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NONCOMMUTATIVE RESOLUTIONS AND RATIONAL SINGULARITIES

J. T. STAFFORD AND M. VAN DEN BERGH

Abstract. Let k be an algebraically closed field of characteristic zero. We show that the centre of a homologically homogeneous, finitely generated k-algebra has rational singularities. In particular if a finitely generated normal commutative k-algebra has a noncommutative crepant resolution, as introduced by the second author, then it has rational singularities.

1. Introduction

Throughout the paper, k will denote a fixed algebraically closed field of characteristic zero and unless otherwise specified all rings will be k-algebras. Suppose that $X = \operatorname{Spec} R$ for an affine (that is, finitely generated) normal Gorenstein k-algebra R. The nicest form of resolution of singularities $f : Y \to X$ occurs when f is *crepant* in the sense that $f^*\omega_X = \omega_Y$. Even when they exist, crepant resolutions need not be unique, but they are related—indeed Bondal and Orlov conjectured in [BO1] (see also [BO2]) that two such resolutions should be derived equivalent.

Bridgeland [Br1] proved the Bondal-Orlov conjecture in dimension 3. The second author observed in [VB3] that Bridgeland's proof could be explained in terms of a *third* crepant resolution of X which is now noncommutative (the definition will be given below) and this had lead to a number of different approaches to the Bondal-Orlov conjecture and related topics—see, for example, [Be, BK, Ch, IR, Kl2, Kw].

It is therefore natural to ask how the existence of a noncommutative crepant resolution affects the original commutative singularity. It is well-known, and follows easily from [KM, Theorem 5.10], that if a Gorenstein singularity has a crepant resolution then it has rational singularities. So it is logical to ask, as was raised in [VB2, Question 3.2], *is this true for a noncommutative crepant resolution*? In this paper we answer this question affirmatively, but before stating the result precisely, we need to define the relevant terms.

Let Δ be a prime affine k-algebra that is finitely generated as a module over its centre $Z(\Delta)$. Mimicking [BH], we say that Δ is *homologically homogeneous of dimension* d if all simple Δ-modules have the same projective dimension d. By [Ra] and [BH] such a ring Δ has global and Krull dimensions equal to d and, as has been shown in [BH], the properties of homologically homogeneous rings

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closely resemble those of commutative regular rings. So the idea is to use such a ring Δ as a noncommutative analogue of a crepant resolution. Formally, following [VB2] we define a *noncommutative crepant resolution* of R to be any homologically homogeneous ring of the form $\Delta = \mathrm{End}_R(M)$, where M is a reflexive, finitely generated R-module. We refer the reader to [VB2, Section 4] for the logic behind this definition.

The main result of the present note is the following:

Theorem 1.1. (Theorem 4.3) *Let Δ be a homologically homogeneous k-algebra. Then the centre $Z(\Delta)$ has rational singularities.*

In particular if a normal affine k-domain R has a noncommutative crepant resolution then it has rational singularities.

In Section 5 we give two examples related to the theorem. The first example shows that if $\Delta = \mathrm{End}_R(M)$ has finite global dimension then it need not be homologically homogeneous even under reasonable hypotheses on M and R. The second shows that Theorem 1.1 can fail in positive characteristic.

Notation. Throughout the paper R will be a normal commutative noetherian k-domain and Δ will be a k-algebra, with centre $Z = Z(\Delta)$ containing R, such that Δ is a finitely generated R-module. We say that R is *essentially affine* if it is a localization of an affine k-algebra. The dimension function used in this paper will be the *Gelfand-Kirillov dimension* of Δ as a k-algebra, written $\mathrm{GKdim}\,\Delta$. By [MR, Proposition 8.2.9(ii) and Theorem 8.2.14(ii)] $\mathrm{GKdim}\,\Delta = \mathrm{GKdim}\,R$ and $\mathrm{GKdim}\,R$ is just the transcendence degree of R over k.

2. Homologically homogeneous rings

In this section we introduce homologically homogeneous rings and prove some basic facts about their structure and their dualizing complexes. Many of these results use the machinery of tame orders and so we start by discussing this concept.

