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Computing invariants and semi-invariants by means of Frobenius Lie algebras

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Abstract. Let $U(\mathfrak{g})$ be the enveloping algebra of a finite dimensional Lie algebra $\mathfrak{g}$ over a field k of characteristic zero, $Z(U(\mathfrak{g}))$ its center and $Sz(U(\mathfrak{g}))$ its semicenter. A sufficient condition is given in order for $Sz(U(\mathfrak{g}))$ to be a polynomial algebra over k. Surprisingly, this condition holds for many Lie algebras, especially among those for which the radical is nilpotent, in which case $Sz(U(\mathfrak{g})) = Z(U(\mathfrak{g}))$. In particular, it allows the explicit description of $Z(U(\mathfrak{g}))$ for more than half of all complex, indecomposable nilpotent Lie algebras of dimension at most 7.

1. INTRODUCTION

Let $U(\mathfrak{g})$ be the enveloping algebra of a nonzero finite dimensional Lie algebra $\mathfrak{g}$ over a field k of characteristic zero, $Z(U(\mathfrak{g}))$ its center and $Sz(U(\mathfrak{g}))$ its semicenter, i.e. the subalgebra of $U(\mathfrak{g})$ generated by the semi-invariants of $U(\mathfrak{g})$. The (Poisson) semicenter $Sz(S(\mathfrak{g}))$ of the symmetric algebra $S(\mathfrak{g})$, which is defined in a similar fashion, is known to be isomorphic to $Sz(U(\mathfrak{g}))$ if k is algebraically closed [RV]. Unlike the center $Z(U(\mathfrak{g}))$ [DNOW, p. 321], the semicenter $Sz(U(\mathfrak{g}))$ is never reduced to k [D1, 4.9.4] and is always factorial [Moe], [LO].

In some important cases $Sz(U(\mathfrak{g}))$ turns out to be a polynomial algebra over k. This happens for instance if $\mathfrak{g}$ is: semisimple [D1, 7.3.8], Frobenius (i.e. the index $i(\mathfrak{g}) = 0$) [DNO, p. 339], square integrable (i.e. $i(\mathfrak{g}) = dim\ Z(\mathfrak{g})$) [DNOW, p. 323], the Lie algebra of strictly lower triangular matrices [D4], any Borel subalgebra (as well as its nilradical) of a complex semisimple Lie algebra [J2, p. 263 and p. 257] and many (but unfortunately not all [Y]) parabolic (and biparabolic) subalgebras of certain complex semisimple Lie algebras [FJ1,2], [J3,4].

The key result of this paper is the following. Let $\mathcal{g}$ be a Lie algebra for which there exists a split torus $T \subset Der\ \mathcal{g}$ (i.e. a commutative Lie algebra consisting of diagonalizable derivations of $\mathcal{g}$) such that the semi-direct product $L = T \oplus \mathcal{g}$ is a Frobenius Lie algebra. Let $x_1, \ldots, x_n$ be a basis of L and let $v_1, \ldots, v_r$ be the irreducible factors of the nonzero determinant $det([x_i, x_j]) \in S(\mathcal{g})$. Then $v_1, \ldots, v_r$ are the only (up to nonzero scalar multipliers) irreducible semi-invariants of $S(L)$ and their weights $\lambda_1, \ldots, \lambda_r \in L^*$ are linearly independent over k [DNO, pp. 339-343]. We will show that $S(\mathcal{g})$ and $S(L)$ have the same semicenter, which coincides with the polynomial algebra $k[v_1, \ldots, v_r]$. A similar result holds for the semicenters of $U(\mathcal{g})$ and $U(L)$. Under these circumstances $dim\ T = i(\mathcal{g})$, which implies that $i(\mathcal{g}) \leq rank\ \mathcal{g}$, the latter being the dimension of a maximal torus inside $Der\ \mathcal{g}$. Also, the sum of the degrees of $v_1, \ldots, v_r$ is at most $c(\mathcal{g}) = (dim\ \mathcal{g} + i(\mathcal{g}))/2$, an inequality which is known to be valid in the cases mentioned above and which has recently been shown to hold in general [OV].

In section 4 we exhibit examples of different types where this simple method can be employed successfully in order to compute algebraically independent generators for $Sz(U(\mathcal{g}))$.

Section 5 is devoted to complex indecomposable nilpotent Lie algebras of dimension at most seven. We use the classification of Carles [Ca] and Magnin[Ma2,3], the latter providing in each case a maximal torus and a basis which it diagonalizes. However, we also include the notation of Morozov ($M\_$) [Mor] and Cerezo ($C\_$) [Ce1] in dimension 6 and of Seeley ($(\ldots)$) [Se] and Romdhani ($R\_$) [R] in dimension 7. We distinguish two classes:

1) $i(\mathcal{g}) \leq rank\ \mathcal{g}$

Remarkably, in this situation our condition is satisfied with very few exceptions (namely, in dimension 6: $\mathcal{g}_{6.15}$ and in dimension 7: $\mathcal{g}_{7.3.1(i_\lambda),\lambda=0,1}$; $\mathcal{g}_{7,3.12}$; $\mathcal{g}_{7,3.13}$ and $\mathcal{g}_{7,3.17}$. For all these $\mathcal{g}$ is coregular, i.e. $Z(U(\mathcal{g}))$ is a polynomial algebra, except for $\mathcal{g}_{7,3.17}$).

It turns out that the generators we find for $Z(U(\mathcal{g}))$ coincide with the algebraic independent generators of its quotient field listed in [Ce1] for dimension 6 and in [R] for dimension 7 (although in the latter quite a few cases are missing [Go, pp. 146-149] and some minor corrections have to be made). Most of the calculations were done with Maple.

As a welcome by-product we obtain all solvable, almost algebraic Frobenius Lie algebras over $\mathbb{C}$ of which the nilradical is indecomposable of dimension at most seven.

In each case a split torus $T \subset Der\ \mathfrak{g}$ for which $L = T \oplus \mathfrak{g}$ is Frobenius is given. Sometimes, the possible solutions for T form a family depending on parameters. See also [CV].

2) $i(\mathfrak{g}) > rank\ \mathfrak{g}$

In this situation, the following methods are employed in order to find the generators of $S(\mathfrak{g})^{\mathfrak{g}}$ (and hence of $Z(U(\mathfrak{g}))$): the procedure Dixmier used for dimension at most 5 [D2; D3, pp. 322-330], Singular [GPS] (in case $\mathfrak{g}$ has an abelian ideal of codimension one), a result by Panyushev [P2, Theorem 1.2] and an effective algorithm recently constructed by Michel Van den Bergh.

We confine ourselves, for now, to finish the seven remaining cases in dimension 6, of which only two are not coregular, namely $\mathfrak{g}_{6,6}$ and $\mathfrak{g}_{6,16}$, the latter being the 6-dimensional standard filiform Lie algebra $L(6)$ (also called generic filiform). In fact, the n-dimensional standard filiform Lie algebra $L(n)$ (i.e. with basis $x_1, \ldots, x_n$ and nonzero Lie brackets $[x_1, x_i] = x_{i+1}, \quad i : 2, \ldots, n-1$) is not coregular if and only if $n \geq 5$ [OV].

2. PRELIMINARIES AND GENERAL RESULTS

(i) The index $i(L)$

Let L be a Lie algebra over a field k of characteristic zero with basis $x_1, \ldots, x_n$. Let $f \in L^*$ and consider the alternating bilinear form B_f on L sending (x, y) into $f([x, y])$, with kernel:

$$L(f) = \{x \in L \mid f([x, y]) = 0 \text{ for all } y \in L\}$$

We put $i(L) = \min_{f \in L^*} \dim L(f)$, the index of L. We recall from [D1, 1.14.13] that

$$i(L) = \dim L - \mathrm{rank}_{R(L)}([x_i, x_j])$$

where $R(L)$ is the quotient field of the symmetric algebra $S(L)$ of L. In particular $\dim L - i(L)$ is an even number. Furthermore, f is called regular if $\dim L(f) = i(L)$. It is well-known that the set L^*_{reg} of all regular elements of L^* is an open dense subset of L^* for the Zariski topology.

Denote by $Z(D(L))$ the center of the quotient division ring $D(L)$ of $U(L)$ and let $R(L)^L$ be the subfield of the invariants of $R(L)$ under the action of $ad\ x$, $x \in L$.

THEOREM 2.1.

$$\mathrm{trdeg}_k Z(D(L)) = \mathrm{trdeg}_k R(L)^L \leq i(L)$$

[O3], [RV]. Moreover, equality occurs if one of the following conditions is satisfied:

(1) L is algebraic [O3], [RV]

(2) k is algebraically closed and L has no proper semi-invariants in $S(L)$ (or equivalently in $U(L)$) [OV, Proposition 4.1]

(ii) The semicenter $Sz(U(L))$

Let $\lambda \in L^*$. We denote by $U(L)_\lambda$ the set of all $u \in U(L)$ such that $ad\ x(u) = \lambda(x)u$ for all $x \in L$. Any nonzero element $u \in U(L)_\lambda$ is called a semi-invariant w.r.t. the weight λ. We call u a proper semi-invariant if $\lambda \neq 0$. Clearly, $U(L)_\lambda U(L)_\mu \subset U(L)_{\lambda+\mu}$ for all $\lambda, \mu \in L^*$. The sum of the $U(L)_\lambda$ is direct and it is a subalgebra $Sz(U(L))$ of $U(L)$, the semicenter of $U(L)$. $Sz(U(L))$ is factorial and any semi-invariant can be written uniquely as a product of irreducible semi-invariants. Suppose $z \in D(L), z \neq 0$. Then, $z \in Z(D(L))$ if and only if z can be written as a quotient of two semi-invariants of $U(L)$ with the same weight. So, if L has no proper semi-invariants (as it is if its radical is nilpotent) then $Z(D(L))$ is the quotient field of $Z(U(L))$. A useful link with $S(L)$ is the symmetrization map, i.e. the canonical linear isomorphism s of $S(L)$ onto $U(L)$, which maps each product $y_1, \ldots, y_m,\ y_i \in L$, into $(1/m!) \sum_p y_{p(1)} \ldots y_{p(m)}$, where p ranges over all permutations of $\{1, \ldots, m\}$.

s is known to commute with each derivation of L and hence maps $S(L)_\lambda$ onto $U(L)_\lambda$ and also $Sz(S(L))$ onto $Sz(U(L))$. This restriction is an algebra isomorphism in case L is nilpotent [D1, 4.8.12] or if L is Frobenius [DNO, p. 399].

The weights of the semi-invariants of $U(L)$ (or $S(L)$) form a semigroup $\Lambda(L)$, which is not necessarily finitely generated [DDV]. However, the additive subgroup $\Lambda_D(L)$ of L^* generated by $\Lambda(L)$, is a finitely generated free abelian group [NO]. Perhaps unaware of this a new proof was recently given [FJ2, p. 1519].

The centralizer of $Sz(U(L))$ in $U(L)$ coincides with $U(L_\Lambda)$, where L_Λ is the intersection of ker λ, $\lambda \in \Lambda(L)$. L_Λ is a characteristic ideal of L for which the following hold [DNO, pp. 332, 343]:

THEOREM 2.2.

$$Z(U(L)) \subset Sz(U(L)) \subset Z(U(L_\Lambda)) = Sz(U(L_\Lambda))$$

and $Z(D(L_\Lambda))$ is the quotient field of $Z(U(L_\Lambda))$. Moreover, $Sz(U(L)) = Z(U(L_\Lambda))$ in case k is algebraically closed and either L is almost algebraic or L is Frobenius. Similar results hold in $S(L)$. See also [FJ2, p. 1518].

(iii) Frobenius Lie algebras

A Lie algebra L is called Frobenius if there is a linear functional $f \in L^*$ such that the alternating bilinear form $B_f(x,y) = f([x,y])$, $x,y \in L$, is nondegenerate, i.e. $i(L) = 0$. This notion was introduced in [O1] in connection with the problem of Jacobson on the characterization of Lie algebras having a primitive universal enveloping algebra. See [O2,3] for the general solution. In particular, $U(L)$ is primitive if L is Frobenius and the converse holds if L is algebraic. It has become clear over the years that these Lie algebras form a large class. They also play an important role in different areas. For instance, many Lie subalgebras of semisimple Lie algebras, such as some parabolic and biparabolic (sea weed) subalgebras, are Frobenius [E1,2,3], [Ge], [P1], including most Borel subalgebras of simple Lie algebras [EO, p. 146]. Other examples are j-algebras which are real, solvable Lie algebras, which play an essential role in the study of bounded homogeneous domains in $\mathbb{C}^n$, admitting a simply transitive solvable group of analytic automorphisms [PS]. Also, Frobenius (and quasi Frobenius) Lie algebras give rise to constant solutions of the classical Yang-Baxter equation [Dr]. More recently, Frobenius Lie algebras appear naturally in the study of simple hypersurface singularities [EK].

