(N+1)p}$ then this nonlinear term in (2.38) is of class $C^{(N+1)p}$.

From Theorem 3.9 in [Bon97] it follows that, given $r \in \mathbb{N}$, if N is large enough in (2.38) then there exists a C^r diffeomorphism h defined on a small neighbourhood of the origin which conjugates system (2.38) to its linear part. Due to the specific expression at hand we can give more properties of h: it is of the form $(\xi, y) = h(\varepsilon, x, y) = (x + \mathcal{O}(y^{(N+1)q}), y)$ and can be obtained as follows. Denote

$$\begin{aligned} X(\xi, y) &= \xi\left(1+\sum_{j=0}^{N-1} \tilde{G}_j(\varepsilon, \xi^p y^q, \xi^p y^q \omega_1(\xi))\xi^{(N+1)p}y^{(N+1)q}\omega_1(\xi)^j\right)\frac{\partial}{\partial \xi} \\ &+ \left(\varepsilon - \frac{p}{q}\right) y \frac{\partial}{\partial y} \end{aligned}$$

and let Φ_t be the flow of X. If we multiply the nonlinear part of X with a cut-off function with a sufficiently small support, and rename the result again X, then we can take h of the form

$$h(x, y) = \left(\lim_{t \to +\infty} \pi_1 \circ \Phi_{-t}\left(e^t x, e^{\left(\varepsilon - \frac{p}{q}\right)t} y\right), y\right)$$

where $\pi_1 \circ \Phi_{-t}$ denotes the first component of the time $-t$ of the flow of X.

In order to fix some ideas, let us illustrate and check the method on a concrete simple example.

Example

Let

$$X(x,y) = x\left(1 + 3xy + x^2y^2\right)\frac{\partial}{\partial x} + (\varepsilon - 1)y\frac{\partial}{\partial y}. \tag{2.39}$$

According to (2.24) we put

$$\omega_1(x_1) = \frac{1 - x_1^{-\varepsilon}}{\varepsilon}$$

and

$$x = x_1 + 3x_1^2 y\omega_1(x_1) =: \varphi(x_1, y);$$

let X^1 denote the vector field in the new coordinates (x_1, y). We calculate X^1, using (2.15) and using for example a computer algebra package, and get

$$X^1(x_1, y) = (x_1 + 18x_1^3y^2\omega_1(x_1) + x_1^3y^2 + \cdots)\frac{\partial}{\partial x_1} + (\varepsilon - 1)y\frac{\partial}{\partial y}. \tag{2.40}$$

Next we apply the transformation according to (2.32) and (2.33) and put

$$x_1 = x_2 + x_2^3y^2\omega_1(x_2) + \left(9 - \frac{\varepsilon}{2}\right)x_2^3y^2\omega_1(x_2)^2 =: \varphi^1(x_2, y);$$

let X^2 denote the vector field written in the new coordinates (x_2, y). We get

$$\begin{aligned} X^2(x_2, y) &= x_2 - \Bigg(162 - 3\varepsilon^3 - 198\varepsilon + \frac{129}{2}\varepsilon^2 \\ &+ \frac{1}{64}(-32\varepsilon^2 - 864 + 624\varepsilon)(-6\varepsilon + 12)\Bigg)x_2^4y^3\omega_1(x_2)^3 \\ &+ \left(-21\varepsilon + 45 + 6\varepsilon^2 + \frac{1}{2}(2\varepsilon - 21)(-6\varepsilon + 12)\right)x_2^4y^3\omega_1(x_2)^2 \\ &+ (3\varepsilon - 51)x_2^4y^3\omega_1(x_2) - 3x_2^4y^3 \\ &+ \sum_{j=0}^{3} G_j(\varepsilon)x_2^5y^4\omega_1(x_2)^j. \end{aligned} \tag{2.41}$$

This can be done for example using the following simple Maple code.

```
> restart;
> readlib(mtaylor);
> X0:=x->x*(1+3*x*y+x^2*y^2);
```

```
> omega:=x1->(x1^(-eps)-1)/(-eps);
> phi:=(x1,y,om)->x1 +3*x1^2*y*om;
> omegax1:=(1-eps*om)/x1;
> phix1:=diff(phi(x1,y,om),x1)+diff(phi(x1,y,om),om)*omegax1;
> X11:=(X0(phi(x1,y,om))
-diff(phi(x1,y,om),y)*(eps-1)*y)/phix1;
> X12:=simplify(X11);
> mtaylor(X12,[x1,y,om]);
> X1:=unapply(X12,x1);
> phi1:=(x2,y,om)-> x2+x2^3*y^2*om
+ (9-(eps/2))*x2^3*y^2*om^2;
> omegax2:=subs(x1=x2,omegax1);
> phi1x2:=diff(phi1(x2,y,om),x2)
+diff(phi1(x2,y,om),om)*omegax2;
> X21:=(X1(phi1(x2,y,om))
-diff(phi1(x2,y,om),y)*(eps-1)*y)/phi1x2;
> X22:=simplify(X21);
> mtaylor(X22,[x2,y,om]);
```

Comments about this code. We have denoted:

```
X0 for the first component of the vector field in (2.39)
phi for the first component of the transformation
(x1,y) ↦ (x,y)
omegax1 for the derivative of omega with respect to x1
phix1 for the partial derivative of phi(x1,y,omega(x1)))
to x1
X1 for the first component of the pullback of X in (2.39)
phi1 for the first component of the transformation
(x2,y) ↦ (x1,y)
phi1x2 for the partial derivative of phi1(x2,y,omega(x2))
to x2
X21 for the first component of the pullback of (2.40)
```

2.3 Conjugacies between saddle vector fields

2.3.1 Settings and compensators

As in Section 2.2 we consider a local C^∞ deformation of a planar saddle singularity. By virtue of the hard work that was done in Subsection 2.2.1 we can start from (2.12). This means we have a system given by

$$X_\varepsilon : \left\{ \begin{array}{rcl} \dot{x} & = & \alpha x \left(1 + P_\varepsilon(x^p y^q)\right) + \mathcal{O}(|(x,y)|^{N+1}) \\ \dot{y} & = & \alpha \left(\varepsilon - \frac{p}{q}\right) y \left(1 + Q_\varepsilon(x^p y^q)\right) + \mathcal{O}(|(x,y)|^{N+1}) \end{array} \right. , \tag{2.42}$$

where P_ε and Q_ε are polynomials of degree at most $\frac{N}{p+q}$. Using invariant manifolds we can and do assume that $\{x=0\}$ and $\{y=0\}$ are invariant. Hence we can start from a local family of the form

$$X_\varepsilon : \begin{cases} \dot{x} &= x\,(1+P_\varepsilon(x^py^q)+R_\varepsilon(x,y)) \\ \dot{y} &= \left(\varepsilon-\frac{p}{q}\right)y\,(1+Q_\varepsilon(x^py^q)+S_\varepsilon(x,y)) \end{cases} \tag{2.43}$$

where $R_\varepsilon(x,y)=\mathcal{O}(|(x,y)|^{N+1})$ and $S_\varepsilon(x,y)=\mathcal{O}(|(x,y)|^{N+1})$. By virtue of Theorem 1.18 or [Bon97] we know that for a given integer $k>0$ and for N large enough (depending on k,p,q) there exists a C^k change of variables eliminating R_ε and S_ε.

Before we start the calculation of the conjugacies, we introduce the variables

$$\omega_1 := \omega(|x|^{-q},-\varepsilon) = \frac{|x|^{-q\varepsilon}-1}{-\varepsilon} \tag{2.44}$$

$$\omega_2 := \omega\left(|y|^{\frac{q}{\frac{p}{q}-\varepsilon}},-\varepsilon\right) = \frac{|y|^{\frac{q\varepsilon}{\frac{p}{q}-\varepsilon}}-1}{-\varepsilon} \tag{2.45}$$

where ω is the Ecalle-Roussarie compensator defined in (1.43). Of course ω_1 and ω_2 depend on ε, p and q but for brevity this dependence is surpressed in the notation. Then a direct calculation gives

$$\frac{\partial\omega_1}{\partial x} = \frac{q(1-\varepsilon\omega_1)}{x} \tag{2.46}$$

$$\frac{\partial\omega_2}{\partial y} = \frac{q}{\varepsilon-\frac{p}{q}}\frac{1-\varepsilon\omega_2}{y} \tag{2.47}$$

and also

$$\lim_{\varepsilon\to0}\omega_1 = q\ln|x| \tag{2.48}$$

$$\lim_{\varepsilon\to0}\omega_2 = -\frac{q^2}{p}\ln|y| \tag{2.49}$$

$$\omega_1(x_1x_2) = \omega_1(x_1)+\omega_2(x_2)-\varepsilon\omega_1(x_1)\omega_1(x_2) \tag{2.50}$$

$$\omega_2(x_1x_2) = \omega_2(x_1)+\omega_2(x_2)-\varepsilon\omega_2(x_1)\omega_2(x_2). \tag{2.51}$$

In the changes of variables in the sequel we will use monomials of the form $(x^py^q)^n\omega_1^j\omega_2^k$; we define a (partial) ordering $\prec$ on them by putting

$$(x^py^q)^n\omega_1^j\omega_2^k \prec (x^py^q)^m\omega_1^\ell\omega_2^s \tag{2.52}$$

iff

$$n<m \text{ or } (n=m \text{ and } j>l \text{ and } k\geq s) \text{ or } (n=m \text{ and } j\geq l \text{ and } k>s). \tag{2.53}$$

We will say that the monomial on the left-hand side of (2.52) is of lower order than the right-hand side.

The proof of the following Proposition is analogous to that of Proposition 2.4 and therefore omitted.

Proposition 2.8 *If* $(x^p y^q)^n \omega_1^j \omega_2^k \prec (x^p y^q)^m \omega_1^\ell \omega_2^s$ *then for* x,y, ε *sufficiently small one has*

$$|(x^p y^q)^n \omega_1^j \omega_2^k| \geq |(x^p y^q)^m \omega_1^\ell \omega_2^s|;$$

that is: lower order terms are "more important".

For all $\eta > 0$ *and* $\mu > 0$ *one also has*

$$(x^p y^q)^n \omega_1^j \omega_2^k = \mathcal{O}(|x|^{np-\eta})\mathcal{O}(|y|^{nq-\mu}).$$

2.3.2 Computation of the conjugacies

By virtue of (2.43) we can start from a system of the form

$$\begin{aligned} X_\varepsilon(x,y) &= \alpha x \left(1 + f_1(\varepsilon) x^p y^q + f_2(\varepsilon, x^p y^q)(x^p y^q)^2\right) \frac{\partial}{\partial x} \\ &+ \alpha \left(\varepsilon - \frac{p}{q}\right) y \left(1 + g_1(\varepsilon) x^p y^q + g_2(\varepsilon, x^p y^q)(x^p y^q)^2\right) \frac{\partial}{\partial y}. \end{aligned} \tag{2.54}$$

We begin by eliminating the first nonlinear term in the first component of X_ε using ω_1.

Lemma 2.9 *The system given by (2.54) is locally conjugate to the system*

$$\left\{ \begin{aligned} \dot{\xi} &= \alpha \xi \left(1 + \tilde{f}_0(\cdot) \xi^{2p} y^{2q} + \tilde{f}_1(\cdot)(\xi^p y^q)^2 \omega_1(\xi)\right) \\ \dot{y} &= \alpha \left(\varepsilon - \tfrac{p}{q}\right) y \left(1 + g_1(\varepsilon) \xi^p y^q + G_0(\cdot)(\xi^p y^q)^2 + G_1(\cdot)(\xi^p y^q)^2 \omega_1(\xi)\right) \end{aligned} \right. \tag{2.55}$$

where

$$\cdot = \varepsilon, \xi^p y^q, \xi^p y^q \omega_1(\xi)$$

by the transformation

$$x = \xi\left(1 + \beta_0(\varepsilon) \xi^p y^q + \beta_1(\varepsilon) \xi^p y^q \omega_1(\xi)\right) \tag{2.56}$$

where β_0 *and* β_1 *are smooth functions in* ε *and the* $\tilde{f}_j$ *are as smooth as* f_2 *and the* G_j *are as smooth as* g_2.

Proof: Substituting (2.56) in the second equation of (2.54) immediately gives us the second equation of (2.55). Differentiating (2.56) with respect to t gives us

$$
\begin{aligned}
\dot{x} &= \dot{\xi}\,(1+((p+1)\beta_0(\varepsilon)+q\beta_1(\varepsilon))\xi^p y^q+(p+1-q\varepsilon)\beta_1(\varepsilon)\xi^p y^q\omega_1(\xi)) \\
&+ \alpha(q\varepsilon-p)\xi\left(\beta_0(\varepsilon)\xi^p y^q+\beta_1(\varepsilon)\xi^p y^q\omega_1(\xi)+\sum_{j=0}^{1}\tilde{G}_j(\cdot)(\xi^p y^q)^2\omega_1(\xi)^j\right)
\end{aligned}
\tag{2.57}
$$

and substituting (2.56) into the right-hand side of the first equation of (2.54) gives us

$$
\begin{aligned}
\dot{x} &= \alpha\xi(1+(\beta_0(\varepsilon)+f_1(\varepsilon))\xi^p y^q+\beta_1(\varepsilon)\xi^p y^q\omega_1(\xi) \\
&+ \sum_{j=0}^{1}\check{G}_j(\cdot)(\xi^p y^q)^2\omega_1(\xi)^j).
\end{aligned}
\tag{2.58}
$$

Equating (2.57) and (2.58) reveals the following equality

$$
\begin{aligned}
\dot{\xi} &= \alpha\xi\left(\frac{1+(1+p-q\varepsilon)\beta_0(\varepsilon)+f_1(\varepsilon))\xi^p y^q}{1+((p+1)\beta_0(\varepsilon)+q\beta_1(\varepsilon))\xi^p y^q+(p+1-q\varepsilon)\beta_1(\varepsilon)\xi^p y^q\omega_1(\xi)}\right. \\
&+ \left.\frac{(1+p-q\varepsilon)\beta_1(\varepsilon)\xi^p y^q\omega_1(\xi)+\sum_{j=0}^{1}F_j(\cdot)(\xi^p y^q)^2\omega_1(\xi)^j}{1+((p+1)\beta_0(\varepsilon)+q\beta_1(\varepsilon))\xi^p y^q+(p+1-q\varepsilon)\beta_1(\varepsilon)\xi^p y^q\omega_1(\xi)}\right).
\end{aligned}
\tag{2.59}
$$

To obtain (2.55) β_0 and β_1 have to fulfill

$$
\beta_1(\varepsilon)+\varepsilon\beta_0(\varepsilon)=\frac{f_1(\varepsilon)}{q}. \tag{2.60}
$$

Choosing $\beta_0(\varepsilon)\equiv 0$, we have that $\beta_1(\varepsilon)=\frac{f_1(\varepsilon)}{q}$. This way β_0 and β_1 are smooth functions of ε. Therefore our transformation is given by

$$
x=\xi\left(1+\frac{f_1(\varepsilon)}{q}\xi^p y^q\omega_1(\xi)\right). \tag{2.61}
$$

By virtue of the Inverse Function Theorem the map $(\xi,y,\varepsilon)\mapsto(x,y,\varepsilon)$ is invertible. □

Remark 2.3 *We like to point out that the transformation given by (2.61) is actually the same transformation given by (2.29) we found in the case of equivalence.*

We proceed by eliminating the first nonlinear term in the second equation of (2.55). This time we will need both ω_1 and ω_2.

Lemma 2.10 *The system given by (2.55) is locally conjugate to the system*

$$\left\{ \begin{array}{rcl} \dot{\xi} & = & \alpha\xi\left(1+\sum_{j=0}^{1}\tilde{F}_j(\star)(\xi^p\eta^q)^2\omega_1(\xi)^j\right) \\ \dot{\eta} & = & \alpha\left(\varepsilon-\frac{p}{q}\right)\eta\left(1+\sum_{j=0}^{1}\sum_{k=0}^{1}G_{jk}(\star)(\xi^p\eta^q)^2\omega_1(\xi)^j\omega_2(\eta)^k\right) \end{array} \right. \tag{2.62}$$

where

$$\star = \varepsilon, \xi^p\eta^q, \xi^p\eta^q\omega_1(\xi), \xi^p\eta^q\omega_2(\eta)$$

by the transformation

$$y = \eta\left(1+\gamma_0(\varepsilon)\xi^p\eta^q+\gamma_1(\varepsilon)\xi^p\eta^q\omega_2(\eta)\right) \tag{2.63}$$

where γ_0 and γ_1 are smooth functions in ε and the functions $\tilde{F}_j$ are as smooth as the functions $\tilde{f}_j$ and the functions G_{jk} are as smooth as the functions G_j.

Proof: Substituting (2.63) in the first equation of (2.55) gives us immediately the first equation of (2.62). Differentiating (2.63) with respect to t gives us

$$\begin{array}{rcl} \dot{y} & = & \dot{\eta}\left[1+\left((q+1)\gamma_0(\varepsilon)+\frac{q}{\varepsilon-\frac{p}{q}}\gamma_1(\varepsilon)\right)\xi^p\eta^q\right. \\ & + & \left.\left(q+1-\frac{q\varepsilon}{\varepsilon-\frac{p}{q}}\right)\gamma_1(\varepsilon)\xi^p\eta^q\omega_2(\eta)\right] \\ & + & \alpha p\eta\left[\gamma_0(\varepsilon)\xi^p\eta^q+\gamma_1(\varepsilon)\xi^p\eta^q\omega_2(\eta)\right. \\ & + & \left.\sum_{j=0}^{2}\sum_{k=0}^{1}\bar{g}_{jk}(\star)(\xi^p\eta^q)^3\omega_1(\xi)^j\omega_2(\eta)^k\right]. \end{array} \tag{2.64}$$

Substituting (2.63) in the right-hand side of the second equation of (2.55) gives us

$$\begin{array}{rcl} \dot{y} & = & \alpha\left(\varepsilon-\frac{p}{q}\right)\eta\left[1+\left(\gamma_0(\varepsilon)+g_1(\varepsilon)\right)\xi^p\eta^q+\gamma_1(\varepsilon)\xi^p\eta^q\omega_2(\eta)\right. \\ & + & \left.\sum_{j=0}^{1}\sum_{k=0}^{1}g_{jk}(\star)(\xi^p\eta^q)^2\omega_1(\xi)^j\omega_2(\eta)^k\right], \end{array} \tag{2.65}$$

where $g_{11}(\star)\equiv 0$. Equating (2.64) and (2.65) gives us

$$\begin{array}{rcl} \dot{\xi} & = & \alpha\left(\varepsilon-\frac{p}{q}\right)\eta\frac{1+\left(\left(1-\frac{p}{\varepsilon-\frac{p}{q}}\right)\gamma_0(\varepsilon)+g_1(\varepsilon)\right)\xi^p\eta^q}{\mathcal{N}} \\ & + & \frac{\left(1-\frac{p}{\varepsilon-\frac{p}{q}}\right)\gamma_1(\varepsilon)\xi^p\eta^q\omega_2(\eta)+\sum_{j=0}^{1}\sum_{k=0}^{1}\tilde{g}_{jk}(\star)(\xi^p\eta^q)^2\omega_1(\xi)^j\omega_2(\eta)^k}{\mathcal{N}} \end{array}$$

with

$$\begin{aligned}\mathcal{N} &= 1 + \left((q+1)\gamma_0(\varepsilon) + \frac{p}{\varepsilon - \frac{p}{q}} \gamma_1(\varepsilon) \right) \xi^p \eta^q \\ &+ \left(q + 1 - \frac{q\varepsilon}{\varepsilon - \frac{p}{q}} \right) \gamma_1(\varepsilon) \xi^p \eta^q \omega_2(\eta).\end{aligned}$$

To obtain (2.62) γ_0 and γ_1 have to fulfill

$$\varepsilon\gamma_0(\varepsilon) + \gamma_1(\varepsilon) = \left(\varepsilon - \frac{p}{q} \right) \frac{g_1(\varepsilon)}{q}. \tag{2.66}$$

We have the freedom to choose $\gamma_0(\varepsilon) \equiv 0$, so we have

$$\gamma_1(\varepsilon) = \left(\varepsilon - \frac{p}{q} \right) \frac{g_1(\varepsilon)}{q}$$

which is a smooth function in ε. This means that our transformation is implicitly given by

$$y = \eta \left(1 + \left(\varepsilon - \frac{p}{q} \right) \frac{g_1(\varepsilon)}{q} \xi^p \eta^q \omega_2(\eta) \right). \tag{2.67}$$

By virtue of the Inverse Function Theorem the map $(\xi, \eta, \varepsilon) \mapsto (\xi, y, \varepsilon)$ is invertible. □

We now develop $\tilde{F}_j$ and G_{jk} in (2.62), this gives us the following system

$$\left\{ \begin{aligned} \dot{\xi} &= \alpha\xi \left(1 + \sum_{j=0}^{1} \bar{f}_j(\varepsilon) (\xi^p\eta^q)^2 \omega_1(\xi)^j \right. \\ &\left. + \sum_{j=0}^{2}\sum_{k=0}^{1} \hat{f}_{jk}(\star)(\xi^p\eta^q)^3 \omega_1(\xi)^j \omega_2(\eta)^k \right) \\ \dot{\eta} &= \alpha \left(\varepsilon - \tfrac{p}{q} \right) \eta \left(1 + \sum_{j=0}^{1}\sum_{k=0}^{1} \bar{g}_{jk}(\varepsilon)(\xi^p\eta^q)^2 \omega_1(\xi)^j \omega_2(\eta)^k \right. \\ &\left. + \sum_{j=0}^{2}\sum_{k=0}^{2} \hat{g}_{jk}(\star)(\xi^p\eta^q)^3 \omega_1(\xi)^j \omega_2(\eta)^k \right) \end{aligned} \right. \tag{2.68}$$

where $\hat{f}_{21}(\star) \equiv 0$, $\bar{g}_{11}(\varepsilon) \equiv 0$ and $\hat{g}_{jk}(\star) \equiv 0$ if $j + k \geq 3$.

In order to avoid problems in the inductive part of the proof, we state the second step of our elimination method seperately. As the proofs are analogous to the previous ones we omit them.

Lemma 2.11 *The system*

$$\left\{\begin{array}{rcl}\dot{x} & = & \alpha x\left(1+\displaystyle\sum_{j=0}^{1} f_j(\varepsilon)(x^p y^q)^2\omega_1(x)^j\right.\\ & + & \left.\displaystyle\sum_{j=0}^{2}\sum_{k=0}^{1} f_{jk}(\cdot)(x^p y^q)^3\omega_1(x)^j\omega_2(y)^k\right)\\ \dot{y} & = & \alpha\left(\varepsilon-\frac{p}{q}\right)y\left(1+\displaystyle\sum_{j=0}^{1}\sum_{k=0}^{1} g_{jk}(\varepsilon)(x^p y^q)^2\omega_1(x)^j\omega_2(y)^k\right.\\ & + & \left.\displaystyle\sum_{j=0}^{2}\sum_{k=0}^{2}\tilde{g}_{jk}(\cdot)(x^p y^q)^3\omega_1(x)^j\omega_2(y)^k\right)\end{array}\right. \tag{2.69}$$

with

$$\cdot = \varepsilon, x^p y^q, x^p y^q\omega_1(x), x^p y^q\omega_2(y)$$

and where $f_{21}(\cdot)\equiv 0$*,* $g_{11}(\varepsilon)\equiv 0$ *and* $\tilde{g}_{jk}(\cdot)\equiv 0$ *if* $j+k\geq 3$*, is locally conjugate to the system*

$$\left\{\begin{array}{rcl}\dot{\xi} & = & \alpha\xi\left(1+\displaystyle\sum_{j=0}^{2}\sum_{k=0}^{1} F_{jk}(\star)(\xi^p y^q)^3\omega_1(\xi)^j\omega_2(y)^k\right)\\ \dot{y} & = & \alpha\left(\varepsilon-\frac{p}{q}\right)y\left(1+\displaystyle\sum_{j=0}^{1}\sum_{k=0}^{1} g_{jk}(\varepsilon)(\xi^p y^q)^2\omega_1(\xi)^j\omega_2(y)^k\right.\\ & + & \left.\displaystyle\sum_{j=0}^{2}\sum_{k=0}^{2}\bar{g}_{jk}(\star)(\xi^p y^q)^3\omega_1(\xi)^j\omega_2(y)^k\right)\end{array}\right. \tag{2.70}$$

with

$$\star = \varepsilon, \xi^p y^q, \xi^p y^q\omega_1(\xi), \xi^p y^q\omega_2(y)$$

and where $F_{21}(\star)\equiv 0$*,* $g_{11}(\varepsilon)\equiv 0$ *and* $\bar{g}_{jk}\equiv 0$ *if* $j+k\geq 3$*, by the transformation*

$$x = \xi\left(1+\sum_{j=0}^{2}\beta_j(\varepsilon)\xi^{2p}y^{2q}\omega_1(\xi)^j\right) \tag{2.71}$$

where the β_j *are smooth functions in* ε *and the* F_{jk} *are as smooth as the* f_{jk} *and the* $\bar{g}_{jk}$ *are as smooth as the* $\tilde{g}_{jk}$*.*

To conclude the second step of the elimination process we need the following result.

Lemma 2.12 *The system given by (2.70) is locally conjugate to the system*

$$\left\{\begin{array}{rcl}\dot{\xi} & = & \alpha\xi\left[1+\sum_{j=0}^{2}\sum_{k=0}^{1}\bar{F}_{jk}(\diamond)(\xi^p\eta^q)^3\omega_1(\xi)^j\omega_2(\eta)^k\right] \\ \dot{\eta} & = & \alpha\left(\varepsilon-\frac{p}{q}\right)\eta\left[1+\sum_{j=0}^{2}\sum_{k=0}^{2}G_{jk}(\diamond)(\xi^p\eta^q)^3\omega_1(\xi)^j\omega_2(\eta)^k\right]\end{array}\right. \tag{2.72}$$

with

$$\diamond=\varepsilon,\xi^p\eta^q,\xi^p\eta^q\omega_1(\xi),\xi^p\eta^q\omega_2(\eta)$$

by the transformation

$$y=\eta\left(1+\sum_{j=0}^{1}\sum_{k=0}^{2}\gamma_{jk}(\varepsilon)\xi^{2p}\eta^{2q}\omega_1(\xi)^j\omega_2(\eta)^k\right) \tag{2.73}$$

where $\bar{F}_{21}(\diamond)\equiv 0$ *and* $G_{jk}(\diamond)\equiv 0$ *if* $j+k\geq 3$*. We also have that the functions* γ_{jk} *are smooth functions and the* $\bar{F}_{jk}$ *are as smooth as the* F_{jk} *and the* G_{jk} *are as smooth as the* $\bar{g}_{jk}$*.*

If we now apply Taylor's Theorem on (2.72) then we find

$$\left\{\begin{array}{rcl}\dot{\xi} & = & \alpha\xi\left(1+\sum_{j=0}^{2}\sum_{k=0}^{1}\hat{F}_{jk}(\varepsilon)(\xi^p\eta^q)^3\omega_1(\xi)^j\omega_2(\eta)^k\right. \\ & + & \left.\sum_{j=0}^{3}\sum_{k=0}^{2}\mathcal{F}_{jk}(\diamond)(\xi^p\eta^q)^4\omega_1(\xi)^j\omega_2(\eta)^k\right) \\ \dot{\eta} & = & \alpha\left(\varepsilon-\frac{p}{q}\right)\eta\left(1+\sum_{j=0}^{2}\sum_{k=0}^{2}\hat{G}_{jk}(\varepsilon)(\xi^p\eta^q)^3\omega_1(\xi)^j\omega_2(\eta)^k\right. \\ & + & \left.\sum_{j=0}^{3}\sum_{k=0}^{3}\mathcal{G}_{jk}(\diamond)(\xi^p\eta^q)^4\omega_1(\xi)\omega_2(\eta)\right)\end{array}\right.$$

where

- $\hat{F}_{21}(\varepsilon)\equiv 0$,
- $\hat{G}_{jk}(\varepsilon)\equiv 0$ if $j+k\geq 3$,
- $\mathcal{F}_{jk}(\diamond)\equiv 0$ if $j+k\geq 4$,
- $\mathcal{G}_{jk}(\diamond)\equiv 0$ if $j+k\geq 4$
- $\diamond=\varepsilon,\xi^p\eta^q,\xi^p\eta^q\omega_1(\xi),\xi^p\eta^q\omega_2(\eta)$.

We now come to the inductive part of the method.

Lemma 2.13 *For $N \geq 2$, the system given by*

$$\begin{cases} \dot{x} &= \alpha x \left(1 + \sum_{j=0}^{N} \sum_{k=0}^{N-1} f_{jk}(\varepsilon)(x^p y^q)^{N+1} \omega_1(x)^j \omega_2(y)^k \right. \\ &+ \left. \sum_{j=0}^{N+1} \sum_{k=0}^{N} \hat{f}_{jk}(\cdot)(x^p y^q)^{N+2} \omega_1(x)^j \omega_2(y)^k \right) \\ \dot{y} &= \alpha \left(\varepsilon - \frac{p}{q} \right) y \left(1 + \sum_{j=0}^{N} \sum_{k=0}^{N} g_{jk}(\varepsilon)(x^p y^q)^{N+1} \omega_1(x)^j \omega_2(y)^k \right. \\ &+ \left. \sum_{j=0}^{N+1} \sum_{k=0}^{N+1} \hat{g}_{jk}(\cdot)(x^p y^q)^{N+2} \omega_1(x)^j \omega_2(y)^k \right) \end{cases} \tag{2.74}$$

where

- $f_{jk}(\varepsilon) \equiv 0$ *if* $j + k \geq N + 1$
- $g_{jk}(\varepsilon) \equiv 0$ *if* $j + k \geq N + 1$
- $\hat{f}_{jk}(\cdot) \equiv 0$ *if* $j + k \geq N + 2$
- $\hat{g}_{jk}(\cdot) \equiv 0$ *if* $j + k \geq N + 2$
- $\cdot = \varepsilon, x^p y^q, x^p y^q \omega_1(x), x^p y^q \omega_2(y)$.

is locally conjugate to the system

$$\begin{cases} \dot{\xi} &= \alpha \xi \left(1 + \sum_{j=0}^{N+1} \sum_{k=0}^{N} \tilde{f}_{jk}(\star)(\xi^p y^q)^{N+2} \omega_1(\xi)^j \omega_2(y)^k \right) \\ \dot{y} &= \alpha \left(\varepsilon - \frac{p}{q} \right) y \left(1 + \sum_{j=0}^{N} \sum_{k=0}^{N} g_{jk}(\varepsilon)(\xi^p y^q)^{N+1} \omega_1(\xi)^j \omega_2(y)^k \right. \\ &+ \left. \sum_{j=0}^{N+1} \sum_{k=0}^{N+1} \tilde{g}_{jk}(\star)(\xi^p y^q)^{N+2} \omega_1(\xi)^j \omega_2(y)^k \right) \end{cases} \tag{2.75}$$

where

- $g_{jk}(\varepsilon) \equiv 0$ *if* $j + k \geq N + 1$
- $\tilde{f}_{jk}(\cdot) \equiv 0$ *if* $j + k \geq N + 2$
- $\tilde{g}_{jk}(\cdot) \equiv 0$ *if* $j + k \geq N + 2$

- $\star = \varepsilon, \xi^p y^q, \xi^p y^q \omega_1(\xi), \xi^p y^q \omega_2(y)$.

by a transformation of the form

$$x = \xi \left(1 + \sum_{j=0}^{N+1} \sum_{k=0}^{N-1} \beta_{jk}(\varepsilon) \xi^{(N+1)p} y^{(N+1)q} \omega_1(\xi)^j \omega_2(y)^k \right) \tag{2.76}$$

where the functions β_{jk} are smooth in ε. Also we have that the functions $\tilde{f}_{jk}$ have the same smoothness as the $\hat{f}_{jk}$ and the functions $\tilde{g}_{jk}$ have the same smoothness as the $\hat{g}_{jk}$.

Proof: Applying (2.76) to the second equation of (2.74) gives us immediately the second equation of (2.75). Differentiating (2.76) with respect to t gives us

$$\begin{aligned}
\dot{x} &= \dot{\xi} \left(1 + \sum_{j=0}^{N+1} \sum_{k=0}^{N-1} ((N+1)p + 1 - jq\varepsilon) \beta_{jk}(\varepsilon) (\xi^p y^q)^{N+1} \omega_1(\xi)^j \omega_2(y)^k \right. \\
&+ \left. \sum_{j=0}^{N} \sum_{k=0}^{N-1} (j+1) q \beta_{j+1,k}(\varepsilon) (\xi^p y^q)^{N+1} \omega_1(\xi)^j \omega_2(y)^k \right) \\
&+ \alpha \left(\varepsilon - \frac{p}{q} \right) \xi \left(\sum_{j=0}^{N+1} \sum_{k=0}^{N-2} \frac{(k+1)q}{\varepsilon - \frac{p}{q}} \beta_{j,k+1}(\varepsilon) (\xi^p y^q)^{N+1} \omega_1(\xi)^j \omega_2(y)^k \right. \\
&+ \sum_{j=0}^{N+1} \sum_{k=0}^{N-1} \left((N+1)q - \frac{kq\varepsilon}{\varepsilon - \frac{p}{q}} \right) \beta_{jk}(\varepsilon) (\xi^p y^q)^{N+1} \omega_1(\xi)^j \omega_2(y)^k \\
&+ \left. \sum_{j=0}^{N+1} \sum_{k=0}^{N} \bar{f}_{jk}(\star) (\xi^p y^q)^{N+1} \omega_1(\xi)^j \omega_2(y)^k \right).
\end{aligned} \tag{2.77}$$

Substituting (2.76) into the right-hand side of the first equation of (2.74) gives us

$$\begin{aligned}
\dot{x} &= \alpha\xi \left(1 + \sum_{j=0}^{N+1} \sum_{k=0}^{N-1} (f_{jk}(\varepsilon) + \beta_{jk}(\varepsilon)) (\xi^p y^q)^{N+1} \omega_1(\xi)^j \omega_2(y)^k \right. \\
&+ \left. \sum_{j=0}^{N+1} \sum_{k=0}^{N} \tilde{f}_{jk}(\star) (\xi^p y^q)^{N+1} \omega_1(\xi)^j \omega_2(y)^k \right),
\end{aligned} \tag{2.78}$$

where $f_{N+1,k}(\varepsilon) \equiv 0$ for all k. Equating (2.77) and (2.78) gives us

$$\dot{\xi} = \alpha\xi \frac{\mathcal{T}}{\mathcal{N}}$$

where

$$
\begin{aligned}
\mathcal{T} &= 1 - \sum_{j=0}^{N+1}\sum_{j=0}^{N-2}(k+1)q\beta_{j,k+1}(\varepsilon)(\xi^p y^q)^{N+1}\omega_1(\xi)^j\omega_2(y)^k \\
&+ \sum_{j=0}^{N+1}\sum_{k=0}^{N-1}(f_{jk}(\varepsilon)+(1+kq\varepsilon-(N+1)(q\varepsilon-p))\beta_{jk}(\varepsilon)) \\
&\quad (\xi^p y^q)^{N+1}\omega_1(\xi)^j\omega_2(y)^k \\
&+ \sum_{j=0}^{N+1}\sum_{k=0}^{N}\check{f}_{jk}(\star)(\xi^p y^q)^{N+2}\omega_1(\xi)^j\omega_2(y)^k,
\end{aligned}
$$

and

$$
\begin{aligned}
\mathcal{N} &= 1 + \sum_{j=0}^{N+1}\sum_{k=0}^{N-1}((N+1)p+1-jq\varepsilon)\beta_{jk}(\varepsilon)(\xi^p y^q)^{N+1}\omega_1(\xi)^j\omega_2(y)^k \\
&+ \sum_{j=0}^{N}\sum_{k=0}^{N-1}(j+1)q\beta_{j+1,k}(\varepsilon)(\xi^p y^q)^{N+1}\omega_1(\xi)^j\omega_2(y)^k.
\end{aligned}
$$

We will obtain system (2.75) iff $\beta_{jk}(\varepsilon) \equiv 0$ for $j+k \geq N+2$ and the system $\mathbf{AB} = \mathbf{F}$ has a solution for all $\varepsilon \sim 0$. Here

$$
\begin{aligned}
\mathbf{B} &= (\beta_{00}, \beta_{01}, \ldots, \beta_{0,N-1}, \beta_{10}, \beta_{11}, \ldots, \beta_{1,N-1}, \ldots, \beta_{N0}, \beta_{N1}), \\
\mathbf{F} &= (f_{00}, f_{01}, \ldots, f_{0,N-1}, f_{10}, f_{11}, \ldots, f_{1,N-1}, \ldots, f_{N0}, f_{N1})
\end{aligned}
$$

and

$$
\mathbf{A} = \left(\begin{array}{c|c|c|c|c|c}
A_0 & qI_N & 0 & \ldots & 0 & 0 \\ \hline
0 & A_1 & 2qI_N & \ldots & 0 & 0 \\ \hline
\vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ \hline
0 & 0 & 0 & \ddots & NqI_2 & 0 \\ \hline
0 & 0 & 0 & \ldots & A_N\ qI_1 & (N+1)qI_1
\end{array}\right)
$$

where I_k is the k-dimensional identity-matrix and A_m (for $m \neq 0$) is the following $m \times m$ -matrix

$$
A_m = \begin{pmatrix}
(N+1-m)q\varepsilon & q & 0 & \ldots & 0 \\
0 & (N-m)q\varepsilon & 2q & \ldots & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
0 & 0 & \ldots & 2q\varepsilon & (N-m)q \\
0 & 0 & \ldots & 0 & q\varepsilon
\end{pmatrix}
$$

and A_0 is the following $N \times N$-matrix

$$A_0 = \begin{pmatrix} (N+1)q\varepsilon & q & 0 & \dots & 0 \\ 0 & Nq\varepsilon & 2q & \dots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \dots & 3q\varepsilon & (N-1)q \\ 0 & 0 & \dots & 0 & 2q\varepsilon \end{pmatrix}.$$

This linear system has N degrees of freedom. We use this freedom to choose $\beta_{0k}(\varepsilon) \equiv 0$ for all $j = 0, 1, \cdots, N-1$. For $\varepsilon = 0$ the reduced linear system, i.e. without consideration of β_{0k}, is diagonal and has a non-zero determinant. So the system has a solution for $\varepsilon = 0$. As the coefficients of $\mathbf{A}$ and $\mathbf{F}$ are smooth with respect to ε we have by continuity that for ε sufficiently close to zero the determinant of the system will be non-zero as well. Hence there exists a solution for ε sufficiently close to zero.

By virtue of the Inverse Function Theorem the map $(\xi, y, \varepsilon) \mapsto (x, y, \varepsilon)$ is invertible. □

We now have reached the final step of this elimination process. As the proof is similar to the proof of Lemma 2.13 we omit this proof.

Lemma 2.14 *The system defined by (2.75) is locally conjugate to the system*

$$\left\{ \begin{array}{rcl} \dot{x} & = & \alpha\xi \left(1 + \sum_{j=0}^{N+1} \sum_{k=0}^{N} F_{jk}(\diamond)(\xi^p\eta^q)^{N+2}\omega_1(\xi)^j\omega_2(\eta)^k\right) \\ \dot{\eta} & = & \alpha\left(\varepsilon - \frac{p}{q}\right)\eta \\ & & \left(1 + \sum_{j=0}^{N+1} \sum_{k=0}^{N+1} G_{jk}(\diamond)(\xi^p\eta^q)^{N+2}\omega_1(\xi)^j\omega_2(\eta)^k\right) \end{array} \right. \qquad (2.79)$$

where

- $F_{jk}(\diamond) \equiv 0$ *if* $j + k \geq N + 2$
- $G_{jk}(\diamond) \equiv 0$ *if* $j + k \geq N + 2$
- $\diamond = \varepsilon, \xi^p\eta^q, \xi^p\eta^q\omega_1(\xi), \xi^p\eta^q\omega_2(\eta),$

by a transformation of the form

$$y = \eta \left(1 + \sum_{j=0}^{N} \sum_{k=0}^{N+1} \gamma_{jk}(\varepsilon)\xi^{(N+1)p}\eta^{(N+1)q}\omega_1(\xi)^j\omega_2(\eta)^k\right) \qquad (2.80)$$

where the functions γ_{jk} are smooth in ε. Also we have that the functions F_{jk} have the same smoothness as the functions $\tilde{f}_{jk}$ and the functions G_{jk} have the same smoothness as the functions $\tilde{g}_{jk}$.

