On the use of bounds on the stop-loss premium for an inventory management decision problem

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Abstract

A specific integral, which is used in insurance mathematics for the determination of a stop-loss premium, corresponds to the definition of a performance characteristic of an inventory management decision problem. It is investigated whether the operational management problem might benefit from specific results obtained in the actuarial world. In the case little information is available on the demand distribution during the lead-time, which is relevant in inventory decision-making, interesting results might be used from the actuarial problem where limited information (e.g., only a few moments of the claim size distribution) is known. With a limited transformation of the actuarial results, upper and lower bounds may be determined for the safety stock in the inventory management problem. Also a numerical approximation using linear programming is shown to be equivalent to the problem under study.

Keywords : *Incomplete information, inventory management, linear programming.*

1. Introduction

In insurance mathematics, an insurance company using the option of re-insurance is confronted with a stop-loss premium. A stop-loss premium limits the risk X of an insurance company to a certain amount t. If the claim size is higher than t the re-insurance company takes over the risk $X - t$.

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The stop-loss premium is based on the expected value of $X - t$, which in case of a known claim size distribution may be defined as:

$$\int_0^\infty (x-t)_+ dF(x) \tag{1}$$

where $F(x)$ represents the claim size distribution (Goovaerts *et al.* [1984]).

The same formula (1) may be useful in the performance evaluation of inventory management in case of uncertain demand during lead time. When a company holds t units of a specific product in inventory starting a period between order and delivery, any demand less than t is satisfied while any demand X greater than t results in a shortage of $X - t$ units. A lesser number of units short results in a better service to the customer. In this way formula (1) is a measure for customer service in inventory management.

This research deals with the case where the demand distribution during lead time is not completely known. This situation is realistic either with products which have been introduced recently to the market or with slow moving products. In both cases not sufficient data are available to decide on the functional form of the demand distribution function. Limited but not full information might exist like the range of the demand, its expected value, its variance and maybe some knowledge about unimodality of the distribution.

In the following sections lower and upper bounds are obtained for the performance measure under study, given various levels of information about the demand distribution. From a production or trading company's point of view, a decision might be formulated to answer the following question: *given an expected number of units short the company wants to face, what should be the safety inventory at least or at most?*

2. The case of a known range, expected value and variance

Let the size of the demand X for a specific product in a finite period have a distribution F with first two moments $\mu_1 = E(X)$ and $\mu_2 = E(X^2)$.

From a mathematical point of view, the problem is to find the following bounds:

$$\sup_{F \in \phi} \int_0^\infty (x-t)_+ dF(x) \tag{2a}$$

and

$$\inf_{F \in \phi} \int_0^\infty (x-t)_+ dF(x) \tag{2b}$$

where Φ is the class of all distribution functions F which have moments μ_1 and μ_2, and which have support in R^+. Let further $\sigma^2 = \mu_2 - \mu_1^2$. We assume t to be strictly positive.

For any polynomial $P(x)$ of degree 2 or less, the integral $\int_0^\infty P(x)dF(x)$ only depends on μ_1 and μ_2, so it takes the same value for all distributions in Φ. In the next sections polynomials P are looked for such that

$$\begin{aligned} -P &\geq (x-t)_+ \text{ on } R^+ \text{ in case of sup,} \\ P &\geq (x-t)_+ \text{ on } R^+ \text{ in case of inf.} \end{aligned}$$

There exists some distribution G in Φ for which the equality holds:

$$\int_0^\infty P(x)dG(x) = \int_0^\infty (x-t)_+ dG(x). \tag{3}$$

As distribution G a two-point or three-point distribution is used. The equality (3) is attained when $P(x)$ and $(x-t)_+$ are equal in the two points of G. The best upper and lower bounds on this term with given moments μ_1 and μ_2 are derived. The method is inspired by papers of Janssen *et al.* (1986), and Heijnen and Goovaerts (1989). In the following we assume the known range of the distribution to be a finite interval $[a, b]$.

A probability distribution F is called n-atomic if all its probability mass is concentrated in n points at most. The points are called the atoms of the distributions. The problem (2a) has a 2-atomic solution and (2b) has a 3-atomic solution.