We now collect some useful facts on semi-invariants from [O4], [DNO]. Let L be a Frobenius Lie algebra with basis $x_1, \dots, x_n$. Then n is even and L has a trivial center. In fact, L is highly noncommutative since $Z(D(L)) = k$. The Pfaffian $Pf([x_i, x_j]) \in S(L)$ is homogeneous of degree $\frac{1}{2}\dim L$ and $(Pf([x_i,x_j]))^2 = \det([x_i, x_j]) \neq 0$. This is well determined by L (up to a nonzero scalar multiple) and for all $g \in \mathrm{Aut}L$:

$$g(Pf([x_i,x_j])) = \det g \; Pf([x_i,x_j])$$

THEOREM 2.3. Let L be Frobenius and let $v_1, \dots, v_r \in S(L)$ be the (pairwise nonassociated) irreducible factors of $Pf([x_i,x_j])$. Then,

(1) $v_1, \dots, v_r$ are the only (up to nonzero scalar multipliers) irreducible semi-invariants of $S(L)$, say with weights $\lambda_1, \dots, \lambda_r \in \Lambda(L)$.

(2) $Sz(S(L)) = k[v_1, \dots, v_r]$, a polynomial algebra over k.

(3) $\lambda_1, \dots, \lambda_r$ are linearly independent over k. They generate the semigroup $\Lambda(L)$ and $L_\Lambda = \cap \ker \lambda_i$, $i : 1, \dots, r$

(4) $r = \dim L - \dim L_\Lambda = i(L_\Lambda)$ (see also Lemma 3.1)

COROLLARY 2.4. Let L be Frobenius. Any semi-invariant of $S(L)$ is homogeneous. It is also a semi-invariant under the action of Der L.

(iv) The Frobenius semiradical

Put $F(L) = \sum_{f \in L^*_{reg}} L(f)$. This is a characteristic ideal of L contained in L_Λ and $F(F(L)) = F(L)$. We have the following characterization: $Z(D(L)) \subset D(F(L))$ and if L is algebraic then $F(L)$ is the smallest Lie subalgebra of L with this property. For example, if L is square integrable then $F(L) = Z(L)$ and so $Z(D(L)) = D(Z(L))$. In particular, $Z(U(L)) = U(Z(L))$, which is a polynomial algebra. This phenomenon occurs quite frequently in section 5. Clearly, $F(L) = 0$ if and only if L is Frobenius. Therefore $F(L)$ is called the Frobenius semiradical of L. At the other end of the spectrum we have the Lie algebras for which $F(L) = L$, which we call quasi quadratic. These are unimodular and they do not possess any proper semi-invariants. They form a large class, which include all quadratic Lie algebras (and hence all abelian and all semisimple Lie algebras) [O6]. This illustrates how far Frobenius and semisimple Lie algebras are removed from each other.

REMARK. Some time ago Mustapha Rais kindly sent us an unpublished manuscript by André Cerezo [Ce2], in which the soul (respectively the rational soul) of a Lie algebra L is introduced and studied. This is the smallest Lie subalgebra of L whose enveloping algebra (resp. enveloping quotient division ring) contains $Z(U(L))$ (resp. $Z(D(L))$). Clearly, the rational soul of L coincides with $F(L)$ in case L is algebraic.

(v) The number $c(L) = (\dim L + i(L))/2$

$c(L)$, which is an integer, occurs frequently in the theory of Lie algebras.

For instance:

THEOREM 2.5. [OV] (k algebraically closed)

Assume that $Sz(S(L))$ is freely generated by homogeneous elements $f_1, \ldots, f_r$. Then $\sum_{i=1}^{r} \deg f_i \leq c(L)$.

The following, which is a minor extension of [OV, Corollary 1.2], is useful as a first test for coregularity.

COROLLARY 2.6. (k algebraically closed)
Assume that L has no proper semi-invariants in $S(L)$ and that $S(L)^L = k[f_1, \ldots, f_r]$ a polynomial algebra for some homogeneous $f_i \in S(L)$. Then

$$3\ i(L) \leq \dim L + 2 \dim Z(L)$$

Moreover, if equality occurs then $\sum\limits_{i=1}^{r} \deg f_i = c(L)$ and $\deg f_i \leq 2$, $i : 1, \ldots, r$.

On the other hand $c(L)$ is also the maximum transcendence degree of a Poisson commutative subfield of $R(L)$ [Sa]. See also [JL], [PY].
Similarly, $c(L)$ is an upperbound for the transcendence degree of maximal subfields of $D(L)$ and this bound can be achieved quite often. However a strictly smaller transcendence degree in $D(L)$ (and also in $R(L)$) is possible as the following example demonstrates.
Let L be the solvable, almost algebraic Lie algebra over k with basis $\{x_0, x_1, \ldots x_n\}$, $n \geq 2$, with nonzero Lie brackets $[x_0, x_i] = \lambda_i x_i$, $i : 1, \ldots, n$, such that $\lambda_1, \ldots, \lambda_n$ are linearly independent over $\mathbb{Q}$. Clearly, $i(L) = n - 1$ and $c(L) = (n + 1 + n - 1)/2 = n$. Then $k(x_0)$ is a maximal (resp. Poisson) commutative subfield of $D(L)$ (resp. of $R(L)$) [O5, Theorems 7, 12] and $\operatorname{tr}\deg_k(k(x_0)) = 1 < n = c(L)$.

Next, let L be an n-dimensional algebraic Lie algebra satisfying the Gelfand-Kirillov conjecture [GK], [BGR], [J1], [AOV1,2], [O7], i.e. $D(L)$ can be generated by elements $z_1, \ldots, z_r, p_1, \ldots, p_s, q_1, \ldots, q_s \in D(L)$ such that $[p_i, p_j] = 0$, $[q_i, q_j] = 0$, $[p_i, q_j] = \delta_{ij}$, $i, j : 1, \ldots, s$ and $Z(D(L)) = k(z_1, \ldots, z_r)$, a purely transcendental extension of k. Then $r = i(L)$, $n = r + 2s$ and $c(L) = r + s$ is the transcendence degree of the maximal commutative subfield $k(z_1, \ldots, z_r, p_1, \ldots, p_s)$ of $D(L)$.

Finally, let P be a commutative Lie subalgebra of L such that $\dim P = c(L)$, i.e. P is a commutative polarization (CP) w.r.t. any $f \in L^*_{reg}$. Then $D(P)$ is a maximal subfield of $D(L)$ [O5, p. 706] with $\operatorname{tr}\deg_k(D(P)) = \dim P = c(L)$.
It is easy to see that $F(L) \subset P$ and hence is commutative. P plays a special role in the construction of the irreducible representations of $U(L)$ and their kernels, the primitive ideals. If in addition, P is an ideal of a completely solvable Lie algebra L, then P is a Vergne polarization of L. Furthermore, $Sz(U(L)) \subset U(P)$ and $P \subset L_\Lambda$ [EO, p. 141]. In section 5 we will provide for each nilpotent Lie algebra a CP-ideal (if it exists).

3. THE MAIN RESULT

LEMMA 3.1. Let L be Frobenius and H an ideal of L such that L/H is nilpotent. Then

$$i(H) = \dim L - \dim H$$

PROOF. Let $f \in L^*_{reg}$ such that its restriction $f|_H \in H^*_{reg}$. Denote by L^∞ the intersection of all terms $C^i(L)$ of the descending central series of L. With respect to the nondegenerate bilinear form $B_f(x,y) = f([x,y])$, $x, y \in L$, we have that $(L^\infty)^\perp \subset L^\infty$ [MO, Corollary 6.2]. Now, L/H being nilpotent implies that $L^\infty \subset H$. Hence,

$$H^\perp \subset (L^\infty)^\perp \subset L^\infty \subset H$$

Consequently,

$$\begin{aligned} i(H) &= \dim H(f|_H) = \dim(H \cap H^\perp) = \dim H^\perp \\ &= \dim L - \dim H \end{aligned}$$

□

PROPOSITION 3.2. Let H be an ideal of L such that L/H is nilpotent. Suppose $v \in S(L)$ is a semi-invariant with weight $\lambda \in L^*$. Then $S(H)$ contains a semi-invariant w of $S(L)$ with the same weight.
In particular, if $U(L)$ is primitive then each semi-invariant v of $S(L)$ is contained in $S(H)$ and

$$Sz(S(L)) \subset Sz(S(H))$$

PROOF. Since L/H is nilpotent, we can find ideals H_i of L with $\dim H_i = i$, $[L, H_i] \subset H_{i-1}$ and such that

$$L = H_n \supset \ldots \supset H_i \supset H_{i-1} \supset \ldots \supset H_d = H$$

If $v \in S(H)$ then put $w = v$. So, we may assume that $v \in S(H_i) \backslash S(H_{i-1})$ with $i > d$. Choose $x \in H_i \backslash H_{i-1}$. Then $S(H_i) = S(H_{i-1})[x]$. So, v can be written as a polynomial in x with coefficients in $S(H_{i-1})$:

$$v = a_m x^m + \ldots + a_1 x + a_0, \ a_m \neq 0$$

Take any $y \in L$. Then $ad\ y(x) \in H_{i-1}$ and
$\lambda(y)v = ad\ y(v) = ad\ y(a_m)x^m +$ terms of lower degree in x.

It follows that $ad\ y(a_m) = \lambda(y)a_m$ for all $y \in L$, i.e. $a_m \in S(L)_\lambda \cap S(H_{i-1})$. After repeating the same reasoning a number of times, we find a semi-invariant w of $S(L)$ with weight $\lambda \in \Lambda(L)$, contained in $S(H)$.
Now, suppose $U(L)$ is primitive, which is equivalent with $R(L)^L = k$. [O3, p. 69]. Since v and w are semi-invariants with the same weight $\lambda \in \Lambda(L)$, we observe that

$$ad\ y(vw^{-1}) = (w\ ad\ y(v) - v\ ad\ y(w))w^{-2} = 0$$

for all $y \in L$, i.e. $vw^{-1} \in R(L)^L = k$ Hence, for some $a \in k$ we have $v = aw \in S(H)$. □

LEMMA 3.3. Let $\lambda \in L^*$ be a weight. Then

$$S(L)_\lambda = \bigoplus_i (S^i(L) \cap S(L)_\lambda)$$

PROOF. The inclusion $\supset$ is obvious. For $\subset$ we take $v \in S(L)_\lambda$ and let

$$v = v_p + \ldots + v_0$$

be its unique decomposition into homogeneous components (i.e. $v_i \in S^i(L)$). Then for any $x \in L$:

$$\sum_{i=0}^{p} ad\ x(v_i) = ad\ x(v) = \lambda(x)v = \sum_{i=0}^{p} \lambda(x)v_i$$

Since $ad\ x(v_i) \in S^i(L)$ for all i we may conclude that $ad\ x(v_i) = \lambda(x)v_i$ for all $x \in L$, i.e. $v_i \in S^i(L) \cap S(L)_\lambda$. □

PROPOSITION 3.4. Let $\mathfrak{g}$ be a finite-dimensional Lie algebra over k. Suppose there exists a split torus $T \subset Der\ \mathfrak{g}$. Consider the semi-direct product $L = T \oplus \mathfrak{g}$. Then,

$$Sz(S(\mathfrak{g})) \subset Sz(S(L)) \quad \text{and} \quad Sz(U(\mathfrak{g})) \subset Sz(U(L))$$

Moreover, each weight $\lambda \in \Lambda(\mathfrak{g})$ can be extended to a weight $\lambda' \in \Lambda(L)$ i.e. $\lambda'|_{\mathfrak{g}} = \lambda$.

