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NEW TESTS FOR EXPONENTIALITY AGAINST NEW BETTER THAN USED IN p-th QUANTILE

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ABSTRACT. A new characterization of the exponential distribution in a wide class of new better than used in p-th quantile ($NBUp$) lifetime distributions is presented. This leads to new classes of scale-free goodness-of-fit tests for exponentiality against $NBUp$ alternatives. The limiting distributions of the test statistics under the null and alternative hypotheses are derived and the tests are shown to be consistent against $NBUp$ alternatives. Pitman efficacies are calculated and a limited Monte Carlo study is conducted to compare the tests with regard to power for small and moderate sample sizes against a range of alternative distributions. On the basis of overall good performance and ease of computation, a member of the class of test statistics which is based on the sample Winsorized mean, is recommended as a scale-free goodness-of-fit test for the exponential distribution.

Key words: Characterization. Exponential distribution. Hazard rate function. New better than used in p-th quantile. Testing. U-quantile. Winsorized mean.

MSC2000 Classification: Primary 62F03, Secondary 62E10, 62N03, 62N05

1. INTRODUCTION

To study stochastic ageing in reliability and survival applications various important classes of distribution functions are defined in terms of monotonicity in t of some functional of the residual lifetime, given survival upon time t. The most well known examples of such functionals are the mean residual lifetime and the median (or some other quantile) of the residual lifetime. See for example Lai and Xie (2006) for a recent survey.

The present paper deals with distribution functions having the property 'new better than used in p-th quantile' ($NBUp$). We develop a new test for the null hypothesis of exponentiality against alternatives in a wide class of $NBUp$ distributions. Let Y denote a nonnegative lifetime variable with continuous distribution function $F(t) = P(Y \leq t)$, supported on $[0, \infty[$ and with $F(0) = 0$. Let $S = 1 - F$ denote the survival function of Y. For $t \geq 0$ we define the residual lifetime distribution $F_t(x) = P(Y - t \leq x \mid Y > t) = (F(t+x) - F(t))/(1 - F(t))$. For $0 < p < 1$ we define the p-th quantile of the residual lifetime distribution:

$$\begin{aligned} \xi_p(t) &= F_t^{-1}(p) = \inf\{x : F_t(x) \geq p\} \\ &= -t + F^{-1}(p + (1-p)F(t)). \end{aligned} \tag{1.1}$$

Note that $\xi_p(0) = \xi_p = F^{-1}(p)$, the p-th quantile of F. The quantity $\xi_p(t)$ was originally introduced by Haines and Singpurwalla (1974). From (1.1) it easily follows that the relation between $\xi_p(t)$ and $F(t)$ is given by

$$S(t + \xi_p(t)) = (1-p)S(t) \tag{1.2}$$

which is a special case of Schröder's functional equation $\psi(\phi(t)) = \delta\psi(t)$.

Joe and Proschan (1984) showed the $\xi_p(t)$ does not uniquely determine the distribution function F. This is in strong contrast with the more popular mean residual lifetime $m(t) = E(Y - t \mid Y > t)$, which characterizes the distribution function F through an inversion formula $S(t) = m(0)^{-1} m(t) \exp(-\int_0^t m(u)^{-1} du)$.

Various ageing classes have been defined in the literature. A well known one is the $NBUE$ class, the new better than used in expectation, which is defined through the condition $m(t) \leq m(0) = E(Y)$, for all $t \geq 0$.

Joe and Proschan (1983, 1984) introduced and studied the $NBUp$ classes ($0 < p < 1$), the new better than used with respect to the p-th quantile. For a fixed $0 < p < 1$, a distribution function F is said to be $NBUp$ if

$$\xi_p(t) \leq \xi_p(0) = \xi_p, \qquad \text{for all } t \geq 0. \tag{1.3}$$

Hence F is $NBUp$ if for all $t \geq 0$, the p-th quantile of the residual lifetime at t is not greater than the p-th quantile of the lifetime of a new item.
It is easily seen that (1.3) is equivalent to $S(t + \xi_p) \leq S(t + \xi_p(t))$, and because of (1.2) also to $S(t + \xi_p) \leq (1 - p)S(t) = S(\xi_p)S(t)$.
Hence,

$$F \text{ is } NBUp \text{ iff } S(t + \xi_p) \leq S(t)S(\xi_p) \text{ for all } t \geq 0. \tag{1.4}$$

We consider some examples.

Example 1. Weibull distribution.
Here we have $S(t) = \exp(-\lambda t^\rho)$, $\rho > 0$. A simple calculation shows the F is $NBUp$ iff $\rho \geq 1$.

Example 2. Linear hazard rate distribution.
This distribution function is characterized by the hazard rate function $\lambda(t) = \lambda + 2\alpha t$ or equivalently by its survival function $S(t) = \exp(-\lambda t - \alpha t^2)$. It easily follows: F is $NBUp$ iff $\alpha \geq 0$.

Example 3. Makeham distribution.
Here the hazard rate function is $\lambda(t) = \lambda + \beta(1 - e^{-t})$ and the survival function is $S(t) = \exp(-(\lambda + \beta)t + \beta - \beta e^{-t})$. We have: F is $NBUp$ iff $\beta \geq 0$.