If α, β are two different atoms of the 2-atomic probability distribution F satisfying the first-order moment constraint $\int x dF = \mu_1$, then the corresponding probability masses p_α and p_β are

$$p_\alpha = \frac{\mu_1 - \beta}{\alpha - \beta}, \qquad p_\beta = \frac{\mu_1 - \alpha}{\beta - \alpha}. \tag{4}$$

If α, β, γ are three different atoms of the 3-atomic probability distribution F satisfying the moment constraints $\int x dF = \mu_1$, $\int x^2 dF = \mu_2$, then the corresponding probability masses p_α, p_β and p_γ are

$$\begin{aligned} p_\alpha &= \frac{\sigma^2 + (\mu_1 - \beta)(\mu_1 - \gamma)}{(\alpha - \beta)(\alpha - \gamma)} \\ p_\beta &= \frac{\sigma^2 + (\mu_1 - \alpha)(\mu_1 - \gamma)}{(\beta - \alpha)(\beta - \gamma)} \\ p_\gamma &= \frac{\sigma^2 + (\mu_1 - \alpha)(\mu_1 - \beta)}{(\gamma - \alpha)(\gamma - \beta)}. \end{aligned} \tag{5}$$

Table 1 shows the results for the problem (2a). The domain of the parameters is $a \leq \mu_1 \leq b, 0 \leq \sigma^2 \leq (\mu_1 - a)(b - \mu_1)$ or $\mu_1^2 \leq \mu_2 \leq \mu_1(a+b) - ab$. Further the following abbreviations are used: $\sigma_{\mu t}^2 = \sigma^2 + (\mu_1 - t)^2$ and $c = 1/2(a+b)$. Further let μ_1 and μ_2 be chosen that the previous inequalities hold, then let $r' = \dfrac{\mu_2 - \mu_1 r}{\mu_1 - r}$ for every $r \in [a, b]$ and $r \neq \mu_1$.

Table 1

Upper bounds for the stop-loss premium in an interval $[a, b]$

Upper bounds	Atoms of solution	Conditions
$\frac{1}{2}(\sigma_{\mu t} + \mu_1 + t)$	$t - \sigma_{\mu t}, t + \sigma_{\mu t}$	$t \leq c, \sigma_{\mu t} \leq t - a$
$(\mu_1 - a)\frac{(\mu_1 - t)(\mu_1 - a) + \sigma^2}{(\mu_1 - a)^2 + \sigma^2}$	$a, \mu_1 + \frac{\sigma^2}{\mu_1 - a}$	$t \leq c, \sigma_{\mu t} \geq t - a$
$\frac{1}{2}(\sigma_{\mu t} + \mu_1 + t)$	$t - \sigma_{\mu t}, t + \sigma_{\mu t}$	$t \geq c, \sigma_{\mu t} \leq b - t$
$\frac{(b-t)\sigma^2}{(b-\mu_1)^2 + \sigma^2}$	$\mu_1 - \frac{\sigma^2}{b - \mu_1}, b$	$t \geq c, \sigma_{\mu t} \leq b - t$

Table 2 shows the results for the problem (2b). The domain of the parameters is the same as in Table 1.

Before moving towards the interdisciplinary application, it should be stated that the bounds and their use in applications can be translated from any distribution defined on $[a, b]$ into the bounds with a distribution defined on $[0, b_0]$, where $b_0 = b - a$. Further let $t_0 = t - a, \mu_{10} = \mu_1 - a$ and $\mu_{20} = \mu_2 - 2a\mu_{10} - a^2$. In the following paragraphs we work, without loss of generalisation, with distributions defined on $[0, b_0]$.