PROOF. Take any weight $\lambda \in \Lambda(\mathfrak{g})$. By the previous lemma and the fact that $Sz(S(\mathfrak{g})) = \bigoplus_\lambda S(\mathfrak{g})_\lambda$, it suffices to show that for all m $V_{m,\lambda} = S^m(\mathfrak{g}) \cap S(\mathfrak{g})_\lambda$

is contained in $Sz(S(L))$. First we notice that $S(\mathfrak{g})_\lambda$ is a T-submodule of $S(\mathfrak{g})$, since $S(\mathfrak{g})_\lambda$ is stable under the derivations of $\mathfrak{g}$ [Mon, p. 265]. Next, we choose a basis $x_1, \ldots, x_n$ of $\mathfrak{g}$ consisting of common eigenvectors for all $t \in T$. Then for each m $S^m(\mathfrak{g})$ is a finite dimensional diagonalizable T-module (indeed the monomials $x_1^{m_1} \ldots x_n^{m_n}$, $\sum_i m_i = m$, form a basis of $S^m(\mathfrak{g})$ consisting of eigenvectors for any $t \in T$). Hence, the same holds for the T-submodule $V_{m,\lambda}$. Therefore if $V_{m,\lambda} \neq 0$ it contains a basis $v_1, \ldots, v_p$ such that $t(v_i) = \mu_i(t)v_i$ for all $t \in T$ for some $\mu_i \in T^*$. On the other hand, $ad\ x(v_i) = \lambda(x)v_i$ for all $x \in \mathfrak{g}$. Hence, each v_i is a semi-invariant for $L = T \oplus \mathfrak{g}$, say with weight $\lambda' \in \Lambda(L)$. Note that λ' is an extension of λ. It follows that $V_{m,\lambda} \subset Sz(S(L))$. Consequently, $Sz(S(\mathfrak{g})) \subset Sz(S(L))$, which implies that $Sz(U(\mathfrak{g})) \subset Sz(U(L))$ (take the image under the symmetrization s). □

NOTATION. Let L be a Frobenius Lie algebra with basis $x_1, \ldots, x_n$. We recall that the Pfaffian $Pf([x_i, x_j]) \in S(L)$ is a homogeneous semi-invariant of degree $\frac{1}{2}\dim L$ with weight $\tau \in \Lambda(L)$ where $\tau(x) = \mathrm{tr}(ad\ x)$, $x \in L$. Also,

$$(Pf([x_i, x_j]))^2 = \det([x_i, x_j])$$

We now put

$$Pf(L) = Pf([x_i, x_j]) \quad \text{and} \quad \Delta(L) = \det([x_i, x_j])$$

These are well-defined, up to a nonzero scalar multiple.

THEOREM 3.5. Let $\mathfrak{g}$ be a finite dimensional Lie algebra over k. Suppose there exists a split torus $T \subset \mathrm{Der}\ \mathfrak{g}$ such that the semi-direct product $L = T \oplus \mathfrak{g}$ is a Frobenius Lie algebra. Let $v_1, \ldots, v_r$ be the irreducible factors of $\Delta(L)$. Then the following hold:

1. $Sz(S(\mathfrak{g})) = Sz(S(L)) = k[v_1, \ldots, v_r]$, a polynomial algebra.
2. $Sz(U(\mathfrak{g})) = Sz(U(L)) = k[s(v_1), \ldots, s(v_r)]$, a polynomial algebra. This coincides with $Z(U(L_\Lambda))$ if k is algebraically closed.
3. $\dim T = i(\mathfrak{g})$ and $r = i(L_\Lambda) = \dim L - \dim L_\Lambda$
4. $\Lambda(\mathfrak{g}) = \{\lambda|_\mathfrak{g} \mid \lambda \in \Lambda(L)\}$ and $\mathfrak{g}_\Lambda = \mathfrak{g} \cap L_\Lambda$
5. $\deg\ v_1 + \ldots + \deg\ v_r \leq c(\mathfrak{g}) = c(L)$

6. If P is a CP of $\mathcal{g}$, then P is also a CP of L.

7. If $\mathcal{g}$ has no proper semi-invariants in $S(\mathcal{g})$ (i.e. $\mathcal{g} = \mathcal{g}_\Lambda$) then

$$S(\mathcal{g})^{\mathcal{g}} = k[v_1, \ldots, v_r] \quad \text{and} \quad Z(U(\mathcal{g})) = k[s(v_1), \ldots, s(v_r)]$$

If in addition k is algebraically closed or if $\mathcal{g}$ is algebraic, then $\mathcal{g} = L_\Lambda$.

PROOF. (1) By the previous proposition $Sz(S(\mathcal{g})) \subset Sz(S(L))$ and each weight $\lambda \in \Lambda(\mathcal{g})$ can be extended to a weight $\lambda' \in \Lambda(L)$. On the other hand, $U(L)$ is primitive as L is Frobenius. Moreover, $\mathcal{g}$ is an ideal of L and $L/\mathcal{g}$ is abelian. Then, by Proposition 3.2 each semi-invariant of $S(L)$ is already contained in $S(\mathcal{g})$ and $Sz(S(L)) \subset Sz(S(\mathcal{g}))$. Consequently, $Sz(S(\mathcal{g})) = Sz(S(L))$, the latter being a polynomial algebra in the variables $v_1, \ldots, v_r$ by Theorem 2.3. Note also that if $\mu \in \Lambda(L)$ then its restriction $\mu|_{\mathcal{g}} \in \Lambda(\mathcal{g})$.

(2) This follows immediately from (1) and the fact that $s : Sz(S(L)) \to Sz(U(L))$ is an algebra isomorphism which maps $Sz(S(\mathcal{g}))$ onto $Sz(U(\mathcal{g}))$.

(3) Both quotients $L/\mathcal{g}$ and L/L_Λ [DNO, p. 330] are abelian. Hence, by Lemma 3.1 and Theorem 2.3:

$$i(\mathcal{g}) = \dim L - \dim \mathcal{g} = \dim T \quad \text{and} \quad i(L_\Lambda) = \dim L - \dim L_\Lambda = r$$

(4) From the proof of (1) we deduce that the map $\Lambda(L) \to \Lambda(\mathcal{g})$, sending λ into its restriction $\lambda|_{\mathcal{g}}$, is surjective. It follows that

$$\Lambda(\mathcal{g}) = \{\lambda|_{\mathcal{g}} \mid \lambda \in \Lambda(L)\} \quad \text{and} \quad \mathcal{g}_\Lambda = \mathcal{g} \cap L_\Lambda$$

(5) $v_1, \ldots, v_r$ are also the irreducible factors of $Pf(L)$ as $\Delta(L) = (Pf(L))^2$. Hence, the sum of their degrees is at most:

$$\begin{aligned} \deg(Pf(L)) &= \frac{1}{2}\dim L = \frac{1}{2}(\dim \mathcal{g} + \dim T) \\ &= \frac{1}{2}(\dim \mathcal{g} + i(\mathcal{g})) = c(\mathcal{g}) \end{aligned}$$

On the other hand, $c(L) = \frac{1}{2}(\dim L + i(L)) = \frac{1}{2}\dim L = c(\mathcal{g})$

(6) Let P be a CP of $\mathcal{g}$, i.e. P is a commutative Lie subalgebra of $\mathcal{g}$ for which $\dim P = c(\mathcal{g})$. Clearly, P is also a CP of L since $c(\mathcal{g}) = c(L)$.

(7) $S(\mathfrak{g})^{\mathfrak{g}} = Sz(S(\mathfrak{g}))$ and $Z(U(\mathfrak{g})) = Sz(U(\mathfrak{g}))$ as $\mathfrak{g}$ does not have proper semi-invariants in $S(\mathfrak{g})$ nor in $U(\mathfrak{g})$. So, it suffices to apply (1) and (2).
Next, we assume in addition that either k is algebraically closed or that $\mathfrak{g}$ is algebraic. Using (4), we deduce from $\mathfrak{g} = \mathfrak{g}_\Lambda$ that $\mathfrak{g} \subset L_\Lambda$. Hence it is enough to show that $\dim \mathfrak{g} = \dim L_\Lambda$. As there are no proper semi-invariants in $U(\mathfrak{g})$ we see that $Z(D(\mathfrak{g}))$ is the quotient field of $Z(U(\mathfrak{g}))$. Therefore,

$$Z(D(\mathfrak{g})) = k(s(v_1), \ldots, s(v_r))$$

In particular, $r = \text{tr deg}_k Z(D(\mathfrak{g})) = i(\mathfrak{g})$ by Theorem 2.1. So,

$$\dim L - \dim L_\Lambda = \dim L - \dim \mathfrak{g}$$

which yields $\dim L_\Lambda = \dim \mathfrak{g}$. □

REMARK. (1) and (2) of the theorem remain valid under the condition that $U(L)$ is primitive (instead of requiring L to be Frobenius). In that case $v_1, \ldots, v_r$ are the irreducible factors of $\Delta_n(L)$, a special semi-invariant of $S(L)$ [DNO, p. 337].

COROLLARY 3.6. See also [J2]. Let k be algebraically closed and let L be a simple Lie algebra over k of rank r with triangular decomposition $L = N^- \oplus H \oplus N$. Consider the Borel subalgebra $B = H \oplus N$. If L is not of one of the following types A_n, $n \geq 2$; D_{2t+1}, $t \geq 2$; E_6, then

(1) B is Frobenius

(2) $i(N) = r$, $c(B) = c(N)$, $N = B_\Lambda$

(3) $Sz(U(B)) = Z(U(N))$, a polynomial algebra in r generators which can be explicitly determined by the method of Theorem 3.5.

PROOF. Clearly, B can be considered as the semi-direct product of the split torus $T = ad_N H \subset \text{Der} N$ with the nilradical N. Since B is Frobenius [EO, p. 146] the previous theorem can be applied. □

4. EXAMPLES

1. Let $\mathcal{g}$ be the 5-dimensional solvable Lie algebra over k with basis $x_1, \dots, x_5$ and with nonvanishing Lie brackets: $[x_1, x_3] = x_3 \quad [x_1, x_4] = x_4 \quad [x_1, x_5] = x_5$ $[x_2, x_3] = x_4 \quad [x_2, x_4] = x_5$.
We want to apply Theorem 3.5. Since $i(\mathcal{g}) = 1$ we need to find a 1-dimensional split torus $T \subset Der\ \mathcal{g}$ such that the semi-direct product $L = T \oplus \mathcal{g}$ is Frobenius. Put $T =< t >$ with $t = \mathrm{diag}(0, -1, 2, 1, 0)$. The matrix of Lie brackets of L is:

	t	x_1	x_2	x_3	x_4	x_5
t	0	0	$-x_2$	$2x_3$	x_4	0
x_1	0	0	0	x_3	x_4	x_5
x_2	x_2	0	0	x_4	x_5	0
x_3	$-2x_3$	$-x_3$	$-x_4$	0	0	0
x_4	$-x_4$	$-x_4$	$-x_5$	0	0	0
x_5	0	$-x_5$	0	0	0	0

Its determinant is $\Delta(L) = x_5^2(x_4^2 - 2x_3x_5)^2$ which is nonzero. So, $L = T \oplus \mathcal{g}$ is Frobenius. Clearly, x_5 and $x_4^2 - 2x_3x_5$ are the irreducible factors of $\Delta(L)$. Hence, by Theorem 3.5:

$$Sz(S(\mathcal{g})) = Sz(S(L)) = k[x_5, x_4^2 - 2x_3x_5]$$

which is a polynomial algebra.
Finally, by symmetrization:

$$Sz(U(\mathcal{g})) = Sz(U(L)) = k[x_5, x_4^2 - 2x_3x_5]$$

2. The following nonsolvable examples, selected from [AOV2], have a Levi decomposition $\mathcal{g} = S \oplus R$, where $S = sl(2, k)$, with standard basis h, x, y and nonzero Lie brackets $[h, x] = 2x$, $[h, y] = -2y$, $[x, y] = h$.
$e_0, e_1, \dots, e_p$ will be a basis of R. W_n will be the $(n+1)$-dimensional irreducible $sl(2, k)$-module, with standard basis $e_0, e_1, \dots, e_n$.

(i) $\mathcal{g} = sl(2, k) \oplus W_1$, with W_1 abelian [AOV2, p. 554]. $\mathcal{g}$ is a 5-dimensional algebraic Lie algebra of index 1. Put $T =< t >$, $t = \mathrm{diag}(0, 0, 0, 1, 1)$ and $L = T \oplus \mathcal{g}$.