2.3.3 Conclusions

We start with a system as in (2.54)

$$\begin{cases} \dot{x} &= \alpha x \left(1 + f_1(\varepsilon) x^p y^q + f_2(\varepsilon, x^p y^q)(x^p y^q)^2\right) \\ \dot{y} &= \alpha \left(\varepsilon - \frac{p}{q}\right) y \left(1 + g_1(\varepsilon) x^p y^q + g_2(\varepsilon, x^p y^q)(x^p y^q)^2\right). \end{cases}$$

Applying successively Lemma 2.9, Lemma 2.10, Lemma 2.11 and Lemma 2.12 we obtain the system

$$\begin{cases} \dot{x}_2 &= \alpha x_2 \left[1 + \sum_{j=0}^{2} \sum_{k=0}^{1} \bar{F}_{jk}(\cdot)(x_2^p y_2^q)^3 \omega_1(x_2)^j \omega_2(y_2)^k\right] \\ \dot{y}_2 &= \alpha \left(\varepsilon - \frac{p}{q}\right) y_2 \left[1 + \sum_{j=0}^{2} \sum_{k=0}^{2} \bar{G}_{jk}(\cdot)(x_2^p y_2^q)^3 \omega_1(x_2)^j \omega_2(y_2)^k\right] \end{cases}$$

where

$$\cdot = \varepsilon, x_2^p y_2^q, x_2^p y_2^q \omega_1(x_2), x_2 y_2 \omega_2(y_2).$$

By multiple successive application of Lemma 2.13 and Lemma 2.14 we obtain, given a positive integer $N \in \mathbb{N} \setminus \{0\}$, a system where all nonlinear terms are of equal order or lower order than $\xi^{Np+1}\eta^{Nq}$ in the first equation and of equal or lower order than $\xi^{Np}\eta^{Nq+1}$ in the second equation where we named the final variables (ξ, η). This system is given by

$$\begin{cases} \dot{\xi} &= \alpha \xi \left(1 + \sum_{j=0}^{N+1} \sum_{k=0}^{N} F_{jk}(\cdot)(\xi^p \eta^q)^{N+2} \omega_1(\xi)^j \omega_2(\eta)^k\right) \\ \dot{\eta} &= \alpha \left(\varepsilon - \frac{p}{q}\right) \eta \\ & \left(1 + \sum_{j=0}^{N+1} \sum_{k=0}^{N+1} G_{jk}(\cdot)(\xi^p \eta^q)^{N+2} \omega_1(\xi)^j \omega_2(\eta)^k\right), \end{cases} \tag{2.81}$$

where $F_{jk}(\cdot) \equiv 0$ if $j + k \geq N + 2$, $G_{jk}(\cdot) \equiv 0$ if $j + k \geq N + 2$ and

$$\cdot = \varepsilon, \xi^p \eta^q, \xi^p \eta^q \omega_1(\xi), \xi^p \eta^q \omega_2(\eta).$$

From Proposition 2.8 it follows that the non-linear term in the first equation of (2.81) is of order $\mathcal{O}(\xi^{(N+1)p+1-\nu}) \cdot \mathcal{O}(\eta^{(N+1)q-\mu})$ for any small $\nu > 0$ and $\mu > 0$ and the non-linear term in the second equation of (2.81) is of order $\mathcal{O}(\xi^{(N+1)p-\nu}) \cdot \mathcal{O}(\eta^{(N+1)q+1-\mu})$ for any small $\nu > 0$ and $\mu > 0$. We want to eliminate those terms, be it in a non-explicit way, by applying the methods in [Bon97, IY91]. For that purpose we need to know that these non-linear terms are sufficiently smooth. From elementary methods of calculus we obtain:

Lemma 2.15 *Let $K, L, M, N \in \mathbb{N} \setminus \{0\}$. The function*

$$\begin{aligned} g(\varepsilon, x, y) &:= x^K y^L \omega_1(x)^M \omega_2(y)^N \text{ for } \varepsilon \neq 0 \\ g(0, x, y) &:= x^K y^L (q \ln |x|)^M \left(-\frac{q^2}{p} \ln |y| \right)^N \end{aligned}$$

is of class C^{P-1} (where $P = \min(K, L)$) in (x, y) for ε sufficiently small; moreover for all $k, \ell \in \{0, 1, \cdots, P-1\}$ one has

$$\lim_{\varepsilon \to 0} \frac{\partial^{k+\ell}}{\partial x^k \partial y^\ell} g(\varepsilon, x, y) = \frac{\partial^{k+\ell}}{\partial x^k \partial y^\ell} g(0, x, y).$$

Moreover, if the functions F_{jk} and G_{jk} in the right-hand side of (2.81) are of class $C^{(N+1)p}$ then the nonlinear terms in (2.81) are of class $C^{(N+1)p}$.

From Lemma 1.19 and Theorem 3.9 in [Bon97] it follows that, given $r \in \mathbb{N}$, if N is large enough in (2.81) then there exists a C^r diffeomorphism h defined on a small neighbourhood of the origin which conjugates system (2.81) to its linear part. Let us denote the system given by (2.81) by X_ε and the linear part of X_ε by X_ε^0. Then X_ε equals X_ε^0 plus N-flat terms. By virtue of Lemma 1.19 we can write these N-flat terms as the sum $X_\varepsilon^s + X_\varepsilon^u$ where X_ε^s contains all terms that are N-flat in the stable direction and X_ε^u contains all terms that are N-flat in the unstable direction. Defining $Y_\varepsilon := X_\varepsilon^0 + X_\varepsilon^s$ and denoting the flow of X_ε, resp. Y_ε by Φ_t, resp. Ψ_t, Theorem 3.9 from [Bon97] gives us that $h = h^{(2)} \circ h^{(1)}$ where $h^{(1)}$ conjugates X_ε with Y_ε and $h^{(2)}$ conjugates Y_ε with X_ε^0. We also have that

$$\begin{aligned} h^{(1)} &:= \lim_{t \to +\infty} \Phi_{-t} \circ \Psi_t(x, y), \\ h^{(2)} &:= \lim_{t \to +\infty} \Psi_{-t} \left(e^t x, e^{\left(\varepsilon - \frac{p}{q}\right)t} y \right). \end{aligned}$$

2.4 C^∞ character of the obtained conjugacies

In Section 2.3 we calculated an explicit form of the conjugacies between local families of planar vector fields with saddle singularities. We found that the conjugacies are finitely smooth. We now want to investigate the smoothness of these conjugacies in terms of x, y and two extra variables z and w. These extra variables are - of course - inspired by the functions ω_1 and ω_2 that were introduced in Section 2.3.

2.4.1 Introduction of new variables

The variables z and w are defined by

$$z := x^p \omega_1(x) \quad (2.82)$$
$$w := y^q \omega_2(y) \quad (2.83)$$

and they have the following properties

$$\frac{\partial z}{\partial x} = \frac{(p-\varepsilon q)z + qx^p}{x} \tag{2.84}$$

$$\frac{\partial w}{\partial y} = \frac{\frac{-p}{\varepsilon-\frac{p}{q}}w + \frac{q}{\varepsilon-\frac{p}{q}}y^q}{y} \tag{2.85}$$

$$\lim_{x\to 0} z = 0 \tag{2.86}$$

$$\lim_{y\to 0} w = 0 \tag{2.87}$$

We define a degree-function which we will denote by $\mathcal{D}$:

$$\mathcal{D}\left(x^a y^b z^c w^d\right) = a + b + cp + dq, \tag{2.88}$$

for all $a, b, c, d \in \mathbb{N}$. Let us give an example:

$$\mathcal{D}\left(x^p y^q\right) = \mathcal{D}\left(y^q z\right) = \mathcal{D}\left(x^p w\right) = \mathcal{D}\left(zw\right) = p + q.$$

So the exponents of z and w are counted with a higher "weight" than those in x and y.

2.4.2 Reformulation of the results from Section 2.3

In Section 2.3 we calculated the transformations needed to eliminate the resonant terms of a vector field with a saddle singularity. We will now reformulate these results in terms of the variables (x, y, z, w). As we interpret z and w as ordinary variables, we will extend our system to a 4-dimensional one. This situation will be only temporary.

We want to indicate that all the transformations will be transformations on x or y, the transformations on z and w will be determined by (2.82) and (2.83).

As in (2.54) we start with the following system

$$\left\{\begin{array}{rcl} \dot{x} &=& \alpha x\left(1 + f_1(\varepsilon)x^p y^q + f_2(\varepsilon, x^p y^q)(x^p y^q)^2\right) \\ \dot{y} &=& \alpha\left(\varepsilon - \frac{p}{q}\right) y \left(1 + g_1(\varepsilon)x^p y^q + g_2(\varepsilon, x^p y^q)(x^p y^q)^2\right) \end{array}\right. . \tag{2.89}$$

Now Lemma 2.9 is reformulated as

Lemma 2.16 *The system*

$$\left\{\begin{array}{rcl} \dot{x} &=& \alpha x\left(1 + f_1(\varepsilon)x^p y^q + f_2(\varepsilon, x^p y^q)(x^p y^q)^2\right) \\ \dot{y} &=& \alpha\left(\varepsilon - \frac{p}{q}\right) y \left(1 + g_1(\varepsilon)x^p y^q + g_2(\varepsilon, x^p y^q)(x^p y^q)^2\right) \\ \dot{z} &=& \alpha\left((p-\varepsilon q)z + qx^p\right)\left(1 + f_1(\varepsilon)x^p y^q + f_2(\varepsilon, x^p y^q)(x^p y^q)^2\right) \\ \dot{w} &=& \alpha(-pw + qy^q)\left(1 + g_1(\varepsilon)x^p y^q + g_2(\varepsilon, x^p y^q)(x^p y^q)^2\right) \end{array}\right. \tag{2.90}$$

is locally conjugate with the system

$$\left\{\begin{array}{rcl} \dot{x}_1 & = & \alpha x_1 \left(1+\tilde{f}_0(\cdot)x_1^{2p}y^{2q}+\tilde{f}_1(\cdot)x_1^p y^{2q}z_1\right) \\ \dot{y} & = & \alpha\left(\varepsilon-\frac{p}{q}\right)y\left(1+g_1(\varepsilon)x_1^p y^q+G_0(\cdot)x_1^p y^{2q}+G_1(\cdot)x_1^p y^{2q}z_1\right) \\ \dot{z}_1 & = & \alpha\left((p-\varepsilon q)z_1+qx_1^p\right)\left(1+\tilde{f}_0(\cdot)x_1^{2p}y^{2q}+\tilde{f}_1(\cdot)x_1^p y^{2q}z_1\right) \\ \dot{w} & = & \alpha(-pw+qy^q)\left(1+g_1(\varepsilon)x_1^p y^q+G_0(\cdot)x_1^p y^{2q}+G_1(\cdot)x_1^p y^{2q}z_1\right) \end{array}\right. \tag{2.91}$$

where

$$\cdot=\varepsilon, x_1^p y^q, y^q z_1$$

by the transformation

$$x=x_1\left(1+\frac{f_1(\varepsilon)}{q}y^q z_1\right) \tag{2.92}$$

with

$$z_1=x_1^p\omega_1(x_1)$$

where the $\tilde{f}_j$ are as smooth as f_2 and the G_j are as smooth as g_2.

The transformation (2.92) is clearly C^∞ in (x_1, y, z_1, w). One might wonder if the transformation from z to z_1 is C^∞, but by some elementary calculations one deduces from the definition of z that

$$z=\left(1+\frac{f_1(\varepsilon)}{q}y^q z_1\right)^p\left(z_1+(x_1^p-\varepsilon z_1)f_1(\varepsilon)y^q z_1\Phi(y^q z_1)\right) \tag{2.93}$$

where Φ is some C^∞ function. Using the Inverse Function Theorem one concludes that the map $(x_1, y, z_1, w)\mapsto(x, y, z, w)$ is a C^∞ function.

In a similar way we can reformulate Lemma 2.10, Lemma 2.11 and Lemma 2.12. We will omit an explicit reformulation of these results and immediately move on to the inductive part that is given by Lemma 2.13 and Lemma 2.14. So after $2N$ steps we can assume that the system is given by

$$\left\{\begin{array}{rcl} \dot{x}_N & = & \alpha x_N\left(1+\sum_{j=0}^{N}\sum_{k=0}^{N-1}f_{jk}(\cdot)x_N^{(N+1-j)p}y_N^{(N+1-k)q}z_N^j w_N^k\right) \\ \dot{y}_N & = & \alpha\left(\varepsilon-\frac{p}{q}\right)y_N\left(1+\sum_{j=0}^{N}\sum_{k=0}^{N}g_{jk}(\cdot)x_N^{(N+1-j)p}y_N^{(N+1-k)q}z_N^j w_N^k\right) \\ \dot{z}_N & = & \alpha((p-\varepsilon q)z_N+qx_N^p) \\ & & \left(1+\sum_{j=0}^{N}\sum_{k=0}^{N-1}f_{jk}(\cdot)x_N^{(N+1-j)p}y_N^{(N+1-k)q}z_N^j w_N^k\right) \\ \dot{w}_N & = & \alpha(-pw_N+qy_N^q)\left(1+\sum_{j=0}^{N}\sum_{k=0}^{N}g_{jk}(\cdot)x_N^{(N+1-j)p}y_N^{(N+1-k)q}z_N^j w_N^k\right) \end{array}\right. \tag{2.94}$$

where $\cdot = \varepsilon, x_N^p y_N^q, y_N^q z_N, x_N^p w_N$.

The transformation given by Lemma 2.13 is now written as

$$x_N = x_{N+1}\left(1 + \sum_{j=1}^{N+1}\sum_{k=0}^{N-1}\beta_{jk}(\varepsilon)x_{N+1}^{(N+1-j)p}y_N^{(N+1-k)q}z_{N+1}^j w_N^k\right), \tag{2.95}$$

hence

$$x_{N+1} = x_N\left(1 - \sum_{j=1}^{N+1}\sum_{k=0}^{N-1}\beta_{jk}(\varepsilon)x_N^{(N+1-j)p}y_N^{(N+1-k)q}z_N^j w_N^k + \cdots\right) \tag{2.96}$$

using the definition given by (2.88) we see that the second term has $\mathcal{D}$-degree $(N+1)(p+q)$ and $\cdots$ denotes term with $\mathcal{D}$-degree at least $(N+2)(p+q)$. The link between z_N and z_{N+1} is now given by

$$\begin{aligned} z_{N+1} &= z_N - q\left(x_N^p + \left(\frac{p}{q} - \varepsilon\right)z_N\right) \\ &\quad \left(\sum_{j=0}^{N}\sum_{k=1}^{N+1}\beta_{jk}(\varepsilon)x_N^{(N+1-j)p}y_N^{(N+1-k)q}z_N^j w_N^k + \cdots\right) \end{aligned} \tag{2.97}$$

we see that the second term of the last factor has $\mathcal{D}$-degree $p + (N+1)(p+q)$ and $\cdots$ denotes terms with $\mathcal{D}$-degree at least $p + (N+2)(p+q)$.

The transformation given by Lemma 2.14 is rewritten as

$$y_N = y_{N+1}\left(1 + \sum_{j=0}^{N}\sum_{k=1}^{N+1}\gamma_{jk}(\varepsilon)x_{N+1}^{(N+1-j)p}y_{N+1}^{(N+1-k)q}z_{N+1}^j w_{N+1}^k\right), \tag{2.98}$$

hence

$$y_{N+1} = y_N\left(1 - \sum_{j=0}^{N}\sum_{k=1}^{N+1}\gamma_{jk}(\varepsilon)x_N^{(N+1-j)p}y_N^{(N+1-k)q}z_N^j w_N^k + \cdots\right) \tag{2.99}$$

where the second term has $\mathcal{D}$-degree $(N+1)(p+q)$, and

$$\begin{aligned} w_{N+1} &= w_N - \left(\frac{-p}{\varepsilon - \frac{p}{q}}w_N + \frac{q}{\varepsilon - \frac{p}{q}}y_N^q\right) \\ &\quad \left(\sum_{j=0}^{N}\sum_{k=1}^{N+1}\gamma_{jk}(\varepsilon)x_N^{(N+1-j)p}y_N^{(N+1-k)q}z_N^j w_N^k + \cdots\right). \end{aligned} \tag{2.100}$$

We see that the $\mathcal{D}$-degree of the first non-linear term equals $q + (N+1)(p+q)$.

2.4.3 Conclusions

Denoting the transformation in (2.96) by T_{N0} and the transformation in (2.99) by T_{N1}, then

$$T_N := T_{N1} \circ T_{N0}$$

is the transformation that maps (x_N, y_N, z_N, w_N) to $(x_{N+1}, y_{N+1}, z_{N+1}, w_{N+1})$. We now make the composition of the first N transformations

$$\mathcal{T}_N := T_N \circ T_{N-1} \circ \cdots \circ T_1.$$

We have that

$$\begin{aligned} T_N(x_N, y_N, z_N, w_N) \quad = \quad & (x_N + \mathcal{D}\left((N+1)(p+q)\right), \\ & y_N + \mathcal{D}\left((N+1)(p+q)\right), \\ & z_N + \mathcal{D}\left(p + (N+1)(p+q)\right), \\ & w_N + \mathcal{D}\left(q + (N+1)(p+q)\right)) \end{aligned}$$

so T_N won't change the coefficients of lower $\mathcal{D}$-degree in the Taylor expansion of $\mathcal{T}_{N-1}$.

This means that for $N \to \infty$ we will have a (not necessarily convergent) formal power series expansion in $(x_\infty, y_\infty, z_\infty, w_\infty)$. By virtue of Borel's Theorem - see for instance [Brö75, Nel70] - we know that there exists a function $\mathcal{T}$ which is C^∞ with respect to $(x_\infty, y_\infty, z_\infty, w_\infty)$ and which has the obtained formal power series expansion as Taylor series expansion. Applying this function $\mathcal{T}$ to (2.90) we obtain the system

$$\left\{ \begin{array}{rcl} \dot{x}_\infty & = & \alpha x_\infty \left(1 + H_1(\varepsilon, x_\infty^p, y_\infty^q, z_\infty, w_\infty)\right) \\ \dot{y}_\infty & = & \alpha \left(\varepsilon - \frac{p}{q}\right) y_\infty \left(1 + H_2(\varepsilon, x_\infty^p, y_\infty^q, z_\infty, w_\infty)\right) \\ \dot{z}_\infty & = & \alpha((p - \varepsilon q) z_\infty + q x_\infty^p) \left(1 + H_1(\varepsilon, x_\infty^p, y_\infty^q, z_\infty, w_\infty)\right) \\ \dot{w}_\infty & = & \alpha(-p w_\infty + q y_\infty^q) \left(1 + H_2(\varepsilon, x_\infty^p, y_\infty^q, z_\infty, w_\infty)\right) \end{array} \right. \tag{2.101}$$

where H_1 and H_2 are C^∞ and infinitely flat, i.e. we have that

$$\frac{\partial^{j_1+j_2+j_3+j_4} H_i}{\partial x_\infty^{j_1} y_\infty^{j_2} z_\infty^{j_3} w_\infty^{j_4}}(\varepsilon, x_\infty, y_\infty, z_\infty, w_\infty) = \mathcal{O}(|(x_\infty, y_\infty, z_\infty, w_\infty)|^n), \forall n \in \mathbb{N}$$

for $i = 1$ or 2 and for all $j_1, j_2, j_3, j_4 \in \mathbb{N}$.

From (2.101) we obtain the following equations (by division of the third equation by the first equation and the fourth equation by the second equation):

$$\frac{dz_\infty}{dx_\infty} = \frac{(p - \varepsilon q) z_\infty + q x_\infty^p}{x_\infty} \tag{2.102}$$

$$\frac{dw_\infty}{dy_\infty} = \frac{-p w_\infty + q y_\infty^q}{(\varepsilon - \frac{p}{q}) y_\infty} \tag{2.103}$$

As (2.102) is the same equation as (2.84) and (2.103) the same as (2.85) we have that $z_\infty = x_\infty^p \omega_1(x_\infty)$ and $w_\infty = y_\infty^q \omega_2(y_\infty)$ on condition we use the initial conditions $z(1) = 0$ and $w(1) = 0$.

It is clear that (2.101) is a hyperbolic system. In [IY91] it is proved that this type of systems meets all requirements of Proposition 1.22 in the case that the non-linear terms are infinitely flat. As (2.101) fulfills all these demands, by virtue of Proposition 1.22 we can conclude there exists some C^∞ function

$$\tilde{T} : (x_\infty, y_\infty, z_\infty, w_\infty) \mapsto (\tilde{x}, \tilde{y}, \tilde{z}, \tilde{w})$$

such that $\tilde{T}$ conjugates (2.101) with

$$\left\{ \begin{array}{rcl} \dot{\tilde{x}} & = & \alpha \tilde{x} \\ \dot{\tilde{y}} & = & \alpha \left(\varepsilon - \frac{p}{q} \right) \tilde{y} \\ \dot{\tilde{z}} & = & \alpha((p - \varepsilon q)\tilde{z} + q\tilde{x}^p) \\ \dot{\tilde{w}} & = & \alpha(-p\tilde{w} + q\tilde{y}^q) \end{array} \right. . \tag{2.104}$$

As we did for z_∞ and w_∞ in (2.102) and (2.103) we can apply the same arguments to conclude that $\tilde{z} = \tilde{x}^p \omega_1(\tilde{x})$ and $\tilde{w} = \tilde{y}^q \omega_2(\tilde{y})$.

It is obvious that the first two equations of (2.104) are independent of z and w. As we introduced these variables as temporary variables, we can delete them from the system. This means we have proved that the system given by (2.89) and

$$\left\{ \begin{array}{rcl} \dot{\tilde{x}} & = & \alpha \tilde{x} \\ \dot{\tilde{y}} & = & \alpha \left(\varepsilon - \frac{p}{q} \right) \tilde{y} \end{array} \right. \tag{2.105}$$

are conjugate by a function $\mathbf{T}$ where $\mathbb{T} : \mathbb{R}^4 \to \mathbb{R}^4$ is a C^∞ function in the variables $(x, y, x^p \omega_1(x), y^q \omega_2(y))$ and

$$\mathbb{T}(x, y, x^p \omega_1(x), y^q \omega_2(y))_i = \mathbf{T}(x, y, x^p \omega_1(x), y^q \omega_2(y))_i$$

for $i = 1, 2$.

2.5 Conjugacies between saddle type diffeomorphisms

2.5.1 Settings, nearly resonance and compensators

We consider a local C^∞ deformation F_λ of a multiplicatively resonant diffeomorphism on $\mathbb{R}^2$. By virtue of the results in Subsection 1.3.5 we can assume that $F_\lambda(0) = 0$ and for the linear part we have that

$$DF_\lambda(0) = \begin{pmatrix} \alpha(\lambda) & 0 \\ 0 & \beta(\lambda) \end{pmatrix}$$

where $\alpha(\lambda), \beta(\lambda) \in \mathbb{R}$, $|\alpha(\lambda)| > 1$ and $|\beta(\lambda)| < 1$. By virtue of Proposition 1.15 we know that the eigenvalues $\alpha(\lambda)$ and $\beta(\lambda)$ are C^∞ functions of λ. Proposition 1.15 and Proposition 1.16 assure us that DF_λ can be diagonalised by a C^∞ change of coordinates.

Let us consider the eigenvalues of the multiplicatively resonant diffeomorphism F_0. As in the 2-dimensional case multiplicatively resonance is the same as multiplicatively one-resonance, we know there exists $p, q \in \mathbb{N}$ with $p + q \geq 1$ such that

$$\alpha(0)^p \beta(0)^q = 1.$$

This means that $\beta(0)$ can be written as a function of $\alpha(0)$, but we have to be aware of the signs of the eigenvalues:

- if $\alpha(0)$ and $\beta(0)$ are positive, then $\beta(0) = \alpha(0)^{-\frac{p}{q}}$,
- if $\alpha(0) < 0$ and $\beta(0) > 0$, then $\beta(0) = (\alpha(0)^{-p})^{\frac{1}{q}} = |\alpha(0)|^{-\frac{p}{q}}$ because p will be even,
- if $\alpha(0) > 0$ and $\beta(0) < 0$, then $\beta(0) = -\alpha(0)^{-\frac{p}{q}}$ because q is even,
- if $\alpha(0) < 0$ and $\beta(0) < 0$, then $\beta(0) = -|\alpha(0)|^{-\frac{p}{q}}$ as p and q have the same parity.

We can summarise these 4 cases by

$$\beta(0) = \sigma_2 |\alpha(0)|^{-\frac{p}{q}},$$

where σ_2 is the sign of $\beta(0)$ and by σ_1 we denote the sign of $\alpha(0)$.

As $\alpha(0)^p \beta(0)^q = 1$ we always have that

$$\sigma_1^p \sigma_2^q = 1.$$

Using a submersion $\lambda \mapsto (\alpha(\lambda), \beta(\lambda))$ we can regard (α, β) as new parameters and omit λ in the notation. As we assume that the system is nearly multiplicatively resonant, we have that

$$\beta = \sigma_2 |\alpha|^{\varepsilon - \frac{p}{q}} \tag{2.106}$$

where ε is sufficiently close to 0 and $p, q \in \mathbb{N}$ with $p + q \geq 1$. Taking (2.106) in consideration it is natural to regard (α, ε) as parameters instead of (α, β) on condition we take ε close to zero. Therefore we can assume that

$$DF_{\alpha,\varepsilon}(0) = \begin{pmatrix} \alpha & 0 \\ 0 & \sigma_2 |\alpha|^{\varepsilon - \frac{p}{q}} \end{pmatrix}. \tag{2.107}$$

From now on we will consider α as a constant, hence we obtain a local deformation F_ε depending on the one-dimensional parameter $\varepsilon \in (\mathbb{R}, 0)$.

For $\varepsilon = 0$ the only resonant monomials that will appear in the formal normal form of F_ε are of the form

$$\left(x(x^p y^q)^k, y(x^p y^q)^k\right), \ k \in \mathbb{N} \setminus \{0\}.$$

This means that all other monomials are non-resonant and thus can be eliminated formally.

The following result will be useful in order to know which monomials are resonant for $\varepsilon \neq 0$.

Proposition 2.17 *There exists a constant $K > 0$ depending on p and q such that for all small $\varepsilon \neq 0$ the resonant monomials of F_ε are all of order $> \frac{K}{|\varepsilon|}$.*

Proof: Observing the resonance equation on the eigenvalues of DF_ε we have

$$\begin{aligned} \alpha^{r_1}\beta^{r_2} = 1 \quad &\Leftrightarrow \quad \alpha^{r_1}\left(\sigma_2|\alpha|^{\varepsilon - \frac{p}{q}}\right)^{r_2} = 1 \\ &\Leftrightarrow \quad |\alpha|^{r_1}|\alpha|^{\varepsilon r_2 - \frac{p}{q}r_2} = 1 \\ &\Leftrightarrow \quad r_1 + \varepsilon r_2 - \frac{p}{q}r_2 = 0 \\ &\Leftrightarrow \quad r_1 + \left(\varepsilon - \frac{p}{q}\right) r_2 = 0. \end{aligned} \tag{2.108}$$

It is clear that (2.5) and (2.6) are equivalent with (2.108), so the statement follows from Proposition 2.1. □

In the first stage we prefer not to eliminate the "near" resonant monomials of low degree. More precisely: we fix a small $\varepsilon_0 > 0$ and let N be the integer part of $\frac{K}{\varepsilon_0}$. Using the results from Subsection 1.3.5 there exists a C^∞ change of coordinates such that F_ε obtains the following form:

$$\begin{aligned} F_\varepsilon(x,y) \quad = \quad &\big(\alpha x(1 + P_\varepsilon(x^p y^q) + \mathcal{O}(|(x,y)|^{N+1}) \ , \\ &\sigma_2|\alpha|^{\varepsilon - \frac{p}{q}} y(1 + Q_\varepsilon(x^p y^q) + \mathcal{O}(|(x,y)|^{N+1})\big) \ , \end{aligned} \tag{2.109}$$

where P_ε and Q_ε are polynomials of degree at most $\frac{N}{p+q}$. Using invariant manifolds we can and do assume that $\{x = 0\}$ and $\{y = 0\}$ are invariant. Hence we can start from a local family of the form

$$\begin{aligned} F_\varepsilon(x,y) \quad = \quad \Big(\quad &\alpha x(1 + P_\varepsilon(x^p y^q) + R_\varepsilon(x,y) \ , \\ &\sigma_2|\alpha|^{\varepsilon - \frac{p}{q}} y(1 + Q_\varepsilon(x^p y^q) + S_\varepsilon(x,y)\Big) \ , \end{aligned} \tag{2.110}$$

where $R_\varepsilon = \mathcal{O}(|(x,y)|^{N+1})$ and $S_\varepsilon(x,y) = \mathcal{O}(|(x,y)|^{N+1})$. By virtue of Theorem 1.24 we know that for a given integer $k > 0$ and for N large enough (depending on k, p, q) there exists a C^k change of variables eliminating R_ε and S_ε.

During the calculations we use the functions ω_1 and ω_2 that were defined by (2.44) and (2.45). We give two properties of these functions that are useful when working with conjugacies between diffeomorphisms. The proof follows from a short straight-forward calculation.

$$\begin{aligned} \omega_1(\alpha x) &= \omega_1(\alpha) + (1-\varepsilon\omega_1(\alpha))\omega_1(x), & (2.111)\\ \omega_2(\pm|\alpha|^{\varepsilon-\frac{p}{q}}y) &= \omega_1(\alpha) + (1-\varepsilon\omega_1(\alpha))\omega_2(y). & (2.112)\end{aligned}$$

2.5.2 Computation of the conjugacies

By virtue of (2.112) we can start with

$$\begin{aligned} F_\varepsilon(x,y) = \Big(& \alpha x(1+f_0(\varepsilon)x^py^q + f_1(\varepsilon,x^py^q)(x^py^q)^2), \qquad (2.113)\\ & \sigma_2|\alpha|^{\varepsilon-\frac{p}{q}}y(1+g_0(\varepsilon)x^py^q + g_1(\varepsilon,x^py^q)(x^py^q)^2)\Big). \end{aligned}$$

We begin by eliminating the first non-linear term in the first component of F_ε using ω_1.

Lemma 2.18 *The system (2.113) is locally conjugate to the system*

$$\begin{aligned} G_\varepsilon(x,y) = & \left(\alpha x\left(1+\bar f_0(\cdot)(x^py^q)^2 + \bar f_1(\cdot)(x^py^q)^2\omega_1(x)\right),\right. \qquad (2.114)\\ & \sigma_2|\alpha|^{\varepsilon-\frac{p}{q}}y\,(1+g_0(\varepsilon)x^py^q+\\ & \left.\bar g_0(\cdot)(x^py^q)^2 + \bar g_1(\cdot)(x^py^q)^2\omega_1(x))\right) \end{aligned}$$

where

$$\cdot = \varepsilon, x^py^q, x^py^q\omega_1(x)$$

by the transformation

$$\phi_\varepsilon(x,y) = (x+\beta_0(\varepsilon)x(x^py^q) + \beta_1(\varepsilon)x(x^py^q)\omega_1(x), y) \qquad (2.115)$$

where β_0 and β_1 are smooth functions in ε and the $\bar f_j$ are as smooth as f_1 and the $\bar g_j$ are as smooth as g_1.

Proof: In order to obtain a conjugacy between F_ε and G_ε we need to prove that

$$F_\varepsilon\circ\phi_\varepsilon = \phi_\varepsilon\circ G_\varepsilon.$$

A short calculation and application of (2.111) and (2.112), gives us

$$\begin{aligned} F_\varepsilon(\phi_\varepsilon(x,y)) = & (\alpha(1+(f_0(\varepsilon)+\beta_0(\varepsilon))x^py^q + \beta_1(\varepsilon)x^py^q\omega_1(x)\\ & +\hat f_0(\cdot)(x^py^q)^2 + \hat f_1(\cdot)(x^py^q)^2\omega_1(x)),\\ & \sigma_2|\alpha|^{\varepsilon-\frac{p}{q}}y(1+g_0(\varepsilon)x^py^q\\ & +\hat g_0(\cdot)(x^py^q)^2 + \hat g_1(\cdot)(x^py^q)^2\omega_1(x))\big) \qquad (2.116)\end{aligned}$$

where $\cdot$ is taken as in the statement and

$$\begin{aligned}\phi_\varepsilon(G_\varepsilon(x,y)) &= \Big(\alpha x\Big(1+\frac{\beta_0(\varepsilon)+\omega_1(\alpha)\beta_1(\varepsilon)}{1-\varepsilon\omega_1(\alpha)}x^py^q+\beta_1(\varepsilon)x^py^q\omega_1(x) \\ &\quad +\check{f}_0(\cdot)(x^py^q)^2+\check{f}_1(\cdot)(x^py^q)^2\Big), \\ &\quad \sigma_2|\alpha|^{\varepsilon-\frac{p}{q}}y\,(1+g_0(\varepsilon)x^py^q \\ &\quad +\bar{g}_0(\cdot)(x^py^q)^2+\bar{g}_1(\cdot)(x^py^q)^2\omega_1(x))\Big). \qquad (2.117)\end{aligned}$$

We have that (2.116) and (2.117) are equal iff

$$f_0(\varepsilon)=\varepsilon\frac{\omega_1(\alpha)}{1-\varepsilon\omega_1(\alpha)}\beta_0(\varepsilon)+\frac{\omega_1(\alpha)}{1-\varepsilon\omega_1(\alpha)}\beta_1(\varepsilon). \qquad (2.118)$$

We have the freedom to choose $\beta_0(\varepsilon)\equiv 0$, so (2.118) gives us that

$$\beta_1(\varepsilon)=\frac{1-\varepsilon\omega_1(\alpha)}{\omega_1(\alpha)}.$$

Hence

$$\phi_\varepsilon(x,y)=\left(x+\frac{1-\varepsilon\omega_1(\alpha)}{\omega_1(\alpha)}f_0(\varepsilon)x(x^py^q)\omega_1(x),y\right). \qquad (2.119)$$

□

Remark 2.4 *Note that in the previous proof σ_2 vanishes in our calculations as $\alpha=\sigma_1|\alpha|$ and $\sigma_1^p\sigma_2^q=1$.*

We continue by eliminating the first non-linear term of the second component of G_ε. This time both ω_1 and ω_2 are involved in the conjugacy. We will omit the proof as it is analogous to the previous proof.

Lemma 2.19 *The system given by (2.114) is locally conjugate to the system*

$$\begin{aligned}H_\varepsilon(x,y) &= \left(\alpha x\left(1+\sum_{j=0}^{1}\tilde{f}_j(\star)(x^py^q)^2\omega_1(x)^j\right),\right. \qquad (2.120) \\ &\quad \left.\sigma_2|\alpha|^{\varepsilon-\frac{p}{q}}y\left(1+\sum_{j=0}^{1}\sum_{k=0}^{1}\tilde{g}_{jk}(\star)(x^py^q)^2\omega_1(x)^j\omega_2(y)^k\right)\right)\end{aligned}$$

with $\tilde{g}_{11}(\star)\equiv 0$ and where

$$\star=\varepsilon,x^py^q,x^py^q\omega_1(x),x^py^q\omega_2(y)$$

by the transformation

$$\psi_\varepsilon(x,y)=(x,y+\gamma_0(\varepsilon)y(x^py^q)+\gamma_1(\varepsilon)y(x^py^q)\omega_2(y)) \qquad (2.121)$$

where γ_0 and γ_1 are smooth functions in ε and the $\tilde{f}_j$ are as smooth as the $\bar{f}_j$ and the $\tilde{g}_{jk}$ are as smooth as the $\bar{g}_j$.

To avoid some technical problems with the inductive part of the proof we will give an explicit proof for the second step in this method. The proofs are similar to the previous ones and therefore omitted.

We now apply Taylor's Theorem to (2.120) and rename the function to F_ε. So we can work with

$$\begin{aligned} F_\varepsilon(x,y) \quad = \quad & \Bigg(\alpha x \Bigg(1 + \sum_{j=0}^{1} f_j(\varepsilon)(x^p y^q)^2 \omega_1(x)^j \\ & + \sum_{j=0}^{2}\sum_{k=0}^{1} \bar{f}_{jk}(\cdot)(x^p y^q)^3 \omega_1(x)^j \omega_2(y)^k \Bigg), \\ & \sigma_2 |\alpha|^{\varepsilon - \frac{p}{q}} y \Bigg(1 + \sum_{j=0}^{1}\sum_{k=0}^{1} g_{jk}(\varepsilon)(x^p y^q)^2 \omega_1(x)^j \omega_2(y)^k \\ & + \sum_{j=0}^{2}\sum_{k=0}^{2} \bar{g}_{jk}(\cdot)(x^p y^q)^3 \omega_1(x)^j \omega_2(y)^k \Bigg)\Bigg) \end{aligned} \tag{2.122}$$

where

$$\begin{aligned} \cdot \quad & = \quad \varepsilon, x^p y^q, x^p y^q \omega_1(x), x^p y^q \omega_2(y) \\ g_{11}(\varepsilon) \quad & \equiv \quad 0 \\ \bar{f}_{jk}(\cdot) \quad & \equiv \quad 0 \text{ if } j+k \geq 3 \\ \bar{g}_{jk}(\cdot) \quad & \equiv \quad 0 \text{ if } j+k \geq 3. \end{aligned}$$

We continue our elimination method with the following lemma.

Lemma 2.20 *The system defined by (2.122) is locally conjugate to*

$$\begin{aligned} G_\varepsilon(x,y) \quad = \quad & \Bigg(\alpha x \Bigg(1 + \sum_{j=0}^{2}\sum_{k=0}^{1} \tilde{f}_{jk}(\cdot)(x^p y^q)^3 \omega_1(x)^j \omega_2(y)^k \Bigg), \quad (2.123) \\ & \sigma_2 |\alpha|^{\varepsilon - \frac{p}{q}} y \Bigg(1 + \sum_{j=0}^{1}\sum_{k=0}^{1} g_{jk}(\varepsilon)(x^p y^q)^2 \omega_1(x)^j \omega_2(y)^k \\ & + \sum_{j=0}^{2}\sum_{k=0}^{2} \tilde{g}_{jk}(\cdot)(x^p y^q)^3 \omega_1(x)^j \omega_2(y)^k \Bigg)\Bigg) \end{aligned}$$

where

$$\begin{aligned} \cdot &= \varepsilon, x^p y^q, x^p y^q \omega_1(x), x^p y^q \omega_2(y) \\ g_{11}(\varepsilon) &\equiv 0 \\ \tilde{f}_{jk}(\cdot) &\equiv 0 \text{ if } j+k \geq 3 \\ \tilde{g}_{jk}(\cdot) &\equiv 0 \text{ if } j+k \geq 3. \end{aligned}$$

by the transformation

$$\phi_\varepsilon(x,y) = \left(x + \sum_{j=0}^{2} \beta_j(\varepsilon) x (x^p y^q)^2 \omega_1(x)^j, y \right) \tag{2.124}$$

where β_{jk} is a smooth function in ε. We also have that the functions $\tilde{f}_{jk}$ are as smooth as the functions $\bar{f}_{jk}$ and the functions $\tilde{g}_{jk}$ are as smooth as the functions $\bar{g}_{jk}$.

To conclude the second step of the elimination process we need the following result.

Lemma 2.21 *The system defined by (2.123) is locally conjugate to*

$$\begin{aligned} H_\varepsilon(x,y) &= \left(\alpha x \left(1 + \sum_{j=0}^{2} \sum_{k=0}^{1} \hat{f}_{jk}(\cdot)(x^p y^q)^3 \omega_1(x)^j \omega_2(y)^k \right), \right. \\ &\qquad \left. \sigma_2 |\alpha|^{\varepsilon - \frac{p}{q}} y \left(1 + \sum_{j=0}^{2} \sum_{k=0}^{2} \hat{g}_{jk}(\cdot)(x^p y^q)^3 \omega_1(x)^j \omega_2(y)^k \right) \right) \end{aligned} \tag{2.125}$$

where

$$\begin{aligned} \cdot &= \varepsilon, x^p y^q, x^p y^q \omega_1(x), x^p y^q \omega_2(y), \\ \hat{f}_{jk}(\cdot) &\equiv 0 \text{ if } j+k \geq 3 \\ \hat{g}_{jk}(\cdot) &\equiv 0 \text{ if } j+k \geq 3 \end{aligned}$$

by the transformation

$$\psi_\varepsilon(x,y) = \left(x, y + \sum_{j=0}^{1} \sum_{k=0}^{2} \gamma_{jk}(\varepsilon) y (x^p y^q)^2 \omega_1(x)^j \omega_2(y)^k \right) \tag{2.126}$$

where the functions γ_{jk} are smooth in ε. We have that the functions $\hat{f}_{jk}$ have the same smoothness as the functions $\tilde{f}_{jk}$ and the functions $\hat{g}_{jk}$ have the same smoothness as the functions $\tilde{g}_{jk}$.

We are now ready to begin the inductive part of the elimination process.

Lemma 2.22 *Take $N \geq 2$. The system*

$$\begin{aligned} F_\varepsilon(x,y) &= \Bigg(\alpha x\Bigg(1+\sum_{j=0}^{N}\sum_{k=0}^{N-1} f_{jk}(\varepsilon)(x^py^q)^{N+1}\omega_1(x)^j\omega_2(y)^k \\ &\quad + \sum_{j=0}^{N+1}\sum_{k=0}^{N}\bar{f}_{jk}(\cdot)(x^py^q)^{N+2}\omega_1(x)^j\omega_2(y)^k\Bigg), \\ &\quad \sigma_2|\alpha|^{\varepsilon-\frac{p}{q}}\Bigg(1+\sum_{j=0}^{N}\sum_{k=0}^{N} g_{jk}(\varepsilon)(x^py^q)^{N+1}\omega_1(x)^j\omega_2(y)^k \\ &\quad \sum_{j=0}^{N+1}\sum_{k=0}^{N+1}\bar{g}_{jk}(\cdot)(x^py^q)^{N+2}\omega_1(x)^j\omega_2(y)^k\Bigg)\Bigg) \end{aligned} \tag{2.127}$$

where

$$\begin{aligned} \cdot &= \varepsilon, x^py^q, x^py^q\omega_1(x), x^py^q\omega_2(y) \\ f_{jk}(\varepsilon) &\equiv 0 \textit{ if } j+k \geq N+1 \\ g_{jk}(\varepsilon) &\equiv 0 \textit{ if } j+k \geq N+1 \\ \bar{f}_{jk}(\cdot) &\equiv 0 \textit{ if } j+k \geq N+2 \\ \bar{g}_{jk}(\cdot) &\equiv 0 \textit{ if } j+k \geq N+2 \end{aligned}$$

is locally conjugate to the system

$$\begin{aligned} G_\varepsilon(x,y) &= \Bigg(\alpha x\Bigg(1+\sum_{j=0}^{N+1}\sum_{k=0}^{N}\tilde{f}_{jk}(\cdot)(x^py^q)^{N+2}\omega_1(x)^j\omega_2(y)^k\Bigg), \\ &\quad \sigma_2|\alpha|^{\varepsilon-\frac{p}{q}}\Bigg(1+\sum_{j=0}^{N}\sum_{k=0}^{N} g_{jk}(\varepsilon)(x^py^q)^{N+1}\omega_1(x)^j\omega_2(y)^k \\ &\quad \sum_{j=0}^{N+1}\sum_{k=0}^{N+1}\tilde{g}_{jk}(\cdot)(x^py^q)^{N+2}\omega_1(x)^j\omega_2(y)^k\Bigg)\Bigg) \end{aligned} \tag{2.128}$$

where

$$\begin{aligned} \tilde{f}_{jk}(\cdot) &\equiv 0 \textit{ if } j+k \geq N+2 \\ \tilde{g}_{jk}(\cdot) &\equiv 0 \textit{ if } j+k \geq N+2 \end{aligned}$$

by the transformation

$$\phi_\varepsilon(x,y) = \left(x + x \sum_{j=0}^{N+1} \sum_{k=0}^{N-1} \beta_{jk}(\varepsilon)(x^p y^q)^{N+1} \omega_1(x)^j \omega_2(y)^k, y \right) \qquad (2.129)$$

where the functions β_{jk} are smooth functions in ε. We also have that the functions $\tilde{f}_{jk}$ are as smooth as the functions $\bar{f}_{jk}$ and the functions $\tilde{g}_{jk}$ are as smooth as the functions $\bar{g}_{jk}$.