In each of the above examples we have the exponential distribution ($\lambda(t) = \lambda, S(t) = \exp(-\lambda t)$) as a special case: $\rho = 1$ or $\alpha = 0$ or $\beta = 0$. The exponential distribution satisfies (1.3) and (1.4) with equality. However the exponential is not the only distribution in this boundary class. For example, take $S(t) = e^{-t}(1 + \varepsilon \sin t)$ with $|\varepsilon| < 1/\sqrt{2}$ and take $p = 1 - e^{-2k\pi}$. Then $\xi_p = 2k\pi$ and $S(t + \xi_p) = S(t)S(\xi_p)$ for all $t \geq 0$ (Song and Cho (1995)).
In this paper we define a wide class $\mathcal{C}$ of $NBUp$ distributions (Section 2) and develop a new characterization of the exponential distribution within that class (Section 3). The new tests for exponentiality are defined in Section 4 and their asymptotic distributions are derived. In Section 5 we obtain Pitman efficacies and in Section 6 we present the result of a small Monte Carlo study on the power of the tests for small and moderate sample sizes.

2. A class of $NBUp$ distributions

We define a large class $\mathcal{C}$ of distribution functions with the $NBUp$ property and having the exponential as the only distribution in the boundary class. The class $\mathcal{C}$ consists of the distribution functions with hazard rate functions given by

$$\lambda(t) = \lambda\rho t^{\rho-1} + 2\alpha t + \beta(1 - e^{-t}), \qquad t \geq 0$$

where $\lambda > 0$, $\rho \geq 1$, $\alpha \geq 0$, $\beta \geq 0$. The corresponding survival function is given by

$$S(t) = \exp(-\lambda t^{\rho} - \alpha t^2 - \beta t - \beta(e^{-t} - 1)), \qquad t \geq 0.$$

Theorem 1 The following statements are equivalent:

(a) F is exponential

(b) $F \in \mathcal{C}$ and $\xi_p(t) = \xi_p$ for all $t \geq 0$ (or equivalently: $S(t + \xi_p) = S(t)S(\xi_p)$ for all $t \geq 0$)

Proof.
Statement (b) can be reformulated as

$$\lambda[(t + \xi_p)^{\rho} - t^{\rho} - \xi_p^{\rho}] + 2\alpha t\xi_p + \beta(e^{-\xi_p} - 1)(e^{-t} - 1) = 0 \tag{1.5}$$

for all $t \geq 0$. We will now show that (1.5) is equivalent to

$$\rho = 1, \text{ and } \alpha = 0, \text{ and } \beta = 0 \tag{1.6}$$

that is, F is exponential.
The implication (1.6) $\Rightarrow$ (1.5) is obvious. For the implication (1.5) $\Rightarrow$ (1.6) we consider the first derivative of (1.5):

$$\lambda\rho[(t + \xi_p)^{\rho-1} - t^{\rho-1}] + 2\alpha\xi_p + \beta(1 - e^{-\xi_p})e^{-t} = 0 \text{ for all } t \geq 0.$$

In particular for $t = 1$ it follows that $\lambda\rho[(1+\xi_p)^{\rho-1} - 1] + 2\alpha\xi_p + \beta(1 - e^{-\xi_p})e^{-1} = 0$. Given the constraints on the parameters, this is only possible if $\rho = 1$ and $\alpha = 0$ and $\beta = 0$.

3. New characterizations of the $NBUp$ property

Since the exponential distribution satisfies (1.4) with equality, it follows that a natural discrepancy measure from exponentiality is given by

$$\Delta(F) = \int_0^\infty [S(x)S(\xi_p) - S(x+\xi_p)]dW(x) \tag{1.7}$$

where $W(x)$ is a positive weight function over the support of $F(x)$. If F is exponential, then $\Delta(F) = 0$. On the other hand, if $F \in \mathcal{C}$ and $\Delta(F) = 0$, then the integrand in (1.7) is zero because $F \in \mathcal{C}$ implies that F is $NBUp$ and hence that the integrand is nonnegative.

Combining this with Theorem 1 leads to the following characterization of the exponential distribution.

Theorem 2 The following statements are equivalent:

(a) F is exponential

(b) $F \in \mathcal{C}$ and $\Delta(F) = 0$

Some interesting choices for the weight function are

(i) $W(x) = 2F(x)$. This then gives

$$\begin{aligned}\Delta(F) &= 2\int_0^\infty [S(x)S(\xi_p) - S(x+\xi_p)]dF(x) \\ &= 2\int_0^\infty [S(x) - S(x+\xi_p)]dF(x) - p \\ &= H_F(\xi_p) - p\end{aligned}$$

where

$$H_F(t) = P(|Y_1 - Y_2| \le t)$$

with Y_1 and Y_2 independent random variables with distribution function F.

(ii) $W(x) = x$. This gives

$$\begin{aligned}\Delta(F) &= \int_0^\infty [S(x)S(\xi_p) - S(x+\xi_p)]dx \\ &= (1-p)\mu - \int_{\xi_p}^\infty S(x)dx = \int_0^{\xi_p} S(x)dx - p\mu\end{aligned}$$

where $\mu = E(Y)$.