Its matrix of Lie brackets

	t	h	x	y	e_0	e_1
t	0	0	0	0	e_0	e_1
h	0	0	$2x$	$-2y$	e_0	$-e_1$
x	0	$-2x$	0	h	0	e_0
y	0	$2y$	$-h$	0	e_1	0
e_0	$-e_0$	$-e_0$	0	$-e_1$	0	0
e_1	$-e_1$	e_1	$-e_0$	0	0	0

has determinant $\Delta(L) = 4(e_1^2 x + e_0 e_1 h - e_0^2 y)^2 \neq 0$.
So, L is Frobenius. On the other hand, $\mathfrak{g}$ has no proper semi-invariants in $S(\mathfrak{g})$ as $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$. By (7) of Theorem 3.5 we conclude that $\mathfrak{g} = L_\Lambda$ and

$$Z(U(\mathfrak{g})) = Sz(U(L)) = k[e_1^2 x + e_0 e_1 h - e_0^2 y]$$

(ii) $\mathfrak{g} = sl(2,k) \oplus R$, where $R = W_2 \oplus W_1$ with standard basis $e_0, e_1, e_2; e_3, e_4$. [AOV2, p. 567], [O7, p. 908]. The nontrivial action of $sl(2,k)$ on R is given by

$[h, e_0] = 2e_0$ $[h, e_2] = -2e_2$ $[h, e_3] = e_3$ $[h, e_4] = -e_4$
$[x, e_1] = 2e_0$ $[x, e_2] = e_1$ $[x, e_4] = e_3$ $[y, e_0] = e_1$ $[y, e_1] = 2e_2$ $[y, e_3] = e_4$.
$\mathfrak{g}$ is an 8-dimensional algebraic Lie algebra of index 2. Consider the split torus $T =< t_1, t_2 >$; $t_1 = \text{diag}(0,0,0,1,1,1,0,0)$, $t_2 = \text{diag}(0,0,0,0,0,0,1,1)$ and $L = T \oplus \mathfrak{g}$.
Then $\Delta(L) = 4(e_1^2 - 4e_0e_2)^2(e_0e_4^2 - e_1e_3e_4 + e_2e_3^2)^2 \neq 0$. Hence, L is a 10-dimensional Frobenius Lie algebra. Clearly, $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$. By (7) of Theorem 3.5 we obtain that $\mathfrak{g} = L_\Lambda$ and

$$Z(U(\mathfrak{g})) = Sz(U(L)) = k[e_1^2 - 4e_0e_2, e_0e_4^2 - e_1e_3e_4 + e_2e_3^2]$$

(iii) Let R be the 5-dimensional nilpotent Lie algebra with basis e_0, e_1, e_2, e_3, e_4 and nonzero Lie brackets $[e_2, e_4] = e_0$ $[e_3, e_4] = e_1$. $sl(2,k)$ acts on R as follows:
$[h, e_0] = e_0$ $[h, e_1] = -e_1$ $[h, e_2] = e_2$ $[h, e_3] = -e_3$ $[x, e_1] = e_0$ $[x, e_3] = e_2$ $[y, e_0] = e_1$ $[y, e_2] = e_3$.
Consider the semi-direct product $\mathfrak{g} = sl(2,k) \oplus R$. $\mathfrak{g}$ is an 8-dimensional algebraic Lie algebra of index 2 [AOV2, p. 573].
Put $T =< t_1, t_2 >$; $t_1 = diag(0,0,0,1,1,0,0,1)$, $t_2 = diag(0,0,0,1,1,1,1,0)$ and $L = T \oplus \mathfrak{g}$. Then,

$$\Delta(L) = 4(e_0e_3 - e_1e_2)^2(e_1^2 x + e_0e_1h - e_0^2 y - e_0e_3e_4 + e_1e_2e_4)^2 \neq 0$$

Hence, L is a 10-dimensional Frobenius Lie algebra. Clearly, $\mathfrak{g}$ has no proper semi-invariants in $S(\mathfrak{g})$ as R is nilpotent. By (7) of Theorem 3.5 we may conclude that $\mathfrak{g} = L_\Lambda$ and after symmetrization

$$Z(U(\mathfrak{g})) = Sz(U(L)) = k[e_0e_3 - e_1e_2, e_1^2x + e_0e_1h - e_0^2y - e_0e_3e_4 + e_1e_2e_4]$$

REMARK. In the last 3 examples we notice that the algebraically independent generators of $Z(D(\mathfrak{g}))$ we obtained in [AOV2] and [O7] (needed for the verification of the GK-conjecture) turn out to be the generators of $Z(U(\mathfrak{g}))$.

5. INDECOMPOSABLE NILPOTENT LIE ALGEBRAS OF DIMENSION ≤ 7 ($k = \mathbb{C}$)

The main purpose is to describe $Z = Z(U(\mathfrak{g}))$ for each Lie algebra $\mathfrak{g}$, but also to give the Frobenius semiradical $F = F(\mathfrak{g})$ and if they exist a CP-ideal (CPI) and a torus $T \subset \mathrm{Der}\ \mathfrak{g}$ for which the semi-direct product $T \oplus \mathfrak{g}$ is Frobenius. Sometimes the possible solutions for T form a family depending on parameters $\in k$. Other abbreviations are: $i = i(\mathfrak{g})$, $r = \mathrm{rank}\ \mathfrak{g}$, $c = c(\mathfrak{g})$, SQ.I. = square integrable, $Q(Z)$ = the quotient field of $Z(U(\mathfrak{g}))$. For dimension ≤ 6 all Lie algebras are listed, while in dimension 7 only those for which $i(\mathfrak{g}) \leq \mathrm{rank}\ \mathfrak{g}$.
These Lie algebras are coregular (i.e. $Z(U(\mathfrak{g}))$ is a polynomial algebra over k), except $\mathfrak{g}_{5,5}$, $\mathfrak{g}_{6,6}$, $\mathfrak{g}_{6,16}$ and $\mathfrak{g}_{7,3.17}$.

Notation: We use the same notation as Magnin [Ma2] and Carles [Ca]. In addition we include the notation of Morozov (M_) [Mor] and Cerezo (C_) [Ce1] in dimension 6 and of Seeley ((...)) [Se] and Romdhani (R_)[R] in dimension 7.

dim $\mathfrak{g} \leq 5$ (See also [D3, pp. 322-330])

1. $\mathfrak{g}_3$ (3-dim Heisenberg Lie algebra)
   $[x_1, x_2] = x_3$.
   SQ.I. $i = 1$ $r = 2$ $c = 2$ $Z = k[x_3]$ $F = < x_3 >$
   $CPI = < x_2, x_3 >$
   $T = < t >$, $t = \mathrm{diag}(\alpha, 1 - \alpha, 1)$.

2. $\mathfrak{g}_4$ (4-dim standard filiform Lie algebra)
   $[x_1, x_2] = x_3$ $[x_1, x_3] = x_4$.
   $i = 2$ $r = 2$ $c = 3$ $Z = k[x_4, x_3^2 - 2x_2x_4]$

$F =< x_2, x_3, x_4 >= CPI$
$T =< t_1, t_2 >, \quad t_1 = \mathrm{diag}(0, 1, 1, 1), \quad t_2 = \mathrm{diag}(1, -2, -1, 0).$

$\dim \mathfrak{g} = 5$, $i(\mathfrak{g}) \leq rank\ \mathfrak{g}$

3. $\mathfrak{g}_{5,1}$ (5-dim. Heisenberg Lie algebra)
   $[x_1, x_3] = x_5 \quad [x_2, x_4] = x_5.$
   SQ.I. $i = 1 \quad r = 3 \quad c = 3 \quad Z = k[x_5]$
   $F =< x_5 > \quad CPI =< x_3, x_4, x_5 >$
   $T =< t >, \quad t = \mathrm{diag}(\alpha, \beta, 1 - \alpha, 1 - \beta, 1).$

4. $\mathfrak{g}_{5,2}$
   $[x_1, x_2] = x_4 \quad [x_1, x_3] = x_5.$
   $i = 3 \quad r = 3 \quad c = 4 \quad Z = k[x_4, x_5, x_2x_5 - x_3x_4]$
   $F =< x_2, x_3, x_4, x_5 >= CPI$
   $T =< t_1, t_2, t_3 >, \quad t_1 = \mathrm{diag}(1, 0, 0, 1, 1),$
   $t_2 = \mathrm{diag}(0, 1, 0, 1, 0), \quad t_3 = \mathrm{diag}(1, -1, -1, 0, 0).$

5. $\mathfrak{g}_{5,3}$
   $[x_1, x_2] = x_4 \quad [x_1, x_4] = x_5 \quad [x_2, x_3] = x_5$
   SQ.I. $i = 1 \quad r = 2 \quad c = 3 \quad Z = k[x_5] \quad F =< x_5 > \quad CPI =< x_3, x_4, x_5 >$
   $T =< t >, \quad t = \mathrm{diag}(\alpha, 1 - 2\alpha, 2\alpha, 1 - \alpha, 1).$

6. $\mathfrak{g}_{5,6}$
   $[x_1, x_2] = x_3 \quad [x_1, x_3] = x_4 \quad [x_1, x_4] = x_5 \quad [x_2, x_3] = x_5$
   SQ.I. $i = 1 \quad r = 1 \quad c = 3 \quad Z = k[x_5] \quad F =< x_5 > \quad CPI =< x_3, x_4, x_5 >$
   $T =< t >, \quad t = \mathrm{diag}(1, 2, 3, 4, 5).$

$\dim \mathfrak{g} = 5$, $i(\mathfrak{g}) >$ rank $\mathfrak{g}$

7. $\mathfrak{g}_{5,4}$
   $[x_1, x_2] = x_3 \quad [x_1, x_3] = x_4 \quad [x_2, x_3] = x_5.$
   $i = 3 \quad r = 2 \quad c = 4 \quad Z = k[x_4, x_5, x_3^2 + 2x_1x_5 - 2x_2x_4]$
   $F = \mathfrak{g}_{5,4}$ No CP's.

8. $\mathfrak{g}_{5,5}$ (5-dim. standard filiform Lie algebra)
   $[x_1, x_2] = x_3 \quad [x_1, x_3] = x_4 \quad [x_1, x_4] = x_5.$
   $i = 3 \quad r = 2 \quad c = 4 \quad F =< x_2, x_3, x_4, x_5 >= CPI$

$Z = k[x_5, f_1, f_2, f_3] \quad f_1 = 2x_3x_5 - x_4^2 \quad f_2 = 3x_2x_5^2 - 3x_3x_4x_5 + x_4^3$
$f_3 = 9x_2^2x_5^2 - 18x_2x_3x_4x_5 + 6x_2x_4^3 + 8x_3^3x_5 - 3x_3^2x_4^2$
Relation: $\quad f_1^3 + f_2^2 - x_5^2 f_3 = 0 \quad Q(Z) = k(x_5, f_1, f_2)$.

$\dim \mathfrak{g} = 6$, $\boldsymbol{i}(\mathfrak{g}) \leq \text{rank } \mathfrak{g}$

9. $\mathfrak{g}_{6,1} \cong M_4 \cong C_{24}$
$[x_1, x_2] = x_5 \quad [x_1, x_4] = x_6 \quad [x_2, x_3] = x_6$.
SQ.I. $i = 2 \quad r = 3 \quad c = 4 \quad Z = k[x_5, x_6]$
$F =< x_5, x_6 > \quad CPI =< x_3, x_4, x_5, x_6 >$
$T =< t_1, t_2 >, \quad t_1 = \text{diag}(\alpha, 1-\alpha, \alpha, 1-\alpha, 1, 1), \quad t_2 = \text{diag}(1, 0, 0, -1, 1, 0)$.

10. $\mathfrak{g}_{6,2} \cong M_{12} \cong C_{22}$
$[x_1, x_2] = x_5 \quad [x_1, x_5] = x_6 \quad [x_3, x_4] = x_6$.
$i = 2 \quad r = 3 \quad c = 4 \quad Z = k[x_6, x_5^2 - 2x_2x_6]$
$F =< x_2, x_5, x_6 > \quad CPI =< x_2, x_4, x_5, x_6 >$
$T =< t_1, t_2 >, \quad t_1 = \text{diag}(0, 1, \alpha, 1-\alpha, 1, 1), \quad t_2 = \text{diag}(1, 0, 2, 0, 1, 2)$.

11. $\mathfrak{g}_{6,4} \cong M_7 \cong C_{18}$
$[x_1, x_2] = x_4 \quad [x_1, x_3] = x_6 \quad [x_2, x_4] = x_5$.
SQ.I. $i = 2 \quad r = 3 \quad c = 4 \quad Z = k[x_5, x_6]$
$F =< x_5, x_6 > \quad CPI =< x_3, x_4, x_5, x_6 >$
$T =< t_1, t_2 >, t_1 = \text{diag}(1-2\alpha, \alpha, 2\alpha, 1-\alpha, 1, 1), \quad t_2 = \text{diag}(0, 1, 0, 1, 2, 0)$.

12. $\mathfrak{g}_{6,5} \cong M_8 \cong C_{12}$
$[x_1, x_2] = x_4 \quad [x_1, x_4] = x_5 \quad [x_2, x_3] = x_6 \quad [x_2, x_4] = x_6$.
SQ.I. $\quad i = 2 \quad r = 2 \quad c = 4 \quad Z = k[x_5, x_6]$
$F =< x_5, x_6 > \quad CPI =< x_3, x_4, x_5, x_6 >$
$T =< t_1, t_2 >, \quad t_1 = \text{diag}(1, 0, 1, 1, 2, 1), \quad t_2 = \text{diag}(2, -1, 1, 1, 3, 0)$.

13. $\mathfrak{g}_{6,7} \cong M_6 \cong C_{19}$
$[x_1, x_2] = x_4 \quad [x_1, x_3] = x_5 \quad [x_1, x_4] = x_6 \quad [x_2, x_3] = -x_6$.
SQ.I. $\quad i = 2 \quad r = 2 \quad c = 4 \quad Z = k[x_5, x_6]$
$F =< x_5, x_6 > \quad CPI =< x_3, x_4, x_5, x_6 >$
$T =< t_1, t_2 >, \quad t_1 = \text{diag}(0, 1, 0, 1, 0, 1), \quad t_2 = \text{diag}(1, -2, 2, -1, 3, 0)$.