Proof: In order to have a conjugacy between F_ε and G_ε we need to check

$$F_\varepsilon \circ \phi_\varepsilon = \phi_\varepsilon \circ G_\varepsilon.$$

A short elementary calculation gives us

$$\begin{aligned} F_\varepsilon(\phi_\varepsilon(x,y)) \quad = \quad & \Bigg(\alpha x \Bigg(1 + \sum_{j=0}^{N} \sum_{k=0}^{N-1} \bar{f}_{jk}(\cdot)(x^p y^q)^{N+1} \omega_1(x)^j \omega_2(y)^k \\ & + \sum_{j+0}^{N+1} \sum_{k=0}^{N-1} (\beta_{jk}(\varepsilon) + f_{jk}(\varepsilon))(x^p y^q)^{N+1} \omega_1(x)^j \omega_2(y)^k \Bigg), \\ & \sigma_2 |\alpha|^{\varepsilon - \frac{p}{q}} \Bigg(1 + \sum_{j=0}^{N} \sum_{k=0}^{N} g_{jk}(\varepsilon)(x^p y^q)^{N+1} \omega_1(x)^j \omega_2(y)^k \\ & + \sum_{j=0}^{N+1} \sum_{k=0}^{N+1} \bar{g}_{jk}(\cdot)(x^p y^q)^{N+2} \omega_1(x)^j \omega_2(y)^k \Bigg) \Bigg) \end{aligned} \qquad (2.130)$$

and

$$\begin{aligned} \phi_\varepsilon(G_\varepsilon(x,y)) \quad = \quad & \Bigg(\alpha x \Bigg(1 + \sum_{j=0}^{N+1} \sum_{k=0}^{N-1} c_{jk}(\varepsilon)(x^p y^q)^{N+1} \omega_1(x)^j \omega_2(y)^k \\ & + \sum_{j=0}^{N+1} \sum_{k=0}^{N} \check{f}_{jk}(\cdot)(x^p y^q)^{N+2} \omega_1(x)^j \omega_2(y)^k \Bigg), \\ & \sigma_2 |\alpha|^{\varepsilon - \frac{p}{q}} \Bigg(1 + \sum_{j=0}^{N} \sum_{k=0}^{N} g_{jk}(\varepsilon)(x^p y^q)^{N+1} \omega_1(x)^j \omega_2(y)^k \\ & + \sum_{j=0}^{N+1} \sum_{k=0}^{N+1} \tilde{g}_{jk}(\cdot)(x^p y^q)^{N+2} \omega_1(x)^j \omega_2(y)^k \Bigg) \Bigg), \end{aligned} \qquad (2.131)$$

where

$$c_{jk}(\varepsilon) := \sum_{s=j}^{N+1}\sum_{m=k}^{N-1} \binom{s}{j}\binom{m}{k} \beta_{sm}(\varepsilon)(1-\varepsilon\omega_1(\alpha))^{j+k-N-1}\omega_1(\alpha)^{s+m-j-k}.$$

We have that (2.130) and (2.131) are equal iff

$$\mathbf{DB} = \mathbf{F}$$

where

$$\begin{aligned}\mathbf{B} &= (\beta_{00}(\varepsilon), \beta_{10}(\varepsilon), \cdots, \beta_{N0}(\varepsilon), \beta_{01}(\varepsilon), \beta_{11}(\varepsilon), \cdots, \beta_{N1}(\varepsilon), \cdots, \\ &\quad \beta_{0,N-1}(\varepsilon), \beta_{1,N-1}(\varepsilon), \beta_{2,N-1})^T \\ \mathbf{F} &= (f_{00}(\varepsilon), f_{10}(\varepsilon), \cdots, f_{N0}(\varepsilon), f_{01}(\varepsilon), \cdots, f_{N-1,1}(\varepsilon), \\ &\quad \cdots, f_{0,N-1}(\varepsilon), f_{1,N-1}(\varepsilon))^T\end{aligned}$$

and all $\beta_{jk}(\varepsilon)$ and $f_{jk}(\varepsilon)$ not appearing in $\mathbf{B}$ or $\mathbf{F}$ are zero, and

$$\mathbf{D} = \left(\begin{array}{c|c|c|c|c} A_0 & B_{01} & B_{02} & \dots & B_{0,N-1} \\ \hline 0 & A_1 & B_{11} & \dots & B_{1,N-1} \\ \hline \vdots & \vdots & \ddots & \ddots & \vdots \\ \hline 0 & 0 & \dots & A_{N-1} & B_{N-2,N-1} \\ \hline 0 & 0 & \dots & 0 & A_N \end{array}\right)$$

with A_k being the $(N+1-k)\times(N+2-k)$ matrix with entries

$$(A_k)_{ij} = \begin{cases} 0 & \text{if } i > j \\ \dfrac{1}{(1-\varepsilon\omega_1(\alpha))^{N+2-k-i}} - 1 & \text{if } i = j \\ \binom{j-1}{i-1} \dfrac{\omega_1(\alpha)^{j-1}}{(1-\varepsilon\omega_1(\alpha))^{N+2-k-i}} & \text{if } i < j \end{cases}$$

and B_{rs} the $(N+1-r)\times(N+2-s)$ matrix with entries

$$(B_{rs})_{ij} = \begin{cases} \binom{s}{r}\binom{j-1}{i-1} \dfrac{\omega_1(\alpha)^{s+j-1}}{(1-\varepsilon\omega_1(\alpha))^{N+2-r-i}} & \text{if } i \leq j \\ 0 & \text{if } i > j \end{cases},$$

where the entry M_{jk} of a $m \times n$ matrix M is located on the jth row and the kth column.

The system of linear equations as N degrees of freedom which we use to choose $\beta_{0k}(\varepsilon) \equiv 0$ for all $k = 0, 1, \cdots, N-1$. We now obtain a system of linear equations where the matrix will be upper-triangular for $\varepsilon = 0$ and the determinant of that matrix equals $C\ln(\alpha)^p$ for some $C \neq 0$ and $p \in \mathbb{N}\setminus\{0\}$

when $\varepsilon = 0$. As the determinant of the matrix is - at least - continuous in ε, by continuity we have that the determinant will be non-zero for ε sufficiently close to zero. Hence we have proved the existence of the transformation ϕ_ε as stated above. □

To finish our elimination method we proceed to the next result which is proved in the same way as Lemma 2.22.

Lemma 2.23 *The system defined by (2.128) is locally conjugate to*

$$\begin{aligned} H_\varepsilon(x,y) &= \left(\alpha x\left(1+\sum_{j=0}^{N+1}\sum_{k=0}^{N}\hat{f}_{jk}(\cdot)(x^py^q)^{N+2}\omega_1(x)^j\omega_2(y)^k\right),\right. \\ &\quad \left.\sigma_2|\alpha|^{\varepsilon-\frac{p}{q}}\left(1+\sum_{j=0}^{N+1}\sum_{k=0}^{N+1}\hat{g}_{jk}(\cdot)(x^py^q)^{N+2}\omega_1(x)^j\omega_2(y)^k\right)\right) \end{aligned} \tag{2.132}$$

where

$$\begin{aligned} \hat{f}_{jk}(\cdot) &\equiv 0 \text{ if } j+k \geq N+2 \\ \hat{g}_{jk}(\cdot) &\equiv 0 \text{ if } j+k \geq N+2 \end{aligned}$$

by the transformation

$$\psi_\varepsilon(x,y) = \left(x, y + y\sum_{j=0}^{N}\sum_{k=0}^{N+1}\gamma_{jk}(\varepsilon)(x^py^q)^{N+1}\omega_1(x)^j\omega_2(y)^k\right) \tag{2.133}$$

where the functions γ_{jk} are smooth functions in ε. We also have that the functions $\hat{f}_{jk}$ are as smooth as the functions $\tilde{f}_{jk}$ and the functions $\hat{g}_{jk}$ are as smooth as the functions $\tilde{g}_{jk}$.

2.5.3 Conclusions

We start with a system as in (2.112)

$$\begin{aligned} F_\varepsilon(x,y) &= \big(\alpha x(1+f_0(\varepsilon)x^py^q + f_1(\varepsilon, x^py^q)(x^py^q)^2), \\ &\quad \sigma_2|\alpha|^{\varepsilon-\frac{p}{q}}y(1+g_0(\varepsilon)x^py^q + g_1(\varepsilon,x^py^q)(x^py^q)^2)\big). \end{aligned}$$

Applying successively Lemma 2.18, Lemma 2.19, Lemma 2.20 and Lemma 2.21 we obtain

$$\begin{aligned} F_\varepsilon(x,y) &= \left(\alpha x\left(1+\sum_{j=0}^{2}\sum_{k=0}^{1}\hat{f}_{jk}(\cdot)(x^py^q)^3\omega_1(x)^j\omega_2(y)^k\right),\right. \\ &\quad \left.\sigma_2|\alpha|^{\varepsilon-\frac{p}{q}}y\left(1+\sum_{j=0}^{2}\sum_{k=0}^{2}\hat{g}_{jk}(\cdot)(x^py^q)^3\omega_1(x)^j\omega_2(y)^k\right)\right) \end{aligned}$$

where

$$\begin{aligned} \cdot &= \varepsilon, x^p y^q, x^p y^q \omega_1(x), x^p y^q \omega_2(y), \\ \hat{f}_{jk}(\cdot) &\equiv 0 \text{ if } j+k \geq 3 \\ \hat{g}_{jk}(\cdot) &\equiv 0 \text{ if } j+k \geq 3. \end{aligned}$$

By multiple successive application of Lemma 2.22 and Lemma 2.23 we obtain, given a positive integer $N \in \mathbb{N}$, a local family of diffeomorphisms F_ε where all nonlinear terms are of equal order or lower order than $x^{Np+1}y^{Nq}$ in the first component and of equal order or lower than $x^{Np}y^{Nq+1}$ in the second component. This family of diffeomorphisms is given by

$$\begin{aligned} H_\varepsilon(x,y) &= \left(\alpha x \left(1 + \sum_{j=0}^{N+1} \sum_{k=0}^{N} \hat{f}_{jk}(\cdot)(x^p y^q)^{N+2} \omega_1(x)^j \omega_2(y)^k \right), \right. \\ &\quad \left. \sigma_2 |\alpha|^{\varepsilon - \frac{p}{q}} \left(1 \sum_{j=0}^{N+1} \sum_{k=0}^{N+1} \hat{g}_{jk}(\cdot)(x^p y^q)^{N+2} \omega_1(x)^j \omega_2(y)^k \right) \right) \end{aligned} \tag{2.134}$$

where

$$\begin{aligned} \cdot &= \varepsilon, x^p y^q, x^p y^q \omega_1(x), x^p y^q \omega_2(y) \\ \hat{f}_{jk}(\cdot) &\equiv 0 \text{ if } j+k \geq N+2 \\ \hat{g}_{jk}(\cdot) &\equiv 0 \text{ if } j+k \geq N+2. \end{aligned}$$

From Proposition 2.8 it follows that the non-linear term in the first component of (2.134) is of order $\mathcal{O}(\xi^{(N+1)p+1-\nu}) \cdot \mathcal{O}(\eta^{(N+1)q-\mu})$ for any small $\nu > 0$ and $\mu > 0$ and the non-linear term in the second equation of (2.134) is of order $\mathcal{O}(\xi^{(N+1)p-\nu}) \cdot \mathcal{O}(\eta^{(N+1)q+1-\mu})$ for any small $\nu > 0$ and $\mu > 0$. We want to eliminate those terms, be it in a non-explicit way, by applying the methods in [IY91]. We can conclude in the same way is Section 2.3 that, given $r \in \mathbb{N}$, if N is large enough in (2.134) then there exists a C^r diffeomorphism h defined on a small neighbourhood of the origin which conjugates (2.134) to its linear part. As there is no discrete equivalent known of Theorem 3.9 in [Bon97], i.e. a version dealing with diffeomorphisms, we cannot give a representation of h by means of flows as we did in Section 2.2 and Section 2.3.

Remark 2.5 *As the conjugacies that were calculated in this section are of the same form as those calculated in Section 2.3, the results form Section 2.4 are also valid for these conjugacies. The proof is almost identical to the one in the case of the vector fields hence we omit the proof.*

2.6 Center vector fields: formal similarity

Since there are formal similarities between normal forms for centers and resonant saddles, we can imitate the foregoing computations for curiosity. For the moment we ignore the meaning of these results and don't have any geometric interpretation of this. We only give the results of these computations.

2.6.1 Settings and complex compensators

Settings

In this section we consider the perturbation of a planar vector field which has a singularity of center type. By virtue of the results from Subsection 1.3.6 we may assume that the local deformation is given by

$$\left\{ \begin{array}{rcl} \dot{x} & = & \varepsilon x - \alpha y + \big(k(\varepsilon, x^2+y^2)x - \ell(\varepsilon, x^2+y^2)y\big)\,(x^2+y^2) + u(\varepsilon, x, y) \\ \dot{y} & = & \alpha x + \varepsilon y + \big(\ell(\varepsilon, x^2+y^2)x + k(\varepsilon, x^2+y^2)y\big)\,(x^2+y^2) + v(\varepsilon, x, y) \end{array} \right. \tag{2.135}$$

where k and ℓ are C^∞ functions and u and v are C^∞ functions that are infinitely flat with respect to x and y.

In Subsection 1.3.6 we deduced a normal form for a local deformation of a planar center in polar coordinates, but it is of course also possible to give a normal form in complex coordinates. Taking $z = x + iy$, we can rewrite (2.135) as

$$\dot{z} = (\varepsilon + i\alpha(\varepsilon))z(1 + f(\varepsilon, z\bar{z})z\bar{z} + g(\varepsilon, z, \bar{z})) \tag{2.136}$$

where

$$f(\varepsilon, z\bar{z}) = k(\varepsilon, z\bar{z}) + i\ell(\varepsilon, z\bar{z})$$

and

$$g(\varepsilon, z, \bar{z}) = u\left(\varepsilon, \frac{z+\bar{z}}{2}, \frac{z-\bar{z}}{2i}\right) + iv\left(\varepsilon, \frac{z+\bar{z}}{2}, \frac{z-\bar{z}}{2i}\right).$$

Essentially the main idea is to find transformations that will eliminate f from (2.136) upto some order in $z\bar{z}$. As f represents terms that are resonant for $\varepsilon = 0$, this cannot be done with a polynomial change of variables.

Complex compensators

In the previous sections we introduced the functions ω_1 and ω_2 in order to construct equivalences or conjugacies between the vector fields with a saddle singularity or between the diffeomorphisms with a saddle fixed point. In order to construct conjugacies between vector fields with a center singularity we will introduce the variable

$$\mathcal{W}(z) := \frac{1 - z^{-\frac{2\varepsilon}{\varepsilon + i\alpha(\varepsilon)}}}{\varepsilon} \tag{2.137}$$

which is a complex-valued function and can be seen as the complex generalisation of the Ecalle–Roussarie compensator. For that reason we will call $\mathcal{W}$ a **complex compensator**. In (2.137) we have used

$$z^{\frac{2\varepsilon}{\varepsilon+i\alpha(\varepsilon)}} = e^{\frac{2\varepsilon}{\varepsilon+i\alpha(\varepsilon)}\ln z}$$

this way it is obvious that $\mathcal{W}(z)$ has the same branching line as the complex logarithm and this for all values of ε.

A direct calculation reveals

$$\lim_{\varepsilon\to 0}\mathcal{W}(z) = -\frac{2i}{\alpha}\ln(z) \tag{2.138}$$

$$\frac{\partial\mathcal{W}}{\partial z}(z) = \frac{2}{\varepsilon+i\alpha(\varepsilon)}\frac{1-\varepsilon\mathcal{W}(z)}{z}. \tag{2.139}$$

In the changes of variables in the sequel we will use monomials of the form $(z^p\bar{z}^q)^n\mathcal{W}^j\overline{\mathcal{W}}^k$; we define a (partial) ordering $\prec$ on them by putting

$$(z^p\bar{z}^q)^n\mathcal{W}^j\overline{\mathcal{W}}^k \prec (z^p\bar{z}^q)^m\mathcal{W}^\ell\overline{\mathcal{W}}^s \tag{2.140}$$

iff

$$n < m \text{ or } (n = m \text{ and } j > l \text{ and } k \geq s) \text{ or } (n = m \text{ and } j \geq l \text{ and } k > s). \tag{2.141}$$

We will say that the monomial on the left-hand side of (2.140) is of lower order than the right-hand side.

We like to remark that one has to be careful when taking the complex conjugate of $\mathcal{W}$. Considering the definition (2.137) it is clear that in general

$$\overline{\mathcal{W}(z)} \neq \mathcal{W}(\bar{z}).$$

2.6.2 Computation of the conjugacies

We start the process by eliminating the term in $z\bar{z}$ from (2.136).

Lemma 2.24 *The system defined by (2.136) is conjugate to*

$$\dot{z} = (\varepsilon+i\alpha(\varepsilon))z \Big(1+G_0(\cdot)(z\bar{z})^2+G_1(\cdot)(z\bar{z})^2\mathcal{W}(z)+G_2(\cdot)(z\bar{z})^2\overline{\mathcal{W}(z)}+\tilde{g}(\cdot,z,\bar{z})\Big) \tag{2.142}$$

with

$$\cdot = \varepsilon, z\bar{z}, z\bar{z}\mathcal{W}(z), z\bar{z}\overline{\mathcal{W}(z)}$$

by the transformation

$$z = v+\beta_0(\varepsilon)v^2\bar{v}+\beta_1(\varepsilon)v^2\bar{v}\mathcal{W}(v) \tag{2.143}$$

where β_0 and β_1 are complex-valued functions that are smooth in ε and where $\tilde{g}$ contains all infinitely flat terms and all terms of higher order than $(z\bar{z})^2$.

Before proceding to the next step of our elimination process, we expand the right hand side of (2.142) and again using z as variable we obtain

$$\begin{aligned}\dot{z} &= (\varepsilon + i\alpha(\varepsilon))z\left(1 + \sum_{j=0}^{1}\sum_{k=0}^{1} f_{jk}(\varepsilon)(z\bar{z})^2\omega(z)^j\overline{\mathcal{W}(z)}^k\right.\\ &\left.+\sum_{j=0}^{2}\sum_{k=0}^{2} g_{jk}(\cdot)(z\bar{z})^3\mathcal{W}(z)^j\overline{\mathcal{W}(z)}^k\right) + \tilde{g}(\cdot, z, \bar{z}) \end{aligned} \tag{2.144}$$

where

$$\begin{aligned}\cdot &= \varepsilon, z\bar{z}, z\bar{z}\mathcal{W}(z), z\bar{z}\overline{\mathcal{W}(z)}\\ f_{jk}(\varepsilon) &\equiv 0 \text{ if } j+k \geq 2\\ g_{jk}(\varepsilon) &\equiv 0 \text{ if } j+k \geq 3.\end{aligned}$$

Lemma 2.25 *The system defined by (2.142) is conjugate to the system*

$$\dot{z} = (\varepsilon + i\alpha(\varepsilon))z\left(1 + \sum_{j=0}^{2}\sum_{k=0}^{2} \hat{g}_{jk}(\cdot)(z\bar{z})^3\mathcal{W}(z)^j\overline{\mathcal{W}(z)}^k + h(\cdot, z, \bar{z})\right) \tag{2.145}$$

by the transformation

$$z = v + \sum_{j=0}^{2}\sum_{k=0}^{2} \beta_{jk}(\varepsilon)v(v\bar{v})^2\mathcal{W}(v)^j\overline{\mathcal{W}(v)}^k \tag{2.146}$$

where the functions β_{jk} are smooth functions in ε and where h contains all infinitely flat terms and all terms of higher order than $(z\bar{z})^3$.

Applying Taylor's Theorem on (2.145) gives us

$$\begin{aligned}\dot{v} &= (\varepsilon + i\alpha(\varepsilon))v\left(1 + \sum_{j=0}^{2}\sum_{k=0}^{2} \tilde{f}_{jk}(\varepsilon)(v\bar{v})^3\mathcal{W}(v)^j\overline{\mathcal{W}(v)}^k\right.\\ &\left.+\sum_{j=0}^{3}\sum_{k=0}^{3} \tilde{g}_{jk}(\cdot)(v\bar{v})^4\mathcal{W}(v)^j\overline{\mathcal{W}(v)}^k + h_N(\cdot, v, \bar{v})\right)\end{aligned} \tag{2.147}$$

We now state the inductive part of the process.

Lemma 2.26 *The system defined by*

$$\begin{aligned}\dot{z} &= (\varepsilon + i\alpha(\varepsilon))z\left(1 + \sum_{j=0}^{N}\sum_{k=0}^{N} f_{jk}(\varepsilon)(z\bar{z})^{N+1}\mathcal{W}(z)^j\overline{\mathcal{W}(z)}^k\right.\\ &\left.+\sum_{j=0}^{N+1}\sum_{k=0}^{N+1} g_{jk}(\cdot)(z\bar{z})^{N+2}\mathcal{W}(z)^j\overline{\mathcal{W}(z)}^k + h(\cdot, z, \bar{z})\right)\end{aligned} \tag{2.148}$$

with $\cdot = \varepsilon, z\bar{z}, z\bar{z}\mathcal{W}(z), z\bar{z}\overline{\mathcal{W}(z)}$ *and*

$$\begin{aligned} f_{jk}(\varepsilon) &\equiv 0 \text{ if } j+k \geq N+1 \\ g_{jk}(\cdot) &\equiv 0 \text{ if } j+k \geq N+2 \end{aligned}$$

is conjugate to the system defined by

$$\dot{z} = (\varepsilon + i\alpha(\varepsilon))z \left(1 + \sum_{j=0}^{N+1} \sum_{k=0}^{N+1} G_{jk}(\cdot)(z\bar{z})^{N+2}\mathcal{W}(z)^j \overline{\mathcal{W}(z)}^k + \tilde{h}(\cdot, z, \bar{z}) \right) \tag{2.149}$$

with $G_{jk} \equiv 0$ *if* $j+k \geq N+2$, *by the transformation*

$$z = v + \sum_{j=0}^{N} \sum_{k=0}^{N} \beta_{jk}(\varepsilon)v(v\bar{v})^{N+1}\mathcal{W}(v)^j \overline{\mathcal{W}(v)}^k \tag{2.150}$$

where β_{jk} *is a smooth function in* ε *and where* h *and* $\tilde{h}$ *contain the high order terms and the infinitely flat terms.*

Chapter 3

Poincaré map near a planar center

3.1 Introduction

In Subsection 1.4.2 we introduced the Poincaré map of a local deformation of a planar center. As in general it is not possible to calculate an explicit form of the Poincaré map, one calculates a formal expansion with respect to ε. The coefficients of this formal expansion are functions of a variable h which describes the level curves of the unperturbed (Hamiltonian) system and are called Melnikov-functions. For the first order Melnikov function we obtained an expression by means of an Abelian integral. As the Abelian integral is a line integral, in most cases it is very difficult to calculate this integral. When the deformation is obtained from a non-Hamiltonian center, there is no expression in terms of Abelian integrals for the first order Melnikov function.

To avoid these problems we introduce a new technique that gives an asymptotic expansion of the Melnikov functions. This technique uses “multi-valued” functions and normal forms. Usually one avoids working with functions that are multi-valued, but for this problem working with these functions will create possibilities that aren't available when working with ordinary (single-valued) functions. The advantage of our technique is that we do not need to calculate Abelian integrals, but a small disadvantage is that our method requires a high number of calculations so one is in need of a computer. Another advantage is that we can also calculate a formal expansion of the Poincaré map with respect to the starting point instead of the parameter ε. All applications to given vector fields were done by means of the computer programme Maple. The topics of this chapter were also discussed in [NN05].

This chapter is structured as follows. In Section 3.2 we discuss the setting in which we will work and introduce some functions we will need to describe the multi-valued normal forms. We also prove some elementary properties of these functions. In Section 3.3 we give a detailed exposition of the multi-valued normal forms and the technique that arises from these normal forms. In Section 3.4 we give an application of our technique to a local deformation of a polynomial Hamiltonian system. This deformation will be chosen in such a way that we can calculate the first order Melnikov function explicitly, hence we can compare the results from both methods. In Section 3.5 we consider a deformation of the so-called "Hamiltonian triangle". We will show that our technique can be adapted to this type of deformations and gives the asymptotic result that was described by [Ili98]. Finally in Section 3.6 we give the Maple source codes of the calculations made in Section 3.4 and Section 3.5.

3.2 Settings and preliminaries

3.2.1 Settings

Let X_ε be a local C^∞ deformation of a planar singularity of center type. By virtue of the results from Subsection 1.3.6, we can assume that this local deformation is written as

$$\begin{aligned} X_\varepsilon(x,y) &= (\varepsilon x - \alpha(\varepsilon)y + xP(x^2+y^2;\varepsilon) - yQ(x^2+y^2;\varepsilon))\frac{\partial}{\partial x} \\ &+ (\alpha(\varepsilon)x + \varepsilon y + xQ(x^2+y^2;\varepsilon) + yP(x^2+y^2;\varepsilon))\frac{\partial}{\partial y} \\ &+ R_1(x,y;\varepsilon)\frac{\partial}{\partial x} + R_2(x,y;\varepsilon)\frac{\partial}{\partial y} \end{aligned} \tag{3.1}$$

where P and Q are C^∞ functions,

$$P(0;\varepsilon)=0, Q(0;\varepsilon)=0 \text{ and } \frac{\partial^{j+k}R_i}{\partial x^j y^k}(0,0;\varepsilon)=0, \forall j,k\in\mathbb{N}, i=1,2.$$

If we delete the infinitely flat functions from (3.1), then we obtain the local deformation

$$\begin{aligned} X_\varepsilon^N(x,y) &= (\varepsilon x - \alpha(\varepsilon)y + xP(x^2+y^2;\varepsilon) - yQ(x^2+y^2;\varepsilon))\frac{\partial}{\partial x} \\ &+ (\alpha(\varepsilon)x + \varepsilon y + xQ(x^2+y^2;\varepsilon) + yP(x^2+y^2;\varepsilon))\frac{\partial}{\partial y}. \end{aligned} \tag{3.2}$$

From the results from Subsection 1.3.6 we know that (3.2) is equivalent with the vector field

$$X_\varepsilon^\pi : \left\{ \begin{array}{lcl} \dot\rho &=& \varepsilon\rho + \rho^3 f(\rho^2;\varepsilon) \\ \dot\theta &=& 1 \end{array} \right. , \tag{3.3}$$

where f is a C^∞ function.

Consider now $\mathcal{M} = \mathbb{R}^+ \times \mathbb{R}$ the universal cover of $\mathbb{R}^+ \times \mathbb{S}^1$ with covering

$$\mathbb{P} : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}^+ \times \mathbb{S}^1 : \ (\rho, \theta) \mapsto (\rho, \theta \bmod 2\pi)$$

Since X_ε^π is (obviously) periodic in θ, the vector field

$$\hat{X}_\varepsilon = \left(\varepsilon\rho + \rho^3 f(\rho^2;\varepsilon)\right)\frac{\partial}{\partial\rho}$$

defined on $\mathcal{M}$ is such that $\mathbb{P}_*(\hat{X}_\varepsilon) = X_\varepsilon^\pi$. Now (3.3) provides the normal form we will use throughout this chapter.

As the origin is an isolated singularity, from (3.2) it follows that

$$\Sigma_\nu = \{(x, y) \mid x \in]0, \nu[, y = 0\} =]0, \nu[\times \{0\} \tag{3.4}$$

is a transversal section to the flow of X_ε^N (hence to the flow of X_ε^π) if ν is sufficiently close to zero. Hence we can define the Poincaré map on this section

$$\mathcal{P}_\varepsilon : \Sigma_\nu \to \Sigma_1 : x \mapsto \mathcal{P}_\varepsilon(x),$$

where - contrary to Subsection 1.4.2 - we take the starting point x as a variable instead of the level curve parameter h. Although this not the Poincaré map we defined in Subsection 1.4.2, we will also call this map the Poincaré map. From the notation it will always be clear which Poincaré map we consider. We have that these Poincaré maps are conjugate. Considering the section Σ_ν given by (3.4) and the Hamiltonian H, we have that the point $(x_0, 0)$ lies on the level curve $H(x, y) = h$ iff $H(x_0, 0) = h$. Introducing the function $H_0(x) := H(x, 0)$, we have that H_0 conjugates $\mathcal{P}_\varepsilon$ and P_ε: $H_0 \circ \mathcal{P}_\varepsilon = \mathsf{P}_\varepsilon \circ H_0$. Taking the larger transversal Σ_1 (which contains Σ_ν), we are assured that for each $h \in \Sigma_\nu$ the Poincaré map will have an image in Σ_1 if ν is taken sufficiently close to zero.

To get the asymptotics of the Poincaré map, the idea is to perform a C^∞ transformation for all integers $n \geq 1$ in the universal cover of the form

$$\Phi_n : \mathbb{R}^+ \times \mathbb{R} \times (\mathbb{R}, 0) \to \mathbb{R}^+ \times \mathbb{R} \times (\mathbb{R}, 0) : (\rho, \theta, \varepsilon) \mapsto (\varphi_n(\rho, \theta; \varepsilon), \theta, \varepsilon) = (\rho_n, \theta, \varepsilon)$$

that conjugates $\hat{X}_\varepsilon$ to $\hat{X}_\varepsilon^{\{n\}}$ where

$$\hat{X}_\varepsilon^{\{n\}} = \rho_n^{2n+1} f_n(\rho_n^2, \theta; \varepsilon)\frac{\partial}{\partial\rho}$$

or in terms of ordinary differential equations

$$\hat{X}_\varepsilon^{\{n\}} : \left\{ \begin{array}{rcl} \dot{\rho}_n & = & \rho_n^{2n+1} f_n(\rho_n^2, \theta; \varepsilon) \\ \dot{\theta} & = & 1 \end{array} \right. , \tag{3.5}$$

where f_n will be defined later on. We claim that Φ_n is no longer 2π-periodic, and therefore the correspondence

$$(\rho, \theta) \mapsto \mathbb{P} \circ \Phi \circ \mathbb{P}^{-1}(\rho, \theta),$$

does not give a well-defined function but a multi-valued function. This normal form will be called a **multi-valued normal form**.

Theorem 3.1 *For each integer $n \geq 2$, there exists a constructive map*

$$\varphi_n : \mathbb{R}^+ \times \mathbb{R} \times (\mathbb{R}, 0) \to \mathbb{R} : \ (\rho, \theta, \varepsilon) \mapsto \varphi_n(\rho, \theta; \varepsilon)$$

of the form

$$\varphi_n(\rho, \theta; \varepsilon) = \rho + \sum_{i=1}^{n-1} \mathbb{B}_i(\theta, \varepsilon)\rho^{2i+1}$$

such that the map

$$\Phi_n : (\mathbb{R}^+, 0) \times \mathbb{R} \to (\mathbb{R}^+, 0) \times \mathbb{R} : (\rho, \theta) \mapsto (\varphi_n(\rho, \theta; \varepsilon), \theta)$$

conjugates (3.3) with

$$\left\{ \begin{array}{rcl} \dot{\rho} & = & \rho^{2n+1} f_{2n+1}(\rho^2, \theta; \varepsilon) \\ \dot{\theta} & = & 1 \end{array} \right. \tag{3.6}$$

where f_{2n+1} is a C^∞ function.

By *constructive*, we mean that for each integer i, $\mathbb{B}_i(\theta, \varepsilon)$ can be calculated explicitly. Note again that $\varphi_n(\rho, \theta; \varepsilon)$ is (in general) not 2π-periodic in the variable θ. However, from the expression of φ_n we can deduce an asymptotic expression of the Poincaré map $\mathcal{P}_\varepsilon$, which takes the form

$$\mathcal{P}_\varepsilon(x) = x + \varphi_n(x, 2\pi; \varepsilon) - \varphi_n(x, 0; \varepsilon) + \mathcal{O}(x^{2n+1}).$$

3.2.2 Angular compensator and Taylor tails

Let $\tau \in \mathbb{R}$. Define $\Omega_\tau : \mathbb{R} \to \mathbb{R} : \theta \mapsto \Omega_\tau(\theta)$ where,

$$\begin{array}{rcl} \Omega_\tau(\theta) & = & \dfrac{1 - e^{-\tau\theta}}{\tau}, \text{ if } \tau \neq 0, \\ \Omega_0(\theta) & = & \theta. \end{array}$$

This latter function is called an **angular compensator**. It verifies

$$\begin{array}{rcl} \lim\limits_{\tau \to 0} \Omega_\tau(\theta) & = & \theta \\ \dfrac{d\Omega_\tau}{d\theta} & = & 1 - \tau\Omega_\tau(\theta). \end{array}$$

It should be noted that, contrary to the Ecalle–Roussarie compensator, the present compensator is analytic as a function of θ. Indeed, we easily see that the following holds

$$\Omega_\tau(\ln|\theta|) = \omega_\tau(\theta).$$

We now introduce the **Taylor tails** of sin and cos. More precisely, we put

$$\begin{aligned} s_0(\theta) &:= \sin(\theta) \\ s_n(\theta) &:= \sin(\theta) - \sum_{j=1}^{n} (-1)^{j-1} \frac{\theta^{2j-1}}{(2j-1)!}, n \geq 1 \\ c_n(\theta) &:= \cos(\theta) - \sum_{j=0}^{n} (-1)^{j} \frac{\theta^{2j}}{(2j)!}, n \geq 0. \end{aligned}$$

By direct calculation one obtains

$$\begin{aligned} c_0(\theta) &= \cos(\theta) - 1, \\ c_0'(\theta) &= -s_0(\theta), \\ s_0'(\theta) &= c_0(\theta) + 1, \\ s_n'(\theta) &= c_{n-1}(\theta), \qquad n \geq 1, \\ c_n'(\theta) &= -s_n(\theta), \qquad n \geq 1, \end{aligned}$$

where $'$ denotes differentiation with respect to θ. Observe that

$$\theta = s_0(\theta) - s_1(\theta).$$

So for all $n \geq 1$ we have

$$s_n(\theta) = s_0(\theta) - \sum_{j=1}^{n} (-1)^{j-1} \frac{(s_0(\theta) - s_1(\theta))^{2j-1}}{(2j-1)!},$$

and for all $n \geq 1$ we have

$$c_n(\theta) = c_0(\theta) - \sum_{j=1}^{n} (-1)^{j} \frac{(s_0(\theta) - s_1(\theta))^{2j}}{(2j)!}.$$

This means that all s_n for $n \geq 2$ and c_n for $n \geq 1$ are polynomials of s_0, s_1 and c_0.

In what follows $\Omega = \Omega_\varepsilon$.

Proposition 3.2 *For any integer $n > 0$ and for any polynomial $P : \mathbb{R}^4 \to \mathbb{R}$ with coefficients that are functions of ε, we have that*

$$\int_0^\theta P(s_0(u), c_0(u), s_1(u), \Omega(u)) e^{2n\varepsilon u} du = e^{2n\varepsilon\theta} Q(s_0(\theta), c_0(\theta), s_1(\theta), \Omega(\theta))$$

where $Q : \mathbb{R}^4 \to \mathbb{R}$ is a polynomial. The degree of Q equals either the degree of P or the degree of P plus one.

Moreover the sequence $(U_n)_{n\in\mathbb{N}}$ defined by

$$U_n = \int_{\theta_1}^{\theta_2} P(s_0(u), c_0(u), s_1(u), \Omega(u)) e^{2n\varepsilon u} du,$$

converges to zero for any $\theta_1 < \theta_2 \leq 0$ *if* $\varepsilon > 0$ *and for any* $0 \leq \theta_1 < \theta_2$ *if* $\varepsilon < 0$*, where* θ_1*,* θ_2 *are taken in a compact interval and* ε *in a sufficiently small neighbourhood of the origin.*

PROOF: It is sufficient to give the proof for a monomial of Taylor tails and an angular compensator, so

$$P(s_0(u), c_0(u), s_1(u), \Omega(u)) = s_0(u)^{a_1} c_0(u)^{a_2} s_1(u)^{a_3} \Omega(u)^{a_4}.$$

Using the definitions of the Taylor tails and the angular compensator and applying Newton's Binomium we have

$$\begin{aligned} P(s_0(u), c_0(u), s_1(u), \Omega(u)) &= \sin(u)^{a_1} (\cos(u) - 1)^{a_2} (\sin(u) - u)^{a_3} \\ &\quad \cdot \left(\frac{1 - e^{-\varepsilon u}}{\varepsilon} \right)^{a_4} \\ &= \sin(u)^{a_1} \sum_{j_2=0}^{a_2} \binom{a_2}{j_2} (-1)^{a_2 - j_2} \cos(u)^{j_2} \\ &\quad \cdot \sum_{j_3=0}^{a_3} \binom{a_3}{j_3} \sin(u)^{j_3} (-1)^{a_3 - j_3} u^{a_3 - j_3} \\ &\quad \cdot \sum_{j_4=0}^{a_4} \binom{a_4}{j_4} (-1)^{j_4} \varepsilon^{-a_4} e^{-\varepsilon j_4 u} \\ &= \sum_{j_2=0}^{a_2} \sum_{j_3=0}^{a_3} \sum_{j_4=0}^{a_4} \binom{a_2}{j_2} \binom{a_3}{j_3} \binom{a_4}{j_4} \\ &\quad \cdot (-1)^{a_2 + a_3 - j_2 - j_3 - j_4} \\ &\quad \cdot \varepsilon^{-a_4} u^{a_3 - j_3} \sin^{a_1 + j_3}(u) \cos^{j_2}(u) e^{-\varepsilon j_4 u}. \end{aligned}$$

Using Euler formulas

$$\begin{aligned} \sin(u) &= \frac{e^{iu} - e^{-iu}}{2i}, \\ \cos(u) &= \frac{e^{iu} + e^{-iu}}{2}, \end{aligned}$$

we can rewrite

$$\int_0^\theta P(s_0(u), c_0(u), s_1(u), \Omega(u)) e^{2n\varepsilon u} du,$$

as a finite linear combination of integrals of the form

$$\int_0^\theta u^m e^{(p+iq)u} du,$$

where $m \in \mathbb{N}$ and $p, q \in \mathbb{R}$. The latter integral can be calculated by induction on m and by means of integration by parts and gives us

$$\left[\sum_{\ell=0}^{m} \frac{m!}{(m-\ell)!}(-1)^{\ell} \frac{u^{m-\ell}}{(p+iq)^{\ell+1}} e^{(p+iq)u}\right]_{u=0}^{u=\theta}$$

for $(p, q) \neq (0, 0)$.

Now the original integral can be written as a linear combination of functions as those computed above. From there one finds again the expression with Taylor tails and the angular compensator. As we integrated a real-valued function, we know that the result will be real as well. The degree of Q comes from the latter integration: if $(p, q) \neq (0, 0)$ then the degrees of P and Q will be equal, if however there is a term where $(p, q) = (0, 0)$, then the degree of Q will be the degree of P plus one.

We now come to the proof of the second statement. We will consider the case where θ_1 and θ_2 are positive and ε is negative. We have that $s_0(u) = \sin(u)$, so it is known that $-u \leq s_0(u) \leq u$, $\forall u > 0$. Integrating the latter inequalities with respect to u over the interval $[0, u]$ gives us

$$-\frac{u^2}{2} \leq c_0(u) = \cos(u) - 1 \leq \frac{u^2}{2},$$

and integrating these inequalities with respect to u over the interval $[0, u]$ gives us

$$-\frac{u^3}{6} \leq s_1(u) = \sin(u) - u \leq \frac{u^3}{6}.$$

In other words we have proved that $s_0(u) = \mathcal{O}(|u|)$, $c_0(u) = \mathcal{O}(|u|^2)$ and $s_1(u) = \mathcal{O}(|u|^3)$ for all $u \in \mathbb{R}$. As we know that the Taylor series of the exponential function converges over $\mathbb{R}$, we have for all $u \in \mathbb{R}$ that

$$\begin{aligned} \Omega(u) &= \frac{1 - e^{-\varepsilon u}}{\varepsilon} \\ &= \frac{1 - \sum_{k=0}^{\infty} (-\varepsilon)^k \frac{u^k}{k!}}{\varepsilon} \\ &= u + \sum_{k=2}^{\infty} (-\varepsilon)^{k-1} \frac{u^k}{k!}, \end{aligned}$$

i.e. $\Omega(u) = \mathcal{O}(|u|)$ for all $u \in \mathbb{R}$. So we have that there exists positive constants K_1, K_2, K_3, K_4, such that

$$\begin{aligned} |U_n| &\leq \int_{\theta_1}^{\theta_2} (K_1 u)^{a_1} (K_2 u^2)^{a_2} (K_3 u^3)^{a_3} (K_4 u)^{a_4} e^{2n\varepsilon u} du \\ &= K \int_{\theta_1}^{\theta_2} u^m e^{2n\varepsilon u} du \end{aligned}$$

where $K = K_1^{a_1} K_2^{a_2} K_3^{a_3} K_4^{a_4}$ and $m = a_1 + 2a_2 + 3a_3 + a_4$. By means of integration by parts and induction on m one proves

$$\int u^m e^{N\varepsilon u} du = e^{N\varepsilon u} \sum_{j=0}^{m} (-1)^j \frac{m!}{(m-j)!} \frac{u^{m-j}}{(N\varepsilon)^{j+1}} + C,$$

$\forall m \in \mathbb{N}$ and $\forall N \in \mathbb{N} \setminus \{0\}$. So for $N = 2n$ we find

$$\int_{\theta_1}^{\theta_2} u^m e^{2n\varepsilon u} du = \left[e^{2n\varepsilon u} \sum_{j=0}^{m} (-1)^j \frac{m!}{(m-j)!} \frac{u^{m-j}}{(2n\varepsilon)^{j+1}} \right]_{u=\theta_1}^{u=\theta_2}. \tag{3.7}$$

As $0 \leq \theta_1 < \theta_2$ and $\varepsilon < 0$, it is clear that the right-hand side of (3.7) will converge to zero for $n \to +\infty$. This means that the sequence $(U_n)_{n \in \mathbb{N}}$ converges to zero with θ_1, θ_2 and ε taken in the stated domains. □

3.3 Multi-valued normal forms

Before stating the main proposition that will describe a sequence of transformations in the universal cover and proving the main theorem, we introduce two transformations that we have to apply in order to get the system described by (3.3) in a suitable form.