4. New tests for exponentiality

Suppose we wish to test the null hypothesis

$$H_0 : F \text{ is exponential}$$

versus

$$H_1 : F \in \mathcal{C}, \text{ but } F \text{ is not exponential.}$$

It follows that, under the alternative hypothesis, $\Delta(F)$ in (1.7) is strictly positive. That is, $H_F(\xi_p) > p$ or equivalently $H_F^{-1}(p) < \xi_p$ and also $\int_0^{\xi_p} S(x)dx > p\mu$. Therefore, appropriate test statistics will be given by empirical versions T_{n1} and T_{n2} of, respectively,

$$\tau_1 = \frac{H_F^{-1}(p)}{\xi_p}$$

or

$$\tau_2 = \frac{\int_0^{\xi_p} S(x)dx}{p\mu} = \frac{\int_0^{p} F^{-1}(t)dt + (1-p)\xi_p}{p\mu}$$

The null hypothesis H_0 will be rejected in favor of H_1 for small values of T_{n1} or for large values of T_{n2}.

Suppose that $Y_1, \ldots, Y_n$ is a random sample from the lifetime Y and that F_n denotes the empirical distribution function

$$F_n(t) = \frac{1}{n} \sum_{i=1}^{n} I(Y_i \leq t).$$

A simple estimator for ξ_p is the p-th empirical quantile

$$\xi_{pn} = F_n^{-1}(p) = \inf\{t : F_n(t) \geq p\}.$$

To estimate $H_F^{-1}(p)$, we note that $H_F(t)$ can be written as

$$H_F(t) = \int_0^{\infty} \int_0^{\infty} I(|y_1 - y_2| \leq t)dF(y_1)dF(y_2).$$

This functional can be estimated by the U-statistic

$$H_n(t) = \binom{n}{2}^{-1} \sum_{1 \le i < j \le n} I(|Y_i - Y_j| \le t)$$

and the quantile $Q_p = H_F^{-1}(p)$ $(0 < p < 1)$ can be estimated by the U-quantile

$$Q_{pn} = H_n^{-1}(p) = \inf\{t : H_n(t) \ge p\}.$$

See Choudhury and Serfling (1988) and Helmers, Janssen and Veraverbeke (1992). Therefore, a natural test statistic based on τ_1 is given by

$$T_{n1} = \frac{Q_{pn}}{\xi_{pn}}. \tag{1.8}$$

A natural estimator for the numerator in τ_2 is given by the sample Winsorized mean

$$W_{pn} = \int_0^p F_n^{-1}(t)dt + (1-p)\xi_{pn}$$

and hence, the empirical version of τ_2 is given by

$$T_{n2} = \frac{W_{pn}}{p\overline{Y}_n}, \tag{1.9}$$

where $\overline{Y}_n$ is the sample mean.

Remark. For the numerator in the expression for τ_2 we note that

$$\int_0^{\xi_p} S(x)dx = E(\min(Y, \xi_p))$$

and hence the estimator for this quantity is also obtainable as

$$\frac{1}{n}\sum_{i=1}^{n} \min(Y_i, \xi_{pn})$$
$$= \begin{cases} \dfrac{1}{n}\left\{\sum_{i=1}^{s} Y_{(i)} + (n-s)Y_{(s)}\right\} & \text{if } np \text{ is integer} \\ \dfrac{1}{n}\left\{\sum_{i=1}^{s} Y_{(i)} + (n-s)Y_{(s+1)}\right\} & \text{if } np \text{ is not integer} \end{cases}$$

where $s = [np]$ and $Y_{(1)} \le \ldots \le Y_{(n)}$ are the order statistics.
This gives an alternative expression for W_{pn}.

Note that both test statistics T_{n1} and T_{n2} are scale invariant. Their asymptotic distributions are given in Theorems 3 and 4 respectively.

Theorem 3.
Assume that F has a density f and that $f(\xi_p) > 0$.
Assume that $H'_F(Q_p) = h_F(Q_p) > 0$.
Then, an $n \to \infty$,

$$n^{1/2}(T_{n1} - \tau_1) \xrightarrow{d} N(0; \sigma_1^2)$$

where

$$\sigma_1^2 = \frac{4\zeta_1}{h_F^2(Q_p)\xi_p^2} + \frac{p(1-p)Q_p^2}{f^2(\xi_p)\xi_p^4} - \frac{4Q_p\zeta_2}{h_F(Q_p)\xi_p^3 f(\xi_p)},$$

$$\zeta_1 = \int_0^\infty [F(x+Q_p) - F(\max(0, x-Q_p))]^2 dF(x) - p^2,$$

and

$$\zeta_2 = \int_0^{\xi_p} [F(x+Q_p) - F(\max(0, x-Q_p))] dF(x) - p^2.$$

Proof.
A simple calculation yields

$$\begin{aligned} T_{n1} - \tau_1 \ &= (Q_{pn} - Q_p)\frac{1}{\xi_p} - (\xi_{pn} - \xi_p)\frac{Q_p}{\xi_p^2} \\ &\quad -(Q_{pn} - Q_p)(\xi_{pn} - \xi_p)\frac{1}{\xi_p\xi_{pn}} + (\xi_{pn} - \xi_p)^2\frac{Q_p}{\xi_{pn}\xi_p^2}. \end{aligned}$$