Table 2

Lower bounds for the stop-loss premium in an interval $[a, b]$

Lower bounds	Atoms of solution	Conditions
0	a, μ_1, t	$\sigma^2 \leq (\mu_1 - a)(t - \mu_1)$
$\mu_1 - t$	t, μ_1, b	$\sigma^2 \leq (\mu_1 - t)(b - \mu_1)$
$\frac{\sigma^2 + (\mu_1 - a)(\mu_1 - t)}{b - a}$	a, t, b	$\sigma^2 \geq (\mu_1 - a)(t - \mu_1)$
$\frac{\sigma^2 + (\mu_1 - a)(\mu_1 - t)}{b - a}$	a, t, b	$\sigma^2 \geq (\mu_1 - a)(t - \mu_1)$

Table 3
Lower bounds on the stop-loss premium in an interval $[0, b_0]$

Lower bounds	Conditions
$\mu_{10} - t_0$	$0 \leq t_0 \leq b_0'$ or $0 \leq t_0 \leq (\mu_{20} - \mu_{10} b_0)/(\mu_{10} - b_0)$
$(\mu_{20} - \mu_{10} t_0)/b_0$	$b_0' \leq t_0 \leq 0'$ or $(\mu_{20} - \mu_{10} b_0)/(\mu_{10} - b_0) \leq t_0 \leq \mu_{20}/\mu_{10}$
0	$0' \leq t_0 \leq b_0$ or $\mu_{20}/\mu_{10} \leq t_0 \leq b_0$

Table 4
Upper bounds on the stop-loss premium in an interval $[0, b_0]$

Upper bounds	Conditions
$\mu_{10}(\mu_{20} - \mu_{10} t_0)/\mu_{20}$	$t_0 \leq 0/2$ or $t_0 \leq \mu_{20}/(2\mu_{10})$
$(\mu_{10} - t_0 + \sqrt{(\mu_{20} - \mu_{10}^2) + (t_0 - \mu_{10})})/2$	$0'/2 \leq t_0 \leq (b_0 + b_0')/2$ or $\mu_{20}/(2\mu_{10}) \leq t_0 \leq (\mu_{20} - b_0^2)/(2(\mu_{10} - b_0))$
$\frac{(\mu_{20} - \mu_{10}^2)(b_0 - t_0)}{((\mu_{20} - \mu_{10}^2) + (b_0 - \mu_{10})^2)}$	$(b_0 + b_0')/2 \leq t_0$ or $(\mu_{20} - b_0^2)/(2(\mu_{10} - b_0)) \leq t_0$

3. Application to an inventory management performance measure

Almost every inventory system contains uncertainty. Some of the uncertainty (such as lead time, quantity and quality) depends on the suppliers. If the suppliers introduce too much uncertainty, corrective action should be taken. Some uncertainty, however, is attributable to customers, especially demand. If insufficient inventory is hold, a stock-out may occur leading to shortage costs. Shortage costs are usually high in relation to holding costs, i.e., the cost of keeping the goods during some time period in the warehouse. Companies are willing to hold additional inventory, above their forecasted needs, to add a margin of safety.

Determination of an inventory replenishment policy, of the quantities to order and of the review period are typical decisions to be taken by logistics managers. Decisions are made making use of optimisation models taking a performance measure into consideration which might be cost-oriented or service-oriented. Performance measures of the service-oriented type may be expressed relatively as a proportion of customer demand met from inventory, or may be expressed absolutely in terms of number of units short, which is a direct indication for lost sales. This study concentrates on the latter type of performance measure.

Note that the problem and its bounds with a demand distribution defined on the finite interval $[a, b]$ can be translated without loss of generalisation to the problem and its bounds with a demand distribution defined on the finite interval $[0, b_0]$.

The use of the bounds is illustrated by means of a numerical example. Let the demand be defined on the interval $[25, 75]$. The demand follows a distribution with only the following characteristics known: $\mu_1 = 45$ and $\mu_2 = 975$. This means that in Tables 3 and 4, the following values have to be used for $\mu_{10} = 20$, $\mu_{20} = 600$, $b_0 = 50$, $b_0' = 13.333$, and $0' = 30$. The values for upper and lower bounds are shown in Tables 5 and 6.