14. $\mathfrak{g}_{6,8} \cong M_9 \cong C_{13}$
$[x_1, x_2] = x_4 \quad [x_1, x_4] = x_5 \quad [x_2, x_3] = x_5 \quad [x_2, x_4] = x_6$.
SQ.I. $\quad i = 2 \quad r = 2 \quad c = 4 \quad Z = k[x_5, x_6]$

$F =< x_5, x_6 > \quad CPI =< x_3, x_4, x_5, x_6 >$
$T =< t_1, t_2 >, \quad t_1 = \mathrm{diag}(1, 0, 2, 1, 2, 1), \quad t_2 = \mathrm{diag}(2, -1, 4, 1, 3, 0).$

15. $\mathfrak{g}_{6,9} \cong M_{14} \cong C_{16}$
$[x_1, x_2] = x_4 \quad [x_1, x_3] = x_5 \quad [x_2, x_5] = x_6 \quad [x_3, x_4] = x_6.$
$i = 2 \quad r = 3 \quad c = 4 \quad Z = k[x_6, x_1x_6 + x_4x_5]$
$F =< x_1, x_4, x_5, x_6 >= CPI$
$T =< t_1, t_2 >, \quad t_1 = \mathrm{diag}(1, \alpha, -\alpha, 1+\alpha, 1-\alpha, 1), \quad t_2 = \mathrm{diag}(1, 0, -1, 1, 0, 0).$

16. $\mathfrak{g}_{6,10} \cong M_{13} \cong C_{17}$
$[x_1, x_2] = x_4 \quad [x_1, x_3] = x_5 \quad [x_1, x_4] = x_6 \quad [x_3, x_5] = x_6.$
$i = 2 \quad r = 2 \quad c = 4 \quad Z = k[x_6, x_4^2 - 2x_2x_6]$
$F =< x_2, x_4, x_6 > \quad CPI =< x_2, x_4, x_5, x_6 >$
$T =< t_1, t_2 >, \quad t_1 = \mathrm{diag}(1, 1, 1, 2, 2, 3), \quad t_2 = \mathrm{diag}(2, -4, -1, -2, 1, 0).$

17. $\mathfrak{g}_{6,11} \cong M_{17} \cong C_{11}$
$[x_1, x_2] = x_4 \quad [x_1, x_4] = x_5 \quad [x_1, x_5] = x_6 \quad [x_2, x_3] = x_6.$
$i = 2 \quad r = 2 \quad c = 4 \quad Z = k[x_6, x_5^2 - 2x_4x_6]$
$F =< x_4, x_5, x_6 > \quad CPI =< x_3, x_4, x_5, x_6 >$
$T =< t_1, t_2 >, \quad t_1 = \mathrm{diag}(1, 0, 3, 1, 2, 3), \quad t_2 = \mathrm{diag}(-1, 3, -3, 2, 1, 0).$

18. $\mathfrak{g}_{6,13} \cong M_{16} \cong C_9$
$[x_1, x_2] = x_4 \quad [x_1, x_4] = x_5 \quad [x_1, x_5] = x_6 \quad [x_2, x_3] = x_5 \quad [x_3, x_4] = -x_6.$
$i = 2 \quad r = 2 \quad c = 4 \quad Z = k[x_6, x_5^3 - 3x_4x_5x_6 + 3x_2x_6^2]$
$F =< x_2, x_4, x_5, x_6 >= CPI$
$T =< t_1, t_2 >, \quad t_1 = \mathrm{diag}(1, -2, 2, -1, 0, 1), \quad t_2 = \mathrm{diag}(1, -3, 2, -2, -1, 0).$

19. $\mathfrak{g}_{6,14} \cong M_{11} \cong C_8$
$[x_1, x_2] = x_3 \quad [x_1, x_3] = x_4 \quad [x_1, x_4] = x_5 \quad [x_2, x_3] = x_6.$
SQ.I. $\quad i = 2 \quad r = 2 \quad c = 4 \quad Z = k[x_5, x_6]$
$F =< x_5, x_6 > \quad CPI =< x_3, x_4, x_5, x_6 >$
$T =< t_1, t_2 >, \quad t_1 = \mathrm{diag}(1, 0, 1, 2, 3, 1), \quad t_2 = \mathrm{diag}(2, -1, 1, 3, 5, 0).$

20. $\mathfrak{g}_{6,15} \cong M_{18}(-1) \cong C_7$
$[x_1, x_2] = x_3 \quad [x_1, x_3] = x_4 \quad [x_1, x_5] = x_6 \quad [x_2, x_3] = x_5 \quad [x_2, x_4] = x_6.$
$i = 2 \quad r = 2 \quad c = 4 \quad Z = k[x_6, x_3x_6 - x_4x_5]$
$F =< x_3, x_4, x_5, x_6 >= CPI$

21. $\mathfrak{g}_{6,18} \cong M_{21} \cong C_2$
$[x_1, x_2] = x_3 \quad [x_1, x_3] = x_4 \quad [x_1, x_4] = x_5 \quad [x_2, x_5] = x_6 \quad [x_3, x_4] = -x_6.$
$i = 2 \quad r = 2 \quad c = 4 \quad Z = k[x_6, x_4^2 - 2x_1x_6 - 2x_3x_5]$
$F =< x_1, x_3, x_4, x_5, x_6 >$ No CP's
$T =< t_1, t_2 >, \quad t_1 = \mathrm{diag}(1, -1, 0, 1, 2, 1), \quad t_2 = \mathrm{diag}(2, -3, -1, 1, 3, 0).$

$\dim \mathfrak{g} = 6, \quad i(\mathfrak{g}) >$ rank $\mathfrak{g}$

22. $\mathfrak{g}_{6,3} \cong M_3 \cong C_{21}$
$[x_1, x_2] = x_4 \quad [x_1, x_3] = x_5 \quad [x_2, x_3] = x_6.$
$i = 4 \quad r = 3 \quad c = 5 \quad Z = k[x_4, x_5, x_6, x_1x_6 - x_2x_5 + x_3x_4]$
$F = \mathfrak{g}_{6,3}$ No CP's.

23. $\mathfrak{g}_{6,6} \cong M_1 \cong C_{20}$
$[x_1, x_2] = x_4 \quad [x_2, x_3] = x_6 \quad [x_2, x_4] = x_5.$
$i = 4 \quad r = 3 \quad c = 5 \quad F =< x_1, x_3, x_4, x_5, x_6 >= CPI$
$Z = k[x_5, x_6, f_1, f_2, f_3], \quad f_1 = x_4^2 + 2x_1x_5, \quad f_2 = x_3x_5 - x_4x_6,$
$f_3 = 2x_1x_6^2 + 2x_3x_4x_6 - x_3^2x_5,$
Relation: $x_6^2 f_1 - f_2^2 - x_5 f_3 = 0 \quad Q(Z) = k(x_5, x_6, f_1, f_2).$

24. $\mathfrak{g}_{6,12} \cong M_{15} \cong C_{10}$
$[x_1, x_2] = x_4 \quad [x_1, x_4] = x_5 \quad [x_1, x_5] = x_6 \quad [x_2, x_3] = x_6 \quad [x_2, x_4] = x_6.$
$i = 2 \quad r = 1 \quad c = 4 \quad Z = k[x_6, x_5^2 + 2x_3x_6 - 2x_4x_6]$
$F =< x_3 - x_4, x_5, x_6 > \quad CPI =< x_3, x_4, x_5, x_6 >.$

25. $\mathfrak{g}_{6,16} \cong M_2 \cong C_5$ (6-dim standard filiform Lie algebra)
$[x_1, x_2] = x_3 \quad [x_1, x_3] = x_4 \quad [x_1, x_4] = x_5 \quad [x_1, x_5] = x_6.$
$i = 4 \quad r = 2 \quad c = 5 \quad F =< x_2, x_3, x_4, x_5, x_6 >= CPI$
$Z = k[x_6, f_1, f_2, f_3, f_4],$
$f_1 = x_5^2 - 2x_4x_6,$
$f_2 = x_5^3 - 3x_4x_5x_6 + 3x_3x_6^2,$
$f_3 = x_4^2 + 2x_2x_6 - 2x_3x_5,$
$f_4 = 2x_4^3 + 6x_2x_5^2 + 9x_3^2x_6 - 12x_2x_4x_6 - 6x_3x_4x_5.$
Relation: $f_1^3 - f_2^2 - 3x_6^2 f_1 f_3 + x_6^3 f_4 = 0$
$Q(Z) = k(x_6, f_1, f_2, f_3).$

26. $\mathfrak{g}_{6,17} \cong M_{19} \cong C_4$
$[x_1, x_2] = x_3 \quad [x_1, x_3] = x_4 \quad [x_1, x_4] = x_5 \quad [x_1, x_5] = x_6 \quad [x_2, x_3] = x_6.$

$i=2 \quad r=1 \quad c=4 \quad Z=k[x_6, x_5^2-2x_4x_6]$
$F=< x_4,x_5,x_6 > \quad CPI=< x_3,x_4,x_5,x_6 >$

27. $\mathfrak{g}_{6,19} \cong M_{20} \cong C_3$
$[x_1,x_2]=x_3 \quad [x_1,x_3]=x_4 \quad [x_1,x_4]=x_5 \quad [x_1,x_5]=x_6$
$[x_2,x_3]=x_5 \; [x_2,x_4]=x_6.$
$i=2 \quad r=1 \quad c=4 \quad Z=k[x_6, x_5^3-3x_4x_5x_6+3x_3x_6^2]$
$F=< x_3,x_4,x_5,x_6 >= CPI.$

28. $\mathfrak{g}_{6,20} \cong M_{22} \cong C_1$
$[x_1,x_2]=x_3 \quad [x_1,x_3]=x_4 \quad [x_1,x_4]=x_5 \quad [x_2,x_3]=x_5 \quad [x_2,x_5]=x_6$
$[x_3,x_4]=-x_6.$
$i=2 \quad r=1 \quad c=4 \quad Z=k[x_6, 2x_5^3+3x_4^2x_6-6x_3x_5x_6-6x_1x_6^2]$
$F=< x_1,x_3,x_4,x_5,x_6 >$ No CP's.

$\dim \mathfrak{g} = 7$, $\quad i(\mathfrak{g}) \leq \mathrm{rank}\, \mathfrak{g}$

29. $\mathfrak{g}_{7,1.03} \cong (13457G) \cong R_{26}$
$[x_1,x_2]=x_3 \quad [x_1,x_3]=x_4 \quad [x_1,x_4]=x_5 \quad [x_1,x_6]=x_7 \quad [x_2,x_3]=x_6$
$[x_2,x_4]=x_7 \quad [x_2,x_5]=x_7 \quad [x_3,x_4]=-x_7.$
SQ.I. $\quad i=1 \quad r=1 \quad c=4 \quad Z=k[x_7]$
$F=< x_7 > \quad CPI=< x_4,x_5,x_6,x_7 >$
$T=< t >, \quad t=\mathrm{diag}(0,1,1,1,1,2,2).$

30. $\mathfrak{g}_{7,1.1(i_\lambda)} \cong (123457I) \cong R_1^\lambda, \lambda \neq 0,1$
$[x_1,x_2]=x_3 \quad [x_1,x_3]=x_4 \quad [x_1,x_4]=x_5 \quad [x_1,x_5]=x_6 \quad [x_1,x_6]=x_7$
$[x_2,x_3]=x_5 \quad [x_2,x_4]=x_6 \quad [x_2,x_5]=\lambda x_7 \quad [x_3,x_4]=(1-\lambda)x_7.$
SQ.I. $\quad i=1 \quad r=1 \quad c=4 \quad Z=k[x_7]$
$F=< x_7 > \quad CPI=< x_4,x_5,x_6,x_7 >$
$T=< t >, \quad t=\mathrm{diag}(1,2,3,4,5,6,7).$

31. $\mathfrak{g}_{7,1.1(ii)} \cong (123457C) \cong R_3$
$[x_1,x_2]=x_3 \quad [x_1,x_3]=x_4 \quad [x_1,x_4]=x_5 \quad [x_1,x_5]=x_6 \quad [x_1,x_6]=x_7$
$[x_2,x_5]=x_7 \quad [x_3,x_4]=-x_7.$
SQ.I. $\quad i=1 \quad r=1 \quad c=4 \quad Z=k[x_7]$
$F=< x_7 > \quad CPI=< x_4,x_5,x_6,x_7 >$
$T=< t >, \quad t=\mathrm{diag}(1,2,3,4,5,6,7).$