From the results on U-quantiles in Choudhurry and Serfling (1988) and Helmers, Janssen and Veraverbeke (1992) we obtain

$$\begin{aligned} T_{n1} - \tau_1 &= (Q_{pn} - Q_p)\frac{1}{\xi_p} - (\xi_{pn} - \xi_p)\frac{Q_p}{\xi_p^2} + o_p(n^{-1/2}) \\ &= \frac{p - H_n(Q_p)}{h_F(Q_p)}\frac{1}{\xi_p} - \frac{p - F_n(\xi_p)}{f(\xi_p)}\frac{Q_p}{\xi_p^2} + o_p(n^{-1/2}) \\ &= \frac{1}{n}\sum_{i=1}^n \left\{ -\frac{2g_p(Y_i)}{h_F(Q_p)\xi_p} - \frac{p - I(Y_i \le \xi_p)}{f(\xi_p)}\frac{Q_p}{\xi_p^2} \right\} + o_p(n^{-1/2}) \end{aligned}$$

where

$$\begin{aligned} g_p(Y_1) &= E[I(|Y_1 - Y_2| \le Q_p)|Y_1] - p \\ &= F(Y_1 + Q_p) - F(\max(0, Y_1 - Q_p)) - p \end{aligned}$$

is the conditional expectation of the kernel of the U-statistic $H_n(Q_p)$. From this asymptotic representation the asymptotic normality follows and the limiting variance σ_1^2 can be calculated using $\zeta_1 = E[g_p^2(Y_1)]$ and $\zeta_2 = E[g_p(Y_1)I(Y_1 \le \xi_p)]$.

If F is exponential with parameter $\lambda > 0$, then is can easily be checked that $H_F(t) = 1 - e^{-\lambda t}$ and hence $Q_p = \xi_p = -\frac{1}{\lambda}\ln(1-p)$. Also, $\zeta_1 = \frac{1}{3}p^2(1-p)$ and $\zeta_2 = \frac{1}{2}p^2(1-p)$.

Corollary 1. If F is exponential with parameter $\lambda > 0$, then, as $n \to \infty$,

$$n^{1/2}(T_{n1} - 1) \xrightarrow{d} N(0; \sigma_{10}^2)$$

where

$$\sigma_{10}^2 = \frac{1}{3}\frac{1}{\ln^2(1-p)}\frac{p(3-2p)}{1-p}.$$

Theorem 4
Assume that F has a density f and that $f(\xi_p) > 0$.
Then, as $n \to \infty$,

$$n^{1/2}(T_{n2} - \tau_2) \xrightarrow{d} N(0; \sigma_2^2)$$

where

$$\begin{aligned}
\sigma_2^2 &= \frac{1}{p^2\mu^2}\left[-\int_0^p\int_0^t F^{-1}(s)ds dF^{-1}(t) + \xi_p\int_0^p t dF^{-1}(t) - \left(\int_0^p t dF^{-1}(t)\right)^2\right] \\
&+ \frac{1}{\mu^2}\frac{1}{f^2(\xi_p)}\frac{(1-p)^3}{p} + \frac{1}{p^2\mu^2}\tau_2^2\sigma^2 + \frac{2}{p^2\mu^2}\frac{(1-p)^2}{f(\xi_p)}\int_0^p t dF^{-1}(t) \\
&+ \frac{2}{p^2\mu^2}\tau_2\int_0^p E(YI(Y \le F^{-1}(t))dF^{-1}(t) + \frac{2\tau_2}{p^2\mu}\int_0^p t dF^{-1}(t) \\
&+ \frac{2}{\mu^2}\frac{1}{f(\xi_p)}\frac{(1-p)}{p}\tau_2 E(YI(Y \le \xi_p)) - \frac{2}{\mu^2}\frac{1}{f(\xi_p)}\tau_2\frac{(1-p)}{p}
\end{aligned}$$

with $\mu = E(Y)$ and $\sigma^2 = \mathrm{Var}(Y)$.

Proof.
From the asymptotic results on quantiles (see e.g. Serfling (1980)) it follows that

$$\begin{aligned} T_{n2} - \tau_2 &= \frac{1}{n}\sum_{i=1}^{n}\left\{\frac{1}{p\mu}\int_0^p \frac{t - I(Y_i \le F^{-1}(t))}{f(F^{-1}(t))}dt \right. \\ &\quad \left. + \frac{1-p}{p\mu}\,\frac{p - I(Y_i \le \xi_p)}{f(\xi_p)} - \frac{\tau_2}{p\mu}(Y_i - \mu)\right\} + o_p(n^{-1/2}) \end{aligned}$$

This representation provides the asymptotic normality result and the limiting variance σ_2^2 is obtained by straightforward calculation.

If F is exponential, we can restrict to the standard exponential due to the scale invariance. For the standard exponential it is easily checked that $\xi_p = -\ln(1-p)$, $\tau_2 = 1$, $\int_0^p t dF^{-1}(t) = -p - \ln(1-p)$, $E(YI(Y \le F^{-1}(t)) = t + (1-t)\ln(1-t)$. Straightforward calculation gives the following corollary.