Tame orders. Assume that Δ is *a prime R-order in A*, by which we mean that Δ is a prime ring with simple artinian ring of fractions A. We write $\mathfrak{P}_1 = \mathfrak{P}_1(R)$ for the set of height one prime ideals of R and say that *a property $\mathcal{P}$ holds for Δ in codimension one* if it holds for all $\Delta_{\mathfrak{p}} = \Delta \otimes_R R_{\mathfrak{p}} : \mathfrak{p} \in \mathfrak{P}_1$. Following [Si], the prime R-order Δ is called a *tame R-order* if Δ is a finitely generated, reflexive R-module that is hereditary in codimension one.

The paper [Si] implicitly assumes that $R = Z(\Delta)$, but we prefer not make this assumption. However, by the following standard result, the question of whether Δ is a tame R-order is independent of the choice of normal central subring R.

Lemma 2.1. *Let Δ be a tame R-order. Then a finitely generated Δ-module is reflexive as an R-module if and only if it is reflexive as a Δ-module.*

Proof. By [Si, Corollary 1.6] (which does not require $R = Z(\Delta)$) a Δ-reflexive module is R-reflexive. Conversely, suppose that M is a finitely generated Δ-module that is R-reflexive. Since M is therefore torsion-free as a Δ-module, $M_{\mathfrak{p}} = M \otimes_R R_{\mathfrak{p}}$ is torsion-free and hence projective over the hereditary prime ring $\Delta_{\mathfrak{p}}$, for all $\mathfrak{p} \in \mathfrak{P}_1$. Thus, by [Si, Lemma 1.1],

$$\begin{aligned} M &= \textstyle\bigcap_{\mathfrak{p}\in\mathfrak{P}_1} M_{\mathfrak{p}} = \bigcap_{\mathfrak{p}\in\mathfrak{P}_1} \mathrm{Hom}_{\Delta_p}(\mathrm{Hom}_{\Delta_p}(M_{\mathfrak{p}}, \Delta_{\mathfrak{p}}), \Delta_{\mathfrak{p}}) \\ &\supseteq \mathrm{Hom}_{\Delta}(\mathrm{Hom}_{\Delta}(M, \Delta), \Delta). \end{aligned}$$

Thus, $M = \mathrm{Hom}_\Delta(\mathrm{Hom}_\Delta(M, \Delta), \Delta)$, as required. □

Let Δ be a tame R-order in A. A *divisorial* fractional Δ-ideal is any reflexive fractional Δ-ideal in A that is invertible in codimension one. By [Si, Theorem 2.3], divisorial fractional ideals form a free abelian group $\mathrm{Div}(\Delta)$ with product $I \cdot J = (IJ)^{**}$ where $K^* = \mathrm{Hom}_R(K, R)$ denotes the R-dual of a fractional ideal K. The nth power $(I^n)^{**}$ of I under this dot-operation is called the nth *symbolic power of* I and written $I^{(n)}$. Write $\operatorname{rad} S$ for the Jacobson radical of a ring S.

Homologically homogeneous rings. Homologically homogeneous ring, as defined in the introduction, have a particularly pleasant structure and the following result provides some of the properties we will need.

Theorem 2.2. *Assume that Δ is homologically homogeneous of dimension d.*

(1) *Δ is CM as a module over its centre Z.*
(2) *Both $\operatorname{GKdim}\Delta$ and the global homological dimension $\operatorname{gl\,dim}\Delta$ of Δ equal d.*
(3) *Z is an affine CM normal domain.*
(4) *Δ is a tame Z-order.*

Proof. (1,2) By [Ra, Theorem 8], $\operatorname{gl\,dim}\Delta = d$. The rest of parts (1) and (2) follow from [BH, Theorem 2.5].

(3) By hypothesis, Δ is finitely generated as both a Z-module and a k-algebra. Thus the Artin-Tate Lemma [MR, Lemma 13.9.10] implies that Z is an affine k-algebra. As $\operatorname{char} k = 0$, the reduced trace map $\Delta \to Z$ is surjective and so Z is a Z-module summand of Δ. Thus Z is CM by part (1). As Δ is prime, Z is a domain, while Z is normal by [BH, Theorem 6.1].