32. $\mathfrak{g}_{7,1.1(iv)} \cong (12457I) \cong R_{22}$
$[x_1, x_2] = x_3 \quad [x_1, x_3] = x_4 \quad [x_1, x_5] = x_6 \quad [x_1, x_6] = x_7 \quad [x_2, x_3] = x_5$
$[x_2, x_4] = x_6 \quad [x_2, x_5] = x_7 \quad [x_3, x_4] = x_7.$
SQ.I. $\quad i = 1 \quad r = 1 \quad c = 4 \quad Z = k[x_7]$
$F =< x_7 > \quad CPI =< x_4, x_5, x_6, x_7 >$
$T =< t >, \quad t = \mathrm{diag}(1, 2, 3, 4, 5, 6, 7).$

33. $\mathfrak{g}_{7,1.1(v)} \cong (12357C) \cong R_{42}$
$[x_1, x_3] = x_4 \quad [x_1, x_4] = x_5 \quad [x_1, x_5] = x_6 \quad [x_1, x_6] = x_7 \quad [x_2, x_3] = x_5$
$[x_2, x_4] = x_6 \quad [x_2, x_5] = x_7 \quad [x_3, x_4] = -x_7.$
SQ.I. $\quad i = 1 \quad r = 1 \quad c = 4 \quad Z = k[x_7]$
$F =< x_7 > \quad CPI =< x_4, x_5, x_6, x_7 >$
$T =< t >, \quad t = \mathrm{diag}(1, 2, 3, 4, 5, 6, 7).$

34. $\mathfrak{g}_{7,1.1(vi)} \cong (13457E) \cong R_{38}$
$[x_1, x_2] = x_3 \quad [x_1, x_3] = x_4 \quad [x_1, x_4] = x_5 \quad [x_1, x_6] = x_7 \quad [x_2, x_3] = x_5$
$[x_2, x_5] = x_7 \quad [x_3, x_4] = -x_7.$
SQ.I. $\quad i = 1 \quad r = 1 \quad c = 4 \quad Z = k[x_7]$
$F =< x_7 > \quad CPI =< x_4, x_5, x_6, x_7 >$
$T =< t >, \quad t = \mathrm{diag}(1, 2, 3, 4, 5, 6, 7).$

35. $\mathfrak{g}_{7,1.2(i_\lambda)} \cong (1357S)(\xi \neq 0) \cong R_{52}^{\lambda}, \lambda \neq 1$
$[x_1, x_2] = x_4 \quad [x_1, x_3] = x_6 \quad [x_1, x_4] = x_5 \quad [x_1, x_5] = x_7 \quad [x_2, x_3] = \lambda x_5$
$[x_2, x_4] = x_6 \quad [x_2, x_6] = x_7 \quad [x_3, x_4] = (1 - \lambda) x_7.$
SQ.I. $\quad i = 1 \quad r = 1 \quad c = 4 \quad Z = k[x_7]$
$F =< x_7 > \quad CPI =< x_4, x_5, x_6, x_7 >$
$T =< t >, \quad t = \mathrm{diag}(1, 1, 2, 2, 3, 3, 4).$

36. $\mathfrak{g}_{7,1.2(ii)} \cong (1357S)(\xi = 0)$
$[x_1, x_2] = x_4 \quad [x_1, x_4] = x_5 \quad [x_1, x_5] = x_7 \quad [x_1, x_6] = x_7 \quad [x_2, x_3] = x_6$
$[x_2, x_4] = x_6 \quad [x_2, x_5] = x_7 \quad [x_3, x_4] = -x_7.$
SQ.I. $\quad i = 1 \quad r = 1 \quad c = 4 \quad Z = k[x_7]$
$F =< x_7 > \quad CPI =< x_4, x_5, x_6, x_7 >$
$T =< t >, \quad t = \mathrm{diag}(1, 1, 2, 2, 3, 3, 4).$

37. $\mathfrak{g}_{7,1.2(iv)} \cong (1357H) \cong R_{72}$
$[x_1, x_2] = x_4 \quad [x_1, x_4] = x_6 \quad [x_1, x_5] = -x_7 \quad [x_1, x_6] = x_7 \quad [x_2, x_3] = x_5$
$[x_2, x_5] = x_7 \quad [x_3, x_4] = x_7.$

SQ.I. $\quad i=1 \quad r=1 \quad c=4 \quad Z=k[x_7]$
$F=< x_7 > \quad CPI=< x_4, x_5, x_6, x_7 >$
$T=< t >, \quad t=\text{diag}(1,1,2,2,3,3,4)$.

38. $\mathfrak{g}_{7,1.3(i_\lambda)} \cong (1357N)(\xi \neq 0) \cong R_{62}, \lambda \neq 0$
$[x_1,x_2]=x_4 \quad [x_1,x_3]=x_5 \quad [x_1,x_4]=x_6 \quad [x_1,x_6]=x_7 \quad [x_2,x_3]=x_6$
$[x_2,x_4]=\lambda x_7 \quad [x_2,x_5]=x_7 \quad [x_3,x_5]=x_7$.
SQ.I. $\quad i=1 \quad r=1 \quad c=4 \quad Z=k[x_7]$
$F=< x_7 > \quad CPI=< x_4, x_5, x_6, x_7 >$
$T=< t >, \quad t=\text{diag}(1,2,2,3,3,4,5)$.

39. $\mathfrak{g}_{7,1.3(ii)} \cong (1357L) \cong R_{63}$
$[x_1,x_2]=x_4 \quad [x_1,x_3]=x_5 \quad [x_1,x_4]=x_6 \quad [x_1,x_6]=x_7 \quad [x_2,x_3]=x_6$
$[x_2,x_4]=x_7 \quad [x_2,x_5]=x_7/2 \quad [x_3,x_4]=-x_7/2$.
SQ.I. $\quad i=1 \quad r=1 \quad c=4 \quad Z=k[x_7]$
$F=< x_7 > \quad CPI=< x_4, x_5, x_6, x_7 >$
$T=< t >, \quad t=\text{diag}(1,2,2,3,3,4,5)$.

40. $\mathfrak{g}_{7,1.3(iii)} \cong (1357F) \cong R_{65}$
$[x_1,x_2]=x_4 \quad [x_1,x_3]=x_5 \quad [x_1,x_4]=x_6 \quad [x_1,x_6]=x_7 \quad [x_2,x_4]=x_7$
$[x_3,x_5]=x_7$.
SQ.I. $\quad i=1 \quad r=1 \quad c=4 \quad Z=k[x_7]$
$F=< x_7 > \quad CPI=< x_4, x_5, x_6, x_7 >$
$T=< t >, \quad t=\text{diag}(1,2,2,3,3,4,5)$.

41. $\mathfrak{g}_{7,1.3(v)} \cong (1357C) \cong R_{99}$
$[x_1,x_2]=x_4 \quad [x_1,x_4]=x_6 \quad [x_1,x_6]=x_7 \quad [x_2,x_3]=x_6 \quad [x_2,x_4]=x_7$
$[x_3,x_4]=-x_7 \quad [x_3,x_5]=-x_7$.
SQ.I. $\quad i=1 \quad r=1 \quad c=4 \quad Z=k[x_7]$
$F=< x_7 > \quad CPI=< x_4, x_5, x_6, x_7 >$
$T=< t >, \ t=\text{diag}(1,2,2,3,3,4,5)$.

42. $\mathfrak{g}_{7,1.8} \cong (1357J) \cong R_{70}$
$[x_1,x_2]=x_4 \quad [x_1,x_4]=x_6 \quad [x_1,x_6]=x_7 \quad [x_2,x_3]=x_5 \quad [x_2,x_4]=x_7$
$[x_3,x_5]=x_7$.
SQ.I. $\quad i=1 \quad r=1 \quad c=4 \quad Z=k[x_7]$
$F=< x_7 > \quad CPI=< x_4, x_5, x_6, x_7 >$

$T =< t >, \quad t = \mathrm{diag}(2, 4, 3, 6, 7, 8, 10).$

43. $\mathfrak{g}_{7,1.11} \cong (12457E) \cong R_{46}$
$[x_1, x_2] = x_4 \quad [x_1, x_4] = x_5 \quad [x_1, x_5] = x_6 \quad [x_1, x_6] = x_7 \quad [x_2, x_3] = x_6$
$[x_2, x_4] = x_6 \quad [x_2, x_5] = x_7 \quad [x_3, x_4] = -x_7.$
SQ.I. $\quad i = 1 \quad r = 1 \quad c = 4 \quad Z = k[x_7]$
$F =< x_7 > \quad CPI =< x_4, x_5, x_6, x_7 >$
$T =< t >, \quad t = \mathrm{diag}(1, 2, 3, 3, 4, 5, 6).$

44. $\mathfrak{g}_{7,1.20} \cong (12457D) \cong R_{35}$
$[x_1, x_2] = x_3 \quad [x_1, x_5] = x_6 \quad [x_1, x_6] = x_7 \quad [x_2, x_3] = x_4 \quad [x_2, x_4] = x_6$
$[x_2, x_5] = x_7 \quad [x_3, x_4] = x_7.$
SQ.I. $\quad i = 1 \quad r = 1 \quad c = 4 \quad Z = k[x_7]$
$F =< x_7 > \quad CPI =< x_4, x_5, x_6, x_7 >$
$T =< t >, \quad t = \mathrm{diag}(1, 2, 3, 5, 6, 7, 8).$

45. $\mathfrak{g}_{7,2.1(i_\lambda)} \cong (1357M) \cong R^{\lambda}_{64}, \lambda \neq 0, 1$
$[x_1, x_2] = x_4 \quad [x_1, x_3] = x_5 \quad [x_1, x_4] = x_6 \quad [x_1, x_6] = x_7 \quad [x_2, x_3] = x_6$
$[x_2, x_5] = \lambda x_7 \quad [x_3, x_4] = (\lambda - 1)x_7.$
SQ.I. $\quad i = 1 \quad r = 2 \quad c = 4 \quad Z = k[x_7]$
$F =< x_7 > \quad CPI =< x_4, x_5, x_6, x_7 >$
$T =< t >, \quad t = \mathrm{diag}(\alpha, 1 - 3\alpha, 2\alpha, 1 - 2\alpha, 3\alpha, 1 - \alpha, 1).$

46. $\mathfrak{g}_{7,2.1(ii)} \cong (1357D) \cong R_{69}$
$[x_1, x_2] = x_4 \quad [x_1, x_3] = x_5 \quad [x_1, x_4] = x_6 \quad [x_1, x_6] = x_7 \quad [x_2, x_5] = x_7$
$[x_3, x_4] = x_7.$
SQ.I. $\quad i = 1 \quad r = 2 \quad c = 4 \quad Z = k[x_7]$
$F =< x_7 > \quad CPI =< x_4, x_5, x_6, x_7 >$
$T =< t >, \quad t = \mathrm{diag}(\alpha, 1 - 3\alpha, 2\alpha, 1 - 2\alpha, 3\alpha, 1 - \alpha, 1).$

47. $\mathfrak{g}_{7,2.1(iii)} \cong (1357A) \cong R_{101}$
$[x_1, x_2] = x_4 \quad [x_1, x_4] = x_6 \quad [x_1, x_6] = x_7 \quad [x_2, x_3] = x_6 \quad [x_2, x_5] = x_7$
$[x_3, x_4] = -x_7.$
SQ.I. $\quad i = 1 \quad r = 2 \quad c = 4 \quad Z = k[x_7]$
$F =< x_7 > \quad CPI =< x_4, x_5, x_6, x_7 >$
$T =< t >, \quad t = \mathrm{diag}(\alpha, 1 - 3\alpha, 2\alpha, 1 - 2\alpha, 3\alpha, 1 - \alpha, 1).$

48. $\mathfrak{g}_{7,2.1(iv)} \cong (137D)$
$[x_1,x_3]=x_5 \quad [x_1,x_4]=x_6 \quad [x_1,x_6]=x_7 \quad [x_2,x_3]=x_6 \quad [x_2,x_5]=x_7$
$[x_3,x_4]=x_7$.
SQ.I. $\quad i=1 \quad r=2 \quad c=4 \quad Z=k[x_7]$
$F=<x_7> \quad CPI=<x_4,x_5,x_6,x_7>$
$T=<t>, \quad t=\mathrm{diag}(\alpha,1-3\alpha,2\alpha,1-2\alpha,3\alpha,1-\alpha,1)$.

49. $\mathfrak{g}_{7,2.2} \cong (147D) \cong R_{95}$
$[x_1,x_2]=x_5 \quad [x_1,x_3]=x_6 \quad [x_1,x_4]=2x_7 \quad [x_2,x_3]=x_4 \quad [x_2,x_6]=x_7$
$[x_3,x_5]=-x_7 \quad [x_3,x_6]=x_7$.
SQ.I. $\quad i=1 \quad r=2 \quad c=4 \quad Z=k[x_7]$
$F=<x_7> \quad CPI=<x_4,x_5,x_6,x_7>$
$T=<t>, \quad t=\mathrm{diag}(1-2\alpha,\alpha,\alpha,2\alpha,1-\alpha,1-\alpha,1)$.