Corollary 2. If F is exponential with parameter $\lambda > 0$, then as $n \to \infty$,

$$n^{1/2}(T_{n2} - 1) \xrightarrow{d} N(0; \sigma_{20}^2)$$

where

$$\sigma_{20}^2 = \frac{1-p}{p}.$$

As a consequence, an approximate level-α test based on T_{n1} of H_0: F is exponential, will have a rejection region given by

$$\widetilde{T}_{n1} := n^{1/2}\frac{T_{n1} - 1}{\sigma_{10}} \le z_\alpha$$

where $z_\alpha = \Phi^{-1}(\alpha)$ and Φ is the standard normal distribution function. Under the alternative hypothesis $H_1 : F \in \mathcal{C}$, but not exponential, we have that

$$P\left(n^{1/2}\frac{T_{n1} - 1}{\sigma_{10}} \le z_\alpha\right) = P\left(n^{1/2}\frac{T_{n1} - \tau_1}{\sigma_1} \le -(\tau_1 - 1)\frac{n^{1/2}}{\sigma_1} + \frac{\sigma_{10}}{\sigma_1}z_\alpha\right)$$

and this tends to 1 as $n \to \infty$ since, under H_1, $n^{1/2}\frac{T_{n1} - \tau_1}{\sigma_1} \xrightarrow{d} N(0;1)$ and $\tau_1 < 1$. Hence, the test is consistent against $F \in \mathcal{C}$, but not exponential.

The same conclusion holds for the test based on T_{n2}. In this case, an approximate level-α test will reject H_0 if

$$\widetilde{T}_{n2} := n^{1/2}\frac{T_{n2}-1}{\sigma_{20}} \geq z_{1-\alpha}.$$

5. Pitman efficacies

For testing the hypothesis that Y is exponential (with parameter 1, without loss of generality), we consider the following alternative distributions:

$$\begin{aligned}
&F_\theta(x) = 1-\exp(-x^\theta),\ \theta \geq 1 \qquad \text{(Weibull)}\\
&F_\theta(x) = 1-\exp(-x-\tfrac{1}{2}\theta x^2),\ \theta \geq 0 \qquad \text{(linear failure rate)}\\
&F_\theta(x) = 1-\exp(-x-\theta x-\theta(e^{-x}-1)),\ \theta \geq 0 \qquad \text{(Makeham).}
\end{aligned}$$

They are particular members of the class $\mathcal{C}$. The null hypothesis of exponentiality corresponds to the choice $\theta = \theta_0 = 1$ for Weibull and $\theta = \theta_0 = 0$ for Linear Failure rate and Makeham. Define

$$\begin{aligned}
\tau_1(\theta) &= \frac{H_{F_\theta}^{-1}(p)}{F_\theta^{-1}(p)}\\
\tau_2(\theta) &= \frac{\int_0^{F_\theta^{-1}(p)} (1-F_\theta(x))dx}{p\mu_\theta}.
\end{aligned}$$

Then Pitman efficacies of the tests based on T_{n1} and T_{n2} are calculated as

$$e_1 = \frac{(\tau_1'(\theta_0))^2}{\sigma_{10}^2} \text{ and } e_2 = \frac{(\tau_2'(\theta_0))^2}{\sigma_{20}^2}$$

where derivative is with respect to the parameter θ and σ_{10}^2 and σ_{20}^2 are given in Corollaries 1 and 2. For comparison we also (re)calculated the efficacy of the test based on W_{2n}^* as proposed in Joe and Proschan (1983). It rejects H_0 if

$$\widetilde{W}_{2n}^* := n^{1/2}\frac{W_{2n}^*}{\sigma_{00}} \geq z_{1-\alpha}$$

where

$$\sigma_{00}^2 = \frac{p(3-2p)}{12(1-p)}.$$

The efficacy is given by

$$e_0 = \frac{(\tau_0'(\theta_0))^2}{\sigma_{00}^2}$$

where

$$\tau_0(\theta) \quad = \quad \frac{\frac{1}{2}F_\theta^{-1}(p) - \int\limits_0^1 F_\theta^{-1}(t)J_2(t)dt}{\mu_\theta}$$

and

$$J_2(t) = \begin{cases} -(1-t) & \text{if } 0 \leq t \leq p \\ ((1-p)^{-2}-1)(1-t) & \text{if } p < t \leq 1. \end{cases}$$

In Figures 1-3 we calculated e_0, e_1 and e_2 as functions of p $(0 < p < 1)$, for the three alternative distributions.

Figures 1–3

Figures 1-3 clearly show the superior performance of T_{n2} with respect to Pitman efficacy. A remarkable fact is also that the efficacies e_0 and e_1 completely coincide, although the statistics T_{n1} and W_{2n}^* are different.

6. Simulations

In this section we present the results of a limited Monte Carlo study in order to compare the power of the tests based on $\widetilde{T}_{n1}$, $\widetilde{T}_{n2}$ and $\widetilde{W}_{2n}^*$ for small and moderate sample sizes $(n = 20, 40, 60, 80)$ and for choices $p = 0.3, 0.5, 0.8$. We use the same alternative distributions as in the previous section: Weibull with $\theta = 1.3, 1.4, 1.5$; linear failure rate with $\theta = 0.5, 1.0, 1.5$; Makeham with $\theta = 1.0, 1.5, 2.0$.
The critical values used for the simulations are given in Table 1 for a significance level $\alpha = 0.05$. Recall that the tests $\widetilde{T}_{n2}$ and $\widetilde{W}_{2n}^*$ reject H_0 for values to the right of the critical value. For the test $\widetilde{T}_{n1}$, rejection of H_0 is for values to the left of the critical point. For the tests $\widetilde{T}_{n2}$ and $\widetilde{W}_{2n}^*$, the critical values were obtained as the 95th percentiles of 1 000 000 simulated test statistics from the exponential distribution. For the test $\widetilde{T}_{n1}$, the simulated 5th percentile was obtained. All standard errors of the estimated critical values were found to be negligibly small and are therefore not reported in Table 1.