Proposition 3.3 *The system defined by (3.3) is conjugate to*

$$\dot{r} = e^{2\varepsilon\theta} r^3 f(r^2 e^{2\varepsilon\theta}; \varepsilon), \tag{3.8}$$

by the transformation

$$r = e^{-\varepsilon\theta} \rho. \tag{3.9}$$

Proof: Differentiating (3.9) with respect to t we find that

$$\dot{r} = -\varepsilon e^{-\varepsilon\theta} \rho \dot{\theta} + e^{-\varepsilon\theta} \dot{\rho},$$

so using (3.3) we find that

$$\begin{aligned} \dot{r} &= -\varepsilon e^{-\varepsilon\theta} \rho + \varepsilon e^{-\varepsilon\theta} \rho + e^{-\varepsilon\theta} \rho^3 f(\rho^2; \varepsilon) \\ &= e^{-\varepsilon\theta} \rho^3 f(\rho^2; \varepsilon) \end{aligned}$$

if we use (3.9), then we have

$$\dot{r} = e^{2\varepsilon\theta} r^3 f(r^2 e^{2\varepsilon\theta}; \varepsilon).$$

□

In what follows it will be easier to work with the formal Taylor series expansion of (3.8). In order to make clear that the expansion is only formal (i.e. up to an infinitely flat function) we will use the notation

$$\begin{aligned} \dot r &\widehat{=} e^{2\varepsilon\theta} r^3 \sum_{n\geq 0} k_n(\varepsilon)(e^{2\varepsilon\theta} r^2)^n \\ \dot r &\widehat{=} \sum_{n\geq 0} k_n(\varepsilon) e^{2(n+1)\varepsilon\theta} r^{2n+3}. \end{aligned} \tag{3.10}$$

Corollary 3.4 *Let $T(\rho,\theta) = e^{-\varepsilon\theta}\rho$, $\mathcal{P}_\varepsilon$ the Poincaré return map of (3.3) defined on Σ_ν and $\tilde{\mathcal{P}}_\varepsilon$ the Poincaré map of (3.8) defined on the same section, then we have*

$$\tilde{\mathcal{P}}_\varepsilon(x_0) = T(\mathcal{P}_\varepsilon(x_0), 2\pi), \forall x_0 > 0. \tag{3.11}$$

Proof: If $\rho(\theta, r_0)$ is the solution of (3.3) with $\rho(0, r_0) = r_0$, then

$$\mathcal{P}_\varepsilon(r_0) = \rho(2\pi, r_0).$$

Similarly $\tilde{\mathcal{P}}_\varepsilon(r_0) = r(2\pi, r_0)$ where $r(\theta, r_0)$ is the solution of (3.8) with initial condition $r(0, r_0) = r_0$. By (3.9) we have

$$r(\theta, r_0) = T(\rho(\theta, r_0), \theta),$$

for all $0 \leq \theta \leq 2\pi$, so for $\theta = 2\pi$ we have $\tilde{\mathcal{P}}_\varepsilon(r_0) = T(\mathcal{P}_\varepsilon(r_0), 2\pi)$. □

Remark 3.1 *For the other similarity transformations we will encounter we always have relation (3.11) between the Poincaré maps of the original and the transformed system.*

In the following proposition we do not eliminate low order terms in r, but we introduce the Taylor tail of $\sin\theta$ in order to put the system in a suitable form so we can describe the inductive part of the technique.

Proposition 3.5 *The system defined by (3.10) is transformed into*

$$\begin{aligned} \dot r_1 &\widehat{=} -k_0(\varepsilon)(2s_0(\theta)\varepsilon + c_0(\theta)) e^{2\varepsilon\theta} r^3 \\ &+ \sum_{n\geq 0}(k_{n+1}(\varepsilon) - 3k_0(\varepsilon)k_n(\varepsilon)s_0(\theta)) e^{2(n+2)\varepsilon\theta} r^{2n+5} \end{aligned} \tag{3.12}$$

by the transformation

$$r_1 = r - k_0(\varepsilon) e^{2\varepsilon\theta} s_0(\theta) r^3. \tag{3.13}$$

Proof: Differentiating (3.13) with respect to t we find that

$$\begin{aligned} \dot r_1 &= \dot r - 2\varepsilon k_0(\varepsilon) e^{2\varepsilon\theta} s_0(\theta) r^3 \dot\theta \\ &- k_0(\varepsilon) e^{2\varepsilon\theta} s_0'(\theta) r^3 \dot\theta - 3k_0(\varepsilon) e^{2\varepsilon\theta} s_0(\theta) r^2 \dot r, \end{aligned}$$

so using (3.10) we find that

$$\begin{aligned}\dot{r}_1 \ &\widehat{=}\ \sum_{n\geq 0} k_n(\varepsilon)e^{2(n+1)\varepsilon\theta}r^{2n+3} - 2\varepsilon k_0(\varepsilon)e^{2\varepsilon\theta}s_0(\theta)r^3 \\ &- \ k_0(\varepsilon)e^{2\varepsilon\theta}(c_0(\theta)+1)r^3 \\ &- \ 3k_0(\varepsilon)e^{2\varepsilon\theta}s_0(\theta)r^2\left(\sum_{n\geq 0} k_n(\varepsilon)e^{2(n+1)\varepsilon\theta}r^{2n+3}\right),\end{aligned}$$

hence

$$\begin{aligned}\dot{r}_1 \ &\widehat{=}\ -k_0(\varepsilon)(2s_0(\theta)\varepsilon + c_0(\theta))e^{2\varepsilon\theta}r^3 \\ &+ \ \sum_{n\geq 0}(k_{n+1}(\varepsilon) - 3k_0(\varepsilon)k_n(\varepsilon)s_0(\theta))e^{2(n+2)\varepsilon\theta}r^{2n+5}\end{aligned}$$

which is exactly (3.12). □

We point out that we do not introduce r_1 in the right-hand side of (3.12). This way it will be easier to describe the "induction step" - i.e. Proposition 3.6 - and as we are only working up to some finite order in r, at the end we can always use (3.13) and (3.15) to replace $\mathcal{O}(r_n^m)$ by $\mathcal{O}(r^m)$ for any $m \in \mathbb{N}$.

Proposition 3.6 *For every integer $n > 0$ and for every polynomial $P_n : \mathbb{R}^4 \to \mathbb{R}$ (with coefficients depending on ε), we consider the system defined by*

$$\dot{R}_1 \widehat{=} \sum_{k\geq n} P_k(s_0(\theta), c_0(\theta), s_1(\theta), \Omega(\theta))e^{2k\varepsilon\theta}r^{2k+1} \tag{3.14}$$

with

$$R_1(r) = r + r^3\mathcal{R}(e^{2\varepsilon\theta}r^2; s_0(\theta), c_0(\theta), s_1(\theta), \Omega(\theta), \varepsilon),$$

where $\mathcal{R}$ is a polynomial in its first variable with coefficients that are functions of the other variables.

By means of the transformation

$$R_2 = R_1 - \tilde{P}_n(s_0(\theta), c_0(\theta), s_1(\theta), \Omega(\theta))e^{2n\varepsilon\theta}R_1^{2n+1}, \tag{3.15}$$

where $\tilde{P}_n : \mathbb{R}^4 \to \mathbb{R}$ is a polynomial (with coefficients depending on ε), system (3.14) is transformed into

$$\dot{R}_2 \widehat{=} \sum_{k\geq n+1} Q_k(s_0(\theta), c_0(\theta), s_1(\theta), \Omega(\theta))e^{2k\varepsilon\theta}r^{2k+1}, \tag{3.16}$$

where each $Q_k : \mathbb{R}^4 \to \mathbb{R}$ is a polynomial (with coefficients depending on ε).

PROOF: Considering the transformation given by (3.15) and applying it to (3.16), then a straightforward calculation gives us

$$\begin{aligned}
\dot{R}_2 &= \dot{R}_1 - \frac{d}{d\theta}\left(\tilde{P}_n(s_0(\theta), c_0(\theta), s_1(\theta), \Omega(\theta))e^{2n\varepsilon\theta}\right) R_1^{2n+1} \\
&- (2n+1)\tilde{P}_n(s_0(\theta), c_0(\theta), s_1(\theta), \Omega(\theta))e^{2n\varepsilon\theta} R_1^{2n}\dot{R}_1 \\
&\widehat{=} \sum_{k\geq n} P_k(s_0(\theta), c_0(\theta), s_1(\theta), \Omega(\theta))e^{2k\varepsilon\theta} r^{2k+1} \\
&- \frac{d}{d\theta}\left(\tilde{P}_n(s_0(\theta), c_0(\theta), s_1(\theta), \Omega(\theta))e^{2n\varepsilon\theta}\right) \\
&\cdot(r + r^3\mathcal{R}(e^{2\varepsilon\theta}r^2; s_0(\theta), c_0(\theta), s_1(\theta), \Omega(\theta), \varepsilon))^{2n+1} \\
&- (2n+1)\tilde{P}_n(s_0(\theta), c_0(\theta), s_1(\theta), \Omega(\theta))e^{2n\varepsilon\theta} \\
&\cdot(r + r^3\mathcal{R}(e^{2\varepsilon\theta}r^2; s_0(\theta), c_0(\theta), s_1(\theta), \Omega(\theta), \varepsilon))^{2n} \\
&\cdot \sum_{k\geq n} P_k(s_0(\theta), c_0(\theta), s_1(\theta), \Omega(\theta))e^{2k\varepsilon\theta} r^{2k+1}
\end{aligned}$$

Hence

$$\begin{aligned}
\dot{R}_2 &\widehat{=} \Big[P_n(s_0(\theta), c_0(\theta), s_1(\theta), \Omega(\theta))e^{2n\varepsilon\theta} \\
&- \frac{d}{d\theta}\left(\tilde{P}_n(s_0(\theta), c_0(\theta), s_1(\theta), \Omega(\theta))e^{2n\varepsilon\theta}\right)\Big] r^{2n+1} \\
&+ \sum_{k\geq n+1} P_k(s_0(\theta), c_0(\theta), s_1(\theta), \Omega(\theta))e^{2k\varepsilon\theta} r^{2k+1} \\
&- \frac{d}{d\theta}\left(\tilde{P}_n(s_0(\theta), c_0(\theta), s_1(\theta), \Omega(\theta))e^{2n\varepsilon\theta}\right) \\
&\cdot \sum_{m=1}^{2n+1} \binom{2n+1}{m} r^{2n+2m+1}\mathcal{R}(e^{2\varepsilon\theta}r^2; s_0(\theta), c_0(\theta), s_1(\theta), \Omega(\theta), \varepsilon)^m \\
&- (2n+1)\tilde{P}_n(s_0(\theta), c_0(\theta), s_1(\theta), \Omega(\theta))e^{2n\varepsilon\theta} \\
&\cdot (r + r^3\mathcal{R}(e^{2\varepsilon\theta}r^2; s_0(\theta), c_0(\theta), s_1(\theta), \Omega(\theta), \varepsilon))^{2n} \\
&\cdot \sum_{k\geq n} P_k(s_0(\theta), c_0(\theta), s_1(\theta), \Omega(\theta))e^{2k\varepsilon\theta} r^{2k+1}. \qquad (3.17)
\end{aligned}$$

Our main target is to obtain $\dot{R}_2 = \mathcal{O}(r^{2n+3})$, so we need that

$$\frac{d}{d\theta}\left(\tilde{P}_n(s_0(\theta), c_0(\theta), s_1(\theta), \Omega(\theta))e^{2n\varepsilon\theta}\right) = P_n(s_0(\theta), c_0(\theta), s_1(\theta), \Omega(\theta))e^{2n\varepsilon\theta}$$

or equivalently (if we take the initial condition $\tilde{P}_n(0,0,0,0) = 0$)

$$\tilde{P}_n(s_0(\theta), c_0(\theta), s_1(\theta), \Omega(\theta))e^{2n\varepsilon\theta} = \int_0^\theta P_n(s_0(u), c_0(u), s_1(u), \Omega(u))e^{2n\varepsilon u}du. \qquad (3.18)$$

By Proposition 3.2 we know that the right-hand side of (3.17) is a polynomial in $(s_0(\theta), c_0(\theta), s_1(\theta), \Omega(\theta))$ multiplied by $e^{2n\varepsilon\theta}$. This means that $\tilde{P}_n$ is a polynomial as well.

From (3.17) and (3.18) it is clear that Q_k is a polynomial as well. □

Proof(of Theorem 3.1): After applying Proposition 3.3 and Proposition 3.5, we can start applying Proposition 3.6 n times, that way we obtain some r_n for which we have

$$\dot{r}_n = e^{2n\varepsilon\theta}\mathcal{O}(r^{2n+1}), \tag{3.19}$$

which permits us to conclude the proof. □

As we gave a constructive proof of Theorem 3.1, Proposition 3.3, Proposition 3.5 and Proposition 3.6 form the backbone of a technique that allows to calculate an asymptotic expression of the Poincaré map. We describe briefly how the technique works, in the next section we will apply this technique to two illustrative examples.

First we start from a system that is written in the form given by (3.3). Second we apply Proposition 3.3 which eliminates the first order term in the phase variable. Next we apply Proposition 3.5 in order to introduce the Taylor tails into the system. As Proposition 3.6 is written in an inductive form one can choose how many times one likes to apply this result. This way one obtains a system given by (3.19). Up to this point we have only repeated the proof of Theorem 3.1. In order to do explicit calculations we transform (3.19) into an equation in r_n, this gives an equation of the form

$$\dot{r}_n = e^{2n\varepsilon\theta}\mathcal{O}(r_n^{2n+1}), \tag{3.20}$$

which we then truncate in order to obtain

$$\dot{r}_n = C(\theta;\varepsilon)r_n^{2n+1} \tag{3.21}$$

where $C(\theta;\varepsilon)$ is a function of θ and ε which can be calculated explicitly. Now (3.21) can by solved explicitly (even by hand). As we want to return to the original variables, we need to invert the transformations we used. The transformation given by (3.9) is very easy to invert. The other transformations (given by (3.13) and (3.15)) are near-identity transformations, i.e.

$$\phi(r) = r + C(\theta;\varepsilon)r^m + \mathcal{O}(r^{m+1})$$

for $m \geq 2$ and some function C. So the inverse is given by

$$\phi^{-1}(r) = r - C(\theta;\varepsilon)r^m + \mathcal{O}(r^{m+1}),$$

which we will truncate to

$$\phi^{-1}(r) = r - C(\theta;\varepsilon)r^m. \tag{3.22}$$

Now applying these truncated inverse transformations we obtain an asymptotic solution of (3.3) which we use to calculate asymptotic expansions of the Poincaré map. As written before, all this will become more clear in the next section.

3.4 Hopf–Takens models

In this section we demonstrate on an "academic" example how our technique works. Choosing the Hamiltonian as

$$H(x, y) = -\frac{1}{2}(x^2 + y^2), \tag{3.23}$$

the Hamiltonian (unperturbed) vector field is given by

$$X_0 = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}. \tag{3.24}$$

The local deformation will be given by a linear perturbation of X_0:

$$X_\varepsilon = X_0 + \varepsilon Y, \tag{3.25}$$

where

$$Y = x\left(1 + \sum_{k=1}^{N} a_k(x^2 + y^2)^k\right)\frac{\partial}{\partial x} + y\left(1 + \sum_{k=1}^{N} a_k(x^2 + y^2)^k\right)\frac{\partial}{\partial y} \tag{3.26}$$

with $a_k \in \mathbb{R}$ and $N \in \mathbb{N} \setminus \{0\}$. So X_ε is given by

$$X_\varepsilon : \left\{ \begin{array}{rcl} \dot{x} & = & \varepsilon x - y + \varepsilon x \sum_{k=1}^{N} a_k(x^2 + y^2)^k \\ \dot{y} & = & x + \varepsilon y + \varepsilon y \sum_{k=1}^{N} a_k(x^2 + y^2)^k \end{array} \right. , \tag{3.27}$$

or written in polar coordinates we have

$$X_\varepsilon : \left\{ \begin{array}{rcl} \dot{\rho} & = & \varepsilon\rho + \varepsilon \sum_{k=1}^{N} a_k \rho^{2k+1} \\ \dot{\theta} & = & 1 \end{array} \right. , \tag{3.28}$$

which is a simplified form of the equation used to study the Hopf–Takens bifurcation of codimension N, see e.g. [BR01, CD04].

One could wonder if the system given by (3.27) is "general" enough to pass for an academic example because the perturbation is linear with respect to ε whilst in (3.2) the perturbation is C^∞ with respect to ε. We demonstrate that (3.26) can be transformed into a system like in (3.2) by means of the rescaling

$$\begin{array}{rcl} x & = & \varepsilon\bar{x}, \\ y & = & \varepsilon\bar{y}. \end{array}$$

This rescaling gives us that $H(x, y) = H(\varepsilon\bar{x}, \varepsilon\bar{y}) = -\varepsilon^2\frac{1}{2}(\bar{x}^2 + \bar{y}^2)$, hence

$$\begin{array}{rcl} \dfrac{\partial H}{\partial x}(x, y) & = & x \\ & = & \varepsilon\bar{x} \\ & = & \varepsilon\dfrac{\partial H}{\partial \bar{x}}(\bar{x}, \bar{y}) \end{array}$$

and

$$\begin{aligned}\frac{\partial H}{\partial y}(x,y) &= y \\ &= \varepsilon\bar{y} \\ &= \varepsilon\frac{\partial H}{\partial \bar{y}}(\bar{x},\bar{y}).\end{aligned}$$

So for $\varepsilon \neq 0$,

$$\begin{aligned}\dot{\bar{x}} &= \frac{1}{\varepsilon}\dot{x} \\ &= \frac{1}{\varepsilon}\left(-y+\varepsilon x(1+\sum_{k=1}^{N} a_k(x^2+y^2)^k)\right) \\ &= \frac{1}{\varepsilon}\left(-\varepsilon\bar{y}+\varepsilon^2\bar{x}(1+\varepsilon\sum_{k=1}^{N} a_k\varepsilon^{2k}(\bar{x}^2+\bar{y}^2)^k)\right) \\ &= \varepsilon\bar{x}-\bar{y}+\bar{x}\sum_{k=1}^{N} R_k(\bar{x},\bar{y})\varepsilon^{2k+1},\end{aligned}$$

where $R_k(\bar{x},\bar{y}) = a_k(\bar{x}^2+\bar{y}^2)^k$. Likewise we obtain

$$\dot{\bar{y}} = \bar{x}+\varepsilon\bar{y}+\bar{y}\sum_{k=1}^{N} R_k(\bar{x},\bar{y})\varepsilon^{2k+1}.$$

This demonstrates that (3.27) is a natural example to consider without loss of generality.

In order to make the calculation of the Melnikov functions less heavy to perform, one wishes to truncate the expression given by (3.28) to obtain a polynomial of a rather low degree. Naturally the question arises if after truncation we find an exact expression of the kth Melnikov function or an asymptotic expression of the kth Melnikov function. The following result will be of great help to conclude when the expression is exact if one calculates the Melnikov functions of (3.28).

Lemma 3.7 *The Poincaré map $\mathcal{P}_\varepsilon$ of (3.28) has the following formal power series expansion $\hat{\mathcal{P}}_\varepsilon$ with respect to ε:*

$$\hat{\mathcal{P}}_\varepsilon(x) = x + x\left(1+\sum_{k=1}^{N} a_k x^{2k}\right)\sum_{m=1}^{\infty} P_{(m-1)N}(x^2)\varepsilon^m \tag{3.29}$$

where - for each integer j - P_{jN} is a polynomial of degree jN.

PROOF: Taking in account the special form of the equations in (3.28), we can consider the following equation

$$\frac{d\rho}{\rho + \sum_{k=1}^{N} a_k \rho^{2k+1}} = \varepsilon d\theta. \tag{3.30}$$

Introducing the function

$$F(\rho) := \int_x^\rho \frac{dr}{r + \sum_{k=1}^{N} a_k r^{2k+1}} \tag{3.31}$$

where $x \neq 0$, we have that (3.30) is equivalent with

$$F(\rho_\varepsilon(\theta)) = \varepsilon\theta \tag{3.32}$$

where $\rho_\varepsilon(\theta)$ is the solution of (3.30) respecting the initial condition $\rho_\varepsilon(0) = x$. Taking $\theta = 2\pi$ gives us

$$F(\mathcal{P}_\varepsilon(x)) = 2\pi\varepsilon. \tag{3.33}$$

As we took $x > 0$, we have that

$$\frac{dF}{d\rho}(\rho) = \frac{1}{\rho + \sum_{k=1}^{N} a_k \rho^{2k+1}}$$

exists and is non-zero for all ρ in a sufficiently small neighbourhood of x. By the Inverse Function Theorem we know that F is invertible. We are now allowed to transform (3.33) into

$$\mathcal{P}_\varepsilon(x) = F^{-1}(2\pi\varepsilon).$$

We now calculate the formal power series expansion of $\mathcal{P}_\varepsilon(x)$ with respect to ε. From (3.31) we get that

$$\mathcal{P}_\varepsilon(x)|_{\varepsilon=0} = F^{-1}(0) = x.$$

We have

$$\begin{aligned}
\frac{\partial}{\partial\varepsilon}\mathcal{P}_\varepsilon(x) &= 2\pi(F^{-1})'(2\pi\varepsilon) \\
&= \frac{2\pi}{F'(F^{-1}(2\pi\varepsilon))} \\
&= \frac{2\pi}{F'(\mathcal{P}_\varepsilon(x))} \\
&= 2\pi\left(\mathcal{P}_\varepsilon(x) + \sum_{k=1}^{N} a_k \mathcal{P}_\varepsilon(x)^{2k+1}\right),
\end{aligned} \tag{3.34}$$

so

$$\left.\frac{\partial}{\partial\varepsilon}\mathcal{P}_\varepsilon(x)\right|_{\varepsilon=0} = 2\pi x\left(1 + \sum_{k=1}^{N} a_k x^{2k}\right). \tag{3.35}$$

We now proceed by induction, so assume that

$$\frac{\partial^m}{\partial\varepsilon^m}\mathcal{P}_\varepsilon(x) = \mathcal{P}_\varepsilon(x)\left(1+\sum_{k=1}^{N}a_k\mathcal{P}_\varepsilon(x)^{2k}\right)P_{(m-1)N}(\mathcal{P}_\varepsilon(x)^2). \tag{3.36}$$

Using (3.36) we obtain

$$\begin{aligned}
\frac{\partial^{m+1}}{\partial\varepsilon^{m+1}}\mathcal{P}_\varepsilon(x) &= \frac{\partial}{\partial\varepsilon}\left[\frac{\partial^m}{\partial\varepsilon^m}\mathcal{P}_\varepsilon(x)\right] \\
&= \frac{\partial}{\partial\varepsilon}\left[\mathcal{P}_\varepsilon(x)\left(1+\sum_{k=1}^{N}a_k\mathcal{P}_\varepsilon(x)^{2k}\right)P_{(m-1)N}(\mathcal{P}_\varepsilon(x)^2)\right] \\
&= \frac{\partial}{\partial\varepsilon}\mathcal{P}_\varepsilon(x)\left(1+\sum_{k=1}^{N}(2k+1)a_k\mathcal{P}_\varepsilon(x)^{2k}\right) \\
&\cdot\ P_{(m-1)N}(\mathcal{P}_\varepsilon(x)^2) \\
&+\ 2\mathcal{P}_\varepsilon(x)\left(1+\sum_{k=1}^{N}a_k\mathcal{P}_\varepsilon(x)^{2k}\right) \\
&\cdot\ P'_{(m-1)N}(\mathcal{P}_\varepsilon(x)^2)\mathcal{P}_\varepsilon(x)\frac{\partial}{\partial\varepsilon}\mathcal{P}_\varepsilon(x) \\
&= \frac{\partial}{\partial\varepsilon}\mathcal{P}_\varepsilon(x) \\
&\cdot\ \left[\left(1+\sum_{k=1}^{N}(2k+1)a_k\mathcal{P}_\varepsilon(x)^{2k}\right)P_{(m-1)N}(\mathcal{P}_\varepsilon(x)^2)\right. \\
&+\ \left.2\left((1+\sum_{k=1}^{N}a_k\mathcal{P}_\varepsilon(x)^{2k})\mathcal{P}_\varepsilon(x)^2\right)P'_{(m-1)N}(\mathcal{P}_\varepsilon(x)^2)\right].
\end{aligned}$$

In the latter formula one observes that in between the square brackets we have a polynomial of degree mN in $\mathcal{P}_\varepsilon(x)^2$ and using (3.34) we have

$$\frac{\partial^{m+1}}{\partial\varepsilon^{m+1}}\mathcal{P}_\varepsilon(x) = \mathcal{P}_\varepsilon(x)\left(1+\sum_{k=1}^{N}a_k\mathcal{P}_\varepsilon(x)^{2k}\right)P_{mN}(\mathcal{P}_\varepsilon(x)^2) \tag{3.37}$$

which permits us to conclude the induction step. Taking $\varepsilon = 0$ in (3.37) gives us

$$\left.\frac{\partial^{m+1}}{\partial\varepsilon^{m+1}}\mathcal{P}_\varepsilon(x)\right|_{\varepsilon=0} = x\left(1+\sum_{k=1}^{N}a_kx^{2k}\right)P_{mN}(x^2). \tag{3.38}$$

This way we can write down the formal power series expansion $\hat{\mathcal{P}}_\varepsilon$ of $\mathcal{P}_\varepsilon$ with respect to ε:

$$\hat{\mathcal{P}}_\varepsilon(x) = x + x\left(1+\sum_{k=1}^{N}a_kx^{2k}\right)\sum_{m=1}^{\infty}P_{(m-1)N}(x^2)\varepsilon^m.$$

□

Remark 3.2 *From (3.29) we can conclude that the kth Melnikov function is a polynomial of degree $2kN+1$ in x, so if we wish to have an exact expression of the kth Melnikov function we need to be sure that the truncations we perform will not affect the terms up to degree $2kN+1$ in x.*

Remark 3.3 *The fact that - by virtue of Lemma 3.7 - we can obtain the degree of the truncated polynomial that will assure us an exact expression of the first k Melnikov functions, is a direct consequence of the fact that we started from a polynomial system. If we would take a more general system, then all Melnikov functions may depend on an infinite number of powers of x. In this situation the Melnikov functions might no longer be polynomials in x.*

We now illustrate our technique by 2 explicit examples.

3.4.1 Neimark–Sacker case

For the rest of this subsection we will take $N = 1$ and $a_1 = -1$ to obtain the following family of vector fields

$$X_\varepsilon : \begin{cases} \dot{\rho} &= \varepsilon\rho - \varepsilon\rho^3 \\ \dot{\theta} &= 1 \end{cases}, \tag{3.39}$$

which is the equation used to study the Neimark-Sacker bifurcation, see e.g. [BR01]. We also take this example as it permits us to calculate the Poincaré map directly and this way we can also compare the results of our technique with the exact results.

We calculate the Poincaré map in 3 different ways: first we solve (3.39) directly, this is of course not possible in more complicated systems, second we calculate an asymptotic expansion of the Poincaré map using an Abelian integral and finally we calculate an asymptotic expansion of the Poincaré map using our technique.

Direct approach

Finding all solutions of (3.39) we can divide the first equation of (3.39) by the second equation, this gives us

$$\rho' := \frac{d\rho}{d\theta} = \varepsilon\rho - \varepsilon\rho^3. \tag{3.40}$$

As we want to calculate the Poincaré map, knowing ρ as function of θ is actually an advantage. It's fairly easy to solve (3.40) as it is an equation of Bernoulli type, so one finds after some elementary calculations that

$$\rho(\theta) = \frac{x_0 e^{\varepsilon\theta}}{\sqrt{1 + x_0^2(e^{2\varepsilon\theta} - 1)}}, \tag{3.41}$$

where $\rho(0) = x_0$.

This means that the Poincaré map of (3.40) is given by

$$\mathcal{P}_\varepsilon(x_0) = \frac{x_0 e^{2\pi\varepsilon}}{\sqrt{1 + x_0^2(e^{4\pi\varepsilon} - 1)}}. \tag{3.42}$$

In order to compare the 2 other approaches we give 2 asymptotic expansions of (3.42), one with respect to ε and one with respect to x_0:

$$\begin{aligned} \mathcal{P}_\varepsilon(x_0) &= x_0 + 2\pi x_0(1 - x_0^2)\varepsilon + 2\pi^2 x_0(1 - 4x_0^2 + 3x_0^4)\varepsilon^2 \\ &+ \mathcal{O}(\varepsilon^3), \qquad (3.43) \\ \mathcal{P}_\varepsilon(x_0) &= e^{2\pi\varepsilon} x_0 + e^{2\pi\varepsilon}\frac{1 - e^{4\pi\varepsilon}}{2} x_0^3 + \mathcal{O}(x_0^5). \qquad (3.44) \end{aligned}$$

We want to stress once more that the direct approach is in general not applicable.

Abelian integral approach

In (1.50) we deduced a formal expansion of the Poincaré map using Abelian integrals. We apply this formula here, so we have

$$\mathsf{P}_\varepsilon(h) = h - \varepsilon I(h) + \mathcal{O}(\varepsilon^2),$$

with

$$I(h) = \oint_{\Gamma_h} \nu,$$

where $\Gamma_h = \{H = h\}$ and ν is the dual form of the perturbation Y, so we have

$$\nu = y(1 - (x^2 + y^2))dx - x(1 - (x^2 + y^2))dy.$$

In this example Γ_h is given by the equation

$$\frac{x^2 + y^2}{2} = h,$$

so

$$\sigma : [0, 2\pi] \to \mathbb{R}^2 : t \mapsto (\sqrt{2h}\cos(t), \sqrt{2h}\sin(t)),$$

is a parametrisation of Γ_h. Therefore the Abelian integral can be calculated directly and we find

$$\begin{aligned} I(h) &= \oint_{\Gamma_h} \nu \\ &= \oint_{\Gamma_h} \big(y(1-(x^2+y^2))dx - x(1-(x^2+y^2))dy\big) \\ &= \int_0^{2\pi} \Big(\sqrt{2h}\sin t(1-2h)(-\sqrt{2h})\sin t \\ &- \sqrt{2h}\cos t(1-2h)(\sqrt{2h})\cos t\Big)\,dt \\ &= -2h(1-2h)\int_0^{2\pi} dt, \end{aligned}$$

hence

$$I(h) = -4\pi h(1-2h). \tag{3.45}$$

Thus the Poincaré map is given by

$$\mathsf{P}_\varepsilon(h) = h + 4\pi h(1-2h)\varepsilon + \mathcal{O}(\varepsilon^2). \tag{3.46}$$

As P_ε and $\mathcal{P}_\varepsilon$ are conjugate by the transformation $H_0(x) = \dfrac{x^2}{2}$, we have that

$$\mathcal{P}_\varepsilon(x_0) = x_0 + 2\pi x_0(1-x_0^2)\varepsilon + \mathcal{O}(\varepsilon^2)$$

which is the asymptotic expansion we had in (3.43) but up to order 1 in ε.

Approach with our technique

We start from (3.40). Applying the first transformation then $k_0(\varepsilon) = -\varepsilon$ and

$$r = T(\rho,\theta) = e^{-\varepsilon\theta}\rho, \tag{3.47}$$

we have

$$r' = -\varepsilon e^{2\varepsilon\theta} r^3. \tag{3.48}$$

Now we apply the second transformation

$$r_1 = T_1(r,\theta) = r + \varepsilon s_0(\theta) e^{2\varepsilon\theta} r^3. \tag{3.49}$$

We get

$$r_1' = \varepsilon(c_0(\theta) + 2\varepsilon s_0(\theta))e^{2\varepsilon\theta} r^3 - 3\varepsilon^2 s_0(\theta) e^{4\varepsilon\theta} r^5. \tag{3.50}$$

By virtue of (3.18) we can calculate the next transformation. As

$$\int_0^\theta (c_0(u) + 2\varepsilon s_0(u))e^{2\varepsilon u} = e^{2\varepsilon\theta}\left(s_0(\theta) - \Omega(\theta) + \frac{\varepsilon}{2}\Omega(\theta)^2\right),$$

we have that the third transformation is given by

$$r_2 = T_2(r_1, \theta) = r_1 - \varepsilon e^{2\varepsilon\theta}\left(s_0(\theta) - \Omega(\theta) + \frac{\varepsilon}{2}\Omega(\theta)^2\right) r_1^3, \tag{3.51}$$

and end up with

$$\begin{aligned} r_2' &= \varepsilon^2[3c_0(\theta)\Omega(\theta) - 3s_0(\theta) + \varepsilon(12c_0^2(\theta) \qquad (3.52)\\ &+ 24c_0(\theta) + 6s_0(\theta)\Omega(\theta) - \frac{3}{2}\Omega(\theta)^2 c_0(\theta)) \\ &\quad -3\varepsilon^3 s_0(\theta)\Omega(\theta)^4]e^{4\varepsilon\theta}r^5 + \mathcal{O}(r^7). \qquad (3.53)\end{aligned}$$

Taking the composition of (3.49) and (3.51) one finds that

$$r_2 = r + \mathcal{O}(r^3),$$

so using this relation between r and r_2 and truncating $\mathcal{O}(r_2^7)$ from (3.53) we find

$$r_2' = \varepsilon^2 C(\theta) e^{4\varepsilon\theta} r_2^5, \tag{3.54}$$

where

$$\begin{aligned} C(\theta) &= 3c_0(\theta)\Omega(\theta) - 3s_0(\theta) + \varepsilon(12c_0^2(\theta) \\ &+ 24c_0(\theta) + 6s_0(\theta)\Omega(\theta) - \frac{3}{2}\Omega(\theta)^2 c_0(\theta)) \\ &- 3\varepsilon^3 s_0(\theta)\Omega(\theta)^4. \qquad (3.55)\end{aligned}$$

One easily solves (3.54) to find

$$r_2(\theta) = \frac{x_0}{\sqrt[4]{1 - 4\varepsilon^2 C_1(\theta) x_0^4}}, \tag{3.56}$$

where

$$C_1(\theta) = \int_0^\theta C(u) e^{4\varepsilon u} du. \tag{3.57}$$

For $\theta = 2\pi$, we have the Poincaré map of (3.54)

$$r_2(2\pi) = \frac{x_0}{\sqrt[4]{1 - 4\varepsilon^2 C_1(2\pi) x_0^4}}. \tag{3.58}$$

We use the inverse transformations of (3.47), (3.49) and (3.51). Using (3.22) we get the inverse of (3.51):

$$r_1 = r_2 + \varepsilon e^{2\varepsilon\theta}(s_0(\theta) - \Omega(\theta) + \frac{\varepsilon}{2}\Omega(\theta)^2) r_2^3 + \mathcal{O}(r_2^5). \tag{3.59}$$

So for $\theta = 2\pi$ we have

$$r_1(2\pi) = r_2(2\pi) + \varepsilon e^{4\pi\varepsilon}(-\Omega(2\pi) + \frac{\varepsilon}{2}\Omega(2\pi)^2) r_2(2\pi)^3 + \mathcal{O}(r_2(2\pi)^5). \tag{3.60}$$

As for $\theta = 2\pi$, (3.60) becomes

$$r_1(2\pi) = r(2\pi),$$

we immediately have

$$r(2\pi) = r_2(2\pi) + \varepsilon e^{4\pi\varepsilon}(-\Omega(2\pi) + \frac{\varepsilon}{2}\Omega(2\pi)^2)r_2(2\pi)^3 + \mathcal{O}(r_2(2\pi)^5). \tag{3.61}$$

Finally by (3.47), (3.58) and (3.61) we have

$$\mathcal{P}_\varepsilon(x_0) = e^{2\pi\varepsilon}r_2(2\pi) + \varepsilon e^{6\pi\varepsilon}(-\Omega(2\pi) + \frac{\varepsilon}{2}\Omega(2\pi)^2)r_2(2\pi)^3 + \mathcal{O}(r_2(2\pi)^5). \tag{3.62}$$

Now we substitute the result of (3.58) in (3.62) and using a computeralgebra programme (e.g. Maple, see Subsection 3.6.1 for the Maple output), we find the following expansions for the Poincaré map:

$$\begin{aligned} \mathcal{P}_\varepsilon(x_0) &= x_0 + (2\pi x_0 - 2\pi x_0^3)\varepsilon + (2\pi^2 x_0 - 8\pi^2 x_0^3 + 6\pi^2 x_0^5)\varepsilon^2 \\ &\quad + \mathcal{O}(\varepsilon^3), & (3.63) \\ \mathcal{P}_\varepsilon(x_0) &= (1 + 2\pi\varepsilon + \mathcal{O}(\varepsilon^2))x_0 + (-2\pi\varepsilon - 8\pi^2\varepsilon^2 - \frac{52}{3}\pi^3\varepsilon^3 + \mathcal{O}(\varepsilon^4))x_0^3 \\ &\quad + \mathcal{O}(x_0^5). & (3.64) \end{aligned}$$

From Remark 3.2 we know that the degree of the polynomial which may not be changed by truncations if we want to obtain an exact expression of the kth Melnikov function, has to be at least $2kN + 1$. In our example we have $N = 1$ and in order to have an exact expression of the first and the second Melnikov function we need to assure ourselves that we didn't delete any terms up to order 5 in x_0 by performing trunctations. This has been done using two transformations and deleting only terms of order $\mathcal{O}(r_2^7)$, moreover one can compare the results from (3.43) and (3.63) to conclude they are equal. In (3.64) we get an expansion of the Poincaré map with respect to x_0.

3.4.2 Chenciner case

We now take $N = 2$ in (3.26) and choose $a_1 = \lambda$ and $a_2 = c$. This way we obtain

$$X_\varepsilon : \begin{cases} \dot{\rho} &= \varepsilon\rho(1 + \lambda\rho^2 + c\rho^4) \\ \dot{\theta} &= 1 \end{cases} \tag{3.65}$$

which is a simplified form of one of the normal forms that are used to study the Chenciner bifurcation, see e.g. [BR01, Che85a, Che85b, Che88].

For this system the direct approach is not a good idea. In order to use the direct approach we need to calculate the following integral

$$\int \frac{d\rho}{\rho(1 + \lambda\rho^2 + c\rho^4)}$$

which is not easy to calculate for unknown values of λ and c. So traditionally one turns to the Abelian integral technique. We calculate

$$I(h) = \oint_{\Gamma_h} \nu,$$

where $\Gamma_h = \{H = h\}$ and ν is the dual form of the perturbation Y, so we have

$$\nu = y(1 + \lambda(x^2 + y^2) + c(x^2 + y^2)^2)dx - x(1 + \lambda(x^2 + y^2) + c(x^2 + y^2)^2)dy.$$

In this example Γ_h is given by the equation

$$\frac{x^2 + y^2}{2} = h,$$

so

$$\sigma : [0, 2\pi] \to \mathbb{R}^2 : t \mapsto (\sqrt{2h}\cos t, \sqrt{2h}\sin t),$$

is a parametrisation of Γ_h. Therefore we can calculate the Abelian integral directly

$$\begin{aligned} I(h) &= \oint_{\Gamma_h} \nu \\ &= \int_0^{2\pi} \Big(\sqrt{2h}\sin(t)(1 + 2\lambda h + c(2h)^2)(-\sqrt{2h}\sin t) \\ &- \sqrt{2h}\cos(t)(1 + 2\lambda h + c(2h)^2)(\sqrt{2h}\cos t)\Big)\, dt \\ &= -2h(1 + 2\lambda h + 4ch^2)\int_0^{2\pi} dt \\ &= -4\pi h(1 + 2\lambda h + 4ch^2). \end{aligned}$$

Thus the Poincaré map is given by

$$\mathsf{P}_\varepsilon(h) = h + 4\pi h(1 + 2\lambda h + 4ch^2)\varepsilon + \mathcal{O}(\varepsilon^2).$$

As P_ε and $\mathcal{P}_\varepsilon$ are conjugate by the transformation $H_0(x) := \dfrac{x^2}{2}$, we have that

$$\mathcal{P}_\varepsilon(x_0) = x_0 + 2\pi x_0(1 + \lambda x_0^2 + c x_0^4)\varepsilon + \mathcal{O}(\varepsilon^2). \tag{3.66}$$

We now apply our technique to (3.65), as the expressions become longer we don't give all the details and refer to the Maple source code in Subsection 3.6.2. This time we apply 4 transformations. The first transformation comes directly from Proposition 3.3 and the second transformation is given by Proposition 3.5. The third and fourth transformation are given by Proposition 3.6 and (3.18).

We apply the following transformations (for the calculations of the functions `Co` and `Co2` we refer to Subsection 3.6.2):

$$r = e^{-\varepsilon\theta}\rho, \tag{3.67}$$
$$r_1 = r - \varepsilon\lambda s_0(\theta)e^{2\varepsilon\theta}r^3, \tag{3.68}$$
$$r_2 = r_1 - \mathtt{Co}(\theta)r_1^3, \tag{3.69}$$
$$r_3 = r_2 - \mathtt{Co2}(\theta)r_2^5. \tag{3.70}$$

After application of the 4 transformations we obtain

$$r_3' = C(\varepsilon,\theta)r_3^7 + \mathcal{O}(r_3^9), \tag{3.71}$$

where $C(\varepsilon,\theta)$ is a function of ε and θ that can be determined explicitly and that in Subsection 3.6.2 is represented by `r34`.

We truncate (3.71) to

$$r_3' = C(\varepsilon,\theta)r_3^7. \tag{3.72}$$

This equation can be solved directly and gives us

$$r_3(\theta) = x_0\left(1 - 6x_0^6\int_0^\theta C(\varepsilon,u)du\right)^{-\frac{1}{6}} \tag{3.73}$$

if we take $r_3(0) = x_0$ as initial condition.

By virtue of (3.22) we use the following truncated inverses of (3.67), (3.68), (3.69) and (3.70):

$$\rho = e^{\varepsilon\theta}r, \tag{3.74}$$
$$r = r_1 + \varepsilon\lambda s_0(\theta)e^{2\varepsilon\theta}r_1^3, \tag{3.75}$$
$$r_1 = r_2 + \mathrm{Co}(\theta)r_2^3, \tag{3.76}$$
$$r_2 = r_3 + \mathrm{Co2}(\theta)r_3^5. \tag{3.77}$$

So applying (3.77), (3.76), (3.75) and (3.74) to (3.73) and then taking $\theta = 2\pi$, we obtain an asymptotic expression of the Poincaré map of (3.65). Expanding this map with respect to ε or x_0 gives us

$$\begin{aligned}\mathcal{P}_\varepsilon(x_0) &= x_0 + 2\pi x_0(1+\lambda x_0^2 + cx_0^4)\varepsilon \\ &+ 2\pi^2 x_0(1 + 4\lambda x_0^2 + (6c - 3\lambda^2)x_0^4 + \mathcal{O}(x_0^6))\varepsilon^2 + \mathcal{O}(\varepsilon^3)\end{aligned} \tag{3.78}$$
$$\begin{aligned}\mathcal{P}_\varepsilon(x_0) &= (1 + 2\pi\varepsilon + 2\pi^2\varepsilon^2 + \mathcal{O}(\varepsilon^3))x_0 + 2\lambda\pi\varepsilon(1 + 4\pi\varepsilon + \mathcal{O}(\varepsilon^2))x_0^3 \\ &+ O(x_0^5).\end{aligned} \tag{3.79}$$

Using the bound given by Remark 3.2 in this case ($N = 2$), we may not have truncated any term up to degree 5 in order to have an exact expression of the first Melnikov function. In order to obtain an exact result of the second

Melnikov function we cannot truncate any term up to degree 9. As we applied 4 transformations all terms up to degree 7 are not subjected to any truncation, so in (3.78) we find an exact result on the first Melnikov function but for the second Melnikov function we only get an asymptotic result. A straightforward calculation using (3.29) shows that the asymptotics of the second Melnikov function is correct up to order 2 in x_0.