50. $\mathfrak{g}_{7,2.10} \cong (13457C) \cong R_{39}$
$[x_1,x_2]=x_4 \quad [x_1,x_3]=x_7 \quad [x_1,x_4]=x_5 \quad [x_1,x_5]=x_6 \quad [x_2,x_6]=x_7$
$[x_4,x_5]=-x_7$.
SQ.I. $\quad i=1 \quad r=2 \quad c=4 \quad Z=k[x_7]$
$F=<x_7> \quad CPI=<x_3,x_5,x_6,x_7>$
$T=<t>, \quad t=\mathrm{diag}(-1+2\alpha,2-3\alpha,2-2\alpha,1-\alpha,\alpha,-1+3\alpha,1)$.

51. $\mathfrak{g}_{7,2.23} \cong (137B) \cong R_{107}$
$[x_1,x_4]=x_6 \quad [x_1,x_6]=x_7 \quad [x_2,x_3]=x_5 \quad [x_2,x_5]=x_7 \quad [x_3,x_4]=x_7$.
SQ.I. $\quad i=1 \quad r=2 \quad c=4 \quad Z=k[x_7]$
$F=<x_7> \quad CPI=<x_4,x_5,x_6,x_7>$
$T=<t>, \quad t=\mathrm{diag}(\alpha,\frac{1}{2}-\alpha,2\alpha,1-2\alpha,\frac{1}{2}+\alpha,1-\alpha,1)$.

52. $\mathfrak{g}_{7,2.28} \cong (147B) \cong R_{112}$
$[x_1,x_2]=x_5 \quad [x_1,x_3]=x_6 \quad [x_1,x_6]=x_7 \quad [x_2,x_5]=x_7 \quad [x_3,x_4]=x_7$.
SQ.I. $\quad i=1 \quad r=2 \quad c=4 \quad Z=k[x_7]$
$F=<x_7> \quad CPI=<x_4,x_5,x_6,x_7>$
$T=<t>, \quad t=\mathrm{diag}(1-2\alpha,\alpha,-1+4\alpha,2-4\alpha,1-\alpha,2\alpha,1)$.

53. $\mathfrak{g}_{7,2.30} \cong (1457B) \cong R_{102}$
$[x_1,x_2]=x_5 \quad [x_1,x_5]=x_6 \quad [x_1,x_6]=x_7 \quad [x_2,x_5]=x_7 \quad [x_3,x_4]=x_7$.
SQ.I. $\quad i=1 \quad r=2 \quad c=4 \quad Z=k[x_7]$
$F=<x_7> \quad CPI=<x_4,x_5,x_6,x_7>$
$T=<t>, \quad t=\mathrm{diag}(\frac{1}{5},\frac{2}{5},\alpha,1-\alpha,\frac{3}{5},\frac{4}{5},1)$.

54. $\mathfrak{g}_{7,2.37} \cong (1357R)$
$[x_1, x_2] = x_4 \quad [x_1, x_3] = x_5 \quad [x_1, x_4] = x_5 \quad [x_1, x_6] = x_7 \quad [x_2, x_4] = x_6$
$[x_2, x_5] = x_7 \quad [x_3, x_4] = x_7.$
SQ.I. $\quad i = 1 \quad r = 2 \quad c = 4 \quad Z = k[x_7]$
$F =< x_7 > \quad CPI =< x_4, x_5, x_6, x_7 >$
$T =< t >, \quad t = \mathrm{diag}(\alpha, \frac{1}{2} - \alpha, \frac{1}{2}, \frac{1}{2}, \frac{1}{2} + \alpha, 1 - \alpha, 1).$

55. $\mathfrak{g}_{7,3.1(i_\lambda)} \cong (147E) \cong R_{93}^{\lambda}, \lambda \neq 0, 1$
$[x_1, x_2] = x_4 \quad [x_1, x_3] = x_5 \quad [x_1, x_6] = x_7 \quad [x_2, x_3] = x_6 \quad [x_2, x_5] = \lambda x_7$
$[x_3, x_4] = (\lambda - 1)x_7.$
SQ.I. $\quad i = 1 \quad r = 3 \quad c = 4 \quad Z = k[x_7]$
$F =< x_7 > \quad CPI =< x_4, x_5, x_6, x_7 >$
$T =< t >, \quad t = \mathrm{diag}(\alpha, \beta, 1 - \alpha - \beta, \alpha + \beta, 1 - \beta, 1 - \alpha, 1).$

56. $\mathfrak{g}_{7,3.1(i_\lambda)}, \lambda = 0 \cong (247P)$
$[x_1, x_2] = x_4 \quad [x_1, x_3] = x_5 \quad [x_1, x_6] = x_7 \quad [x_2, x_3] = x_6 \quad [x_3, x_4] = -x_7.$
$i = 3 \quad r = 3 \quad c = 5 \quad Z = k[x_5, x_7, x_2x_7 - x_4x_6]$
$F =< x_2, x_4, x_5, x_6, x_7 >= CPI$

57. $\mathfrak{g}_{7,3.1(i_\lambda)}, \lambda = 1$
$[x_1, x_2] = x_4 \quad [x_1, x_3] = x_5 \quad [x_1, x_6] = x_7 \quad [x_2, x_3] = x_6 \quad [x_2, x_5] = x_7.$
$i = 3 \quad r = 3 \quad c = 5 \quad Z = k[x_4, x_7, x_3x_7 - x_5x_6]$
$F =< x_3, x_4, x_5, x_6, x_7 >= CPI$

58. $\mathfrak{g}_{7,3.1(iii)} \cong (147A) \cong R_{113}$
$[x_1, x_2] = x_4 \quad [x_1, x_3] = x_5 \quad [x_1, x_6] = x_7 \quad [x_2, x_5] = x_7 \quad [x_3, x_4] = x_7.$
SQ.I. $\quad i = 1 \quad r = 3 \quad c = 4 \quad Z = k[x_7]$
$F =< x_7 > \quad CPI =< x_4, x_5, x_6, x_7 >$
$T =< t >, \quad t = \mathrm{diag}(\alpha, \beta, 1 - \alpha - \beta, \alpha + \beta, 1 - \beta, 1 - \alpha, 1).$

59. $\mathfrak{g}_{7,3.3} \cong (2457B) \cong R_{80}$
$[x_1, x_2] = x_4 \quad [x_1, x_4] = x_6 \quad [x_1, x_6] = x_7 \quad [x_2, x_3] = x_5.$
$i = 3 \quad r = 3 \quad c = 5 \quad Z = k[x_5, x_7, x_6^2 - 2x_4x_7]$
$F =< x_4, x_5, x_6, x_7 > \quad CPI =< x_3, x_4, x_5, x_6, x_7 >$
$T =< t_1, t_2, t_3 >, \quad t_1 = \mathrm{diag}(0, 1, 0, 1, 1, 1, 1),$
$t_2 = \mathrm{diag}(-1, 3, 0, 2, 3, 1, 0), \quad t_3 = \mathrm{diag}(0, 0, 1, 0, 1, 0, 0).$

60. $\mathfrak{g}_{7,3.4} \cong (247F) \cong R_{83}$
$[x_1, x_2] = x_4 \quad [x_1, x_3] = x_5 \quad [x_2, x_4] = x_6 \quad [x_3, x_5] = x_7.$

$i = 3 \quad r = 3 \quad c = 5 \quad Z = k[x_6, x_7, x_4^2 x_7 + 2x_1x_6x_7 + x_5^2 x_6]$
$F =< x_1, x_4, x_5, x_6, x_7 >= CPI$
$T =< t_1, t_2, t_3 >, \quad t_1 = \mathrm{diag}(1, 0, 0, 1, 1, 1, 1),$
$t_2 = \mathrm{diag}(0, 1, 0, 1, 0, 2, 0), \quad t_3 = \mathrm{diag}(2, -1, -1, 1, 1, 0, 0).$

61. $\mathfrak{g}_{7,3.5} \cong (247I) \cong R_{86}$
$[x_1, x_2] = x_4 \quad [x_1, x_3] = x_5 \quad [x_2, x_4] = x_6 \quad [x_2, x_5] = x_7 \quad [x_3, x_4] = x_7.$
$i = 3 \quad r = 3 \quad c = 5 \quad Z = k[x_6, x_7, 2x_1x_7^2 + 2x_4x_5x_7 - x_5^2 x_6]$
$F =< x_1, x_4, x_5, x_6, x_7 >= CPI$
$T =< t_1, t_2, t_3 >, \quad t_1 = \mathrm{diag}(1, 0, 0, 1, 1, 1, 1),$
$t_2 = \mathrm{diag}(1, -1, 0, 0, 1, -1, 0), \quad t_3 = \mathrm{diag}(2, -1, -1, 1, 1, 0, 0).$

62. $\mathfrak{g}_{7,3.6} \cong (357A) \cong R_{98}$
$[x_1, x_2] = x_4 \quad [x_1, x_3] = x_5 \quad [x_1, x_5] = x_7 \quad [x_2, x_3] = x_6.$
SQ.I. $\quad i = 3 \quad r = 3 \quad c = 5 \quad Z = k[x_4, x_6, x_7]$
$F =< x_4, x_6, x_7 > \quad CPI =< x_3, x_4, x_5, x_6, x_7 >$
$T =< t_1, t_2, t_3 >, \quad t_1 = \mathrm{diag}(0, 0, 1, 0, 1, 1, 1),$
$t_2 = \mathrm{diag}(0, 1, 0, 1, 0, 1, 0), \quad t_3 = \mathrm{diag}(1, 2, -2, 3, -1, 0, 0).$

63. $\mathfrak{g}_{7,3.7} \cong (257H) \cong R_{117}$
$[x_1, x_2] = x_5 \quad [x_1, x_5] = x_6 \quad [x_2, x_4] = x_6 \quad [x_3, x_4] = -x_7.$
$i = 3 \quad r = 3 \quad c = 5 \quad Z = k[x_6, x_7, 2x_3x_6^2 + 2x_2x_6x_7 - x_5^2 x_7]$
$F =< x_2, x_3, x_5, x_6, x_7 >= CPI$
$T =< t_1, t_2, t_3 >, \quad t_1 = diag(0, 0, 1, 0, 0, 0, 1),$
$t_2 = \mathrm{diag}(0, 1, 0, 0, 1, 1, 0), \quad t_3 = \mathrm{diag}(1, -2, -2, 2, -1, 0, 0).$

64. $\mathfrak{g}_{7,3.8} \cong (257A) \cong R_{121}$
$[x_1, x_2] = x_5 \quad [x_1, x_3] = x_6 \quad [x_1, x_5] = x_7 \quad [x_2, x_4] = x_7.$
$i = 3 \quad r = 3 \quad c = 5 \quad Z = k[x_6, x_7, x_3x_7 - x_5x_6]$
$F =< x_3, x_5, x_6, x_7 > \quad CPI =< x_3, x_4, x_5, x_6, x_7 >$
$T =< t_1, t_2, t_3 >, \quad t_1 = \mathrm{diag}(1, 0, 0, 2, 1, 1, 2)$
$t_2 = \mathrm{diag}(0, 0, 1, 0, 0, 1, 0), \quad t_3 = \mathrm{diag}(1, -2, -1, 2, -1, 0, 0).$

65. $\mathfrak{g}_{7,3.9} \cong (257C) \cong R_{123}$
$[x_1, x_2] = x_5 \quad [x_1, x_5] = -x_7 \quad [x_2, x_3] = x_6 \quad [x_2, x_4] = x_7.$
$i = 3 \quad r = 3 \quad c = 5 \quad Z = k[x_6, x_7, x_3x_7 - x_4x_6]$
$F =< x_3, x_4, x_6, x_7 > \quad CPI =< x_3, x_4, x_5, x_6, x_7 >$

$T =< t_1, t_2, t_3 >, \quad t_1 = \text{diag}(0,1,0,0,1,1,1),$
$t_2 = \text{diag}(0,0,1,0,0,1,0), \quad t_3 = \text{diag}(1,-2,2,2,-1,0,0).$

66. $\mathfrak{g}_{7,3.10} \cong (137C) \cong R_{111}$
$[x_1,x_2]=x_5 \quad [x_1,x_3]=x_6 \quad [x_2,x_4]=x_6 \quad [x_2,x_6]=x_7 \quad [x_3,x_5]=x_7.$
$i=3 \quad r=3 \quad c=5 \quad Z=k[x_7, x_6^2-2x_4x_7, x_1x_7+x_5x_6]$
$F =< x_1,x_4,x_5,x_6,x_7 >= CPI$
$T =< t_1, t_2, t_3 >, \quad t_1 = \text{diag}(1,0,0,1,1,1,1),$
$t_2 = \text{diag}(0,-1,1,2,-1,1,0), \quad t_3 = \text{diag}(1,0,-1,0,1,0,0).$