Tables 1–2

From Table 1 it is clear the the critical values of $\widetilde{T}_{n2}$ for $p = 0.3$ and $p = 0.5$ converge rather fast to the normal critical value 1.65, which seems not to be the case with $\widetilde{T}_{n1}$.

Further evidence of the rapid convergence of the critical values of $\widetilde{T}_{n2}$ can also be seen from Table 2, which contains the estimated sizes of $\widetilde{T}_{n2}$ when using the normal critical values 2.33, 1.65 and 1.28, corresponding to the nominal significance levels $\alpha = 1\%$, $\alpha = 5\%$ and $\alpha = 10\%$, respectively. These estimated sizes were calculated as the proportion of 1 000 000 Monte Carlo samples that resulted in rejection of H_0 using the above mentioned normal critical values. The standard errors of the estimated sizes are less than or equal to $\sqrt{0.25/1\ 000\ 000} = 0.0005$. The close correspondence between the estimated sizes of $\widetilde{T}_{n2}$ using normal critical values and the α-values is quite remarkable.

Power estimates were calculated as the proportion of 20 000 Monte Carlo samples that resulted in rejection of H_0 at significance level $\alpha = 5\%$ for the alternative distributions considered. In Tables 3-5 we present power comparisons for sample sizes $n = 20, 40, 60, 80$. The standard errors of the estimated probabilities are less than or equal to $\sqrt{0.25/20\ 000} = 0.0035$.

Tables 3–5

Tables 3-5 also provide the powers of $\widetilde{T}_{n2}$ (displayed in parentheses) for different sample sizes and the various alternatives when normal critical values are used. The close correspondence between the powers using the exact critical values and the normal critical values (for $p = 0.3$ and $p = 0.5$) also provides evidence of the correspondence between these two types of critical values.

From Tables 3–5 we can conclude the following:

(1) For all alternatives at $p = 0.3$ the powers of $\widetilde{W}^*_{2n}$ and $\widetilde{T}_{n1}$ are almost identical

(2) For all alternatives at $p = 0.5$ the powers of $\widetilde{T}_{n1}$ are slightly better than those of $\widetilde{W}^*_{2n}$

(3) For all alternatives at $p = 0.8$ the powers of $\widetilde{T}_{n1}$ are considerably larger than those of $\widetilde{W}^*_{2n}$

(4) For all alternatives at $p = 0.3$ the powers of $\widetilde{T}_{n2}$ are considerably larger than those of $\widetilde{T}_{n1}$

(5) For $p = 0.5$ at the Weibull alternative the powers of $\widetilde{T}_{n1}$ and $\widetilde{T}_{n2}$ are approximately the same, but for the linear failure rate and Makeham alternatives the powers of $\widetilde{T}_{n2}$ are considerably larger than those of $\widetilde{T}_{n1}$

(6) For all alternatives at $p = 0.8$, the powers of $\widetilde{T}_{n1}$ are considerably larger than those of $\widetilde{T}_{n2}$

Based on the overall good performance with regard to the Pitman efficacies and based on good power performance for $p \leq 0.5$, we recommend $\widetilde{T}_{n2}$ for $p \leq 0.5$ (especially a choice around $p = 0.3$) as an effective procedure for testing exponentiality against $NBUp$ alternatives in the class $\mathcal{C}$.

From the discussions above, it is clear that from a *practical* point of view, the use of normal critical values when using $\widetilde{T}_{n2}$ for $p = 0.3$ and $p = 0.5$ can be recommended even for small sample sizes. If, however, one would prefer to make use of the exact critical values (calculated from 1 000 000 independent Monte carlo trials), these are provided in Table 6 for small samples and different significance levels.

Table 6

The calculations were done using double precision arithmetic in FORTRAN and routines from the IMSL library.

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References

Choudhury, J. and Serfling, R. (1988). Generalized order statistics, Bahadur representations and sequential nonparametric fixed-width confidence intervals. J. Statist. Planning Inf., 19, 269-282.

Haines, A.L. and Singpurwalla, N.D. (1974). Some contributions to the stochastic characterization of wear. Reliability and Biometry (eds. F. Proschan and R.J. Serfling), Siam, Philadelphia, 47-80.

Helmers, R., Janssen, P. and Veraverbeke, N. (1992). Bootstrapping U-quantiles.

Exploring the limits of bootstrap (eds. E. Le Page and L. Billard), Wiley, New York, 145-155.

Joe, H. and Proschan, F. (1983). Tests for properties of the percentile residual life function. Communications in Statistics-Theory and Methods, 12, 1087-1119.

Joe, H. and Proschan, F. (1984). Percentile residual life functions. Operations Research, 32, 668-678.