Table 5

Lower bounds on the number of units short for the illustrative example

Lower bounds	Conditions
$20 - t_0$	$0 \le t_0 \le 13.333$
$12 - 2/5t_0$	$13.333 \le t_0 \le 30$
0	$30 \le t_0 \le 50$

Table 6

Upper bounds on the number of units short for the illustrative example

Upper bounds	Conditions
$20 - 2/3t_0$	$0 \le t_0 \le 15$
$(20 - t_0 + \sqrt{200 + (t_0 - 20)})/2$	$15 \le t_0 \le 31.667$
$(100 - 2t_0)/11$	$31.667 \le t_0 \le 50$

The upper and lower bounds, as expressed in Tables 5 and 6, related to the safety inventory level for the numerical example under study is shown in Figure 1.

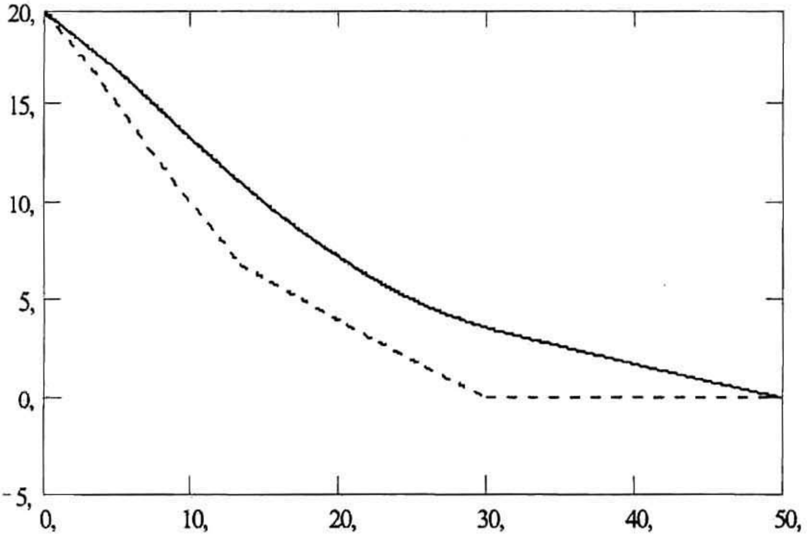

Figure 1

Upper and lower bounds on the number of units short W (vertical axis) in relation to the safety inventory level t_0 (horizontal axis)

From Tables 5 and 6, a decision-maker may be decide which level of inventory to hold, given a target value on the number of units short W as a performance measure. From these tables he can derive upper bounds on t_0, which correspond to a pessimistic viewpoint and lower bounds on t_0, which correspond to an optimistic viewpoint. The values corresponding to both viewpoints for the numerical example under study are given in Tables 7 and 8.

Table 7

Lower bounds on the safety inventory level Upper bounds

Lower bounds	Requirements
$30 - 5/2W$	$W \leq 6.667$
$20 - W$	$6.667 \leq W$

Table 8

Upper bounds on the safety inventory level

Lower bounds	Requirements
$50 - 11/2W$	$W \leq 3.333$
$(50 - W^2 + 20W)/W$	$3.333 \leq W \leq 10$
$(60 - 3W)/2$	$10 \leq W$

4. A linear program formulation for the problem under study

In this section we consider, of the relevant integral, the supremum version. Let $f_1, f_2, \ldots, f_n$ be functions or R. Let, for any $z' = (z_1, \ldots, z_{n-1}) \in R^{n-1}$ consider the primal maximisation problem:

$$P(z') = \sup_{F \in \phi} \left[\int_0^{\infty} (x-t)_+ dF(x) | I(F) | \right] \tag{6}$$

where $I(F)$ is a set of integral equality constraints of the type $\int f_i(x) dF(x) = z_i$, $(i = 1, \ldots, n-1)$ and $f_n = (x-t)_+$. In our application, the constraints are moment constraints, i.e., the first and second moment equalities and the obvious constraint because any member of ϕ is a probability distribution.

$$\int dF(x) = 1, \qquad \int x dF(x) = \mu_1, \qquad \int x^2 dF(x) = \mu_2 \tag{7}$$

which means that

$$n = 3$$
$$f_1(x) = x$$
$$f_2(x) = x^2$$
$$f_3(x) = (x-t)_+$$
$$z_1 = \mu_1$$
$$z_2 = \mu_2$$