We point out that in practice, since we already know that the first Melnikov function does not vanish, we usually do not need to compute the second Melnikov function. In the next section, this problem is treated on a particular example.

3.5 The Hamiltonian triangle

3.5.1 Properties of the Hamiltonian triangle

We consider the Hamiltonian triangle, which is given by the complex ordinary differential equation

$$\dot{z} = -iz + \bar{z}^2, \tag{3.80}$$

or equivalently by the system of real ordinary differential equations

$$\left\{ \begin{array}{lcl} \dot{x} & = & y + x^2 - y^2 \\ \dot{y} & = & -x - 2xy \end{array} \right. . \tag{3.81}$$

The Hamiltonian of (3.81) is given by

$$H(x,y) = \frac{x^2+y^2}{2} + x^2 y - \frac{y^3}{3}. \tag{3.82}$$

As (3.81) is a Hamiltonian system, the level curves of H contain the orbits of (3.81).

In Figure 3.1 one sees 3 level curves that are straight lines. These lines are the level curves for $H(x,y) = \frac{1}{6}$ and they have the following cartesian equations

$$\begin{array}{lcl} y & = & -\frac{1}{2}, \\ x & = & \frac{y-1}{\sqrt{3}}, \\ x & = & -\frac{y-1}{\sqrt{3}}. \end{array}$$

Inside this "triangle" the level curves of (3.82) seem to be closed. This is equivalent with the fact that the Poincaré map of (3.81) coincides with the identity map in the neighbourhood $\mathcal{U}$ of the origin bounded by the "triangle". To prove

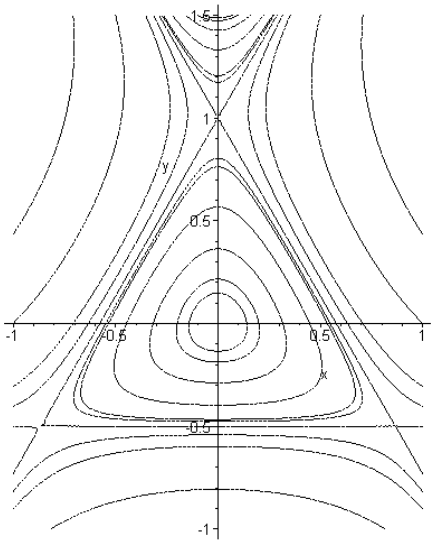

Figure 3.1: Level curves of (3.82).

this we transform (3.81) into an equivalent system in polar coordinates as we did in (1.37). So we obtain

$$\begin{cases} \dot{\rho} &= \rho^2 \frac{\cos(3\theta)}{1-\rho\sin(3\theta)} \\ \dot{\theta} &= 1 \end{cases} . \tag{3.83}$$

Lemma 3.8 *The Poincaré map of (3.83) coincides with the identity map in the - previously defined - neighbourhood $\mathcal{U}$ of the origin.*

Proof: From (3.83) we have:

$$\frac{d\rho}{d\theta} = \rho^2 \frac{\cos(3\theta)}{1-\rho\sin(3\theta)}. \tag{3.84}$$

Taking $u = \sin(3\theta)$ and $v = \dfrac{1}{\rho}$, (3.84) becomes:

$$\frac{du}{dv} = 3\frac{u}{v} - 3. \tag{3.85}$$

Solving (3.85) one finds

$$u = Cv^3 + \frac{3}{2}v$$

where C is some real constant, so returning to polar coordinates (ρ, θ) we have

$$\sin(3\theta) = \frac{C}{\rho(\theta)^3} + \frac{3}{2\rho(\theta)}. \tag{3.86}$$

Using the initial condition $\rho(0) = x_0$ gives us

$$0 = \frac{C}{x_0^3} + \frac{3}{2x_0},$$

so

$$C = -\frac{3}{2}x_0^2,$$

thus giving

$$\sin(3\theta) = \frac{3}{2}\left(\frac{1}{\rho(\theta)} - \frac{x_0^2}{\rho(\theta)^3}\right). \tag{3.87}$$

For $\theta = 2\pi$, we have $\rho(2\pi) = \mathcal{P}(x_0)$ and (3.87) becomes

$$0 = \frac{3}{2}\left(\frac{1}{\mathcal{P}(x_0)} - \frac{x_0^2}{\mathcal{P}(x_0)^3}\right). \tag{3.88}$$

This means that $\mathcal{P}(x_0) = \pm x_0$. But as $x_0 > 0$, the only possible solution to (3.88) is $\mathcal{P}(x_0) = x_0$. In other words the Poincaré map of (3.83) is the identity map in $\mathcal{U}$. □

Corollary 3.9 *The Poincaré map of a C^∞ local deformation of (3.83) is given by*

$$\mathcal{P}_\varepsilon(x_0) = x_0 + \sum_{j=1}^{k} M_j(x_0)\varepsilon^j + \mathcal{O}(\varepsilon^{k+1}). \tag{3.89}$$

In the next subsection we will introduce the deformation of (3.83) on which we want to apply our technique.

3.5.2 "Essential perturbation" of the Hamiltonian triangle

In [Ili98] the following local deformation of (3.83) is considered (we kept the same notations)

$$\dot{z} = (\lambda_1\varepsilon^3 - i)z + (\lambda_2\varepsilon^2 + i\lambda_3\varepsilon)z^2 + i\lambda_5\varepsilon|z|^2 + \bar{z}^2 \tag{3.90}$$

where λ_1, λ_2, λ_3 and λ_5 are real constants independent of ε. In [Ili98] the system given by (3.90) is called an "essential perturbation" of (3.83).

In real coordinates (3.90) becomes

$$\begin{cases} \dot{x} &= \lambda_1\varepsilon^3 x + y + (\lambda_2\varepsilon^2+1)(x^2-y^2) - 2\lambda_3\varepsilon xy \\ \dot{y} &= -x + \lambda_1\varepsilon^3 y + (\lambda_3+\lambda_5)\varepsilon x^2 + \varepsilon(\lambda_5-\lambda_3)y^2 + 2(\lambda_2\varepsilon^2-1)xy \end{cases}. \tag{3.91}$$

According to [Ili98] the first and the second order Melnikov functions in the formal expansion of the Poincaré map of (3.91) are zero and the third order Melnikov function is given by

$$M_3(h) = \int\int_{H(x,y)<h} \left[\mu_1 + \mu_2 x^{-1} + \mu_3 h^{-1} + \mu_4 h^{-1}(x-1)\ln x\right] dxdy$$

where H is the Hamiltonian (3.82) and μ_1, μ_2, μ_3 and μ_4 are cubic polynomials of λ_1, λ_2, λ_3 and λ_5.

In the next subsection we apply our technique to (3.91) and refind the asymptotics of the Poincaré map that are given by [Ili98].

3.5.3 Application of our method to the essential perturbation of the Hamiltonian triangle

Before applying our method to (3.91) it is necesarry to remark that (3.91) is not written in the form that is given by (3.2), which is the form we need to apply Theorem 3.1. We could of course use results on normal form theory to obtain a form like (3.2) for (3.91) but as the system is already written in a polynomial normal form we will not perform any normal form results in order to preserve this polynomial normal form. Therefore we will need to extend our technique to this type of normal form.

Using once again polar coordinates and (1.37), one has that (3.91) is equivalent with

$$\begin{cases} \dot{\rho} &= \rho\dfrac{\lambda_1\varepsilon^3 + \left(\lambda_2\varepsilon^2\cos\theta + \lambda_3\varepsilon\sin\theta - \lambda_5\varepsilon\sin\theta + \cos(3\theta)\right)\rho}{1 + (\lambda_2\varepsilon^2\sin\theta - \lambda_3\varepsilon\cos\theta - \lambda_5\varepsilon\cos\theta - \sin(3\theta))\rho} \\ \dot{\theta} &= 1 \end{cases}. \tag{3.92}$$

The presence of $\cos(3\theta)$ and $\sin(3\theta)$ in (3.92) forms no obstruction to our technique as it is straightforward to prove

$$\begin{aligned} \cos(3\theta) &= (c_0(\theta)+1)(c_0(\theta)^2 - 3s_0(\theta)^2 + 2c_0(\theta) + 1) \\ \sin(3\theta) &= s_0(\theta)(3c_0(\theta)^2 - s_0(\theta)^2 + 6c_0(\theta) + 3). \end{aligned}$$

A minor obstacle is that if we take the Taylor series of (3.92) for $\rho = 0$, then the coefficient of ρ is $\lambda_1\varepsilon^3$ and not ε, that is why during the application of our technique we will use the parameter δ instead of ε where

$$\delta = \lambda_1\varepsilon^3. \tag{3.93}$$

Substituting (3.93) into (3.92) we have

$$\begin{aligned}\dot{\rho} &= \rho\frac{\delta+\left(\lambda_2\sqrt[3]{\left(\frac{\delta}{\lambda_1}\right)^2}\cos\theta+\lambda_3\sqrt[3]{\frac{\delta}{\lambda_1}}\sin\theta-\lambda_5\sqrt[3]{\frac{\delta}{\lambda_1}}\sin\theta+\cos(3\theta)\right)\rho}{1+\left(\lambda_2\sqrt[3]{\left(\frac{\delta}{\lambda_1}\right)^2}\sin\theta-\lambda_3\sqrt[3]{\frac{\delta}{\lambda_1}}\cos\theta-\lambda_5\sqrt[3]{\frac{\delta}{\lambda_1}}\cos\theta-\sin(3\theta)\right)\rho}\\ \dot{\theta} &= 1. \end{aligned} \tag{3.94}$$

On (3.94) we apply the following transformation (cfr. Proposition 3.3)

$$r = e^{-\delta\theta}\rho. \tag{3.95}$$

This leads to

$$\dot{r} = C_1(\theta,\delta)r^2 + \mathcal{O}(r^3), \tag{3.96}$$

where

$$\begin{aligned} C_1(\theta,\delta) &= \lambda_2\left(\frac{\delta}{\lambda_1}\right)^{\frac{2}{3}}\cos(\theta)e^{\delta\theta} + (\lambda_3-\lambda_5)\left(\frac{\delta}{\lambda_1}\right)^{\frac{1}{3}}\sin(\theta)e^{\delta\theta}\\ &- \lambda_2\left(\frac{\delta}{\lambda_1}\right)^{\frac{2}{3}}\delta\sin(\theta)e^{\delta\theta} + \left(\frac{\delta}{\lambda_1}\right)^{\frac{1}{3}}\delta(\lambda_3+\lambda_5)\cos(\theta)e^{\delta\theta}\\ &+ \cos(3\theta)e^{\delta\theta} + \delta\sin(3\theta)e^{\delta\theta}. \end{aligned} \tag{3.97}$$

Similar to Proposition 3.6 we now apply the transformation

$$r_1 = r - \left(\int_0^\theta C_1(u,\delta)du\right)r^2. \tag{3.98}$$

This gives us

$$\dot{r}_1 = C_2(\theta,\delta)r^3 + \mathcal{O}(r^4), \tag{3.99}$$

where one can find an explicit expression for $C_2(\theta,\delta)$ with aid of a computeralgebra programme. For a Maple output we refer to Subsection 3.6.3. We know that $r_1 = r+\mathcal{O}(r^2)$, so also $r = r_1+\mathcal{O}(r_1^2)$ holds. Using the latter equality and deleting all terms of $\mathcal{O}(r_1^4)$ in (3.99) by truncation, we obtain:

$$\dot{r}_1 = C_2(\theta,\delta)r_1^3. \tag{3.100}$$

The solution of (3.100) with initial condition $r_1(0) = x_0$ is given by

$$r_1(\theta) = \frac{x_0}{\sqrt{1-2x_0\int_0^\theta C_2(u,\delta)du}}. \tag{3.101}$$

Inverting (3.98) gives us (see (3.22))

$$r = r_1 + \left(\int_0^\theta C_1(u, \delta) du \right) r_1^2 + \mathcal{O}(r_1^3), \tag{3.102}$$

and as the inverse of (3.95) is given by

$$\rho = e^{\delta\theta} r, \tag{3.103}$$

applying (3.102) and (3.103) to (3.101) gives us an asymptotic expansion of the solution of (3.94). Replacing δ by $\lambda_1 \varepsilon^3$ and filling in $\theta = 2\pi$, gives us an asymptotic expansion of the Poincaré map of (3.92). After two steps we found the following asymptotic expression

$$\begin{aligned} \mathcal{P}_\varepsilon(x_0) &= x_0 + 2\pi x_0 (\frac{1}{3} x_0^2 \lambda_1 + x_0^2 \lambda_2 \lambda_5 - \lambda_1) \varepsilon^3 \\ &+ 2 x_0^2 \lambda_1 \pi (\lambda_5 - \lambda_3) \varepsilon^4 + \mathcal{O}(\varepsilon^5). \end{aligned} \tag{3.104}$$

One should notice that in (3.104) we find that the first and second order Melnikov functions are zero and the asymptotic expression of the third order Melnikov function is non-zero.

A few remarks on this result need to be made. As the Hamiltonian triangle doesn't meet the criterions of Lemma 3.7, we cannot apply Remark 3.2 and thus cannot decide if we have obtained an exact expression of the Melnikov functions. We also do not know up to which order in x_0 the calculated Melnikov function is equal to the exact - but unknown - Melnikov function. Also looking at (3.104) one might think that the third order Melnikov function will vanish if λ_1 and λ_2 are zero. But as we only obtain an asymptotic expansion of this Melnikov function (up to an order which remains unspecified) it would be very unwise to jump to such conclusions. Also starting from a system that is not in Poincaré–Dulac normal form has the disadvantage that we need to eliminate much more terms. This leads to more calculations which can have a considerable negative effect on our resulting asymptotic expansion of the Poincaré map. For instance, if one should decide to perform three steps of the method - which is theoretically possible as the method is clearly applicable for any finite number of steps - one has to take in account that the expression of $C_n(\theta, \delta)$ for large values of n becomes extremely long. For $C_3(\theta, \delta)$ Maple found that it already contained over 7000 terms. So if one wishes to perform more steps, it will be necessary to have access to more developped software and sufficiently strong computer equipment.

3.6 Maple source codes

3.6.1 Neimark-Sacker case

```
> restart:
```

```
> c0:= t-> cos(t) -1:
> s0:=t-> sin(t):
> s1:=t-> sin(t)-t:
> omega:= t-> (1-exp(-e*t))/e:
> C:=t-> 3*c0(t)*omega(t) - 3*s0(t) + e*(12*c0(t)^2 + 24*c0(t)
 + 6*s0(t)*omega(t) - 3/2 *omega(t)^2 *c0(t))
 - 3*e^3*s0(t)*omega(t)^4:
> C1:=int(C(u)*exp(4*e*u),u=0..2*Pi):
> P2:= x-> x/(1-4*e^2 *C1*x^4)^(1/4):
> Poin:= x -> exp(2*Pi*e) * P2(x)+ e*exp(6*Pi*e)
*(-omega(2*Pi) + e/2 * omega(2*Pi)^2) *P2(x)^3:
> Poine:=taylor(Poin(x),e=0,10):
> Poinx:=taylor(Poin(x),x=0,10):
> Poine1:=unapply(convert(Poine,polynom),e):
> Poinx1:=unapply(convert(Poinx,polynom),x):
> readlib(coeftayl):
> for i from 0 to 4 do
taylor(expand(coeftayl(Poine1(e),e=0,i)),x=0, 2*i+2); od:
> for i from 0 to 4 do
taylor(expand(coeftayl(Poinx1(x),x=0,i)),e=0, 2*i+2); od:
```

3.6.2 Chenciner case

```
> restart:
> readlib(coeftayl):
> s0:= t-> sin(t):
> c0:= t -> cos(t) - 1:
> s1:= t-> sin(t) -t:
> r:= t-> exp(-e*t) * rho(t):
> r01:= diff(r(t),t):
> r02:=subs(diff(rho(t),t) =
e * rho(t) *(1 + l1 * rho(t)^2 + c*rho(t)^4), r01):
> r03:=simplify(r02):
> r04:=subs(rho(t) = exp(e*t)*R(t), r03):
> r05:=simplify(r04):
> k0:= e-> l1 * e:
> r1:= t-> R(t) - k0(e)*exp(2*e*t)*s0(t)*R(t)^3:
> r11:=diff(r1(t),t):
> r12:=subs(diff(R(t),t)= r05,r11):
> r13:=sort(simplify(r12),R(t)):
> r14:=unapply(simplify(coeftayl(subs(R(t)=X,r13),X=0,3)),t):
> Co:=unapply( int(r14(u), u=0..t),t):
> r2:= t -> r1(t) -Co(t)*r1(t)^3:
```

```
> r21:=diff(r2(t),t):
> r22:=subs(diff(R(t),t)= r05, r21):
> r23:=simplify(coeftayl(subs(R(t)=X,r22),X=0,3)):
> r24:=simplify(coeftayl(subs(R(t)=X,r22),X=0,5)):
> r25:=unapply(r24,t):
> Co2:=unapply(int(r25(u),u=0..t),t):
> r3:= t -> r2(t) - Co2(t)*r2(t)^5:
> r31:=diff(r3(t),t):
> r32:=subs(diff(R(t),t) = r05, r31):
> r33:=simplify(expand(coeftayl(subs(R(t)=X,r32),X=0,5))):
> r34:=unapply(simplify(coeftayl(subs(R(t)=X,r32),X=0,7)),t):
> r3_trunc:= t -> x0/(1 - 6 * x0^6 *int(r34(u),u=0..t))^(1/6):
> r2inv:= r3_trunc(2*Pi) + Co2(2*Pi)*r3_trunc(2*Pi)^5:
> r1inv:= r2inv+Co(2*Pi)*r2inv^3:
> rhoinv:= exp(2*Pi*e)*r1inv:
> poin:= unapply(rhoinv, (e,x0)):
> taylor(poin(e,x0),e=0, 3):
> taylor(poin(e,x0), x0=0, 5):
> Te:=convert(taylor(poin(e,x0),e=0,5),polynom):
> Tx:=convert(taylor(poin(e,x0),x0=0,7), polynom):
> for i from 0 to 4 do taylor(coeftayl(Te, e=0, i),x0=0,8) od:
> for i from 0 to 6 do taylor(coeftayl(Tx,x0=0,i), e=0, 5) od:
```

3.6.3 Hamiltonian triangle

```
> restart;
> readlib(coeftayl):
> s0:= t-> sin(t):
> c0:= t -> cos(t) - 1:
> s1:= t-> sin(t) -t:
> r:= t-> exp(-delta *t)*rho(t):
> r01:=diff(r(t),t):
> r02:=subs(e= (delta/l1)^(1/3),subs(diff(rho(t),t)=
(l1*e^3*rho(t) + (l2*e^2*cos(t) + l3*e*sin(t) - l5*e*sin(t)
+ cos(3*t))*rho(t)^2)/ (1 + (l2*e^2*sin(t) - l3*e*cos(t)
- l5*e*cos(t) - sin(3*t))*rho(t)), r01)):
> r03:=subs(rho(t)=exp(delta*t)*R(t), r02):
> r04:= taylor(subs(R(t)=X, r03),X=0,3):
> Co1:=unapply(simplify(coeftayl(r04, X=0,2)),t):
> r1:= t-> R(t) - int(Co1(u),u=0..t)*R(t)^2:
> r11:=diff(r1(t),t):
> r12:=subs(diff(R(t),t) = r03, r11):
> r13:=subs(R(t)=X, r12):
```

```
> coeftayl(r13,X=0,1):
> expand(simplify(coeftayl(r13,X=0,2))):
> Co2:= unapply(simplify(coeftayl(r13,X=0,3)),t):
> r1_trunc:= t-> x/sqrt(1 -2* x^2 *int(Co2(u),u=0..t)):
> r_trunc:=unapply(r1_trunc(t)
+ int(Co1(u),u=0..t)*r1_trunc(t)^2,t):
> rho_trunc:=unapply(r_trunc(t)* exp(-delta*t),t):
> Poin:= simplify(subs(delta = l1*e^3,rho_trunc(2*Pi))):
> T:=taylor(Poin,e=0,7):
> TT:=convert(T,polynom):
> for j from 0 to 6 do simplify(coeftayl(TT,e=0,j)) od:
```

Chapter 4

Local analytic models for hyperbolic families

4.1 Introduction

In Subsection 1.3.1 we discussed the linearisation of a hyperbolic vector field. If the hyperbolic vector field is analytic, then Poincaré's Theorem and Siegel's Theorem are telling us when the linearisation is analytic as well. In Subsection 1.3.2 we discussed the analogous situation for a hyperbolic diffeomorphism. The main difference between the two theorems is that Poincaré's Theorem deals with systems that are either repelling in all directions (called a *source*), either attracting in all directions (called a *sink*), while Siegel's Theorem deals with saddles.

We now want to look at analytic families of hyperbolic vector fields or diffeomorphisms. First we consider with the saddle case. Given an analytic family of hyperbolic vector fields X_ε or diffeomorphisms F_ε where X_0 is non-resonant or F_0 is multiplicatively non-resonant, it cannot be expected that the analytic family X_ε or F_ε can be linearised analytically. This is caused by the lower bound given by (1.13) for vector fields and (1.18) for diffeomorphisms. Even if (1.13) is valid for X_0, it is possible that the lower bound is no longer valid for X_ε as a small perturbation in the eigenvalues of a saddle can cause very high order resonances. We remind that we encountered a similar situation for planar resonant saddles in Proposition 2.1. Analogous remarks can be made for a family of diffeomorphisms.

Here we propose not to linearise the vector field or diffeomorphism in the neighourhood of the singular point or fixed point, but we will prove that it is conjugate to its linear part plus some high-order terms which have high order in as well the stable as the unstable directions. This conjugacy will be proven to be analytic. A similar approach for vector fields was done in [Tuc99, Tuc04] in the case that all eigenvalues are real and in [Hom02] for a three-dimensional saddle with 1 real eigenvalue and a pair of complex conjugate eigenvalues (with high-order terms of order 3). The advantage of this approach is that the normal form will be *robust*, i.e. under a small perturbation of the system the normal form will remain valid in the analytic category. We will "model" this perturbation by adding a parameter to the vector field or diffeomorphism and demanding that this family of vector fields or diffeomorphisms is analytic in its variables and the parameter.

The techniques and results that we will use in the saddle case, will also be very useful to prove Poincaré's Theorem for a family of vector fields or diffeomorphisms. As in this case we will work with sinks or sources, it will be possible to obtain an analytic linearisation.

The rest of this chapter is organised as follows. In Section 4.2 we introduce the necessary concepts in order to give a good description of the results we want to prove. In Section 4.3 we demonstrate why - contrary to Section 1.3 and Section 1.4 - we don't have to worry about the possible appearance of small divisors in our calculations. Although this section may seem very technical it is one of the key-elements to obtain the results that are presented in Section 4.2. In Section 4.4 we prove the main result for vector fields and in Section 4.5 we do the same for diffeomorphisms. It is natural to expect that both proofs will have similarities. In Section 4.6 we consider the case where symmetry appears in the family of vector fields or diffeomorphisms. We will prove that our analytic model respects the symmetry of the original system. Finally in Section 4.7 we apply the techniques used in Section 4.4 and Section 4.5 to prove Poincaré's Theorem for a family of vector fields or diffeomorphisms (see also Theorem 1.9 and Theorem 1.12). In our opinion the method we use to prove this result is easier than the ones found in the literature.

4.2 Settings and preliminaries

4.2.1 Choice of basis

Family of vector fields

Consider a family of n-dimensional real vector fields X_ε with a singularity of hyperbolic type. We assume that the vector field can be written as a convergent power series in its variables and the parameter ε such that if we extend this vector field to $\mathbb{C}^n$, i.e. we replace each real variable by a complex one, we obtain a complex power series that converges on a poly-disk $\mathbb{D}(0,R)\times\mathbb{D}(0,r)\subset\mathbb{C}^n\times\mathbb{C}^p$. We will say that this type of vector field is **real analytic**. The complex extension of a real analytic function is of course a (complex) analytic function. We also assume that all eigenvalues of the linear part at the singularity have multiplicity 1 for $\varepsilon=0$. By virtue of Proposition 1.15, Proposition 1.16 and the results from Subsection 1.3.4 we may assume that the singular point is the origin and that X_ε is given by

$$X_\varepsilon : \dot{x} = A_\varepsilon x + f_\varepsilon(x), \tag{4.1}$$

where $f_\varepsilon(x) = \mathcal{O}(|x|^2)$ is an analytic function of (x, ε) on a poly-disk $\mathbb{D}(0, R)$ and A_ε is in Jordan Normal Form:

$$A_\varepsilon = \begin{pmatrix} A_\varepsilon^{(1)} & 0 & 0 & 0 \\ 0 & A_\varepsilon^{(2)} & 0 & 0 \\ 0 & 0 & A_\varepsilon^{(3)} & 0 \\ 0 & 0 & 0 & A_\varepsilon^{(4)} \end{pmatrix} \tag{4.2}$$

where

$$A_\varepsilon^{(1)} = \begin{pmatrix} \nu_1(\varepsilon) & 0 & \cdots & 0 \\ 0 & \nu_2(\varepsilon) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \nu_a(\varepsilon) \end{pmatrix}$$

where $\nu_1(\varepsilon) < \nu_2(\varepsilon) < \cdots < \nu_a(\varepsilon) < 0$ are the negative real eigenvalues of A_ε;

$$A_\varepsilon^{(2)} = \begin{pmatrix} \alpha_1(\varepsilon) & -\beta_1(\varepsilon) & 0 & 0 & \cdots & 0 & 0 \\ \beta_1(\varepsilon) & \alpha_1(\varepsilon) & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \alpha_2(\varepsilon) & -\beta_2(\varepsilon) & \cdots & 0 & 0 \\ 0 & 0 & \beta_2(\varepsilon) & \alpha_2(\varepsilon) & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \alpha_b(\varepsilon) & -\beta_b(\varepsilon) \\ 0 & 0 & 0 & 0 & \cdots & \beta_b(\varepsilon) & \alpha_b(\varepsilon) \end{pmatrix}$$

is the matrix containing all complex eigenvalues with negative real part such that $\alpha_1(0) \le \alpha_2(0) \le \cdots \le \alpha_b(0) < 0$ and $\beta_j(\varepsilon) > 0$ for all $j = 1, \cdots, b$;

$$A_\varepsilon^{(3)} = \begin{pmatrix} \mu_1(\varepsilon) & 0 & \cdots & 0 \\ 0 & \mu_2(\varepsilon) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mu_c(\varepsilon) \end{pmatrix}$$

where $0 < \mu_1(\varepsilon) < \mu_2(\varepsilon) < \cdots < \mu_c(\varepsilon)$ are the positive real eigenvalues of A_ε;

$$A_\varepsilon^{(4)} = \begin{pmatrix} \gamma_1(\varepsilon) & -\delta_1(\varepsilon) & 0 & 0 & \cdots & 0 & 0 \\ \delta_1(\varepsilon) & \gamma_1(\varepsilon) & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \gamma_2(\varepsilon) & -\delta_2(\varepsilon) & \cdots & 0 & 0 \\ 0 & 0 & \delta_2(\varepsilon) & \gamma_2(\varepsilon) & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \gamma_d(\varepsilon) & -\delta_d(\varepsilon) \\ 0 & 0 & 0 & 0 & \cdots & \delta_d(\varepsilon) & \gamma_d(\varepsilon) \end{pmatrix}$$

is the matrix containing all complex eigenvalues with positive real part such that $0 < \gamma_1(0) \leq \gamma_2(0) \leq \cdots \leq \gamma_d(0)$ and $\delta_j(\varepsilon) > 0$ for all $j = 1, \cdots, d$. We have that $a + 2b + c + 2d = n$.

In order to calculate the formal normal form it would have been more convenient to have a diagonal linear part at the origin, therefore we will use complex coordinates. We define the matrix

$$Q = \begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix},$$

then we obtain the change of coordinates $z = Px$ where P is the $n \times n$ matrix defined by

$$P = \left(\begin{array}{c|ccc|c|ccc} I_a & & 0 & & 0 & & 0 & \\ \hline & Q & \cdots & 0 & & & & \\ 0 & \vdots & \ddots & \vdots & 0 & & 0 & \\ & 0 & \cdots & Q & & & & \\ \hline 0 & & 0 & & I_c & & 0 & \\ \hline & & & & & Q & \cdots & 0 \\ 0 & & 0 & & 0 & \vdots & \ddots & \vdots \\ & & & & & 0 & \cdots & Q \end{array}\right). \tag{4.3}$$

Applying this change of coordinates (4.1) is transformed into

$$Y_\varepsilon : \dot{z} = B_\varepsilon z + F_\varepsilon(z) \tag{4.4}$$

where

$$\begin{aligned} B_\varepsilon \quad = \quad & \mathrm{diag}(\nu_1(\varepsilon), \cdots, \nu_a(\varepsilon), \alpha_1(\varepsilon) + i\beta_1(\varepsilon), \alpha_1(\varepsilon) - i\beta_1(\varepsilon), \cdots, \\ & \alpha_b(\varepsilon) + i\beta_b(\varepsilon), \alpha_b(\varepsilon) - i\beta_b(\varepsilon), \mu_1(\varepsilon), \cdots, \mu_c(\varepsilon), \\ & \gamma_1(\varepsilon) + i\delta_1(\varepsilon), \gamma_1(\varepsilon) - i\delta_1(\varepsilon), \cdots, \gamma_d(\varepsilon) + i\delta_d(\varepsilon), \gamma_d(\varepsilon) - i\delta_d(\varepsilon)). \end{aligned}$$

As f_ε already is a real analytic function with a complex extension converging on a poly-disk $\mathbb{D}(0, R)$, we have that F_ε is an analytic function of (z, ε) where z has the following properties:

- if $\lambda_j(\varepsilon)$ is a real eigenvalue of A_ε, then $\overline{z_j} = z_j = x_j$, in other words z_j is a real variable,
- if $\lambda_j(\varepsilon)$ and $\lambda_{j+1}(\varepsilon)$ form a pair of complex conjugate eigenvalues of A_ε, then $\overline{z_j} = z_{j+1}$. So $x_j = \frac{z_j + z_{j+1}}{2}$ and $x_{j+1} = \frac{z_j - z_{j+1}}{2i}$.

Therefore F_ε will be analytic for

- $|z_j| = |x_j| < R_j$, if z_j is real,

- $|z_j + z_{j+1}| = 2|x_j| < 2R_j$ and $|z_j - z_{j+1}| < 2R_{j+1}$, if $\overline{z_j} = z_{j+1}$.

This immediately gives the following properties of F_ε:

- if z_j is real, then $\overline{F_{\varepsilon,j}(z)} = F_{\varepsilon,j}(z)$,
- if $\overline{z_j} = z_{j+1}$, then $\overline{F_{\varepsilon,j}(z)} = F_{\varepsilon,j+1}(z)$.

Family of diffeomorphisms

Consider a real analytic family of diffeomorphisms f_ε with a hyperbolic fixed point. We assume that the eigenvalues of the linear part at the origin have multiplicity 1 for $\varepsilon = 0$. In an analogous way to the case of the vector fields we may assume that we can obtain - without loss of analyticity - that the origin is the fixed point and

$$f_\varepsilon : \mathbb{R}^n \to \mathbb{R}^n : x \mapsto A_\varepsilon x + F_\varepsilon(x) \tag{4.5}$$

where A_ε is analytic in ε, $F_\varepsilon(x)$ is real analytic in x and $F_\varepsilon(x) = \mathcal{O}(|x|^2)$ for $x \to 0$, and A_ε is in Jordan Normal Form given by (4.2) where $A_\varepsilon^{(1)}$ and $A_\varepsilon^{(2)}$ contain all eigenvalues with modulus strictly smaller than 1 and $A_\varepsilon^{(3)}$ and $A_\varepsilon^{(4)}$ contain all eigenvalues with modulus strictly greater than 1. Employing the change of coordinates given by (4.3) we obtain the complexified diffeomorphism $f_\varepsilon^\star$ given by

$$f_\varepsilon^\star : \mathbb{C}^n \to \mathbb{C}^n : z \mapsto B_\varepsilon z + F_\varepsilon^\star(z) \tag{4.6}$$

where B_ε is the same matrix is in the case of the vector fields and $F_\varepsilon^\star$ is analytic in (z, ε) with the same remarks on z as we made for the vector field case.

4.2.2 Resonance conditions

In Chapter 1 we already encountered the impact of resonance on the normal form of a vector field or diffeomorphism. Here we want to relax the notion of non-resonance and multiplicatively non-resonance a bit. Therefore we consider a complex $n \times n$ matrix A with $\mathrm{Spec}(A) = \{\lambda_1, \cdots, \lambda_n\}$. In the case of a vector field we have that $\lambda_1, \cdots, \lambda_s$ have negative real part and $\lambda_{s+1}, \cdots, \lambda_n$ have positive real part. In the case of a diffeomorphism we have that $\lambda_1, \cdots, \lambda_s$ have modulus strictly smaller than 1 and $\lambda_{s+1}, \cdots, \lambda_n$ have modulus strictly larger than 1. We now introduce (for any integer $\ell \geq 1$):

$$\mathcal{S}_{\ell,n,s} := \left\{ m \in \mathbb{N}^n | \min\left(\sum_{j=1}^{s} m_j, \sum_{j=s+1}^{n} m_j \right) < \ell \right\} \tag{4.7}$$

$$\mathcal{T}_{\ell,n,s} := \left\{ m \in \mathbb{N}^n | \min\left(\sum_{j=1}^{s} m_j, \sum_{j=s+1}^{n} m_j \right) \geq \ell \right\} \tag{4.8}$$

so $\mathbb{N}^n = \mathcal{S}_{\ell,n,s} \cup \mathcal{T}_{\ell,n,s}$ and $\mathcal{S}_{\ell,n,s} \cap \mathcal{T}_{\ell,n,s} = \emptyset$. For any formal power series $F(x) = \sum_{m\in\mathbb{N}^n} F_m x^m$ we have

$$\begin{aligned} F(x) &= \sum_{m\in\mathcal{S}_{\ell,n,s}} F_m x^m + \sum_{m\in\mathcal{T}_{\ell,n,s}} F_m x^m \\ &=: [F(x)]^{\mathcal{S}_{\ell,n,s}} + [F(x)]^{\mathcal{T}_{\ell,n,s}}. \end{aligned}$$

We recall that $\mathrm{Spec}(A)$ is a **resonant** set if there exists a $m \in \mathbb{N}^n$ with $|m| \geq 2$ such that

$$\sum_{j=0}^{n} m_j \lambda_j = \lambda_k, \tag{4.9}$$

with $k = 1, \cdots, n$. $\mathrm{Spec}(A)$ is a **multiplicatively resonant** set if there exists a $m \in \mathbb{N}^n$ with $|m| \geq 2$ such that

$$\prod_{j=0}^{n} \lambda_j^{m_j} = \lambda_k, \tag{4.10}$$

with $k = 1, \cdots, n$.

In what follows we will fix an integer ℓ (which one wants to take as large as possible in applications) and demand that no element of $\mathcal{S}_{\ell,n,s}$ is a solution of (4.9), resp. (4.10), if we work with a vector field, resp. diffeomorphism. In such a case we will say that $\mathcal{S}_{\ell,n,s}$ **causes no resonances** in $\mathrm{Spec}(A)$. It is obvious that if $\mathrm{Spec}(A)$ is non-resonant or multiplicatively non-resonant then $\mathcal{S}_{\ell,n,s}$ will cause no resonance in $\mathrm{Spec}(A)$. To make this more clear we give an example of a hyperbolic singularity which is resonant but no element of $\mathcal{S}_{\ell,n,s}$ (for given ℓ and n) satisfies (4.9). Take $\ell = 19$, $n = 3$, $s = 1$ and eigenvalues -11, $9+i$ and $9-i$, then the first resonance equation becomes

$$-11m_1 + (9+i)m_2 + (9-i)m_3 = -11.$$

The solution with smallest "length" (i.e. with minimal $|m| \geq 2$) of this equation is $(19, 11, 11)$ so $m_1 = 19$ and $m_2 + m_3 = 22$ so even though the system is resonant, $\mathcal{S}_{19,3,1}$ causes no resonances in $\{-11, 9+i, 9-i\}$.

4.2.3 Results

Before stating the results of this chapter we introduce the following norms:

$$\begin{aligned} |y| &= \max_{1\leq j\leq n} |y_j|, \forall y \in \mathbb{C}^n \\ \|F\|_r &= \max_{|x|\leq r} |F(x)| \end{aligned}$$

for any continuous function F on $\mathbb{D}(0, r\mathbf{e})$.

Theorem 4.1 *Consider a fixed integer $\ell \geq 1$ and an n-dimensional real vector field*

$$X_\varepsilon : \dot{x} = A_\varepsilon x + f_\varepsilon(x) \tag{4.11}$$

such that A_ε is a real $n \times n$ matrix in Jordan Normal Form, $f_\varepsilon(x) = \mathcal{O}(|x|^2)$ for $x \to 0$ where f_ε is a real analytic function of x and ε such that $f_\varepsilon^\diamond$, the complex extension of $f_\varepsilon(x)$, is analytic on a poly-disk $\mathbb{D}(0,R) \times \mathbb{D}(0,r)$ and $\mathcal{S}_{\ell,n,s}$ causes no resonances in Spec(A_0)*, where s denotes the number of eigenvalues of A_0 that have a negative real part. Then there exists positive constants r_0, r_1, K_0, K_1, ρ and a change of coordinates*

$$x = y + \phi_\varepsilon(y) \tag{4.12}$$

which is real analytic in (y, ε) such that $\phi_\varepsilon^\diamond$ is analytic on $\mathbb{D}(0, r_1\mathbf{e}) \times \mathbb{D}(0, \rho\mathbf{e})$, such that $\|\phi_\varepsilon\|_q \leq K_0 q^2$ for $q < r_0$ and $\varepsilon \in \mathbb{D}(0, \rho\mathbf{e})$, and (4.12) conjugates (4.11) to

$$Y_\varepsilon : \dot{y} = A_\varepsilon y + g_\varepsilon(y), \tag{4.13}$$

where $g_\varepsilon(y)$ is real analytic in (y, ε), $[g_\varepsilon(y)]^{\mathcal{S}_{\ell,n,s}} = 0$ and

$$|g_\varepsilon(y)| \leq K_1 |(y_1, \cdots, y_s)|^\ell |(y_{s+1}, \cdots, y_n)|^\ell$$

for $y \in \mathbb{D}(0, r_1\mathbf{e})$.

Theorem 4.2 *Consider a fixed integer $\ell \geq 1$ and a diffeomorphism*

$$f_\varepsilon : \mathbb{R}^n \to \mathbb{R}^n : x \mapsto A_\varepsilon x + F_\varepsilon(x) \tag{4.14}$$

such that A_ε is a real $n \times n$ matrix in Jordan Normal Form, $F_\varepsilon(x) = \mathcal{O}(|x|^2)$ for $x \to 0$, f_ε is a real analytic function of x and ε such that $f_\varepsilon^\diamond$, the complex extension of f_ε, is analytic on a poly-disk $\mathbb{D}(0,R) \times \mathbb{D}(0,r)$ and $\mathcal{S}_{\ell,n,s}$ causes no resonances in Spec(A_0)*, where s denotes the number of eigenvalues of A_0 with modulus strictly smaller than one. Then there exist positive constants r_0, r_1, K_0, K_1, ρ and a change of coordinates*

$$x = y + \phi_\varepsilon(y) \tag{4.15}$$

which is real analytic in (y, ε) such that $\phi_\varepsilon^\diamond$ is analytic on $\mathbb{D}(0, r_1\mathbf{e}) \times \mathbb{D}(0, \rho\mathbf{e})$, such that $\|\phi_\varepsilon\|_q \leq K_0 q^2$ for $q < r_0$ and $\varepsilon \in \mathbb{D}(0, \rho\mathbf{e})$, and (4.15) conjugates (4.14) to

$$g_\varepsilon : \mathbb{R}^n \to \mathbb{R}^n : x \mapsto A_\varepsilon x + G_\varepsilon(x), \tag{4.16}$$

where $G_\varepsilon(y)$ is real analytic in (y, ε), $[G_\varepsilon]^{\mathcal{S}_{\ell,n,s}} = 0$ and

$$|G_\varepsilon(y)| \leq K_1 |(y_1, \cdots, y_s)|^\ell |(y_{s+1}, \cdots, y_n)|^\ell$$

for $y \in \mathbb{D}(0, r_1\mathbf{e})$.

From the properties of g_ε in (4.13) and G_ε in (4.16) and the fact that the transformations given by (4.12) and (4.15) are analytic in the variable and the parameter gives us the following results as corollaries of Theorem 4.1 and Theorem 4.2.

Corollary 4.3 *Under the conditions of Theorem 4.1 we have that the stable and unstable manifold of X_ε at the origin are real analytic manifolds depending in a real analytic way on the parameter ε.*

Corollary 4.4 *Under the conditions of Theorem 4.2 we have that the stable and unstable manifold of f_ε at the origin are real analytic manifolds depending in a real analytic way on the parameter ε.*

If the original family admits symmetry, then we have the following results.