Lai, C.-D. and Xie, M. (2006). Stochastic Ageing and Dependence for Reliability. Springer Verlag.

Serfling, R.J. (1980), Approximation Theorems of Mathematical Statistics. Wiley, New York.

Song, J.-K. and Cho, G.-Y. (1995). A note on percentile residual life. Sankhya, Series A, 57, 333-335.

Figure 1. Pitman efficacies for the LFR alternative

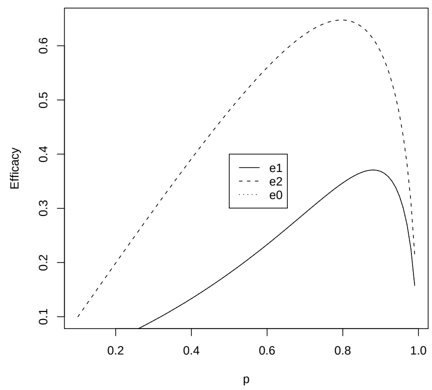

Figure 2. Pitman efficacies for the Makeham alternative

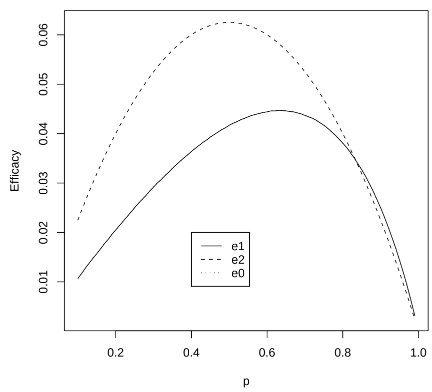

Figure 3. Pitman efficacies for the Weibull alternative

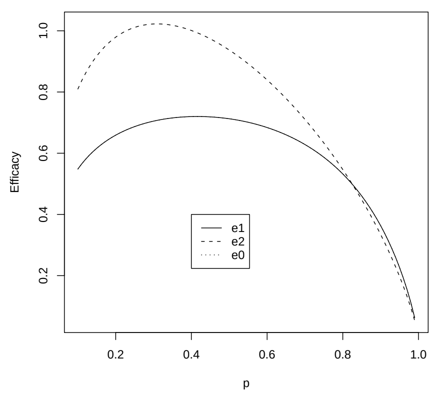

Table 1. Estimated critical values for $\alpha = 5\%$

	$p = 0.3$			$p = 0.5$			$p = 0.8$		
n	$\widetilde{W}^*_{2n}$	$\widetilde{T}_{n1}$	$\widetilde{T}_{n2}$	$\widetilde{W}^*_{2n}$	$\widetilde{T}_{n1}$	$\widetilde{T}_{n2}$	$\widetilde{W}^*_{2n}$	$\widetilde{T}_{n1}$	$\widetilde{T}_{n2}$
10	1.702	-0.975	1.712	1.504	-0.999	1.573	1.141	-0.956	1.257
20	1.674	-1.070	1.710	1.521	-1.093	1.610	1.227	-0.985	1.395
30	1.657	-1.115	1.706	1.537	-1.133	1.622	1.277	-1.036	1.447
40	1.647	-1.161	1.699	1.542	-1.179	1.627	1.312	-1.080	1.481
50	1.649	-1.185	1.698	1.551	-1.203	1.635	1.347	-1.111	1.499
60	1.646	-1.219	1.695	1.556	-1.237	1.636	1.365	-1.140	1.517
70	1.658	-1.239	1.693	1.568	-1.253	1.638	1.379	-1.161	1.527
80	1.652	-1.255	1.690	1.565	-1.274	1.640	1.395	-1.185	1.534

Table 2. Estimated size of $\widetilde{T}_{n2}$ using normal critical values

	Nominal significance level α					
	$\alpha = 1\%$		$\alpha = 5\%$		$\alpha = 10\%$	
n	$p = 0.3$	$p = 0.5$	$p = 0.3$	$p = 0.5$	$p = 0.3$	$p = 0.5$
10	.014	.004	.058	.042	.104	.096
20	.014	.007	.057	.046	.104	.098
30	.014	.008	.056	.048	.103	.098
40	.014	.008	.055	.048	.103	.099
50	.013	.009	.055	.049	.102	.099
60	.013	.009	.055	.049	.102	.099
70	.013	.009	.054	.049	.102	.099
80	.012	.009	.053	.050	.101	.100