The optimisation problem (6) has a dual program of the type:

$$Q(z') = \inf(y_1 z_1 + y_2 z_2 + y_3 | y_1 \hat{f}_1(\theta) + y_2 \hat{f}_2(\theta) + y_3 \geq \hat{f}_3(\theta)) \quad (\theta \in J) \tag{8}$$

where the infimum is over all $y = (y_1, y_2, y_3) \in R^3$ satisfying the constraints indicated after the slash. The functions $\hat{f}_i(\theta)$ $(i = 1, \ldots, n)$ are defined on a, mostly infinite, set J by

$$\hat{f}_i(\theta) = \int f_i dH_\theta(x) \ \text{ with } \theta \in J, \tag{9}$$

where H_θ is a subset of functions depending on the family of distributions under consideration.

The family of distributions considered in this study concerns distributions on a finite interval $[0, b]$, where b is a fixed positive number. In this case (Goovaerts *et al.* (1982)):

$$H_\theta(x) = 1_{\theta \leq x} \quad (0 \leq \theta \leq b), \tag{10}$$

which means in our case that

$$\hat{f}_1(\theta) = \theta \quad (0 \leq \theta \leq b)$$

$$\hat{f}_2(\theta) = \theta^2 \quad (0 \leq \theta \leq b)\,.$$

Mostly the set J is infinite, so the number of linear constraints on y is infinite. In Goovaerts *et al.* [1982] an idea is launched to replace J by a large finite subset of J and then to solve the so obtained linear program.

It means that the optimisation problem (8) can be approximated by the problem Q^A:

$$Q^A(z') = \inf_{y \in R^3} (y_1\mu_1 + y_2\mu_2 + y_3 | y_1\theta_j + y_2\theta_j^2 + y_3 \geq (\theta_j - t)_+),$$
$$\left(\theta_j = j \cdot \frac{b}{k}, \ (j = 0, 1, \ldots, k)\right). \quad (11)$$

The problem in section 3 will be approximated by the linear program as discussed above. The convergence to the optimal solution obtained from the results in section 3 will be shown.

As an illustration, 11 evaluation points are used ($k = 10$) for various values of t in the three ranges mentioned in Table 6.

For a value $t = 12$, the optimisation model can be written as:

$$\min 20Y1 + 600Y2 + Y3$$

subject to

$$0Y1 + 0Y2 + Y3 + \geq 0$$
$$5Y1 + 25Y2 + Y3 \geq 0$$
$$10Y1 + 100Y2 + Y3 \geq 0$$
$$15Y1 + 225Y2 + Y3 \geq 3$$
$$20Y1 + 400Y2 + Y3 \geq 8$$
$$25Y1 + 625Y2 + Y3 \geq 13$$
$$30Y1 + 900Y2 + Y3 \geq 18$$
$$35Y1 + 1225Y2 + Y3 \geq 23$$
$$40Y1 + 1600Y2 + Y3 \geq 28$$
$$45Y1 + 2025Y2 + Y3 \geq 33$$
$$50Y1 + 2500Y2 + Y3 \geq 38$$

The objective function value of this optimisation problem is 12, which corresponds to the value which should be obtained by using the formula from Table 6, with mass in two points $\alpha = 0$ and $\beta = 30$, with $p_\alpha = 1/3$ and $p_\beta = 2/3$. The values of the decision variables are $Y1 = 0.12$, $Y2 = 0.016$ and $Y3 = 0$.

For a value $t = 24$, the optimisation model can be written as:

$$\min 20Y1 + 600Y2 + Y3$$

subject to

$$0Y1 + 0Y2 + Y3 \geq 0$$
$$5Y1 + 25Y2 + Y3 \geq 0$$
$$10Y1 + 100Y2 + Y3 \geq 0$$
$$15Y1 + 225Y2 + Y3 \geq 0$$
$$20Y1 + 400Y2 + Y3 \geq 0$$
$$25Y1 + 625Y2 + Y3 \geq 1$$
$$30Y1 + 900Y2 + Y3 \geq 6$$
$$35Y1 + 1225Y2 + Y3 \geq 11$$
$$40Y1 + 1600Y2 + Y3 \geq 16$$
$$45Y1 + 2025Y2 + Y3 \geq 21$$
$$50Y1 + 2500Y2 + Y3 \geq 26$$

The objective function value of this optimisation problem is 5.333, which is close to the value 5.348 which should be obtained by using the formula from Table 6. The values of the decision variables are $Y1 = -0.229$, $Y2 = 0.015$ and $Y3 = 0.762$.