Theorem 4.5 *If - under the conditions of Theorem 4.1 - the family of vector fields X_ε admits an analytic family of symmetries S_ε (i.e. S_ε is an analytic family of linear maps such that $(S_\varepsilon)_* X_\varepsilon = X_\varepsilon$), then the transformation given by (4.12) commutes with S_ε and the resulting family of vector fields given by (4.13) admits the same family of symmetries.*

Theorem 4.6 *If - under the conditions of Theorem 4.2 - the family of diffeomorphisms f_ε admits an analytic family of symmetries S_ε (i.e. S_ε is an analytic family of linear maps and $S_\varepsilon \circ f_\varepsilon = f_\varepsilon \circ S_\varepsilon$), then the transformation given by (4.15) commutes with S_ε and the resulting family of diffeomorphisms given by (4.16) admits the same family of symmetries.*

4.3 Absence of small divisors

In this section we want to show that if $\mathcal{S}_{\ell,n,s}$ causes no resonances in $\mathrm{Spec}(A_0)$, then there exists a constant $\rho > 0$ such that for all $\varepsilon \in B(0,\rho) \subset \mathbb{C}^p$ we have that $\mathcal{S}_{\ell,n,s}$ causes no resonances in $\mathrm{Spec}(A_\varepsilon)$. Although the proofs for vector fields and diffeomorphisms are similar, we prefer to give both proofs separately to avoid confusions.

4.3.1 Family of vector fields

We consider the kth resonance equation (for $k = 1, \cdots, n$):

$$\begin{aligned}\sum_{j=1}^{a} r_j \nu_j(0) + \sum_{j=1}^{b} s_j(\alpha_j(0) + i\beta_j(0)) + \sum_{j=1}^{b} \tilde{s}_j(\alpha_j(0) - i\beta_j(0)) \quad &+ \\ \sum_{j=1}^{c} t_j \mu_j(0) + \sum_{j=1}^{d} u_j(\gamma_j(0) + i\delta_j(0)) + \sum_{j=1}^{d} \tilde{u}_j(\gamma_j(0) - i\delta_j(0)) \quad &= \quad \lambda_k(0)\end{aligned} \tag{4.17}$$

where $\lambda_j(\varepsilon)$ is the jth component of Λ_ε. Looking more closely at this equation, one should note that there are actually 2 equations to consider: one coming from the real parts and one coming from the imaginary parts. It would of course be much easier to solve the equation if one shouldn't worry about the imaginary parts. That is why we prove the following result that says that we only have to consider the real parts of the eigenvalues if one is concerned about the existence of resonance in the system.

Proposition 4.7 *Let A_ε be as in (4.11). Then the eigenvalues of A_0 are resonant iff the eigenvalues of $\tilde{A}_0$ are resonant, where $\tilde{A}_\varepsilon$ is the $(a+b+c+d)\times(a+b+c+d)$ matrix defined by*

$$\tilde{A}_\varepsilon = \begin{pmatrix} A_\varepsilon^{(1)} & 0 & 0 & 0 \\ 0 & \tilde{A}_\varepsilon^{(2)} & 0 & 0 \\ 0 & 0 & A_\varepsilon^{(3)} & 0 \\ 0 & 0 & 0 & \tilde{A}_\varepsilon^{(4)} \end{pmatrix}$$

where $A_\varepsilon^{(1)}$ and $A_\varepsilon^{(3)}$ are defined in (4.1) and $A_\varepsilon^{(2)} = diag(\alpha_1(\varepsilon),\cdots,\alpha_b(\varepsilon))$ and $A_\varepsilon^{(4)} = diag(\gamma_1(\varepsilon),\cdots,\gamma_d(\varepsilon))$.

Proof: The eigenvalues of A_0 form a resonant set iff (4.9) has a solution. Looking at the real and the imaginary part of this equation, we obtain the following two equations

$$\begin{aligned} \sum_{j=1}^{a} r_j\nu_j(0) + \sum_{j=1}^{b}(s_j+\tilde{s}_j)\alpha_j(0) + \sum_{j=1}^{c} t_j\mu_j(0) \\ + \sum_{j=1}^{d}(u_j+\tilde{u}_j)\gamma j(0) &= \Re(\lambda_k(0)), \qquad (4.18) \\ \sum_{j=1}^{b}(s_j-\tilde{s}_j)\beta_j(0) + \sum_{j=1}^{d}(u_j-\tilde{u}_j)\delta_j(0) &= \Im(\lambda_k(0)). \qquad (4.19) \end{aligned}$$

If $\Im(\lambda_k(0)) = 0$, a solution of (4.19) is given by taking $\tilde{s}_j = s_j$ for $j = 1,\cdots,b$ and $\tilde{u}_j = u_j$ for $j = 1,\cdots,d$. If $\Im(\lambda_k(0)) \neq 0$, then we have to look at the sign of $\Re(\lambda_k(0))$. In the positive case there is a $q \in \{1,\cdots,d\}$ such that we take $\tilde{u}_q(0) = \tilde{u}_q(0) \pm 1$ (the $\pm$ is determined by the sign of $\Im(\lambda_k(0))$). Taking $\tilde{s}_j = s_j$ for $j = 1,\cdots,b$ and $\tilde{u}_j = u_j$ for $j \neq q$, we find a solution of (4.19). In the negative case there is a $q \in \{1,\cdots,d\}$ such that we take $\tilde{s}_q(0) = \tilde{s}_q(0) \pm 1$ (the $\pm$ is determined by the sign of $\Im(\lambda_k(0))$). Taking $\tilde{s}_j = s_j$ for $j \neq q$ and $\tilde{u}_j = u_j$ for $j = 1,\cdots,d$, we find a solution of (4.19). In all of these cases (4.18) is reduced to

$$\sum_{j=1}^{a} r_j\nu_j(0) + \sum_{j=1}^{b} s_j(2\alpha_j(0)) + \sum_{j=1}^{c} t_j\mu_j(0) + \sum_{j=1}^{d} u_j(2\gamma_j(0)) = \Re(\lambda_k(0)).$$

This latter equation is equivalent with saying that there is resonance between the eigenvalues of

$$\hat{A} = \begin{pmatrix} A_0^{(1)} & 0 & 0 & 0 \\ 0 & 2\tilde{A}_0^{(2)} & 0 & 0 \\ 0 & 0 & A_0^{(3)} & 0 \\ 0 & 0 & 0 & 2\tilde{A}_0^{(4)} \end{pmatrix}.$$

We now prove that the eigenvalues of $\tilde{A}_0$ are resonant iff the eigenvalues of $\hat{A}$ are resonant. So, resonance between the eigenvalues of $\tilde{A}_0$ can happen iff there exists a solution of

$$\sum_{j=1}^{a} r_j\nu_j(0) + \sum_{j=1}^{b} s_j\alpha_j(0) + \sum_{j=1}^{c} t_j\mu_j(0) + \sum_{j=1}^{d} u_j\gamma_j(0) = \tilde{\lambda}_k(0).$$

This equation is obviously equivalent with

$$\sum_{j=1}^{a}(2r_j)\nu_j(0) + \sum_{j=1}^{b} s_j(2\alpha_j(0)) + \sum_{j=1}^{c}(2t_j)\mu_j(0) + \sum_{j=1}^{d} u_j(2\gamma_j(0)) = 2\tilde{\lambda}_k(0).$$

If $\tilde{\lambda}_k(0)$ equals one of the $\alpha_j(0)$ or $\gamma_j(0)$ then we introduce $\tilde{r}_j = 2r_j$ and $\tilde{t}_j = 2t_j$; if $\tilde{\lambda}_k(0) = \nu_q(0)$ for some $q \in \{0, \cdots, a\}$ then we introduce $\tilde{r}_q = 2r_q - 1$, $\tilde{r}_j = 2r_j$ for $j \neq q$ and $\tilde{t}_j = 2t_j$; if $\lambda_k(0) = \mu_q(0)$ for some $q \in \{0, \cdots, c\}$ then we introduce $\tilde{r}_j = 2r_j$, $\tilde{t}_q = 2t_q - 1$ and $\tilde{t}_j = 2t_j$ if $j \neq q$, then in all of these cases the latter equation is equivalent with

$$\sum_{j=1}^{a} \tilde{r}_j\nu_j(0) + \sum_{j=1}^{b} s_j(2\alpha_j(0)) + \sum_{j=1}^{c} \tilde{t}_j\mu_j(0) + \sum_{j=1}^{d} u_j(2\gamma_j(0)) = \tilde{\lambda}_k(0),$$

which is the resonance equation on the eigenvalues of $\hat{A}$. □

From the proof of Proposition 4.7 we obtain the following result.

Corollary 4.8 *Let A_ε be as in (4.11). Then $\mathcal{S}_{\ell,n,s}$ causes no resonances in* Spec(A_0) *iff $\mathcal{S}_{\tilde{\ell},\tilde{n},\tilde{s}}$ causes no resonances in* Spec$(\tilde{A}_0)$*, where $\tilde{A}_\varepsilon$ is defined in Proposition 4.7 and*

$$\begin{aligned} \tilde{n} &= a+b+c+d, \\ \tilde{s} &= a+b, \end{aligned}$$

$$\frac{\ell}{2} - \max_{1\le k\le n}\frac{\tilde{\lambda}_k(0)}{2} \le \tilde{\ell} \le \frac{\ell}{2} - \min_{1\le k\le n}\frac{\tilde{\lambda}_k(0)}{2}.$$

The exact value of $\tilde{\ell}$ depends on Spec$(\tilde{A}_0)$.

Let us give some examples of this situation:

- Consider $\mathrm{Spec}(A_0) = \{-3, 5+i, 5-i\}$, then $\mathcal{S}_{6,3,1}$ causes no resonances in $\mathrm{Spec}(A_0)$ (and 6 is the maximal value of ℓ with this property). As $\mathrm{Spec}(\tilde{A}_0) = \{-3, 5\}$, we have that $\mathcal{S}_{3,2,1}$ causes no resonances in $\mathrm{Spec}(\tilde{A}_0)$, hence $\tilde{\ell} = 3 = \frac{\ell}{2}$.

- Consider $\mathrm{Spec}(A_0) = \{-2, 5+i, 5-i\}$, then $\mathcal{S}_{2,3,1}$ causes no resonances in $\mathrm{Spec}(A_0)$. As $\mathrm{Spec}(\tilde{A}_0) = \{-2, 5\}$, we have that $\mathcal{S}_{2,2,1}$ causes no resonances in $\mathrm{Spec}(\tilde{A}_0)$, hence $\tilde{\ell} = 2 = \frac{\ell}{2} - \frac{(-2)}{2}$.

To facilitate the notations, we use the constants $\tilde{\ell}$, $\tilde{n}$ and $\tilde{s}$ defined in Corollary 4.8, this way we can write that $\tilde{A}_\varepsilon$ is an $\tilde{n} \times \tilde{n}$ matrix and that there are $\tilde{s}$ stable directions. Also we introduce $\tilde{\Lambda}_\varepsilon$ as the $\tilde{n}$-tuple of eigenvalues of $\tilde{A}_\varepsilon$.

Now we look at the kth resonance equations on the eigenvalues of $\tilde{A}_0$:

$$\sum_{j=1}^{a} r_j \nu_j(0) + \sum_{j=1}^{b} s_j \alpha_j(0) + \sum_{j=1}^{c} t_j \mu_j(0) + \sum_{j=1}^{d} u_j \gamma_j(0) = \tilde{\lambda}_k(0). \tag{4.20}$$

As we assume that the eigenvalues are non-resonant, (4.20) has no non-trivial solutions in $\mathcal{S}_{\tilde{\ell},\tilde{n},\tilde{s}}$. We can interpret this non-resonance in the following geometrical way. Consider the $\tilde{n}$-tuple

$$(r_1, \cdots, r_a, s_1, \cdots, s_b, t_1, \cdots, t_c, u_1, \cdots, u_d)$$

as a point on the grid $\mathbb{Z}^{\tilde{n}}$, then the non-resonance of the eigenvalues of $\tilde{A}_0$ means that the hyperplane H with equation given by (4.20) contains only one of the "grid points" in $\mathcal{S}_{\tilde{\ell},\tilde{n},\tilde{s}}$. This point is the intersection of H with the x_k-axis (the kth axis in $\mathbb{R}^{\tilde{n}}$) and it has coordinates $e_k := (0, \cdots, 0, 1, 0, \cdots, 0)$ with a 1 on the kth position. The hyperplane H will intersect the x_j-axis (for $j \neq k$) in the point $\frac{\tilde{\lambda}_k(0)}{\tilde{\lambda}_j(0)} e_j$. For each point P of $\mathcal{S}_{\tilde{\ell},\tilde{n},\tilde{s}}$ (with $|P| \geq 2$) we consider the hyperplanes through the points P and e_k. These hyperplanes will intersect each axis in a point of the form $\left(\frac{\tilde{\lambda}_k(0)}{\tilde{\lambda}_j(0)} + \eta_{H',P}\right) e_j$ where $\eta_{H',P} \in \mathbb{R} \setminus \{0\}$ depends on the hyperplane H' and the point P. As we are working in $\mathcal{S}_{\tilde{\ell},\tilde{n},\tilde{s}}$ we know that $\min_{P \in \mathcal{S}_{\tilde{\ell},\tilde{n},\tilde{s}}} |\eta_{H',P}| > 0$, so there exists a $\theta > 0$ such that $\theta = \min_{P \in \mathcal{S}_{\tilde{\ell},\tilde{n},\tilde{s}}} |\eta_{H',P}| > 0$. Let us denote the hyperplane that gives this θ by $\hat{H}$, the intersection of this hyperplane with the axis will give us the "closest" resonance. This way we have obtained a bound for the ratio of the eigenvalues of $\tilde{A}_\varepsilon$:

$$\frac{\tilde{\lambda}_k(0)}{\tilde{\lambda}_j(0)} - \theta < \frac{\tilde{\lambda}_k(\varepsilon)}{\tilde{\lambda}_j(\varepsilon)} < \frac{\tilde{\lambda}_k(0)}{\tilde{\lambda}_j(0)} + \theta, \tag{4.21}$$

for $j = 1, \cdots, \tilde{n}$. The region U of $\mathbb{R}^{\tilde{n}}$ defined by the bounds

$$\frac{\tilde{\lambda}_k(0)}{\tilde{\lambda}_j(0)} - \theta < \frac{x_k}{x_j} < \frac{\tilde{\lambda}_k(0)}{\tilde{\lambda}_j(0)} + \theta,$$

for $j = 1, \cdots, \tilde{n}$, is an open subset of $\mathbb{R}^{\tilde{n}}$ containing $\tilde{\Lambda}_0$. We know that for each $k = 1, \cdots, \tilde{n}$, $\varepsilon \mapsto \tilde{\lambda}_k(\varepsilon)$ is a continuous map that is either strictly positive either strictly negative in a neighbourhood of the origin. As U is open, the continuity of the mappings $\varepsilon \mapsto \frac{\tilde{\lambda}_k(\varepsilon)}{\tilde{\lambda}_j(\varepsilon)}$ gives us the existence of a $\rho_k > 0$ such that (4.21) is fulfilled for all $\varepsilon \in B(0, \rho_k)$. Taking ρ as minimum of all ρ_k (as there only a finite number of ρ_k, we have that $\rho > 0$), we have that $\mathcal{S}_{\tilde{\ell},\tilde{n},\tilde{s}}$ causes no resonances on the eigenvalues of $\tilde{A}_\varepsilon$ for all $\varepsilon \in B(0, \rho) = \mathbb{D}(0, \rho\mathbf{e})$. By virtue of Proposition 4.7 we have that $\mathcal{S}_{\ell,n,s}$ causes no resonances on the eigenvalues of A_ε if $\varepsilon \in B(0, \rho) = \mathbb{D}(0, \rho\mathbf{e})$.

Remark 4.1 *During the argumentation we did up until now, we never used that the eigenvalues of $\tilde{A}_0$ have multiplicity* 1*. Actually the result remains valid if some eigenvalues have multiplicity higher than* 1 *as long as* $\mathrm{Spec}(\tilde{A}_0)$ *is a non-resonant set.*

We now prove that small divisors cannot appear.

Proposition 4.9 *If $\mathcal{S}_{\ell,n,s}$ causes no resonances in* $\mathrm{Spec}(A_0)$*, then there exists a positive constant κ such that $\forall m \in \mathcal{S}_{\ell,n,s}$ and $\forall \varepsilon \in B(0, \rho) = \mathbb{D}(0, \rho\mathbf{e})$ (where ρ was determined in the previous argumentation):*

$$|\langle \Lambda_\varepsilon, m\rangle - \lambda_j(\varepsilon)| \geq \kappa |m| \tag{4.22}$$

where $1 \leq j \leq n$ and $\lambda_j(\varepsilon)$ denotes the jth eigenvalue of A_ε.

As we have that

$$\begin{aligned} |\langle \Lambda_\varepsilon, m\rangle - \lambda_j(\varepsilon)| &\geq |\Re(\langle \Lambda_\varepsilon, m\rangle - \lambda_j(\varepsilon))| \\ &\geq \left|\left\langle \tilde{\Lambda}_\varepsilon, \tilde{m}\right\rangle - \Re(\lambda_j(\varepsilon))\right| \end{aligned}$$

where $\tilde{m} \in \mathcal{S}_{\ell,\tilde{n},\tilde{s}}$ is related to m as follows:

$$\begin{array}{ll} \tilde{m}_j = m_j & \text{for} \quad 1 \leq j \leq a \\ \tilde{m}_{a+j} = m_{a+2j-1} + m_{a+2j} & \text{for} \quad 1 \leq j \leq b \\ \tilde{m}_{a+b+j} = m_{a+2b+j} & \text{for} \quad 1 \leq j \leq c \\ \tilde{m}_{a+b+c+j} = m_{a+2b+c+2j-1} + m_{a+2b+c+2j} & \text{for} \quad 1 \leq j \leq d, \end{array}$$

and

$$|\tilde{m}| = |m|.$$

Proposition 4.9 will be a consequence of

Proposition 4.10 *There exists a positive constant K such that for the eigenvalues of $\tilde{A}_\varepsilon$ we have that $\forall \varepsilon \in B(0,\rho) = \mathbb{D}(0,\rho\mathbf{e})$:*

$$\left|\left\langle \tilde{\Lambda}_\varepsilon, m \right\rangle - \tilde{\lambda}_j(\varepsilon)\right| \geq K|m| \tag{4.23}$$

for all $m \in \mathcal{S}_{\tilde{\ell},\tilde{n},\tilde{s}}$ and $j = 1,\cdots,\tilde{n}$.

To prove Proposition 4.10 we need another result. To make the proof a bit clearer, we will assume that the eigenvalues of $\tilde{A}_\varepsilon$ meet

$$\tilde{\lambda}_1(0) \leq \cdots \leq \tilde{\lambda}_{\tilde{s}}(0) < 0 < \tilde{\lambda}_{\tilde{s}+1}(0) \leq \cdots \leq \tilde{\lambda}_{\tilde{n}}(0).$$

This can be achieved by a permutation of the basis vectors, so it won't effect the result given in (4.23).

Before stating and proving the lemma, we need to introduce the following notations

$$\begin{aligned}
q_0(\varepsilon) &:= \begin{cases} \max\limits_{1\leq j\leq \tilde{s}} \tilde{\lambda}_j(\varepsilon) & \text{if} \quad \min\limits_{1\leq j\leq \tilde{s}} |\tilde{\lambda}_j(\varepsilon)| < \min\limits_{\tilde{s}+1\leq j\leq \tilde{n}} |\tilde{\lambda}_j(\varepsilon)| \\ \min\limits_{\tilde{s}+1\leq j\leq \tilde{n}} \tilde{\lambda}_j(\varepsilon) & \text{if} \quad \min\limits_{1\leq j\leq \tilde{s}} |\tilde{\lambda}_j(\varepsilon)| > \min\limits_{\tilde{s}+1\leq j\leq \tilde{n}} |\tilde{\lambda}_j(\varepsilon)| \end{cases} \\
q_+(\varepsilon) &:= \begin{cases} \min\limits_{1\leq j\leq \tilde{s}} \tilde{\lambda}_j(\varepsilon) & \text{if} \quad q_0(\varepsilon) < 0 \\ \max\limits_{\tilde{s}+1\leq j\leq \tilde{n}} \tilde{\lambda}_j(\varepsilon) & \text{if} \quad q_0(\varepsilon) > 0 \end{cases} \\
q_-(\varepsilon) &:= \begin{cases} \max\limits_{\tilde{s}+1\leq j\leq \tilde{n}} \tilde{\lambda}_j(\varepsilon) & \text{if} \quad q_+(\varepsilon) < 0 \\ \min\limits_{1\leq j\leq \tilde{s}} \tilde{\lambda}_j(\varepsilon) & \text{if} \quad q_+(\varepsilon) > 0 \end{cases} \\
\lceil x \rceil &:= \min\{k \in \mathbb{Z} | x \leq k\}.
\end{aligned}$$

We remark that q_0, q_+ and q_- are always continuous functions of ε but not necessarily analytic functions of ε.

For each $m \in \mathcal{S}_{\ell,n,s}$ we use the following notations which denotes the splitting with respect to the stable and the unstable directions:

$$\begin{aligned} M_s &:= (m_1,\cdots,m_s) \\ M_u &:= (m_{s+1},\cdots,m_n). \end{aligned}$$

Lemma 4.11 *As $\mathcal{S}_{\tilde{\ell},\tilde{n},\tilde{s}}$ causes no resonances in* $\operatorname{Spec}(\tilde{A}_\varepsilon)$*, we have for all $m \in \mathcal{S}_{\tilde{\ell},\tilde{n},\tilde{s}}$ satisfying*

$$|m| \geq \left\lceil \frac{q_+(\varepsilon)}{q_0(\varepsilon)} - (\tilde{\ell}-1)\frac{q_-(\varepsilon)}{q_0(\varepsilon)} + (\tilde{\ell}-1) \right\rceil := \Xi(\varepsilon,\tilde{\ell}) \tag{4.24}$$

the following inequality

$$\left|\left\langle \tilde{\Lambda}_\varepsilon, m \right\rangle - \tilde{\lambda}_j\right| \geq |(\tilde{\ell}-1)q_-(\varepsilon) + (|m| - \tilde{\ell}+1)q_0(\varepsilon) - q_+(\varepsilon)| \tag{4.25}$$

for all $j = 1,\cdots,\tilde{n}$.

PROOF: First we establish the inequality for those m for which $|m|$ is "sufficiently" large, afterwards we show that these $|m|$ are bounded below by $\Xi(\varepsilon, \tilde{\ell})$.

First we consider the case where $|M_{\tilde{s}}| < \tilde{\ell}$. For $|M_{\tilde{u}}|$ sufficiently large $\left\langle \tilde{\Lambda}_\varepsilon, m \right\rangle - \tilde{\lambda}_j(\varepsilon)$ will be positive. So taking $|M_{\tilde{s}}| = \tilde{\ell} - 1$, we have

$$\begin{aligned}
\left\langle \tilde{\Lambda}_\varepsilon^{\tilde{s}}, M_{\tilde{s}} \right\rangle &\geq (\tilde{\ell} - 1)\tilde{\lambda}_1(\varepsilon), \\
\left\langle \tilde{\Lambda}_\varepsilon^{\tilde{u}}, M_{\tilde{u}} \right\rangle &\geq \tilde{\lambda}_{\tilde{s}+1}(\varepsilon)(|m| - \tilde{\ell} + 1), \\
-\tilde{\lambda}_j(\varepsilon) &\geq -\tilde{\lambda}_{\tilde{n}}(\varepsilon),
\end{aligned}$$

where

$$\begin{aligned}
\tilde{\Lambda}_\varepsilon^{\tilde{s}} &:= (\tilde{\lambda}_1(\varepsilon), \cdots, \tilde{\lambda}_{\tilde{s}}(\varepsilon)), \\
\tilde{\Lambda}_\varepsilon^{\tilde{u}} &:= (\tilde{\lambda}_{\tilde{s}+1}(\varepsilon), \cdots, \tilde{\lambda}_{\tilde{n}}(\varepsilon)).
\end{aligned}$$

So we can conclude

$$\left\langle \tilde{\Lambda}, m \right\rangle - \tilde{\lambda}_j(\varepsilon) \geq (\tilde{\ell} - 1)\tilde{\lambda}_1(\varepsilon) + \tilde{\lambda}_{s+1}(\varepsilon)(|m| - \tilde{\ell} + 1) - \tilde{\lambda}_{\tilde{n}}(\varepsilon) > 0. \tag{4.26}$$

Second we consider the case where $|M_{\tilde{u}}| < \tilde{\ell}$. For $|M_{\tilde{s}}|$ sufficiently large $\left\langle \tilde{\Lambda}_\varepsilon, m \right\rangle - \tilde{\lambda}_j(\varepsilon)$ will be negative. So taking $|M_{\tilde{u}}| = \tilde{\ell} - 1$, we have

$$\begin{aligned}
\left\langle \tilde{\Lambda}_\varepsilon^{\tilde{s}}, M_{\tilde{s}} \right\rangle &\leq (|m| - \tilde{\ell} + 1)\tilde{\lambda}_{\tilde{s}}(\varepsilon), \\
\left\langle \tilde{\Lambda}_\varepsilon^{\tilde{u}}, M_{\tilde{u}} \right\rangle &\leq (\tilde{\ell} - 1)\tilde{\lambda}_{\tilde{n}}(\varepsilon), \\
-\tilde{\lambda}_j(\varepsilon) &\leq -\tilde{\lambda}_1(\varepsilon).
\end{aligned}$$

So we can conclude

$$\left\langle \tilde{\Lambda}, m \right\rangle - \tilde{\lambda}_j(\varepsilon) \leq (\tilde{\ell} - 1)\tilde{\lambda}_{\tilde{n}}(\varepsilon) + \tilde{\lambda}_s(\varepsilon)(|m| - \tilde{\ell} + 1) - \tilde{\lambda}_1(\varepsilon) < 0. \tag{4.27}$$

Combining (4.26) and (4.27) we find the inequality stated in (4.25).

We now come to the point where we determine a lower bound on $|m|$ such that (4.25) is valid. One observes that the right-hand side of (4.25) is actually linear in $|m|$ if one forgets about the absolute value sign. The absolute value sign gives that the right-hand side of (4.25) will be increasing after the unique zero of this function. From (4.25) it is a very short, straightforward calculation to deduce this unique zero which will give us the lower bound $\Xi(\varepsilon, \tilde{\ell})$ as stated in (4.24). □

PROOF(of Proposition 4.10): From (4.25) we deduce that there exists a constant K_* for all $m \in \mathcal{S}_{\tilde{\ell},\tilde{n},\tilde{s}}$ with $|m| \geq \Xi(\varepsilon, \tilde{\ell})$ such that $\left| \left\langle \tilde{\Lambda}_\varepsilon, m \right\rangle - \tilde{\lambda}_j(\varepsilon) \right| \geq K_*|m|$.

Starting from the right-hand side of (4.25) we have that

$$\begin{array}{rl}
|(\tilde{\ell}-1)q_-(\varepsilon)+(|m|-\tilde{\ell}+1)q_0(\varepsilon)-q_+(\varepsilon)| & = \\
|q_0(\varepsilon)|m|+(\tilde{\ell}-1)q_-(\varepsilon)+(1-\tilde{\ell})q_0(\varepsilon)-q_+(\varepsilon)| & \geq \\
|q_0(\varepsilon)|.|m|-|(\tilde{\ell}-1)q_-(\varepsilon)+(1-\tilde{\ell})q_0(\varepsilon)-q_+(\varepsilon)|. &
\end{array}$$

The latter expression will be positive for $|m| \geq \Xi_1(\varepsilon,\tilde{\ell})$. Take $|m| \geq \xi_{\varepsilon,\tilde{\ell}} := \max\{\Xi(\varepsilon,\tilde{\ell}),\Xi_1(\varepsilon,\tilde{\ell})\}$, then

$$\begin{array}{rl}
|q_0(\varepsilon)| - \frac{|(\tilde{\ell}-1)q_-(\varepsilon)+(1-\tilde{\ell})q_0(\varepsilon)-q_+(\varepsilon)|}{|m|} & \geq \\
|q_0(\varepsilon)| - \frac{|(\tilde{\ell}-1)q_-(\varepsilon)+(1-\tilde{\ell})q_0(\varepsilon)-q_+(\varepsilon)|}{\xi_{\varepsilon,\tilde{\ell}}} & \geq \\
K_* := \inf\limits_{\varepsilon\in B(0,\rho)} \left(|q_0(\varepsilon)| - \dfrac{|(\tilde{\ell}-1)q_-(\varepsilon)+(1-\tilde{\ell})q_0(\varepsilon)-q_+(\varepsilon)|}{\xi_{\varepsilon,\tilde{\ell}}}\right), &
\end{array}$$

hence

$$|(\tilde{\ell}-1)q_-(\varepsilon)+(|m|-\tilde{\ell}+1)q_0(\varepsilon)-q_+(\varepsilon)| \geq K_*|m|$$

and

$$K_* \leq \inf_{\varepsilon\in B(0,\rho)} |q_0(\varepsilon)|.$$

For each r with $1 \leq |m| < \Xi(\varepsilon,\tilde{\ell})$ we can find a constant $K_m > 0$ such that $\left|\left\langle\tilde{\Lambda}_\varepsilon, m\right\rangle - \tilde{\lambda}_j(\varepsilon)\right| \geq K_m|m|$: we just take

$$K_m := \inf_{\varepsilon\in B(0,\rho)} \frac{\left|\left\langle\tilde{\Lambda}_\varepsilon, m\right\rangle - \tilde{\lambda}_j(\varepsilon)\right|}{|m|}.$$

Defining

$$K := \min\left(\{K_r \mid |r| < \Xi(\varepsilon,\tilde{\ell})\} \cup \{K_*\}\right),$$

we have the wanted constant. K will be strictly positive as it is the minimum of a finite set of strictly positive numbers. □

4.3.2 Family of diffeomorphisms

We look at the kth resonance equation ($1 \leq k \leq n$):

$$\prod_{1\leq j\leq n} \lambda_j(\varepsilon)^{m_j} = \lambda_k(\varepsilon). \tag{4.28}$$

As not all eigenvalues are real, we will work with the modulus of each eigenvalue. This way we obtain:

$$\prod_{1\leq j\leq n} |\lambda_j(\varepsilon)^{m_j}| = |\lambda_k(\varepsilon)|. \tag{4.29}$$

Equation (4.29) is equivalent with

$$\sum_{j=1}^{n} m_j \ln|\lambda_j(\varepsilon)| = \ln|\lambda_k(\varepsilon)|. \tag{4.30}$$

Now (4.30) is the resonance equation one would obtain when studying a vector field X_ε on $\mathbb{R}^n$ where the linear part has eigenvalues

$$\{\ln|\lambda_1(\varepsilon)|, \cdots, \ln|\lambda_n(\varepsilon)|\}$$

at the origin. This means that the vector field we obtain has the origin as a hyperbolic singularity. As some eigenvalues can have the same modulus, it is possible that the set of eigenvalues contain (for some value of ε) an eigenvalue with multiplicity greater than 1. But as the set of eigenvalues of the diffeomorphism was multiplicatively non-resonant, it is clear that the eigenvalues of the vector field will be non-resonant. This is the situation described in Remark 4.1. Therefore we can apply the results we obtained in Subsection 4.3.1 to conclude the existence of a $\rho_k > 0$ for which (4.28) will have no solutions if $\varepsilon \in B(0, \rho_k)$. For each resonance equation we obtain a $\rho_j > 0$, so taking

$$\rho := \min_{1\le j\le n} \rho_j$$

provides the radius we are looking for. We now prove that small divisors cannot appear.

Proposition 4.12 *If $\mathcal{S}_{\ell,n,s}$ causes no resonances in* $\mathrm{Spec}(A_0)$*, then there exist positive constants K, C_1, C_2 with $C_1 < 1$ and $C_2 < 1$ such that $\forall m \in \mathcal{S}_{\ell,n,s}$ and $\forall \varepsilon \in B(0,\rho)$ (where ρ was determined in the previous argumentation):*

$$|\Lambda_\varepsilon^m - \lambda_j(\varepsilon)| \ge C_1 - KC_2^{|m|} > 0 \tag{4.31}$$

where $1 \le j \le n$ and $\lambda_j(\varepsilon)$ denotes the jth eigenvalue of $A(\varepsilon)$.

In (4.31) we used the notation

$$\Lambda_\varepsilon^m := \prod_{1\le j\le n} \lambda_j(\varepsilon)^{m_j}.$$

To prove Proposition 4.12 we need another lemma. But before stating and proving this lemma, we need to introduce the following constants:

$$\begin{aligned}
\mu_{--} &:= \inf_{\varepsilon\in B(0,\rho)} \min_{1\le j\le s} |\lambda_j(\varepsilon)|,\\
\mu_{-+} &:= \sup_{\varepsilon\in B(0,\rho)} \max_{1\le j\le s} |\lambda_j(\varepsilon)|,\\
\mu_{+-} &:= \inf_{\varepsilon\in B(0,\rho)} \min_{s+1\le j\le n} |\lambda_j(\varepsilon)|,\\
\mu_{++} &:= \sup_{\varepsilon\in B(0,\rho)} \max_{s+1\le j\le n} |\lambda_j(\varepsilon)|,
\end{aligned}$$

obviously we have

$$\mu_{--} < \mu_{-+} < \mu_{+-} < \mu_{++}.$$

Now we state and prove a crucial lemma

Lemma 4.13 *As* $\mathcal{S}_{\ell,n,s}$ *causes no resonances in* $\mathrm{Spec}(A_\varepsilon)$*, there exists a* $N_\ell \in \mathbb{N}$ *such that* $\forall m \in \mathcal{S}_{\ell,n,s}$ *satisfying* $|m| \geq N_\ell$ *we have that*

$$|\Lambda_\varepsilon^m - \lambda_j(\varepsilon)| \geq \mu_{--} - \left(\frac{\mu_{++}}{\mu_{-+}}\right)^\ell \mu_{-+}^{|m|} \tag{4.32}$$

for $j \in \{1, 2, \cdots, n\}$.

Proof: First of all we like to point out that we have the following inequality

$$|\Lambda_\varepsilon^m - \lambda_j(\varepsilon)| \geq \left| \prod_{1\leq k\leq n} |\lambda_k(\varepsilon)|^{m_k} - |\lambda_j(\varepsilon)| \right|$$

for all $j = 1, \cdots, n$. We will work with the right-hand side of the latter inequality throughout the rest of the proof.

We start by establishing the bound for those m for which $|m|$ is "sufficiently large", afterwards we will show how N_ℓ can be determined.

To make the proof more clear we adopt the following notations: for given $m \in \mathbb{N}^n$ we have

$$\begin{aligned} M_s &:= (m_1, \cdots, m_s), \\ M_u &:= (m_{s+1}, \cdots, m_n), \end{aligned}$$

so $m = (M_s, M_u)$.

First we consider the case where $|M_s| < \ell$. For $|M_u|$ sufficiently large $\left|(\Lambda_\varepsilon^s)^{M_s}\right| \cdot \left|(\Lambda_\varepsilon^u)^{M_u}\right| - |\lambda_j(\varepsilon)|$ will be positive. So taking $|M_s| = \ell - 1$, we have that $|M_u| = |m| - \ell + 1$. From this it follows that $\left|(\Lambda_\varepsilon^s)^{M_s}\right| \geq \mu_{--}^{\ell-1}$ and $\left|(\Lambda_\varepsilon^u)^{M_u}\right| \geq \mu_{+-}^{|m|-\ell+1}$. It is obvious that $|\lambda_j(\varepsilon)| \leq \mu_{++}$. From these results one obtains that

$$|\Lambda_\varepsilon^m| - |\lambda_j(\varepsilon)| \geq \left(\frac{\mu_{--}}{\mu_{+-}}\right)^{\ell-1} \mu_{+-}^{|m|} - \mu_{++}. \tag{4.33}$$

Second we consider the case where $|M_u| < \ell$. For $|M_s|$ sufficiently large $\left|(\Lambda_\varepsilon^s)^{M_s}\right| \cdot \left|(\Lambda_\varepsilon^u)^{M_u}\right| - |\lambda_j(\varepsilon)|$ is negative. Taking $|M_u| = \ell - 1$, we have that $|M_s| = |m| - \ell + 1$. From this it follows that $\left|(\Lambda_\varepsilon^s)^{M_s}\right| \leq \mu_{-+}^{|m|-\ell+1}$ and $\left|(\Lambda_\varepsilon^u)^{M_u}\right| \leq \mu_{++}^{\ell-1}$. It is obvious that $|\lambda_j(\varepsilon)| \geq \mu_{--}$. From these results one obtains that

$$|\Lambda_\varepsilon^m| - |\lambda_j(\varepsilon)| \leq \left(\frac{\mu_{++}}{\mu_{-+}}\right)^{\ell-1} \mu_{-+}^{|m|} - \mu_{--}. \tag{4.34}$$

As $\mu_{+-} > 1$ the right-hand side of (4.33) will diverge for $|m| \to \infty$ and as $0 < \mu_{-+} < 1$ the absolute value of the right-hand side of (4.34) will be increasing and converging to μ_{--} as $|m| \to \infty$. Thus there exists a $N_\ell \in \mathbb{N}$ such that for $|m| \geq N_\ell$ (4.32) is fulfilled. □

Proof(of Proposition 4.12): From Lemma 4.13 we have that there exists a constant K_* for all $m \in \mathbb{N}^n$ with $|m| \geq N_\ell$ such that

$$|\Lambda_\varepsilon^m - \lambda_j(\varepsilon)| \geq \mu_{--} - K_* \mu_{-+}^{|m|},$$

we simply take

$$K_* = \left(\frac{\mu_{++}}{\mu_{-+}}\right)^\ell.$$

For each m with $0 < |m| < N_\ell$ we can find a constant $K_m > 0$ such that

$$|\Lambda_\varepsilon^m - \lambda_j(\varepsilon)| \geq \mu_{--} - K_m \mu_{-+}^{|m|} > 0 :$$

we simply take

$$K_m := \sup_{\varepsilon \in B(0,\rho)} \frac{\mu_{--} - |\Lambda_\varepsilon^m - \lambda_j(\varepsilon)|}{\mu_{-+}^{|m|}}.$$

Take

$$K := \min\{K_m |\ |m| < N_\ell\} \cup \{K_*\}$$

then we have the wanted constant. K will be strictly positive as the minimum is taken over a finite set of strictly positive numbers. □

4.4 Proof of Theorem 4.1

The proof of Theorem 4.1 consists of 3 parts: first we determine 2 equations that will give us (4.12), second we show that there exists a formal solution and finally we show that the formal solution converges, i.e. there exists an analytic solution.

4.4.1 Determining the change of coordinates

In this subsection we want to establish an equation which will allow us to determine the transformation (4.12) we are seeking. From now on we will work in the complexified setting given by (4.4), this will make it easier to determine a formal solution. This means that we will need to "complexify" the function ϕ_ε given by (4.12).

So we have a vector field given by

$$\dot{z} = B_\varepsilon z + F_\varepsilon(z)$$

where $z = Px$ with P given by (4.3), which we want to transform into a vector field

$$\dot{w} = B_\varepsilon w + G_\varepsilon(w)$$

where $P^{-1}w \in \mathbb{R}^n$ (as we wish to return to a real vector field at the end) and $[G_\varepsilon(w)]^{\mathcal{S}_{\ell,n,s}} = 0$, by a transformation

$$z = w + \varphi_\varepsilon(w).$$

Of course we want that this transformation puts the vector field into a vector field that is also the complexification of a real vector field. The following result gives us that an analytic transformation will put an analytic complexified real vector field into a complexified real vector field.

Proposition 4.14 *Given the analytic complexified real vector field*

$$\left\{ \begin{array}{rcl} \dot{z} & = & f(z, \bar{z}, r) \\ \dot{\bar{z}} & = & \overline{f(z, \bar{z}, r)} \\ \dot{r} & = & g(z, \bar{z}, r) \end{array} \right. , \tag{4.35}$$

with z, $\bar{z} \in \mathbb{C}^p$ ($p \geq 1$), $r \in \mathbb{C}^q$ ($q \geq 1$) and $\overline{g(z, \bar{z}, r)} = g(z, \bar{z}, r)$, and given an analytic transformation

$$(z, \bar{z}, r) = \Phi(w, \bar{w}, s),$$

with

$$\Phi(w, \bar{w}, s) = (\varphi(w, \bar{w}, s), \bar{\varphi}(w, \bar{w}, s), \psi(w, \bar{w}, s))$$

in which we denoted $\bar{\varphi}(w, \bar{w}, s) = \overline{\varphi(w, \bar{w}, s)}$ and where $\overline{\psi(w, \bar{w}, s)} = \psi(w, \bar{w}, s)$, then the transformed system

$$\left\{ \begin{array}{rcl} \dot{w} & = & F_1(w, \bar{w}, s) \\ \dot{\bar{w}} & = & F_2(w, \bar{w}, s) \\ \dot{s} & = & F_3(w, \bar{w}, s) \end{array} \right. \tag{4.36}$$

is a complexified real vector field.

Proof: Straightforward. □

So if we can obtain an analytic transformation, we are assured of the fact that the complexified real vector field is transformed into a complexified real vector field.

In order to return to a real vector field we will have

$$\phi_\varepsilon(w) = P^{-1} \cdot \varphi_\varepsilon(Pw) \tag{4.37}$$

and

$$G_\varepsilon(w) = P^{-1} \cdot g_\varepsilon(Pw)$$

where P is the matrix defined by (4.3), that gives the change of basis.

Performing this transformation we find the following two equalities

$$\begin{aligned} \dot{z} &= (I_n + D_w\varphi_\varepsilon(w))(B_\varepsilon w + G_\varepsilon(w)) \\ \dot{z} &= B_\varepsilon(w + \varphi_\varepsilon(w)) + F_\varepsilon(w + \varphi_\varepsilon(w)) \end{aligned}$$

If we introduce the operator L_{B_ε}

$$L_{B_\varepsilon}\varphi_\varepsilon(w) = D_w\varphi_\varepsilon(w)B_\varepsilon w - B_\varepsilon\varphi_\varepsilon(w) \tag{4.38}$$

then these equations can be combined to obtain

$$L_{B_\varepsilon}\varphi_\varepsilon(w) = F_\varepsilon(w + \varphi_\varepsilon(w)) - G_\varepsilon(w) - D_w\varphi_\varepsilon(w)G_\varepsilon(w) \tag{4.39}$$

We split (4.39) up into two separate equations. This splitting up will be done with respect to $\mathcal{S}_{\ell,n,s}$ and $\mathcal{T}_{\ell,n,s}$. Thus we will solve

$$L_{B_\varepsilon}\varphi_\varepsilon(w) = [F_\varepsilon(w + \varphi_\varepsilon(w))]^{\mathcal{S}_{\ell,n,s}} \tag{4.40}$$

$$[F_\varepsilon(w + \varphi_\varepsilon(w))]^{\mathcal{T}_{\ell,n,s}} = (I_n + D_w\varphi_\varepsilon(w))G_\varepsilon(w) \tag{4.41}$$

If we can solve (4.40), then we can determine $G_\varepsilon(w)$ directly as $(I_n + D_w\varphi_\varepsilon(w))$ is invertible in a sufficiently small neighbourhood of the origin. We know that the formal expansion of φ_ε starts with terms of degree 2 in w, so multiplying $[F_\varepsilon(w + \varphi_\varepsilon(w))]^{\mathcal{T}_{\ell,n,s}}$ with $(I_n + D_w\varphi_\varepsilon(w))^{-1}$ will only increase the degree of each term in w, hence

$$\left[(I_n + D_w\varphi_\varepsilon(w))^{-1}\left[F_\varepsilon(w) + \varphi_\varepsilon(w)\right)]^{\mathcal{T}_{\ell,n,s}}\right]^{\mathcal{S}_{\ell,n,s}} = 0,$$

so $[G_\varepsilon(w)]^{\mathcal{S}_{\ell,n,s}} = 0$.