67. $\mathfrak{g}_{7,3.11} \cong (257B) \cong R_{119}$
$[x_1,x_2]=x_5 \quad [x_1,x_3]=x_6 \quad [x_1,x_5]=x_7 \quad [x_2,x_4]=x_6.$
$i=3 \quad r=3 \quad c=5 \quad Z=k[x_6,x_7,x_3x_7-x_5x_6]$
$F =< x_3,x_5,x_6,x_7 > \quad CPI =< x_3,x_4,x_5,x_6,x_7 >$
$T =< t_1, t_2, t_3 >, \quad t_1 = \text{diag}(0,1,0,-1,1,0,1),$
$t_2 = \text{diag}(0,0,1,1,0,1,0), \quad t_3 = \text{diag}(-1,2,1,-2,1,0,0).$

68. $\mathfrak{g}_{7,3.12} \cong (37D) \cong R_{124}$
$[x_1,x_2]=x_5 \quad [x_1,x_3]=x_6 \quad [x_2,x_4]=x_6 \quad [x_3,x_4]=x_7.$
SQ.I. $\quad i=3 \quad r=3 \quad c=5 \quad Z=k[x_5,x_6,x_7]$
$F =< x_5,x_6,x_7 > \quad CPI =< x_1,x_4,x_5,x_6,x_7 >.$

69. $\mathfrak{g}_{7,3.13} \cong (257K) \cong R_{105}$
$[x_1,x_2]=x_5 \quad [x_1,x_5]=x_6 \quad [x_2,x_5]=x_7 \quad [x_3,x_4]=x_7.$
$i=3 \quad r=3 \quad c=5 \quad Z=k[x_6,x_7,x_5^2+2x_1x_7-2x_2x_6]$
$F =< x_1,x_2,x_5,x_6,x_7 >$ No CP's.

70. $\mathfrak{g}_{7,3.14} \cong (257F) \cong R_{120}$
$[x_1,x_2]=x_5 \quad [x_1,x_3]=x_6 \quad [x_1,x_6]=x_7 \quad [x_2,x_4]=x_7.$
$i=3 \quad r=3 \quad c=5 \quad Z=k[x_5,x_7,x_6^2-2x_3x_7]$
$F =< x_3,x_5,x_6,x_7 > \quad CPI =< x_3,x_4,x_5,x_6,x_7 >$
$T =< t_1, t_2, t_3 >, \quad t_1 = \text{diag}(0,0,1,1,0,1,1),$
$t_2 = \text{diag}(-1,0,2,0,-1,1,0), \quad t_3 = \text{diag}(0,1,0,-1,1,0,0).$

71. $\mathfrak{g}_{7,3.15} \cong (257E) \cong R_{122}$
$[x_1,x_2]=x_5 \quad [x_1,x_3]=x_6 \quad [x_2,x_5]=x_7 \quad [x_3,x_4]=x_7.$
$i=3 \quad r=3 \quad c=5 \quad Z=k[x_6,x_7,x_5^2+2x_1x_7+2x_4x_6]$
$F =< x_1,x_4,x_5,x_6,x_7 >= CPI$

$T =< t_1, t_2, t_3 >, \quad t_1 = \mathrm{diag}(1, 0, 0, 1, 1, 1, 1),$
$t_2 = \mathrm{diag}(0, 0, 1, -1, 0, 1, 0), \quad t_3 = \mathrm{diag}(2, -1, -2, 2, 1, 0, 0).$

72. $\mathfrak{g}_{7,3.16} \cong (137A) \cong R_{108}$
$[x_1, x_2] = x_5 \quad [x_1, x_5] = x_7 \quad [x_3, x_4] = x_6 \quad [x_3, x_6] = x_7.$
$i = 3 \quad r = 3 \quad c = 5 \quad Z = k[x_7, x_5^2 - 2x_2x_7, x_6^2 - 2x_4x_7]$
$F =< x_2, x_4, x_5, x_6, x_7 >= CPI$
$T =< t_1, t_2, t_3 >, \quad t_1 = \mathrm{diag}(0, 1, 0, 1, 1, 1, 1),$
$t_2 = \mathrm{diag}(0, 0, -1, 2, 0, 1, 0), \quad t_3 = \mathrm{diag}(1, -2, 0, 0, -1, 0, 0).$

73. $\mathfrak{g}_{7,3.17} \cong (1457A) \cong R_{103}$
$[x_1, x_2] = x_5 \quad [x_1, x_5] = x_6 \quad [x_1, x_6] = x_7 \quad [x_3, x_4] = x_7.$
$i = 3 \quad r = 3 \quad c = 5 \quad F =< x_2, x_5, x_6, x_7 >$
$CPI =< x_2, x_4, x_5, x_6, x_7 > \quad Z = k[x_7, f_1, f_2, f_3]$
$f_1 = 2x_5x_7 - x_6^2,$
$f_2 = 3x_2x_7^2 - 3x_5x_6x_7 + x_6^3,$
$f_3 = 9x_2^2x_7^2 - 18x_2x_5x_6x_7 + 6x_2x_6^3 + 8x_5^3x_7 - 3x_5^2x_6^2.$
Relation: $f_1^3 + f_2^2 - x_7^2 f_3 = 0$
$Q(Z) = k(x_7, f_1, f_2).$ Note that Z is isomorphic with $Z(U(\mathfrak{g}_{5,5}))$.

74. $\mathfrak{g}_{7,3.18} \cong (157) \cong R_{129}$
$[x_1, x_2] = x_6 \quad [x_1, x_6] = x_7 \quad [x_2, x_5] = x_7 \quad [x_3, x_4] = x_7.$
SQ.I. $i = 1 \quad r = 3 \quad c = 4 \quad Z = k[x_7]$
$F =< x_7 > \quad CPI =< x_4, x_5, x_6, x_7 >$
$T =< t >, \quad t = \mathrm{diag}(\alpha, 1 - 2\alpha, \beta, 1 - \beta, 2\alpha, 1 - \alpha, 1).$

75. $\mathfrak{g}_{7,3.19} \cong (27B) \cong R_{130}$
$[x_1, x_2] = x_6 \quad [x_1, x_3] = x_7 \quad [x_3, x_4] = x_6 \quad [x_4, x_5] = x_7.$
$i = 3 \quad r = 3 \quad c = 5 \quad Z = k[x_6, x_7, x_2x_7^2 - x_3x_6x_7 - x_5x_6^2]$
$F =< x_2, x_3, x_5, x_6, x_7 >= CPI$
$T =< t_1, t_2, t_3 >, \quad t_1 = \mathrm{diag}(0, 0, 1, -1, 2, 0, 1),$
$t_2 = \mathrm{diag}(0, 1, 0, 1, -1, 1, 0), \quad t_3 = \mathrm{diag}(1, -1, -1, 1, -1, 0, 0).$

76. $\mathfrak{g}_{7,3.21} \cong (247B)$
$[x_1, x_2] = x_4 \quad [x_1, x_3] = x_5 \quad [x_1, x_4] = x_6 \quad [x_3, x_5] = x_7.$
$i = 3 \quad r = 3 \quad c = 5 \quad Z = k[x_6, x_7, x_4^2 - 2x_2x_6]$
$F =< x_2, x_4, x_6, x_7 > \quad CPI =< x_2, x_4, x_5, x_6, x_7 >$

$T =< t_1, t_2, t_3 >, \quad t_1 = \mathrm{diag}(1, 0, 0, 1, 1, 2, 1),$
$t_2 = \mathrm{diag}(0, 1, 0, 1, 0, 1, 0), \quad t_3 = \mathrm{diag}(2, -4, -1, -2, 1, 0, 0).$

77. $\mathfrak{g}_{7,3.22} \cong (247D)$
$[x_1, x_2] = x_4 \quad [x_1, x_3] = x_5 \quad [x_1, x_5] = x_7 \quad [x_2, x_5] = x_6 \quad [x_3, x_4] = x_6.$
$i = 3 \quad r = 3 \quad c = 5 \quad Z = k[x_6, x_7, x_1x_6 - x_2x_7 + x_4x_5]$
$F =< x_1, x_2, x_4, x_5, x_6, x_7 >$ No CP's
$T =< t_1, t_2, t_3 >, \quad t_1 = \mathrm{diag}(0, 0, 1, 0, 1, 1, 1),$
$t_2 = \mathrm{diag}(0, 1, 0, 1, 0, 1, 0), \quad t_3 = \mathrm{diag}(1, 1, -2, 2, -1, 0, 0).$

78. $\mathfrak{g}_{7,3.23} \cong (357B)$
$[x_1, x_2] = x_3 \quad [x_1, x_3] = x_5 \quad [x_1, x_4] = x_7 \quad [x_2, x_3] = x_6.$
SQ.I. $i = 3 \quad r = 3 \quad c = 5 \quad Z = k[x_5, x_6, x_7]$
$F =< x_5, x_6, x_7 > \quad CPI =< x_3, x_4, x_5, x_6, x_7 >$
$T =< t_1, t_2, t_3 >, \quad t_1 = \mathrm{diag}(0, 0, 0, 1, 0, 0, 1),$
$t_2 = \mathrm{diag}(0, 1, 1, 0, 1, 2, 0), \quad t_3 = \mathrm{diag}(2, -1, 1, -2, 3, 0, 0).$

79. $\mathfrak{g}_{7,3.24} \cong (37C)$
$[x_1, x_2] = x_5 \quad [x_2, x_3] = x_6 \quad [x_2, x_4] = x_7 \quad [x_3, x_4] = x_5.$
SQ.I. $i = 3 \quad r = 3 \quad c = 5 \quad Z = k[x_5, x_6, x_7]$
$F =< x_5, x_6, x_7 > \quad CPI =< x_1, x_4, x_5, x_6, x_7 >$
$T =< t_1, t_2, t_3 >, \quad t_1 = \mathrm{diag}(0, 0, -1, 1, 0, -1, 1),$
$t_2 = \mathrm{diag}(1, 0, 1, 0, 1, 1, 0), \quad t_3 = \mathrm{diag}(3, -1, 1, 1, 2, 0, 0).$

80. $\mathfrak{g}_{7,4.1} \cong (37B) \cong R_{126}$
$[x_1, x_2] = x_5 \quad [x_1, x_3] = x_6 \quad [x_3, x_4] = x_7.$
SQ.I. $i = 3 \quad r = 4 \quad c = 5 \quad Z = k[x_5, x_6, x_7]$
$F =< x_5, x_6, x_7 > \quad CPI =< x_2, x_4, x_5, x_6, x_7 >$
$T =< t_1, t_2, t_3 >, \quad t_1 = \mathrm{diag}(\alpha, 1 - \alpha, 1 - \alpha, \alpha, 1, 1, 1),$
$t_2 = \mathrm{diag}(1, 0, 0, 0, 1, 1, 0), \quad t_3 = \mathrm{diag}(1, 0, -1, 1, 1, 0, 0).$

81. $\mathfrak{g}_{7,4.3} \cong (27A) \cong R_{131}$
$[x_1, x_2] = x_6 \quad [x_3, x_5] = x_6 \quad [x_4, x_5] = x_7.$
$i = 3 \quad r = 4 \quad c = 5 \quad Z = k[x_6, x_7, x_3x_7 - x_4x_6]$
$F =< x_3, x_4, x_6, x_7 > \quad CPI =< x_1, x_3, x_4, x_6, x_7 >$
$T =< t_1, t_2, t_3 >, \quad t_1 = \mathrm{diag}(\alpha, 1 - \alpha, 1, 1, 0, 1, 1),$
$t_2 = \mathrm{diag}(1, 0, 1, 0, 0, 1, 0), \quad t_3 = \mathrm{diag}(0, 0, 1, 1, -1, 0, 0).$

82. $\mathfrak{g}_{7,4.4} \cong (17) \cong R_{132}$ (7-dim Heisenberg Lie algebra)
$[x_1, x_4] = x_7 \quad [x_2, x_5] = x_7 \quad [x_3, x_6] = x_7.$
SQ.I. $\quad i = 1 \quad r = 4 \quad c = 4 \quad Z = k[x_7]$
$F =< x_7 > \quad CPI =< x_4, x_5, x_6, x_7 >$
$T =< t >, \quad t = \mathrm{diag}(\alpha, \beta, \gamma, 1 - \alpha, 1 - \beta, 1 - \gamma, 1).$

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http://math.unice.fr/∼frou/AC.html

created by François Rouvière.

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