Table 3. Estimated power functions for the Weibull alternative

θ	n	$p = 0.3$			$p = 0.5$			$p = 0.8$		
		$\widetilde{W}^*_{2n}$	$\widetilde{T}_{n1}$	$\widetilde{T}_{n2}$	$\widetilde{W}^*_{2n}$	$\widetilde{T}_{n1}$	$\widetilde{T}_{n2}$	$\widetilde{W}^*_{2n}$	$\widetilde{T}_{n1}$	$\widetilde{T}_{n2}$
1.3	20	.25	.24	.28(.30)	.25	.27	.25(.24)	.20	.27	.15(.05)
	40	.41	.40	.47(.49)	.41	.43	.41(.42)	.34	.42	.26(.17)
	60	.54	.54	.62(.63)	.54	.57	.57(.57)	.44	.55	.35(.29)
	80	.65	.65	.74(.75)	.65	.68	.70(.69)	.55	.64	.46(.39)
1.4	20	.35	.34	.39(.42)	.35	.37	.35(.33)	.29	.37	.19(.07)
	40	.58	.57	.65(.67)	.58	.61	.60(.59)	.48	.60	.36(.26)
	60	.74	.74	.81(.82)	.74	.77	.78(.77)	.62	.75	.53(.43)
	80	.84	.84	.91(.91)	.84	.87	.88(.88)	.74	.83	.65(.58)
1.5	20	.45	.44	.50(.53)	.45	.47	.46(.43)	.37	.48	.24(.10)
	40	.72	.73	.79(.81)	.71	.76	.75(.74)	.61	.74	.43(.35)
	60	.87	.88	.93(.93)	.87	.90	.90(.90)	.77	.87	.67(.58)
	80	.93	.95	.98(.98)	.94	.96	.96(.96)	.87	.94	.80(.75)

Table 4. Estimated power functions for the LFR alternative

θ	n	$p = 0.3$			$p = 0.5$			$p = 0.8$		
		$\widetilde{W}^*_{2n}$	$\widetilde{T}_{n1}$	$\widetilde{T}_{n2}$	$\widetilde{W}^*_{2n}$	$\widetilde{T}_{n1}$	$\widetilde{T}_{n2}$	$\widetilde{W}^*_{2n}$	$\widetilde{T}_{n1}$	$\widetilde{T}_{n2}$
0.5	20	.12	.11	.15(.17)	.14	.14	.16(.15)	.14	.15	.12(.04)
	40	.18	.16	.23(.23)	.21	.21	.26(.25)	.23	.25	.21(.14)
	60	.22	.21	.29(.30)	.27	.27	.35(.34)	.29	.34	.30(.23)
	80	.25	.24	.36(.36)	.32	.33	.43(.42)	.35	.41	.37(.31)
1.0	20	.18	.17	.23(.24)	.21	.21	.24(.22)	.21	.22	.16(.06)
	40	.28	.27	.37(.37)	.34	.35	.41(.39)	.35	.40	.31(.21)
	60	.35	.35	.48(.49)	.44	.46	.55(.55)	.47	.54	.46(.38)
	80	.42	.42	.58(.58)	.53	.55	.67(.66)	.56	.65	.58(.51)
1.5	20	.22	.21	.28(.30)	.26	.26	.30(.28)	.25	.29	.20(.08)
	40	.36	.35	.47(.49)	.43	.45	.52(.51)	.43	.51	.39(.28)
	60	.46	.47	.61(.62)	.56	.59	.68(.67)	.58	.66	.57(.47)
	80	.56	.56	.72(.73)	.67	.69	.79(.79)	.70	.77	.69(.64)

Table 5. Estimated power functions for the Makeham alternative

		$p = 0.3$			$p = 0.5$			$p = 0.8$		
θ	n	$\widetilde{W}^*_{2n}$	$\widetilde{T}_{n1}$	$\widetilde{T}_{n2}$	$\widetilde{W}^*_{2n}$	$\widetilde{T}_{n1}$	$\widetilde{T}_{n2}$	$\widetilde{W}^*_{2n}$	$\widetilde{T}_{n1}$	$\widetilde{T}_{n2}$
1.0	20	.13	.12	.15(.16)	.13	.13	.15(.14)	.12	.14	.10(.04)
	40	.19	.18	.22(.24)	.21	.21	.23(.22)	.19	.22	.15(.10)
	60	.23	.24	.29(.30)	.26	.27	.30(.30)	.23	.28	.21(.16)
	80	.28	.28	.35(.37)	.32	.32	.37(.37)	.30	.33	.26(.21)
1.5	20	.16	.15	.19(.20)	.17	.18	.18(.17)	.15	.18	.12(.04)
	40	.26	.24	.31(.32)	.28	.29	.31(.30)	.25	.30	.20(.14)
	60	.32	.32	.42(.42)	.36	.39	.42(.42)	.32	.38	.28(.22)
	80	.40	.39	.50(.52)	.45	.46	.52(.52)	.40	.47	.35(.29)
2.0	20	.20	.19	.23(.24)	.21	.22	.22(.22)	.18	.22	.15(.05)
	40	.32	.30	.38(.39)	.34	.36	.39(.37)	.30	.37	.25(.17)
	60	.41	.40	.50(.52)	.45	.47	.52(.51)	.41	.48	.34(.28)
	80	.49	.49	.61(.62)	.55	.57	.63(.63)	.49	.58	.44(.38)

Table 6. Estimated critical values for $\widetilde{T}_{n2}$

	Nominal significance level α					
	$\alpha = 1\%$		$\alpha = 5\%$		$\alpha = 10\%$	
n	$p = 0.3$	$p = 0.5$	$p = 0.3$	$p = 0.5$	$p = 0.3$	$p = 0.5$
10	2.456	2.085	1.712	1.573	1.314	1.257
20	2.478	2.200	1.710	1.610	1.309	1.269
30	2.477	2.241	1.706	1.622	1.305	1.272
40	2.464	2.266	1.699	1.627	1.302	1.275
50	2.448	2.273	1.698	1.635	1.297	1.278