Also from (4.41) we immediately have that $|G_\varepsilon(w)| \leq K_1|W_s|^\ell|W_u|^\ell$, where $W_s = (w_1, \cdots, w_s)$ and $W_u = (w_{s+1}, \cdots, w_n)$. Hence by virtue of (4.37) we have the same bounds for $g_\varepsilon(y)$.

In the next subsections we will solve (4.40) and show that the solution has all properties as stated in Theorem 4.1.

4.4.2 Formal solution of (4.40)

A direct calculation shows that

$$L_{B_{\varepsilon,j}}(vw^m) = v(\langle\Lambda_\varepsilon, m\rangle - \lambda_j(\varepsilon))w^m,$$

$1 \leq j \leq n$, for any $m \in \mathbb{N}^n$ and any $v \in \mathbb{C}^n$. This means that if we want to have a formal solution $\varphi_\varepsilon(w) = \sum_{|m|\geq 2} a_m(\varepsilon)w^m$, then (4.40) becomes

$$\sum_{|m|\geq 2} a_{m,j}(\varepsilon)(\langle\Lambda_\varepsilon, m\rangle - \lambda_j(\varepsilon))w^m = \left[\sum_{|m|\geq 2} F_{m,j}(\varepsilon)\left(w + \sum_{|k|\geq 2} a_k(\varepsilon)w^k\right)^m\right]^{\mathcal{S}_{\ell,n,s}} \tag{4.42}$$

where $1 \leq j \leq n$ and

$$F_\varepsilon(w) = \sum_{|m|\geq 2} F_m(\varepsilon) w^m.$$

We now show how (4.42) can be solved formally. First we take the coefficient of w^M for $M \in \mathbb{N}^n$ with $|M| = 2$, then (4.42) gives

$$a_{M,j}(\varepsilon) = \frac{F_{M,j}(\varepsilon)}{\langle \Lambda_\varepsilon, M\rangle - \lambda_j(\varepsilon)}$$

thus $a_{M,j}(\varepsilon)$ is an analytic function as in the right-hand side both the numerator and the denominator are analytic functions and by non-resonance we know that the denominator is bounded away from zero. Each $a_{M,j}(\varepsilon)$ will be analytic on the same poly-disk as $F_\varepsilon(w)$ is analytic on a poly-disk, so by Theorem 1.4 its coefficients in the formal power series expansion with respect to w are analytic on a fixed poly-disk. We now proceed by induction, so assume that $a_{m,j}$ is an analytic function of ε for all $m \in \mathbb{N}^n$ with $2 \leq |m| \leq N-1$. Now take a $m \in \mathbb{N}^n$ with $|m| = N$. Taking the coefficients of w^m in (4.42) we find

$$a_{m,j}(\varepsilon) = \sigma_m \frac{F_{m,j}(\varepsilon) + \displaystyle\sum_{\substack{r\in\mathbb{N}^n \\ |r|\leq N-1}} P_r^m\big((a_k(\varepsilon))|_{|k|\leq N-1}\big) F_{r,j}(\varepsilon)}{\langle \Lambda_\varepsilon, m\rangle - \lambda_j(\varepsilon)} \tag{4.43}$$

where σ_m is defined by

$$\sigma_m := \begin{cases} 1 & \text{if} \quad m \in \mathcal{S}_{\ell,n,s} \\ 0 & \text{if} \quad m \in \mathcal{T}_{\ell,n,s} \end{cases}$$

and where P_r^m is polynomial with positive integer coefficients. Before proving this, we need the following definition.

Definition 4.1 *Consider two formal power series $\sum_{m\in\mathbb{N}^n} a_m z^m$ and $\sum_{m\in\mathbb{N}^n} b_m z^m$ where $a_m, b_m \in \mathbb{C}$, $\forall m \in \mathbb{N}^n$ and $z \in \mathbb{C}^n$, then the* **product** *of these two series is defined as*

$$\left(\sum_{m\in\mathbb{N}^n} a_m z^m\right)\left(\sum_{m\in\mathbb{N}^n} b_m z^m\right) = \sum_{m\in\mathbb{N}^n} (a*b)_m z^m$$

where

$$(a*b)_m = \sum_{r_1=0}^{m_1} \cdots \sum_{r_n=0}^{m_n} a_r b_{m-r}.$$

The expression of $a_{m,j}$ in (4.43) will follow from the following result.

Proposition 4.15 *Assume* $(a_k)_{k\in\mathbb{N}^n}$ *is a sequence in* $\mathbb{C}$, $p \in \mathbb{N}\setminus\{0,1\}$ *and* $z \in \mathbb{C}^n$, *then*

$$\left(z_j + \sum_{|k|\geq 2} a_k z^k\right)^p = \sum_{|k|\geq p} P_k^p\left((a_r)|_{|r|<|k|}\right) z^k$$

where $1 \leq j \leq n$ *and* P_k^p *is a polynomial with positive integer coefficients.*

Proof: First we consider the case where $p = 2$. We now have

$$\left(z_j + \sum_{|k|\geq 2} a_k z^k\right)^2 = \left(z_j + \sum_{|k|\geq 2} a_k z^k\right)\left(z_j + \sum_{|k|\geq 2} a_k z^k\right).$$

Taking the convention that $a_{e_s} = \delta_{sj}$ for $s = 1,\cdots,n$ where e_s is the vector with a 1 on the sth place and zeros elsewhere, we obtain

$$\left(z_j + \sum_{|k|\geq 2} a_k z^k\right)^2 = \sum_{|k|\geq 2}\left(\sum_{m_1=0}^{k_1}\cdots\sum_{m_n=0}^{k_n} a_m a_{k-m}\right) z^k \qquad (4.44)$$

where the coefficient of z^k in the right-hand side of (4.44) does not contain the terms for $m = 0$ or $m = k$ as $a_0 = 0$. This means that

$$P_k^2\left((a_r)|_{|r|<|k|}\right) = \sum_{m_1=0}^{k_1}\cdots\sum_{m_n=0}^{k_n} a_m a_{k-m}$$

is a polynomial with positive integer coefficients.

We proceed by induction. Assume that the result holds for $p = q-1$, then we have

$$\begin{aligned}\left(z_j + \sum_{|k|\geq 2} a_k z^k\right)^q &= \left(z_j + \sum_{|k|\geq 2} a_k z^k\right)^{q-1}\left(z_j + \sum_{|k|\geq 2} a_k z^k\right)\\ &= \left(\sum_{|k|\geq q-1} P_k^{q-1}((a_r)|_{|r|<|k|})z^k\right)\left(\sum_{|k|\geq 1} a_k z^k\right)\end{aligned}$$

$$\left(z_j + \sum_{|k|\geq 2} a_k z^k\right)^q = \sum_{|k|\geq q}\left(\sum_{m_1=0}^{k_1}\cdots\sum_{m_n=0}^{k_n} P_m^{q-1}((a_r)|_{|r|<|k|})a_{k-m}\right) z^k \quad (4.45)$$

where the coefficient of z^k in the right-hand side of (4.45) does not contain the terms for $m = 0$ or $m = k$ as $a_0 = 0$. We also used the notations $a_{e_s} = \delta_{sj}$ for

$s = 1, \cdots, n$. This means that

$$P_k^q((a_r)|_{|r|<|k|}) = \sum_{m_1=0}^{k_1} \cdots \sum_{m_n=0}^{k_n} P_m^{q-1}((a_r)|_{|r|<|k|-1}) a_{k-m}$$

is polynomial with positive integer coefficients. □

As we know that all $F_{r,j}$ are analytic on the same poly-disk and the denominator is non-zero, the induction hypothesis will give us that $a_{M,j}(\varepsilon)$ is an analytic function of ε in a poly-disk independent of M and j.

4.4.3 Convergence of the formal solution

We now want to prove that this formal solution converges, i.e. we have an analytic solution in w. For this we will use the technique of *majorants*, this technique is very common in the study of complex systems, see for instance [Mos56, Sto94]. Using this technique in combination with Proposition 1.6 will give us that $\varphi_\varepsilon(w)$ is analytic in (w, ε).

Definition 4.2 *Given two formal power series* $f(z) = \sum_{m \in \mathbb{N}^n} f_m z^m$ *and* $g(z) = \sum_{m \in \mathbb{N}^n} g_m z^m$*, we say that* g *is a* **majorant** *of* f *if we have that* $|f_m| \leq g_m$*,* $\forall m \in \mathbb{N}^n$.

One should note that in the latter definition the coefficients of $g(z)$ must be real and positive whilst the coefficients of $f(z)$ may be complex.

Given $m \in \mathcal{S}_{\ell,n,s}$ with $|m| \geq 2$, we have that

$$\nu(m) := \inf_{\varepsilon \in B(0,\rho)} \min_{1 \leq k \leq n} |\langle \Lambda_\varepsilon, m \rangle - \lambda_k(\varepsilon)|$$

is bounded away from zero by virtue of Proposition 4.9. If we use the notation

$$\tilde{c}_m := \sup_{\varepsilon \in B(0,\tilde{\rho})} \max_{1 \leq k \leq n} |F_{m,k}(\varepsilon)|,$$

for a fixed $\tilde{\rho}$ with $0 < \tilde{\rho} < \rho$, then we can define

$$\tilde{F}(w) = \sum_{|m| \geq 2} \tilde{c}_m w^m \mathbf{e}$$

so $\tilde{F}_1 = \cdots = \tilde{F}_n$ and $\tilde{F}_j$ is a majorant of $F_{\varepsilon,j}$ for $j = 1, \cdots, n$.

We know that $F_\varepsilon(w)$ is analytic on $\mathbb{D}(0, R) \times \mathbb{D}(0, \rho\mathbf{e})$, so by Theorem 1.4 its Taylor series is normally convergent on $\mathbb{D}(0, R) \times \mathbb{D}(0, \rho\mathbf{e})$. Writing $F_\varepsilon(w) =$

$\sum\limits_{\substack{|m|\geq 2\\|\tilde{m}|\geq 0}} f_{m,\tilde{m}}\varepsilon^{\tilde{m}}w^m$, we have

$$\sum_{\substack{|m|\geq 2\\|\tilde{m}|\geq 0}} \sup_{(w,\varepsilon)\in K} |f_{m,\tilde{m}}\varepsilon^{\tilde{m}}w^m| < +\infty \tag{4.46}$$

for all compact subsets K in $\mathbb{D}(0,R)\times\mathbb{D}(0,\rho\mathbf{e})$. The normal convergence gives us that for all ρ_* and R_* with $\rho_* < \rho$ and $|R_*| < |R|$ we have that

$$\sum_{\substack{|m|\geq 2\\|\tilde{m}|\geq 0}} |f_{m,\tilde{m}}|\rho_*^{|\tilde{m}|}R_*^m < \infty.$$

As $F_\varepsilon(w) = \sum_{|m|\geq 2} F_m(\varepsilon)w^m$, we have that

$$|F_m(\varepsilon)| \leq \sum_{|\tilde{m}|\geq 0} |f_{m,\tilde{m}}|\tilde{\rho}^{|\tilde{m}|} < \infty, \forall \varepsilon \in \mathbb{D}(0,\tilde{\rho}\mathbf{e}),$$

hence

$$\tilde{c}_m = \sup_{\varepsilon\in B(0,\tilde{\rho})} |F_m(\varepsilon)| \leq \sum_{|\tilde{m}|\geq 0} |f_{m,\tilde{m}}|\tilde{\rho}^{|\tilde{m}|}.$$

Define $\tilde{R} = \min\limits_{1\leq j\leq n} R_j$, then for each compact $\tilde{K}\subset\mathbb{D}(0,\tilde{R}\mathbf{e})$ we have

$$\begin{aligned}
\sum_{|m|\geq 2} \sup_{w\in\tilde{K}} |\tilde{c}_m w^m \mathbf{e}| &\leq \sum_{|m|\geq 2} |\tilde{c}_m| R_\star^{|m|} \\
&\leq \sum_{\substack{|m|\geq 2\\|\tilde{m}|\geq 0}} |f_{m,\tilde{m}}|\tilde{\rho}^{|\tilde{m}|}R_\star^{|m|} \\
&< \infty
\end{aligned}$$

by virtue of (4.46) where $R_\star$ is the radius of the smallest closed ball for which we have that $\tilde{K}\subset\overline{B(0,R_\star)}\subset\mathbb{D}(0,\tilde{R}\mathbf{e})$. This means that the series $\sum_{|m|\geq 2}\tilde{c}_m w^m$ converges normally on $\mathbb{D}(0,\tilde{R}\mathbf{e})$. As $\tilde{c}_m w^m$ is clearly analytic in w for each m we now have that $\tilde{F}$ is analytic on $\mathbb{D}(0,\tilde{R}\mathbf{e})$.

Let $\tilde{\varphi}(w) = \sum_{|m|\geq 2}\tilde{a}_m w^m$ be the solution of

$$\sum_{\substack{|m|\geq 2\\m\in\mathcal{S}_{\ell,n,s}}} \nu(m)\tilde{a}_m w^m = \left[\tilde{F}(w+\tilde{\varphi}(w))\right]^{\mathcal{S}_{\ell,n,s}}. \tag{4.47}$$

As the coefficients on the right-hand side of (4.40) are majorised by the coefficients on the right-hand side of (4.47) and the moduli of the coefficients on the left-hand side of (4.40) are majorising the coefficients on the left-hand side of (4.47), hence by division and (4.43) we obtain that $\tilde{\varphi}$ is a majorant of φ_ε for all $\varepsilon \in B(0,\tilde{\rho})$, in other words

$$|a_{m,j}(\varepsilon)| \leq \tilde{a}_{m,j}, j = 1,\cdots,n \tag{4.48}$$

we also have that $\tilde{\varphi}_1 = \cdots = \tilde{\varphi}_n$.

We would like to reduce the problem of convergence to a 1-dimensional problem. Therefore we will need another majorant. We define

$$c_k := \sum_{|m|=k} \tilde{c}_m$$

and

$$\hat{F}(Z) := \sum_{k\geq 2} c_k Z^k, Z \in \mathbb{C}$$

then $\hat{F}(Z)$ equals $\tilde{F}_j(Z\mathbf{e})$ for each $j = 1,\cdots,n$, so $\hat{F}(Z)$ is obviously a majorant for each component of $\tilde{F}(Z\mathbf{e})$. As $\hat{F}(Z)\mathbf{e} = \tilde{F}(Z\mathbf{e})$, $\hat{F}$ is analytic iff $|Z| \leq R_j$ for all $j = 1,\cdots,n$. Hence $\hat{F}(Z)$ is analytic on $B(0,\hat{R}) = \mathbb{D}(0,\hat{R}\mathbf{e})$ where $\hat{R} = \min\limits_{1\leq j\leq n} R_j$. In the same line of arguments we introduce

$$\nu_k := \min_{\substack{|m|=k \\ m\in\mathcal{S}_{\ell,n,s}}} \nu(m)$$

then by Proposition 4.9 we know there exists a constant $\kappa > 0$ for which we have

$$\nu_k \geq \kappa k.$$

We can look at the solution $\hat{\varphi}(Z) = \sum_{k\geq 2} \hat{a}_k Z^k$ of

$$\sum_{k\geq 2} \kappa k \hat{a}_k Z^k = \hat{F}(Z + \hat{\varphi}(Z)). \tag{4.49}$$

As the coefficients on the right-hand side of (4.47) are majorised by the coefficients on the right-hand side of (4.49) and the coefficients on the left-hand side of (4.47) are majorising the coefficients of the left-hand side of (4.49), hence by division, (4.43) and (4.46) we obtain that $\hat{\varphi}(Z)$ is a majorant of each component of $\tilde{\varphi}(Z\mathbf{e})$, i.e.

$$\tilde{a}_{m,j} \leq \hat{a}_k$$

for all $m \in \mathcal{S}_{\ell,n,s}$ with $|m| = k$ and $1 \leq j \leq n$.

As $k \geq 2$, it is obvious that $\sum_{k\geq 2} \kappa k \hat{a}_k Z^k$ is a majorant for $\sum_{k\geq 2} \kappa \hat{a}_k Z^k$. We know that $\hat{F}$ is analytic on $B(0,\hat{R})$, so we have that

$$\overline{\lim_{k\to\infty}} \sqrt[k]{c_k} = \frac{1}{\hat{R}}. \tag{4.50}$$

Take a small but fixed $\delta > 0$, then (4.50) implies that there exists a $K \in \mathbb{N}$ such that for all $k \geq K$ we have

$$\sqrt[k]{c_k} \leq \frac{1+\delta}{\hat{R}},$$

whence

$$c_k \leq \left(\frac{1+\delta}{\hat{R}}\right)^k, \forall k \geq K.$$

For $2 \leq k \leq K-1$ we obviously have

$$c_k \leq c_k \left(\frac{\hat{R}}{1+\delta}\right)^k \left(\frac{1+\delta}{\hat{R}}\right)^k.$$

Defining $\check{R} = \frac{\hat{R}}{1+\delta}$ and

$$\check{c} := \max\left(\left\{ c_k \left(\frac{\hat{R}}{1+\delta}\right)^k \middle| 2 \leq k \leq K-1 \right\} \cup \{1\}\right),$$

we have that

$$c_k \leq \check{c}\left(\frac{1}{\check{R}}\right)^k, \forall k \geq 2.$$

As $\check{F}(Z) := \check{c}\sum_{k\geq 2}\left(\frac{Z}{\check{R}}\right)^k$ is a geometrical series, we know that $\check{F}(Z)$ is analytic on $B(0,\check{R})$ and $\check{F}$ is a majorant of $\hat{F}$.

Let $\Phi(Z) = \sum_{k\geq 2} \check{a}_k Z^k$ be the solution of

$$\kappa\Phi(Z) = \check{F}(Z + \Phi(Z)), \tag{4.51}$$

then Φ will be a majorant of $\hat{\varphi}$.

As $\check{F}$ is given by a geometrical series (4.51) becomes

$$\begin{aligned} \kappa\Phi(z) &= \check{c}\left(\frac{Z+\Phi(Z)}{\check{R}}\right)^2 \sum_{k\geq 2}\left(\frac{Z+\Phi(Z)}{\check{R}}\right)^{k-2} \\ &= \frac{\check{c}}{\check{R}^2}\frac{(Z+\Phi(Z))^2}{1-\frac{Z+\Phi(Z)}{\check{R}}}, \end{aligned}$$

which gives the following quadratic equation in $\Phi(Z)$:

$$\left(\check{c}+\kappa\check{R}\right)\Phi(Z)^2 + \left(\left(2\check{c}+\kappa\check{R}\right)Z - \kappa\check{R}^2\right)\Phi(Z) + \check{c}Z^2 = 0. \tag{4.52}$$

The discriminant of (4.52) is given by

$$D(Z) = \left(\left(2\check{c}+\kappa\check{R}\right)Z - \kappa\check{R}^2\right)^2 - 4\left(\check{c}+\kappa\check{R}\right)\check{c}Z^2.$$

Now Φ is given by

$$\Phi(Z) = \frac{\check{R}^2\kappa - \left(2\check{c} + \kappa\check{R}\right) Z - \sqrt{D(Z)}}{2\left(\check{c} + \kappa\check{R}\right)},$$

where we take the solution with $-\sqrt{D(Z)}$ as we need the solution without constant and linear terms in its formal series expansion. Now it is clear that Φ is analytic in $B(0, \check{R})$. From the series expansion it is clear that $\|\Phi\|_r \leq K_0 r^2$ for any $r < r_0 := \check{R}$, and by virtue of the majorisation we have the same bound for φ_ε. Writing down everything in its real components we obtain the properties stated in Theorem 4.1.

4.5 Proof of Theorem 4.2

The proof of Theorem 4.1 consisted of three parts and also the proof of Theorem 4.2 will consist of three parts. First we determine the 2 equations that determine ϕ_ε as given in (4.15), second we show that these equations have a formal solution and finally we show that this formal solution is convergent, i.e. there exists an analytic solution.

4.5.1 Determining the change of coordinates

In this subsection we want to establish an equation which will allow us to determine the transformation (4.15) we are seeking. From now on we will work in the complexified setting given by (4.6), this will make it easier to determine a formal solution. This means that we will need to "complexify" the function ϕ_ε given by (4.15), but as we started in a real-valued setting, we can always get the "real" form of ϕ_ε by returning to the real coordinates x.

So we have a diffeomorphism given by

$$f^\star_\varepsilon(z) = B_\varepsilon z + F^\star_\varepsilon(z) \tag{4.53}$$

where $z = Px$ with P given by (4.3), which we want to conjugate to a diffeomorphism

$$g^\star_\varepsilon(w) = B_\varepsilon w + G^\star_\varepsilon(w) \tag{4.54}$$

where $P^{-1}w \in \mathbb{R}^n$ (as we wish to return to the real diffeomorphism at the end) and $[G^\star_\varepsilon]^{\mathcal{S}_{\ell,n,s}} = 0$, by a transformation

$$z = w + \varphi_\varepsilon(w). \tag{4.55}$$

Of course we want that (4.55) transforms the diffeomorphism into a diffeomorphism that is the complexication of a real diffeomorphism. It is straightforward to prove that the following result.

Proposition 4.16 *Given the analytic complexified real diffeomorphism*

$$F : \mathbb{C}^n \to \mathbb{C}^n : (z, \bar{z}, r) \mapsto \left(f(z, \bar{z}, r), \overline{f(z, \bar{z}, r)}, g(z, \bar{z}, r) \right)$$

with $z \in \mathbb{C}^p$, $r \in \mathbb{C}^q$, $2p + q = n$ *and* $\overline{g(z, \bar{z}, r)} = g(z, \bar{z}, r)$, *and given an analytic transformation*

$$\Phi : \mathbb{C}^n \to \mathbb{C}^n : (w, \bar{w}, s) \mapsto \left(\phi(w, \bar{w}, s), \overline{\phi(w, \bar{w}, s)}, \psi(w, \bar{w}, s) \right)$$

with $\phi(w, \bar{w}, s) \in \mathbb{C}^p$, $\psi(w, \bar{w}, s) \in \mathbb{C}^q$, *then the diffeomorphism* $G = \Phi^{-1} \circ F \circ \Phi$ *is a complexified real diffeomorphism.*

In order to return to a real diffeomorphism we will have

$$\phi_\varepsilon(x) = P^{-1} \cdot \varphi_\varepsilon(Px)$$

and

$$G^\star_\varepsilon(w) = P^{-1} \cdot g_\varepsilon(Pw)$$

where P is the matrix defined by (4.3), that gives the change of basis.

(4.53) and (4.54) are conjugate by (4.55) iff

$$f^\star_\varepsilon \circ \Phi_\varepsilon = \Phi_\varepsilon \circ g^\star_\varepsilon \tag{4.56}$$

where $\Phi_\varepsilon(z) = z + \varphi_\varepsilon(z)$. A direct calculation gives us that (4.56) is equivalent with

$$B_\varepsilon \varphi_\varepsilon(y) + F^\star_\varepsilon(z + \varphi_\varepsilon(z)) = G^\star_\varepsilon(z) + \varphi_\varepsilon(B_\varepsilon z + G^\star_\varepsilon(z)), \tag{4.57}$$

If we introduce the operator L_{B_ε}

$$L_{B_\varepsilon} \varphi_\varepsilon(z) = \varphi_\varepsilon(B_\varepsilon z) - B_\varepsilon \varphi_\varepsilon(z) \tag{4.58}$$

then (4.57) can be written as

$$L_{B_\varepsilon} \varphi_\varepsilon(z) = F^\star_\varepsilon(z + \varphi_\varepsilon(z)) - G^\star_\varepsilon(z) - (\varphi_\varepsilon(B_\varepsilon z + G^\star_\varepsilon(z)) - \varphi_\varepsilon(B_\varepsilon z)) . \tag{4.59}$$

We will first split (4.59) up into two separate equations. This splitting will be done with respect to $\mathcal{S}_{\ell,n,s}$ and $\mathcal{T}_{\ell,n,s}$. Thus we will solve

$$L_{B_\varepsilon} \varphi_\varepsilon(z) = [F^\star_\varepsilon(z + \varphi_\varepsilon(z))]^{\mathcal{S}_{\ell,n,s}} \tag{4.60}$$

$$[F^\star_\varepsilon(z + \varphi_\varepsilon(z))]^{\mathcal{T}_{\ell,n,s}} = G^\star_\varepsilon(z) + (\varphi_\varepsilon(B_\varepsilon z + G^\star_\varepsilon(z)) - \varphi_\varepsilon(B_\varepsilon z)) . \tag{4.61}$$

First we will discuss the solution of (4.60). This discussion is analogous to the one we did in Subsection 4.4.3, so we will be very brief. Once we have established the analytic nature of φ_ε we will tackle (4.61) and prove that

$$\begin{aligned} [G^\star_\varepsilon(z)]^{\mathcal{S}_{\ell,n,s}} &= 0, \\ [\varphi_\varepsilon(B_\varepsilon z + G^\star_\varepsilon(z)) - \varphi_\varepsilon(z)]^{\mathcal{S}_{\ell,n,s}} &= 0 \end{aligned}$$

and that $G^\star_\varepsilon$ is an analytic function with the stated properties.

4.5.2 Solution of (4.60)

Formal solution

First we prove that (4.60) has a *formal* solution. A direct calculation shows that

$$L_{B_\varepsilon,j}(vz^m) = v\,(\Lambda_\varepsilon^m - \lambda_j(\varepsilon))\,z^m,$$

$1 \le j \le n$, for any $m \in \mathbb{N}^n$ and any $v \in \mathbb{C}^n$. This means that if we want to have a formal solution $\varphi_\varepsilon(z) = \sum_{|m|\ge 2} a_m(\varepsilon)z^m$, then (4.60) becomes

$$\sum_{|m|\ge 2} a_{m,j}(\varepsilon)(\Lambda_\varepsilon^m - \lambda_j(\varepsilon))z^m = \left[\sum_{|m|\ge 2} F^\star_{m,j}(\varepsilon)\left(z + \sum_{|k|\ge 2} a_k(\varepsilon)z^k\right)^m\right]^{\mathcal{S}_{\ell,n,s}} \tag{4.62}$$

where $1 \le j \le n$ and

$$F^\star_\varepsilon(z) = \sum_{|m|\ge 2} F^\star_{m,j}(\varepsilon)z^m.$$

As (4.42) and (4.62) are almost equal, we can repeat the technique used to solve (4.42) on (4.62) and obtain the same conlusions.

Convergence of the formal solution

We now want to prove that the formal solution converges, i.e. we have an analytic solution. Therefore we will use the technique of the majorants as we did in Subsection 4.4.3. As the proofs are very similar, we skip most of the details and focus on the main differences.

Given $m \in \mathcal{S}_{\ell,n,s}$ with $|m| \ge 2$, we have that

$$\nu(m) := \inf_{\varepsilon\in B(0,\rho)} \min_{1\le j\le n} |\Lambda_\varepsilon^m - \lambda_j(\varepsilon)|$$

is bounded away from zero by virtue of Proposition 4.12. If we use the notation

$$\tilde{c}_m := \sup_{\varepsilon\in B(0,\rho)} \max_{1\le j\le n} |F^\star_{m,j}(\varepsilon)|$$

then we can define

$$\tilde{F}(y) = \sum_{|m|\ge 2} \tilde{c}_m z^m \mathbf{e}$$

so $\tilde{F}_1 = \tilde{F}_2 = \cdots = \tilde{F}_n$ and $\tilde{F}_j$ is a majorant of $F^\star_{\varepsilon,j}$ for $j = 1, \cdots, n$.

We know that $F^\star_\varepsilon(w)$ is analytic on $\mathbb{D}(0,R)\times\mathbb{D}(0,\rho\mathbf{e})$. Repeating the proof of the vector field case we conclude that $\tilde{F}$ is analytic on $\mathbb{D}(0,\tilde{R}\mathbf{e})$ where $\tilde{R} = \min_{1\le j\le n} R_j$.

Let $\tilde{\varphi}(z) = \sum_{|m|\geq 2} \tilde{a}_m z^m$ be the solution of

$$\sum_{\substack{|m|\geq 2 \\ m\in\mathcal{S}_{\ell,n,s}}} \nu(m)\tilde{a}_m z^m = \left[\tilde{F}(z+\tilde{\varphi}(z))\right]^{\mathcal{S}_{\ell,n,s}}. \tag{4.63}$$

We have that $\tilde{\varphi}$ is a majorant of φ_ε for all $\varepsilon \in B(0,\tilde{\rho})$, in other words

$$|a_{m,j}| \leq \tilde{a}_{m,j} \tag{4.64}$$

for $j = 1, \cdots, n$, we also have that $\tilde{a}_1 = \cdots = \tilde{a}_n$.

We want to reduce the problem of convergence to a 1-dimensional problem. Therefore we need another majorant. We define

$$c_k := \sum_{|m|=k} \tilde{c}_m$$

and

$$\hat{F}(Z) := \sum_{k\geq 2} c_k Z^k, Z \in \mathbb{C}$$

then $\hat{F}(Z) = \tilde{F}_j(Z\mathbf{e})$ for each $j = 1, \cdots, n$, so $\hat{F}$ is obviously a majorant for each component of $\tilde{F}(Z\mathbf{e})$. As $\hat{F}(Z)\mathbf{e} = \tilde{F}(Z\mathbf{e})$, $\hat{F}$ is analytic iff $|Z| < R_j$ for all $j = 1, \cdots, n$. Hence $\hat{F}(Z)$ is analytic on $B(0,\hat{R}) = \mathbb{D}(0,\hat{R}\mathbf{e})$ where $\hat{R} = \min\limits_{1\leq j\leq n} R_j$. In the same line of arguments we introduce

$$\nu_k := \min_{\substack{|m|=k \\ m\in\mathcal{S}_{\ell,n,s}}} \nu(m)$$

then by Proposition 4.12 we know there exists a constant $K > 0$ for which we have

$$\nu_k \geq \mu_{--} - K\mu_{-+}^k > 0.$$

We look at the solution $\hat{\varphi}(Z) = \sum_{k\geq 2} \hat{a}_k Z^k$ of

$$\sum_{k\geq 2} (\mu_{--} - K\mu_{-+}^k)\hat{a}_k Z^k = \hat{F}(Z + \hat{\varphi}(Z)). \tag{4.65}$$

We have that $\hat{\varphi}(Z)\mathbf{e}$ is a majorant of $\tilde{\varphi}(Z\mathbf{e})$, i.e.

$$\tilde{a}_{m,j} \leq \hat{a}_k$$

for all $m \in \mathcal{S}_{\ell,n,s}$ with $|m| = k$ and $1 \leq j \leq n$. As $k \geq 2$ and the sequence $(\mu_{--} - K\mu_{-+}^k)_{k\geq 2}$ increases monotone, we have that $\sum_{k\geq 2}(\mu_{--} - K\mu_{-+}^k)\hat{a}_k Z^k$ is a majorant for $\sum_{k\geq 2} \kappa\hat{a}_k Z^k$ where $\kappa = \mu_{--} - K\mu_{-+}^2$.

From this point on we can repeat the techniques from Subsection 4.4.3 to conclude that the formal solution of (4.60) converges, i.e. (4.60) has an analytic solution.

4.5.3 Solution of (4.61)

In the previous subsection we have proved that φ_ε is analytic in (z, ε). Contrary to the case of the vector fields where the second equation - given by (4.41) - was easily solved, we have to do a greater effort to solve (4.61). We start by proving that

$$[\varphi_\varepsilon(B_\varepsilon z + G_\varepsilon^\star(z)) - \varphi_\varepsilon(B_\varepsilon z)]^{\mathcal{S}_{\ell,n,s}} = 0.$$

Therefore we will use the power series expansion of $\varphi_\varepsilon(z) = \sum_{|m|\geq 2} a_m z^m$, this gives us

$$\varphi_\varepsilon(B_\varepsilon z + G_\varepsilon^\star(z)) - \varphi_\varepsilon(B_\varepsilon z) = \sum_{|m|\geq 2} a_m \left((B_\varepsilon z + G_\varepsilon^\star(z))^m - (B_\varepsilon z)^m\right). \quad (4.66)$$

We now look at the right-hand side of (4.66) on the level of the components, this gives (for $j = 1, \cdots, n$):

$$\begin{aligned}
(B_{\varepsilon,j} z + G_{\varepsilon,j}^\star(z))^{m_j} - (B_{\varepsilon,j} z)^{m_j} &= \sum_{k_j=0}^{m_j} \binom{m_j}{k_j} (B_{\varepsilon,j} z)^{k_j} (G_{\varepsilon,j}^\star(z))^{m_j - k_j} \\
&\quad -(B_{\varepsilon,j} z)^{m_j} \\
&= \sum_{k_j=0}^{m_j-1} \binom{m_j}{k_j} (B_{\varepsilon,j} z)^{k_j} (G_{\varepsilon,j}^\star(z))^{m_j - k_j} \\
&= G_{\varepsilon,j}^\star(z) \sum_{k_j=0}^{m_j-1} \binom{m_j}{k_j} \\
&\quad (B_{\varepsilon,j} z)^{k_j} (G_{\varepsilon,j}^\star(z))^{m_j - k_j - 1}.
\end{aligned}$$

Applying this result to (4.66) gives us

$$\varphi_\varepsilon(B_\varepsilon z + G_\varepsilon^\star(z)) - \varphi_\varepsilon(B_\varepsilon z) = G_\varepsilon^\star(z)^{\mathbf{e}} H(B_\varepsilon z, G_\varepsilon^\star(z)), \quad (4.67)$$

where H is an analytic function that is defined as

$$H(X, Y) := \sum_{|m|\geq 2} a_m \sum_{j_1=0}^{m_1-1} \cdots \sum_{j_n=0}^{m_n-1} X^j Y^{m-j-\mathbf{e}} \quad (4.68)$$

for $X, Y \in \mathbb{C}^n$. From (4.67) and (4.68) it now is clear that the series expansion of $G_\varepsilon^\star$ will contain terms of a lower degree than those in the series expansion of $\varphi_\varepsilon(B_\varepsilon z + G_\varepsilon^\star(z)) - \varphi_\varepsilon(B_\varepsilon z)$, so (4.61) gives that $[G_\varepsilon^\star(z)]^{\mathcal{S}_{\ell,n,s}} = 0$, hence

$$[\varphi_\varepsilon(B_\varepsilon z + G_\varepsilon^\star(z)) - \varphi_\varepsilon(B_\varepsilon z)]^{\mathcal{S}_{\ell,n,s}} = 0,$$

because $[G_\varepsilon^\star(z)]^{\mathcal{S}_{\ell,n,s}} = 0$ implies that $[G_\varepsilon^\star(z)^{\mathbf{e}}]^{\mathcal{S}_{\ell,n,s}} = 0$.

Next we prove that G_ε is an analytic function. Therefore we apply (4.67) on (4.61) to obtain

$$G_\varepsilon^\star(z) + G_\varepsilon^\star(z)^{\mathbf{e}} H(B_\varepsilon z, G_\varepsilon^\star(z)) = \left[F_\varepsilon^\star(z+\varphi_\varepsilon(z))\right]^{\mathcal{T}_{\ell,n,s}}. \tag{4.69}$$

Introducing

$$\tilde{H}(X,Y) := Y^{\mathbf{e}} H(X,Y),$$

we can rewrite (4.69) as

$$(Id + \tilde{H}(B_\varepsilon z, \cdot)) \circ G_\varepsilon^\star(z) = \left[F_\varepsilon^\star(z+\varphi_\varepsilon(z))\right]^{\mathcal{T}_{\ell,n,s}}. \tag{4.70}$$

As the function between brackets in the left-hand side of (4.70) is invertible, we have that

$$G_\varepsilon^\star(z) = (Id + \tilde{H}(B_\varepsilon z, \cdot))^{-1} \circ \left[F_\varepsilon^\star(z+\varphi_\varepsilon(z))\right]^{\mathcal{T}_{\ell,n,s}}. \tag{4.71}$$

As all functions in the right-hand side of (4.71) are analytic functions of (z,ε) also $G_\varepsilon^\star(z)$ is analytic in (z,ε).

From (4.71) one easily deduces

$$|G_\varepsilon^\star(z)| \leq C \left| \left[F_\varepsilon^\star(z+\varphi_\varepsilon(z))\right]^{\mathcal{T}_{\ell,n,s}} \right|$$

for some $C > 0$. So there exist a $r_1 > 0$ and $K_1 > 0$ such that

$$|G_\varepsilon^\star| \leq K_1 |(z_1, \cdots, z_s)|^\ell |(z_{s+1}, \cdots, z_n)|^\ell, \text{ for } r < r_1,$$

hence

$$\|G_\varepsilon^\star\|_r \leq K_1 r^{2\ell}, \text{ for } r < r_1.$$

This implies

$$\|G_\varepsilon\|_r \leq K_1 r^{2\ell}, \text{ for } r < r_1.$$

4.6 Symmetric case

In this section we want to prove Theorem 4.5 and Theorem 4.6. First we consider the case of a family of symmetries of a family of vector fields and after that we consider the case of a family of symmetries of a family of diffeomorphisms.

4.6.1 Proof of Theorem 4.5

Consider an analytic family of linear maps $S_\varepsilon : \mathbb{R}^n \to \mathbb{R}^n$ (so the components of S_ε are analytic functions of ε), then S_ε is a **symmetry** of the family of vector fields X_ε if $(S_\varepsilon)_* X_\varepsilon = X_\varepsilon$.

Consider a family of real vector fields X_ε with a symmetry S_ε. First of all we put $DX_\varepsilon(0)$ in its Jordan Normal Form A_ε, this can be done with a suitable matrix M_ε such that $A_\varepsilon = M_\varepsilon^{-1} DX_\varepsilon(0) M_\varepsilon$, hence the vector field becomes $\tilde{X}_\varepsilon = M_\varepsilon^{-1} \cdot X_\varepsilon \circ M_\varepsilon$. It is well-known, see for instance [CLW94], that under a linear change of coordinates the symmetry S_ε of X_ε is transformed into the symmetry $\tilde{S}_\varepsilon = M_\varepsilon^{-1} \cdot S_\varepsilon \cdot M_\varepsilon$.

So from now on we will assume that $DX_\varepsilon(0) = A_\varepsilon$ is already in its Jordan Normal Form. As S_ε is a symmetry of the family of real vector fields X_ε, we have that $T_\varepsilon := P^{-1} S_\varepsilon P$ is a symmetry of the complexified system where P is given by (4.3). Given the fact that T_ε is a symmetry of $\dot{z} = B_\varepsilon z + F_\varepsilon(z)$, we necessarily have that T_ε commutes with B_ε and F_ε. As B_ε is diagonal and all its eigenvalues are non-zero and have multiplicity 1, T_ε will be diagonal as well. Therefore T_ε cannot "mix up" stable directions with unstable directions. This gives us that T_ε also commutes with $[\cdot]^{\mathcal{S}_{\ell,n,s}}$ and $[\cdot]^{\mathcal{T}_{\ell,n,s}}$: let $h_\varepsilon(z) = \sum_{m \in \mathbb{N}^n} h_m(\varepsilon) z^m$ be a formal power series, as $T_\varepsilon := \operatorname{diag}(t_1(\varepsilon), \cdots, t_n(\varepsilon))$ we have

$$\begin{aligned}
T_\varepsilon \left[h_\varepsilon(z)\right]^{\mathcal{S}_{\ell,n,s}} &= T_\varepsilon \sum_{m \in \mathcal{S}_{\ell,n,s}} h_m(\varepsilon) z^m \\
&= \sum_{m \in \mathcal{S}_{\ell,n,s}} T_\varepsilon h_m(\varepsilon) z^m \\
&= \left(\sum_{m \in \mathcal{S}_{\ell,n,s}} t_1(\varepsilon) h_{m,1}(\varepsilon) z^m, \cdots, \sum_{m \in \mathcal{S}_{\ell,n,s}} t_n(\varepsilon) h_{m,n}(\varepsilon) z^m \right)
\end{aligned}$$

and

$$\begin{aligned}
\left[T_\varepsilon h_\varepsilon(z)\right]^{\mathcal{S}_{\ell,n,s}} &= \left[T_\varepsilon \sum_{m \in \mathbb{N}^n} h_m(\varepsilon) z^m \right]^{\mathcal{S}_{\ell,n,s}} \\
&= \left[\sum_{m \in \mathbb{N}^n} T_\varepsilon h_m(\varepsilon) z^m \right]^{\mathcal{S}_{\ell,n,s}} \\
&= \left[\left(\sum_{m \in \mathbb{N}^n} t_1(\varepsilon) h_{m,1}(\varepsilon) z^m, \cdots, \sum_{m \in \mathbb{N}^n} t_n(\varepsilon) h_{m,n}(\varepsilon) z^m \right) \right]^{\mathcal{S}_{\ell,n,s}} \\
&= \left(\sum_{m \in \mathcal{S}_{\ell,n,s}} t_1(\varepsilon) h_{m,1}(\varepsilon) z^m, \cdots, \sum_{m \in \mathcal{S}_{\ell,n,s}} t_n(\varepsilon) h_{m,n}(\varepsilon) z^m \right)
\end{aligned}$$

so

$$T_\varepsilon \left[h_\varepsilon(z)\right]^{\mathcal{S}_{\ell,n,s}} = \left[T_\varepsilon h_\varepsilon(z)\right]^{\mathcal{S}_{\ell,n,s}},$$

hence T_ε and $[\cdot]^{\mathcal{S}_{\ell,n,s}}$ commute. The proof that T_ε and $[\cdot]^{\mathcal{T}_{\ell,n,s}}$ commute is analogous.