For a value $t = 36$, the optimisation model can be written as:

$$\min 20Y1 + 600Y2 + Y3$$

subject to

$$0Y1 + 0Y2 + Y3 \geq 0$$
$$5Y1 + 25Y2 + Y3 \geq 0$$
$$10Y1 + 100Y2 + Y3 \geq 0$$
$$15Y1 + 225Y2 + Y3 \geq 0$$
$$20Y1 + 400Y2 + Y3 \geq 0$$
$$25Y1 + 625Y2 + Y3 \geq 0$$
$$30Y1 + 900Y2 + Y3 \geq 0$$
$$35Y1 + 1225Y2 + Y3 \geq 0$$
$$40Y1 + 1600Y2 + Y3 \geq 4$$
$$45Y1 + 2025Y2 + Y3 \geq 9$$
$$50Y1 + 2500Y2 + Y3 \geq 14$$

The objective function value of this optimisation problem is 2.5, which is close to the value 2.545 which should be obtained by using the formula from Table 6. The values of the decision variables are $Y1 = -0.25$, $Y2 = 0.01$ and $Y3 = 1.5$.

Table 9 shows how the objective function value and the solution change with increasing values of k in the approximation problem (11). It can be observed that the objective function value does not increase monotonically with k, but is does with k versus k' where $k = n \cdot k'$ with n a positive integer ($n > 1$).

Table 9

Evolution of the objective junction value in terms of the number of constraints for various values of the safety inventory level

		Objective			
t	k	value	$Y1$	$Y2$	$Y3$
12	10	12.000	0.120	0.0160	0.000
	20	12.000	0.164	0.0145	0.000
	30	12.000	0.176	0.0141	0.000
	40	12.000	0.183	0.0139	0.000
	50	12.000	0.186	0.0138	0.000
	100	12.000	0.193	0.0136	0.000
24	10	5.333	–0.229	0.0152	0.762
	20	5.333	–0.287	0.0164	1.231
	30	5.340	–0.309	0.0169	1.405
	40	5.344	–0.321	0.0171	1.496
	50	5.345	–0.327	0.0172	1.552
	100	5.347	–0.314	0.0169	1.449
36	10	2.500	–0.250	0.0100	1.500
	20	2.533	–0.293	0.0110	2.000
	30	2.545	–0.249	0.0099	1.549
	40	2.543	–0.270	0.0103	1.770
	50	2.544	–0.284	0.0110	1.913
	100	2.545	–0.275	0.0104	1.819

5. Conclusion and further work

It is shown that, in the case of limited information on the demand distribution during lead time, upper and lower bounds might be obtained for the safety stock level based on an analogy between an inventory management performance measure and the stop-loss premium. The information in this study is limited to the range and to the first and second moments of the distribution.

This idea offers opportunities of computing bounds in case other types of limited information are available, for example, the unimodality of the distribution, the knowledge of a tail probability, or the knowledge that the distribution might be written as a mixture of exponential distributions.

References

[1] M. J. Goovaerts, J. Haezendonck and F. De Vylder (1982), Numerical best bounds on stop-loss premiums, *Insurance: Mathematics and Economics*, Vol. 1, pp. 287–302.

[2] M. J. Goovaerts, F. De Vylder and J. Haezendonck (1984), *Insurance Premiums*, North Holland, Amsterdam.

[3] B. Heijnen and M. J. Goovaerts (1989), Best upper bounds on risks altered by deductibles under incomplete information, *Scandinavian Actuarial Journal*, pp. 23–46.

[4] K. Janssen, J. Haezendonck and M. J. Goovaerts (1986), Upper bounds on stop-loss premiums in case of known moments up to the fourth order, *Insurance Mathematics and Economics*, Vol. 5, pp. 315–334.

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