ϕ_ε commutes with S_ε

First we show that T_ε commutes with φ_ε. To obtain this result we need to look at (4.40). We know that φ_ε is the unique solution of (4.40), so if we prove that $T_\varepsilon^{-1} \circ \varphi_\varepsilon \circ T_\varepsilon$ is also a solution of (4.40) then by unicity we have that $\varphi_\varepsilon = T_\varepsilon^{-1} \circ \varphi_\varepsilon \circ T_\varepsilon$ or in other words $T_\varepsilon \circ \varphi_\varepsilon = \varphi_\varepsilon \circ T_\varepsilon$.

Let us define $\psi_\varepsilon := T_\varepsilon^{-1} \circ \psi_\varepsilon \circ T_\varepsilon$, then ψ_ε is a solution of (4.40) iff

$$D\psi_\varepsilon(w)B_\varepsilon w - B_\varepsilon \psi_\varepsilon(w) = [F_\varepsilon(w + \psi_\varepsilon(w))]^{\mathcal{S}_{\ell,n,s}}$$

or equivalently

$$\begin{aligned} D\left(T_\varepsilon^{-1} \circ \varphi_\varepsilon \circ T_\varepsilon\right)(w)B_\varepsilon w \quad &- \quad B_\varepsilon\left(T_\varepsilon^{-1} \circ \varphi_\varepsilon \circ T_\varepsilon\right)(w) \\ &= \quad \left[F_\varepsilon(w + (T_\varepsilon^{-1} \circ \varphi_\varepsilon \circ T_\varepsilon)(w)\right]^{\mathcal{S}_{\ell,n,s}}. \end{aligned} \tag{4.72}$$

As T_ε commutes with B_ε, F_ε and $[\cdot]^{\mathcal{S}_{\ell,n,s}}$, (4.72) is equivalent with

$$T_\varepsilon^{-1} \cdot D\varphi_\varepsilon(T_\varepsilon w)T_\varepsilon B_\varepsilon w - T_\varepsilon^{-1} B_\varepsilon \varphi_\varepsilon(T_\varepsilon w) = T_\varepsilon^{-1}\left[F_\varepsilon(T_\varepsilon w + \varphi_\varepsilon(T_\varepsilon w))\right]^{\mathcal{S}_{\ell,n,s}}. \tag{4.73}$$

As T_ε is invertible, it is also bijective. This means we can put $T_\varepsilon w =: z$ for all $z \in \mathbb{C}^n$. Hence (4.73) is equivalent with

$$D\varphi_\varepsilon(z)B_\varepsilon z - B_\varepsilon \varphi_\varepsilon(z) = [F_\varepsilon(z + \varphi_\varepsilon(z))]^{\mathcal{S}_{\ell,n,s}}. \tag{4.74}$$

Obviously (4.74) is equivalent with the demand that φ_ε is a solution of (4.40). So if φ_ε is a solution of (4.40) also ψ_ε will be a solution of (4.40) and vice versa. As we have proved that (4.40) has a unique solution, necessarily $\psi_\varepsilon = \varphi_\varepsilon$.

From this it is easy to prove that ϕ_ε and S_ε commute. We know that $T_\varepsilon = P^{-1} \circ S_\varepsilon \circ P$ and $\varphi_\varepsilon = P^{-1} \circ \phi_\varepsilon \circ P$, so $T_\varepsilon \circ \varphi_\varepsilon = \varphi_\varepsilon \circ T_\varepsilon$ is equivalent with

$$(P^{-1} \circ S_\varepsilon \circ P) \circ (P^{-1} \circ \phi_\varepsilon \circ P) = (P^{-1} \circ \phi_\varepsilon \circ P) \circ (P^{-1} \circ S_\varepsilon \circ P)$$

which is equivalent with $S_\varepsilon \circ \phi_\varepsilon = \phi_\varepsilon \circ S_\varepsilon$, hence ϕ_ε and S_ε commute.

g_ε commutes with S_ε

To obtain the desired commutation result we need (4.41) and prove that G_ε commutes with T_ε. We use the same line of arguments as in the previous section. So define $\Gamma_\varepsilon := T_\varepsilon^{-1} \circ G_\varepsilon \circ T_\varepsilon$, then Γ_ε is a solution of (4.41) iff

$$[F_\varepsilon(w + \varphi_\varepsilon(w))]^{\mathcal{T}_{\ell,n,s}} = (I_n + D\varphi_\varepsilon(w))\Gamma_\varepsilon(w)$$

or equivalently

$$[F_\varepsilon(w + \varphi_\varepsilon(w))]^{\mathcal{T}_{\ell,n,s}} = (I_n + D\varphi_\varepsilon(w))(T_\varepsilon^{-1} \circ G_\varepsilon \circ T_\varepsilon)(w). \tag{4.75}$$

As T_ε commutes with φ_ε we have by differentiating both sides of $\varphi_\varepsilon \circ T_\varepsilon(w) = T_\varepsilon \circ \varphi_\varepsilon(w)$ that $D\varphi_\varepsilon(T_\varepsilon w)T_\varepsilon w = T_\varepsilon D\varphi_\varepsilon(w)$, which is equivalent with

$$T_\varepsilon^{-1} D\varphi_\varepsilon(T_\varepsilon w)T_\varepsilon w = D\varphi_\varepsilon(w). \tag{4.76}$$

Applying (4.76) on (4.75) gives us

$$[F_\varepsilon(w + \varphi_\varepsilon(w))]^{\mathcal{T}_{\ell,n,s}} = (I_n + T_\varepsilon^{-1} D\varphi_\varepsilon(T_\varepsilon w)T_\varepsilon)(T_\varepsilon^{-1} G_\varepsilon(T_\varepsilon w)). \tag{4.77}$$

As T_ε commutes with B_ε, φ_ε, F_ε and $[\cdot]^{\mathcal{T}_{\ell,n,s}}$, (4.77) is equivalent with

$$T_\varepsilon^{-1} [F_\varepsilon(T_\varepsilon w + \varphi_\varepsilon(T_\varepsilon w))]^{\mathcal{T}_{\ell,n,s}} = T_\varepsilon^{-1}(I_n + D\varphi_\varepsilon(T_\varepsilon w))G_\varepsilon(T_\varepsilon w). \tag{4.78}$$

T_ε is invertible, so it is also bijective. This means we can put $T_\varepsilon w =: z$ for all $z \in \mathbb{C}^n$. Hence (4.78) is equivalent with

$$[F_\varepsilon(z + \varphi_\varepsilon(z))]^{\mathcal{T}_{\ell,n,s}} = (I_n + D\varphi_\varepsilon(w))G_\varepsilon(w). \tag{4.79}$$

Obviously (4.79) is equivalent with the demand that G_ε is a solution of (4.41). This means that Γ_ε is a solution of (4.41) iff G_ε is a solution of (4.41). As (4.41) has a unique solution, necessarily $\Gamma_\varepsilon = G_\varepsilon$, hence $T_\varepsilon \circ G_\varepsilon = G_\varepsilon \circ T_\varepsilon$. As we did before one derives that this latter equality is equivalent with $S_\varepsilon \circ g_\varepsilon = g_\varepsilon \circ S_\varepsilon$.

This means we have proved Theorem 4.5.

4.6.2 Proof of Theorem 4.6

Consider a family of linear maps $S_\varepsilon : \mathbb{R}^n \to \mathbb{R}^n$ (so the components of S_ε are analytic functions of ε), then S_ε is a **symmetry** of the family of diffeomorphisms f_ε if $S_\varepsilon \circ f_\varepsilon = f_\varepsilon \circ S_\varepsilon$.

As in Subsection 4.6.1 we put $Df_\varepsilon(0)$ in its Jordan Normal Form. This transforms the symmetry S_ε into another symmetry $\tilde{S}_\varepsilon$. The relation between S_ε and $\tilde{S}_\varepsilon$ is the same as in Subsection 4.6.1.

So from now on we will assume that $Df_\varepsilon(0) = A_\varepsilon$ is already in its Jordan Normal Form. As S_ε is a symmetry of the family of real diffeomorphisms f_ε, we have that $T_\varepsilon := P^{-1} S_\varepsilon P$ is a symmetry of the complexified family of diffeomorphisms $f_\varepsilon^\star$ where P is given by (4.3). Given the fact that T_ε is a symmetry of $f_\varepsilon^\star(z) = B_\varepsilon z + F_\varepsilon^\star(z)$, we necessarily have that T_ε commutes with B_ε and $F_\varepsilon^\star$. As B_ε is diagonal and its eigenvalues are non-zero and have multiplicity 1, T_ε will be diagonal as well. Therefore T_ε cannot "mix up" stable directions with unstable directions. This gives us that T_ε also commutes with $[\cdot]^{\mathcal{S}_{\ell,n,s}}$ and $[\cdot]^{\mathcal{T}_{\ell,n,s}}$. The proof has already been given in Subsection 4.6.1 and is therefore omitted here.

ϕ_ε commutes with S_ε

First we show that T_ε commutes with φ_ε. To obtain this result we need to look at (4.60). We know that φ_ε is the unique solution of (4.60), so if we prove that $T_\varepsilon^{-1} \circ \varphi_\varepsilon \circ T_\varepsilon$ is also a solution of (4.60) then by unicity we have that $\varphi_\varepsilon = T_\varepsilon^{-1} \circ \varphi_\varepsilon \circ T_\varepsilon$ or in other words $T_\varepsilon \circ \varphi_\varepsilon = \varphi_\varepsilon \circ T_\varepsilon$.

Let us define $\psi_\varepsilon := T_\varepsilon^{-1} \circ \psi_\varepsilon \circ T_\varepsilon$, then analogous as in the previous subsection we prove that ψ_ε is a solution of (4.60) iff φ_ε is a solution of (4.60). So if φ_ε is a solution of (4.60) also ψ_ε will be a solution of (4.60) and vice versa. As we have proved that (4.60) has a unique analytic solution, necessarily $\psi_\varepsilon = \varphi_\varepsilon$.

From this it is easy to prove that ϕ_ε and S_ε commute.

G_ε commutes with S_ε

To obtain the desired commutation result we need (4.61) and prove that $G_\varepsilon^\star$ commutes with T_ε. We use the same line of arguments as in the previous section. So define $\Gamma_\varepsilon := T_\varepsilon^{-1} \circ G_\varepsilon^\star \circ T_\varepsilon$, then we find that Γ_ε is a solution of (4.61) iff $G_\varepsilon^\star$ is a solution of (4.61). This means that Γ_ε is a solution of (4.61) iff $G_\varepsilon^\star$ is a solution of (4.61). As (4.61) has a unique solution, necessarily $\Gamma_\varepsilon = G_\varepsilon^\star$, hence $T_\varepsilon \circ G_\varepsilon^\star = G_\varepsilon^\star \circ T_\varepsilon$. As in the previous subsection one derives that the latter equality is equivalent with $S_\varepsilon \circ G_\varepsilon = G_\varepsilon \circ S_\varepsilon$.

This means that we have proved Theorem 4.6.

4.7 Poincaré's Theorem for a family of vector fields or diffeomorphisms

In this section we want to illustrate that the method we used to prove Theorem 4.1 and Theorem 4.2 also works in the proof of Poincaré's Theorem for an analytic family of vector fields or diffeomorphisms. We will be able to prove that the transformation is analytic and that it depends analytically on the parameter. This way we will also cover the case of one vector field, i.e. Theorem 1.9, or one diffeomorphism, i.e. Theorem 1.12. In comparison with the original statements our version will be weaker on one point: in order to be able to put the linear part at the singularity or fixed point into its Jordan Normal Form we need to assume that all eigenvalues have multiplicity 1 for $\varepsilon = 0$. The reason of this is given by Proposition 1.15.

It needs to be mentioned that this version of Poincaré's Theorem is not new at all. In [Brus71] a similar result is proved. The main difference lies in the method that is used: in [Brus71] the existence of an analytic transformation is given by means of a contraction in a suitably chosen Banach space while we give a proof using the method of the majorants.

First we will the case of a family of vector fields and afterwards we look into the case of a family of diffeomorphisms.

4.7.1 Family of vector fields

Consider a family of vector fields X_ε on $\mathbb{C}^n$ that is analytic in its variables and the parameter ε having a hyperbolic singularity x_0. Assuming that the eigenvalues of $DX_0(x_0)$ have multiplicity 1, we can use the results from Subsection 1.3.4. This gives us the following normal form

$$X_\varepsilon : \dot{x} = A_\varepsilon x + f_\varepsilon(x),$$

where $x \in \mathbb{C}^n$, $\varepsilon \in \mathbb{C}^p$, $A_\varepsilon = \text{diag}(\lambda_1(\varepsilon), \cdots, \lambda_n(\varepsilon))$ and $f_\varepsilon(x) = \mathcal{O}(|x|^2)$.

We state the main result of this subsection.

Theorem 4.17 *Let* $A_\varepsilon = \text{diag}(\lambda_1(\varepsilon), \cdots, \lambda_n(\varepsilon))$. *If* $\text{Spec}(A_0)$ *forms a non-resonant set that lies in the Poincaré domain, then there exist positive constants* ρ_0, R_0 *and an analytic change of variables*

$$x = y + \xi_\varepsilon(y), \ (y, \varepsilon) \in \mathbb{D}(0, R_0) \times \mathbb{D}(0, \rho_0), \tag{4.80}$$

where $\xi_\varepsilon(y) = \mathcal{O}(|y|^2)$ *as* $y \to 0$, *which transforms the analytic vector field*

$$\dot{x} = A_\varepsilon x + f_\varepsilon(x), \ (x, \varepsilon) \in \mathbb{D}(0, R) \times \mathbb{D}(0, \rho), \tag{4.81}$$

with $f_\varepsilon(x) = \mathcal{O}(|x|^2)$ *if* $x \to 0$, *into the linear system*

$$\dot{y} = A_\varepsilon y, \ y \in \mathbb{C}^n.$$

One should note that in Theorem 4.17 we assume that $\text{Spec}(A_0)$ forms a non-resonant set that lies in the Poincaré domain. So in order to linearise (4.81) it will be crucial to prove that there exists a $\hat{\rho} > 0$ such that for all $\varepsilon \in B(0, \hat{\rho}) \subset \mathbb{C}^p$ we have that $\text{Spec}(A_\varepsilon)$ forms a non-resonant set that lies in the Poincaré domain. We prove this in the following lemma.

Lemma 4.18 *Given* $A_\varepsilon = \text{diag}(\lambda_1(\varepsilon), \cdots, \lambda_n(\varepsilon))$. *If* $\text{Spec}(A_0)$ *forms a non-resonant set that lies in the Poincaré domain, then there exist positive constants* ρ, C *such that for all* $\varepsilon \in B(0, \rho)$ *we have that* $\text{Spec}(A_\varepsilon)$ *forms a non-resonant set that lies in the Poincaré domain and for all* $m \in \mathbb{N}^n$ *with* $|m| \geq 2$ *we have*

$$|\langle \Lambda_\varepsilon, m \rangle - \lambda_j(\varepsilon)| \geq C|m|. \tag{4.82}$$

PROOF: The proof consists of 3 parts: first we prove that $\text{Spec}(A_\varepsilon)$ is in the Poincaré domain, second we prove that for sufficiently large $|m|$ there are no resonances and (4.82) is fulfilled and finally we prove that for all m with $|m| \geq 2$ there are no resonances and (4.82) is fulfilled.

As $\text{Spec}(A_0)$ is in the Poincaré domain, $0 \notin \text{Conv}(\text{Spec}(A_0))$, i.e. for all $m \in \mathbb{R}^n$ with $\sum_{j=1}^{n} m_j = 1$ and all $m_j \geq 0$:

$$\sum_{j=1}^{n} m_j \lambda_j(0) \neq 0. \tag{4.83}$$

As $\lambda_j(\varepsilon)$ is analytic in ε, we have that

$$\varepsilon \mapsto \sum_{j=1}^{n} m_j \lambda_j(\varepsilon)$$

is (at least) a continuous function of ε. Combining this with (4.83) we know there exists a $\hat{\rho} > 0$ such that for all $\varepsilon \in B(0, \hat{\rho})$ we have that

$$\sum_{j=1}^{n} m_j \lambda_j(\varepsilon) \neq 0,$$

hence for all $\varepsilon \in B(0, \hat{\rho})$ we have that $0 \notin \mathrm{Conv}(\mathrm{Spec}(A_\varepsilon))$, i.e. $\mathrm{Spec}(A_\varepsilon)$ is in the Poincaré domain.

As $0 \notin \mathrm{Conv}(\mathrm{Spec}(A_\varepsilon))$ for all $\varepsilon \in B(0, \hat{\rho})$ we know that

$$d_\varepsilon := \mathrm{dist}(0, \mathrm{Conv}(\mathrm{Spec}(A_\varepsilon))) > 0, \forall \varepsilon \in B(0, \hat{\rho}),$$

hence for a fixed ρ_1 with $0 < \rho_1 < \hat{\rho}$ we have

$$\delta := \inf_{\varepsilon \in B(0,\rho_1)} d_\varepsilon > 0.$$

For all $m \in \mathbb{N}^n$ with $|m| \geq 2$ we have that

$$\frac{\langle \Lambda_\varepsilon, m \rangle}{|m|} \in \mathrm{Conv}(\mathrm{Spec}(A_\varepsilon)),$$

hence

$$\left| \frac{\langle \Lambda_\varepsilon, m \rangle}{|m|} \right| \geq \delta.$$

Introducing the constant μ

$$\mu := \sup_{\varepsilon \in B(0,\rho_1)} \max_{1 \leq j \leq n} |\lambda_j(\varepsilon)|,$$

we have for all $m \in \mathbb{N}^n$ with $|m| \geq \frac{2\mu}{\delta}$ and for all $j = 1, \cdots, n$:

$$\begin{aligned} \frac{|\langle \Lambda_\varepsilon, m \rangle - \lambda_j(\varepsilon)|}{|m|} &\geq \frac{|\langle \Lambda_\varepsilon, m \rangle|}{|m|} - \frac{|\lambda_j(\varepsilon)|}{|m|} \\ &\geq \frac{\delta}{2}, \end{aligned}$$

hence

$$|\langle \Lambda_\varepsilon, m \rangle - \lambda_j(\varepsilon)| \geq \frac{\delta}{2}|m|, \tag{4.84}$$

for all $m \in \mathbb{N}^n$ with $|m| \geq \frac{2\mu}{\delta}$. From (4.84) we also obtain that there are no resonances with $|m| \geq \frac{2\mu}{\delta}$.

We know that $\mathrm{Spec}(A_0)$ is non-resonant, so for all $m \in \mathbb{N}^n$ with $|m| \geq 2$ we have that $\langle \Lambda_0, m\rangle - \lambda_j(0) \neq 0$. Obviously we have for all $m \in \mathbb{N}^n$ with $2 \leq |m| < \frac{2\mu}{\delta}$ that $\langle \Lambda_0, m\rangle - \lambda_j(0) \neq 0$. Now consider the set $\mathcal{M} := \{m \in \mathbb{N}^n | 2 \leq |m| \leq \frac{2\mu}{\delta}\}$. For each $m \in \mathcal{M}$ there exists a $\rho_m > 0$ such that $\langle \Lambda_\varepsilon, m\rangle - \lambda_j(\varepsilon) \neq 0$, $\forall \varepsilon \in B(0, \rho_m)$ and ρ_m is maximal with this property. As $\mathcal{M}$ is a finite set, we have that

$$\rho_2 := \min_{m \in \mathcal{M}} \rho_m > 0.$$

So for all $m \in \mathcal{M}$ and for all $\varepsilon \in B(0, \rho_2)$ we have that

$$\langle \Lambda_\varepsilon, m\rangle - \lambda_j(\varepsilon) \neq 0,$$

hence

$$\frac{|\langle \Lambda_\varepsilon, m\rangle - \lambda_j(\varepsilon)|}{|m|} \geq C_m > 0.$$

Thus

$$|\langle \Lambda_\varepsilon, m\rangle - \lambda_j(\varepsilon)| \geq C_m |m| > 0.$$

Taking

$$\begin{aligned} \rho &:= \min(\rho_1, \rho_2), \\ C &:= \min\left(\{C_m | m \in \mathcal{M}\} \cup \left\{\frac{\delta}{2}\right\}\right), \end{aligned}$$

we obtain for all $m \in \mathbb{N}^n$ with $|m| \geq 2$ and for all $\varepsilon \in B(0, \rho)$ that

$$|\langle \Lambda_\varepsilon, m\rangle - \lambda_j(\varepsilon)| \geq C|m|.$$

This inequality implies that $\mathrm{Spec}(A_\varepsilon)$ is non-resonant for all $\varepsilon \in B(0, \rho)$. □

We now will establish an equation which will allow us to determine the transformation (4.80) we are seeking. Performing the transformation (4.80) on (4.81) we find the following two equalities

$$\begin{aligned} \dot{x} &= A_\varepsilon y + D\varphi_\varepsilon(y) A_\varepsilon y, \\ \dot{x} &= A_\varepsilon y + A_\varepsilon \varphi_\varepsilon(y) + f_\varepsilon(y + \varphi_\varepsilon(y)). \end{aligned}$$

If we introduce the operator L_{A_ε}

$$L_{A_\varepsilon} \varphi_\varepsilon(y) = D\varphi_\varepsilon(y) A_\varepsilon y - A_\varepsilon \varphi_\varepsilon(y) \tag{4.85}$$

then these equalities can be combined to obtain

$$L_{A_\varepsilon} \varphi_\varepsilon(y) = f_\varepsilon(y + \varphi_\varepsilon(y)). \tag{4.86}$$

We now want to prove that (4.86) has a formal solution which converges on a poly-disk $\mathbb{D}(0, R) \times \mathbb{D}(0, \rho) \subset \mathbb{C}^n \times \mathbb{C}^p$ for some $R \in (\mathbb{R}^+ \setminus \{0\})^n$ and $\rho \in (\mathbb{R}^+ \setminus \{0\})^p$. As (4.40) and (4.86) are essentially the same equations we can repeat the same proof as given in Subsection 4.4.2 and Subsection 4.4.3 up to some minor changes in notations. Hence we have proved Theorem 4.17.

4.7.2 Family of diffeomorphisms

Consider a family of diffeomorphisms $F_\varepsilon : \mathbb{C}^n \to \mathbb{C}^n$ that is analytic in its variables and the parameter ε having a hyperbolic fixed point x_0. Assuming that the eigenvalues of $DF_0(x_0)$ have multiplicity 1, we can use the results from Subsection 1.3.5. This gives us the following normal form

$$F_\varepsilon(x) = A_\varepsilon x + f_\varepsilon(x),$$

where $x \in \mathbb{C}^n$, $\varepsilon \in \mathbb{C}^p$, $A_\varepsilon = \text{diag}(\lambda_1(\varepsilon), \cdots, \lambda_n(\varepsilon))$ and $f_\varepsilon(x) = \mathcal{O}(|x|^2)$.

We state the main result of this subsection.

Theorem 4.19 *Let $A_\varepsilon = \text{diag}(\lambda_1(\varepsilon), \cdots, \lambda_n(\varepsilon))$. If $\text{Spec}(A_0)$ forms a multiplicatively non-resonant set and $0 < |\lambda_j(\varepsilon)| < 1$ for all $j = 1, \cdots, n$, then there exist positive constants ρ_0, R_0 and an analytic change of variables*

$$x = y + \xi_\varepsilon(y), \ (y, \varepsilon) \in \mathbb{D}(0, R_0) \times \mathbb{D}(0, \rho_0), \tag{4.87}$$

where $\xi_\varepsilon(y) = \mathcal{O}(|y|^2)$ as $y \to 0$, which transforms the analytic family of diffeomorphisms

$$F_\varepsilon : \mathbb{C}^n \to \mathbb{C}^n : x \mapsto A_\varepsilon x + f_\varepsilon(x), \ (x, \varepsilon) \in \mathbb{D}(0, R) \times \mathbb{D}(0, \rho), \tag{4.88}$$

with $f_\varepsilon(x) = \mathcal{O}(|x|^2)$ if $x \to 0$, into the linear family of diffeomorphisms

$$G_\varepsilon : \mathbb{C}^n \to \mathbb{C}^n : y \mapsto A_\varepsilon y, \ y \in \mathbb{C}^n.$$

Corollary 4.20 *Let $A_\varepsilon = \text{diag}(\lambda_1(\varepsilon), \cdots, \lambda_n(\varepsilon))$. If $\text{Spec}(A_0)$ forms a multiplicatively non-resonant set and $|\lambda_j(\varepsilon)| > 1$ for all $j = 1, \cdots, n$, then there exist positive constants ρ_0, R_0 and an analytic change of variables*

$$x = y + \xi_\varepsilon(y), \ (y, \varepsilon) \in \mathbb{D}(0, R_0) \times \mathbb{D}(0, \rho_0), \tag{4.89}$$

where $\xi_\varepsilon(y) = \mathcal{O}(|y|^2)$ as $y \to 0$, which transforms the analytic family of diffeomorphisms

$$F_\varepsilon : \mathbb{C}^n \to \mathbb{C}^n : x \mapsto A_\varepsilon x + f_\varepsilon(x), \ (x, \varepsilon) \in \mathbb{D}(0, R) \times \mathbb{D}(0, \rho), \tag{4.90}$$

with $f_\varepsilon(x) = \mathcal{O}(|x|^2)$ if $x \to 0$, into the linear family of diffeomorphisms

$$G_\varepsilon : \mathbb{C}^n \to \mathbb{C}^n : y \mapsto A_\varepsilon y, \ y \in \mathbb{C}^n.$$

PROOF: The family of inverse diffeomorphisms F_ε^{-1} fulfills all criteria that are posed by Theorem 4.19. Hence there exists an analytic change of variables that conjugates F_ε^{-1} to G_ε^{-1}. As the transformation given by (4.87) is bijective its inverse will be the transformation (4.89) we were looking for. □

It is important to note that in Theorem 4.19 we assume that $\text{Spec}(A_0)$ forms a multiplicatively non-resonant set and the modulus of each eigenvalue of A_0 lies strictly between 0 and 1. So if we want to linearise (4.88) we need to prove that A_ε has these properties for ε in some sufficiently small ball in $\mathbb{C}^p$. We prove this in the following lemma.

Lemma 4.21 *Given* $A_\varepsilon = \mathrm{diag}(\lambda_1(\varepsilon), \cdots, \lambda_n(\varepsilon))$. *If* $\mathrm{Spec}(A_0)$ *forms a multiplicatively non-resonant set such that* $0 < |\lambda_j(0)| < 1$, *then there exist positive constants* ρ, C *such that for all* $\varepsilon \in B(0, \rho)$ *we have that* $\mathrm{Spec}(A_\varepsilon)$ *forms a multiplicatively non-resonant set such that* $0 < |\lambda_j(\varepsilon)| < 1$ *and for all* $m \in \mathbb{N}^n$ *with* $|m| \geq 2$ *we have*

$$|\Lambda_\varepsilon^m - \lambda_j(\varepsilon)| \geq C. \tag{4.91}$$

PROOF: As all eigenvalues of A_ε are analytic functions of ε, they are clearly also continuous. As for all $j = 1, \cdots, n$ we have that $0 < |\lambda_j(0)| < 1$ the continuity of $\lambda_j(\varepsilon)$ assures us the existence of a constant $\rho_1 > 0$ such that for all $\varepsilon \in B(0, \rho_1)$ we obtain that

$$0 < |\lambda_j(\varepsilon)| < 1. \tag{4.92}$$

From (4.92) it follows immediately that

$$\lim_{|m| \to +\infty} \Lambda_\varepsilon^m = \lim_{|m| \to +\infty} \prod_{j=1}^{n} \lambda_j(\varepsilon)^{m_j} = 0.$$

Hence there exists an integer M such that for all $m \in \mathbb{N}^n$ with $|m| \geq M$ we have that

$$|\Lambda_\varepsilon^m - \lambda_j(\varepsilon)| \geq \frac{\mu(\varepsilon)}{2}, \tag{4.93}$$

where

$$\mu(\varepsilon) := \min_{1 \leq j \leq n} |\lambda_j(\varepsilon)| > 0.$$

From (4.93) we also obtain that there are no resonances for $|m| \geq M$.

The rest of the proof is analogous to the proof of Lemma 4.18 and therefore omitted. □

We now will establish an equation which will allow us to determine the transformation (4.87) we are seeking. The transformation (4.87) will conjugate (4.88) with its linear part if the following two expressions are equal

$$\begin{aligned} F_\varepsilon \circ (Id + \varphi_\varepsilon)(y) &= A_\varepsilon y + A_\varepsilon \varphi_\varepsilon(y), \\ (Id + \varphi_\varepsilon) \circ G_\varepsilon(y) &= A_\varepsilon y + \varphi_\varepsilon(A_\varepsilon y). \end{aligned}$$

If we introduce the operator L_{A_ε}

$$L_{A_\varepsilon} \varphi_\varepsilon(y) = \varphi_\varepsilon(A_\varepsilon y) - A_\varepsilon \varphi_\varepsilon(y) \tag{4.94}$$

then these equalities can be combined to obtain

$$L_{A_\varepsilon} \varphi_\varepsilon(y) = f_\varepsilon(y + \varphi_\varepsilon(y)). \tag{4.95}$$

We now want to prove that (4.95) has a formal solution which converges on a poly-disk $\mathbb{D}(0, R) \times \mathbb{D}(0, \rho) \subset \mathbb{C}^n \times \mathbb{C}^p$ for some $R \in (\mathbb{R}^+ \setminus \{0\})^n$ and $\rho \in (\mathbb{R}^+ \setminus \{0\})^p$. As (4.60) and (4.95) are essentially the same equations we can repeat the same proof as given in Subsection 4.5.2 up to some minor changes in notations. Hence we have proved Theorem 4.19.

Locale equivalentie en conjugatie van families van vectorvelden en diffeomorfismes

In deze thesis werken we met C^∞ of analytische families van vectorvelden of diffeomorfismes. We zijn geïnteresseerd in locale equivalenties en conjugaties tussen dergelijke families en families in een “eenvoudige” vorm. Deze vorm wordt soms een *normaalvorm* genoemd. Gewoonlijk kiest men voor een lineaire normaalvorm, maar soms verhindert deze keuze ons tot het bereiken van een analytische equivalentie of conjugatie. Daarom zullen we in deze gevallen toestaan dat er niet-lineaire termen voorkomen in de normaalvorm.

Er zijn al veel resultaten bereikt voor individuele vectorvelden en diffeomorfismes. Het blijkt dat de eigenwaarden van het lineair deel van het vectorveld, resp. diffeomorfisme in het singulier, resp. vast punt bepalen of het vectorveld of diffeomorfisme equivalent of geconjugeerd is met zijn lineair deel. Als de eigenwaarden een hyperbolische niet-resonante verzameling vormen dan zijn er de gekende resultaten van Poincaré en Siegel, die ons zeggen wanneer een analytische conjugatie met het lineair deel mogelijk is. Als de eigenwaarden een hyperbolische resonante verzameling vormen, dan bestaan er soms eindig gladde conjugaties. In het niet-hyperbolische geval wordt het stukken moeilijker om het bestaan van gladde equivalenties en conjugaties aan te tonen.

We werken in deze thesis met families van vectorvelden en diffeomorfismes. Bijgevolg zullen we dezelfde problemen ontmoeten aangaande hyperboliciteit en resonantie als bij de individuele systemen. Een bijkomend probleem kan veroorzaakt worden door de parameters die de familie bepalen. Aangezien de parameter de eigenwaarden verstoord, kan er een resonantie opduiken die niet bestaat voor het onverstoorde systeem. Hierdoor wordt de gladheid van de equivalentie of de conjugatie sterk beïnvloed.

Vereiste voorkennis en technische eigenschappen

We beginnen met een definitie van de objecten die we het vaakst zullen gebruiken in deze thesis: vectorveld, stroom, singulariteit (ook singulier punt genoemd), vast punt, conjugatie en equivalentie. Een eerste (topologische) classificatie op basis van het linear deel in het singulier of vast punt wordt gegeven. We geven een korte inleiding over analytische functies in verscheidene veranderlijken.

Hierna volgt een grondige bespreking van normaalvormen. Eerst worden zowel de formele als de gladde normaalvorm van een vectorveld en een diffeomorfisme besproken. Dit wordt gevolgd door een paar algemene definities van families en deformaties van vectorvelden en diffeomorfismes. Ook twee resultaten in verband met de gladheid van de eigenwaarden en eigenvectoren ten opzichte van de parameter worden aangetoond. Deze resultaten zijn noodzakelijk om de normaalvorm van een familie van hyperbolische vectorvelden en diffeomorfismes te bespreken. Belangrijk is dat de normaalvormen worden bepaald zonder verlies aan gladheid. De laatste normaalvorm, die gedetailleerd besproken wordt, is die van een locale deformatie van een vlakke centrum-singulariteit.

Het hoofdstuk wordt afgesloten met een korte bespreking van transitie-afbeeldingen van vlakke vectorvelden. De klemtoon ligt hier vooral op de toepassing. Als eerste toepassing bespreken we de Dulac-afbeelding. Deze is de transitie-afbeelding in de omgeving van een zadel. In de formele ontwikkeling van de Dulac-afbeelding vinden we de Ecalle–Roussarie compensatoren terug die een belangrijke rol zullen spelen in Hoofdstuk 2. Een tweede toepassing is de Poincaré-afbeelding in de omgeving van een vlak centrum. Indien de Poincaré-afbeelding van de deformatie van een vlak centrum wordt bestudeerd, ziet men dat een formele ontwikkeling van de Poincaré-afbeelding ten opzichte van de parameter kan gegeven worden. De coëfficiënten van deze formele machtreeks zijn functies van het beginpunt x_0 en worden de *Melnikov-functies* genoemd. Er wordt kort aangetoond hoe de Melnikov-functie van eerste orde kan geschreven worden als een Abelse integraal.

Bijna-resonante zadels

In dit hoofdstuk is het de bedoeling een expliciete beschrijving van equivalenties en conjugaties tussen bijna-resonante zadels en hun lineaire delen te geven. We zullen dit doen door de equivalentie of conjugatie te schrijven als de samenstelling van een eindig aantal transformaties, waarbij enkel de laatste op een niet-expliciete manier zal gegeven worden. We beperken ons tot vlakke vectorvelden en diffeomorfismes op $\mathbb{R}^2$, aangezien in [BK94, Har60a] door middel van een tegenvoorbeeld wordt aangetoond dat enkel in het tweedimensionale geval er C^1 equivalenties en conjugaties verwacht mogen worden.

We beginnen met het bewijs van een ondergrens op de graad van de resonante termen die ontstaan door verstoring van de eigenwaarden door de parameter.

Dit resultaat is van groot belang opdat de laatste - niet-expliciete - transformatie C^1 zal zijn. Hierna beginnen we met het geval van een C^1 equivalentie. In tegenstelling tot de formele normaalvorm waarbij alle transformaties de som waren van de identieke afbeelding met een homogene veelterm, zullen de transformaties hier de som van de identieke afbeelding met een homogene veelterm in x, y en een Ecalle–Roussarie compensator in x zijn. Hierdoor is de afbeelding C^1 en gedefinieerd voor $\varepsilon \to 0$. Dit resultaat wordt daarna uitgebreid voor C^1 conjugaties. In dit geval zal met homogene veeltermen in x, y en Ecalle–Roussarie compensatoren in x en y gewerkt worden. De bewijstechnieken zijn gelijkaardig met die voor een C^1 equivalentie en vergen enkel wat meer rekenwerk.

Door twee extra veranderlijken in te voeren, kunnen we bewijzen dat deze C^1 conjugatie eigenlijk C^∞ is ten opzichte van de twee oorspronkelijke veranderlijken en de twee extra veranderlijken. Deze extra veranderlijken zullen geïnspireerd zijn door de Ecalle–Roussarie compensatoren. Om dit resultaat aan te tonen zullen de stelling van Borel en de homotopische methode gebruikt worden.

Hierna beschouwen we het geval van een C^1 conjugatie tussen bijna-resonante zadel-diffeomorfismes en hun lineair deel. We bewijzen een analoog resultaat als in het geval van de vectorvelden en ook de bewijzen zijn analoog. Enkel de interpretatie van de laatste transformatie als de limiet van een samenstelling van twee stromen is afwezig daar er geen versie van [Bon97] voor diffeomorfismes bekend is.

Wegens de formele gelijkenissen tussen het berekenen van de formele normaalvorm van een zadel en een centrum proberen we de berekeningen te herhalen voor een deformatie van een vlak centrum. Enkel de resultaten van de berekeningen worden gegeven. Een meetkundige interpretatie van deze resultaten is niet gekend.

Poincaré-afbeelding nabij een vlak centrum

In Hoofdstuk 1 voerden we de Poincaré-afbeelding nabij de deformatie van een vlak centrum in en we gaven een uitdrukking van de Melnikov-functie van eerste orde door middel van een Abelse integraal. Aangezien deze Abelse integraal een lijnintegraal is die over het algemeen erg moeilijk te berekenen is, proberen we een andere methode te vinden om deze Melnikov-functies te berekenen. De manier waarop dit aangepakt is, is via *meerwaardige normaalvormen*. Om deze normaalvormen op een goede manier te beschrijven, hebben we een aantal hulpfuncties ingevoerd. De eerste hulpfunctie is de *hoekcompensator* die op natuurlijke manier verwant is met de Ecalle–Roussarie compensator. De andere hulpfuncties zijn de *Taylor-staarten* van sin en cos. Nadat een aantal elementaire eigenschappen van deze functies zijn aangetoond, beschrijven we de techniek van de meerwaardige normaalvormen. De hele methode, alsook het bewijs, is algoritmisch. Hierdoor kunnen we de methode toepassen op een aantal voorbeelden.

Het eerste voorbeeld dat besproken wordt, zijn de Hopf–Takens modellen. Dit zijn vereenvoudigde versies van de vectorvelden waarin de Hopf–Takens bifurcatie voorkomt. We hebben voor deze vectorvelden gekozen aangezien deze ons de unieke kans geven om onze resultaten te vergelijken met de traditionele techniek via de Abelse integraal. Voor deze modellen is de Abelse integraal relatief eenvoudig uit te rekenen. Het blijkt dat we met onze methode dezelfde resultaten terugvinden. Daarenboven kunnen we ook asymptotische uitdrukkingen berekenen voor de Melnikov-functies van hogere orde. Het berekenen van een asymptotische ontwikkeling van de Poincaré-afbeelding ten opzichte van het beginpunt x_0 is eveneens mogelijk via onze methode.

Het tweede voorbeeld dat besproken wordt, is de Hamiltoniaanse driehoek. Geïnspireerd door [Ili98] bekijken we een "essentiële perturbatie" van de Hamiltoniaanse driehoek. Daar dit een deformatie van een ontaard centrum is, bieden de traditionele rechtstreekse technieken maar weinig soelaas. Door kleine aanpassingen te doen aan onze methode met meerwaardige normaalvormen kunnen we de Poincaré-afbeelding van dit systeem berekenen en vinden we dezelfde asymptotische ontwikkeling terug als in [Ili98].

Tot slot worden de Maple-broncodes gegeven die gebruikt werden bij de berekeningen in verband met de Hopf–Takens modellen en de Hamiltoniaanse driehoek.

Locale analytische modellen voor hyperbolische families

In Hoofdstuk 1 kwamen de stellingen van Poincaré en Siegel ter sprake. Deze stellingen geven weer wanneer er een analytische conjugatie tussen een analytisch hyperbolisch vectorveld en zijn linear deel mogelijk is. Voor analytische hyperbolische diffeomorfismes werden varianten van deze stellingen besproken. Het is nu onze bedoeling om deze stellingen uit te breiden naar analytische families van vectorvelden en diffeomorfismes. Voor de stelling van Siegel is dit niet mogelijk omdat een van de voorwaarden die cruciaal zijn voor deze stelling, in het algemeen niet bewaard blijft onder verstoring van de eigenwaarden. Daarom zullen we geen analytische conjugatie met het lineair deel proberen te bewijzen, maar wel een analytische conjugatie met een analytisch systeem bestaande uit het lineaire deel plus hogere orde termen die zowel in de stabiele als in de instabiele richting van voldoende hoge graad zijn. De stelling van Poincaré is wel uitbreidbaar voor families van vectorvelden en diffeomorfismes op voorwaarde dat we kunnen aantonen dat er geen *kleine noemers* optreden.

Dit "kleine noemer"-probleem is een dermate groot probleem om analytische conjugaties te verkrijgen dat we daar eerst grondig op in gaan. We bewijzen: als de parameter voldoende klein is, dan kunnen er geen kleine noemers optreden binnen de deelverzameling $\mathcal{S}_{\ell,n,s}$ van $\mathbb{N}^n$ waarbij ℓ op voorhand vastgelegd wordt en s de dimensie van de stabiele eigenruimte is. Het bewijs voor de vectorvelden verloopt algoritmisch en is meetkundig geïnspireerd.

Hierna volgen de bewijzen van de analytische conjugaties voor analytische families van hyperbolische vectorvelden en diffeomorfismes. Eerst wordt een vergelijking bepaald waarvan de conjugatie een oplossing is. In tegenstelling tot de formele normaalvormen zullen we deze vergelijking opsplitsen in 2 vergelijkingen door de componenten te nemen van de formele machtreeksen ten opzichte van $\mathcal{S}_{\ell,n,s}$ en $\mathcal{T}_{\ell,n,s}$. Eerst wordt een formele oplossing van de eerste vergelijking gezocht en via de methode van de *majoranten* wordt dan bewezen dat de formele oplossing convergent is. Dit betekent dat er een analytische oplossing is. Voor vectorvelden volgt hier bijna onmiddellijk uit dat de oplossing van de tweede vergelijking eveneens analytisch is. In het geval van de diffeomorfismen volgt dit ook na aanzienlijk meer moeite. Een onmiddellijk gevolg van deze resultaten is dat de stabiele en instabiele variëteiten van deze systemen analytisch zijn en analytisch afhangen van de parameter.

Tevens wordt een uitbreiding van dit resultaat naar symmetrische systemen gegeven. Daar we enkel met niet-resonante systemen werken en de opslitsing ten opzichte van $\mathcal{S}_{\ell,n,s}$ en $\mathcal{T}_{\ell,n,s}$ niet wordt verstoord door de symmetrie (want deze is lineair), kunnen we op een vrij eenvoudige manier aantonen dat het resulterende systeem dezelfde symmetrie heeft en dat de analytische conjugatie commuteert met de symmetrie.

Als toepassing van de bewijsmethode tonen we de stelling van Poincaré voor een familie van vectorvelden en diffeomorfismes aan. Dit resultaat werd reeds eerder bewijzen, onder andere in [Brus71], maar met onze methode krijgen we een elegant bewijs waarin het gebruik van contracties en Banach-ruimten vermeden wordt.

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