(4) As Δ is CM as a Z-module, it is certainly reflexive. By [BH, Corollary 2.2 and Theorem 2.5], Δ is hereditary in codimension one. □

The standing assumption that k have characteristic zero is crucial for the proof of part (3) of the theorem. Indeed, [BHM, Example 7.3] shows that the centre $Z(\Gamma)$ of a homologically homogeneous ring Γ need not be CM in bad characteristic.

The following criterion for a ring to be homologically homogeneous will be useful.

Lemma 2.3. *Suppose that R is an affine k-algebra and that Δ is a prime ring. If Δ is a CM R-module with $\operatorname{GKdim}\Delta = \operatorname{gl\,dim}\Delta$, then Δ is homologically homogeneous.*

Proof. This is, essentially, [BH, Proposition 7.2], but here is a direct proof. Suppose that S is a simple Δ-module with projective dimension $u < d = \operatorname{gl\,dim}\Delta$ and consider a projective Δ-resolution of S:

$$0 \to P_u \to \cdots \to P_1 \to P_0 \to S \to 0.$$

Viewed as a complex over R this is a resolution of length $< d$ of a finite length R-module by CM modules of dimension d. An easy depth argument shows that this is impossible. □

Dualizing modules and complexes. In order relate properties of a homologically homogeneous ring to those of its centre we will use the machinery of dualizing complexes and we discuss their structure in this subsection. Most of the background material comes from [VB1, Ye2, YZ1, YZ2] and the reader is referred to those papers for more details. Throughout this discussion, in addition to our standing assumptions, we assume that R is essentially affine.

Write $\Delta^e = \Delta \otimes_k \Delta^{\rm op}$ and denote the derived category of left Δ^e-modules by $D(\Delta^e)$. Following [Ye1], a *dualizing complex* for Δ is a complex of Δ-bimodules D, with finite injective dimension on both sides, such that

(1) the cohomology of D is given by Δ-bimodules that are finitely generated on both sides, and
(2) in $D(\Delta^e)$ the pair of natural morphisms $\Phi : \Delta \to \mathrm{RHom}_\Delta(D, D)$ and $\Phi^o : \Delta \to \mathrm{RHom}_{\Delta^{\rm op}}(D, D)$ are isomorphisms.

Following [VB1, Definition 8.1], the dualizing complex D_Δ is called *rigid* if there is an isomorphism $\chi : D_\Delta \cong \mathrm{RHom}_{\Delta^e}(\Delta, D_\Delta \otimes D_\Delta)$ in $D(\Delta^e)$. The significance of rigidity is that, although dualizing complexes are not unique, rigid dualizing complexes are, in the sense that the pair (D_Δ, χ) is unique up to a unique isomorphism [VB1, Proposition 8.2] [YZ1, Theorem 3.2].

Although dualizing complexes (rigid or otherwise) do not exist for all finitely generated noncommutative noetherian rings [KRS, p. 529], by [Ye2, Proposition 5.7] and [YZ2, Theorem 3.8] they do exist for our rings R and Δ.

Write $d = \mathrm{GKdim}\,\Delta = \mathrm{GKdim}\,R$. The cohomology of D_R and D_Δ lies in degrees $\geq -d$ and we define $\omega_R = H^{-d}(D_R)$ and $\omega_\Delta = H^{-d}(D_\Delta)$. An important fact [YZ1, Corollary 3.6] is that the cohomology of D_Δ is *Z-central* in the sense that the left and right actions of Z agree. In particular, ω_Δ is Z-central.

The following results gives some basic properties that we will need about these objects. If M is Δ-bimodule then $Z(M) = \{w \in M : \delta w = w\delta \text{ for all } \delta \in \Delta\}$ is called *the centre of M*.

Lemma 2.4. *Assume that R is an essentially affine k-algebra. Then:*

(1) $D_\Delta \cong \mathrm{RHom}_R(\Delta, D_R)$ *in* $D(\Delta^e)$.
(2) $\omega_\Delta \cong \mathrm{Hom}_R(\Delta, \omega_R)$ *as* Δ^e*-modules.*
(3) *If* $\mathcal{C} \subset Z$ *is multiplicatively closed then* $\omega_{\Delta_\mathcal{C}} \cong (\omega_\Delta)_\mathcal{C}$ *as* Δ*-bimodules.*

Assume in addition that Δ is a tame R-order. Then:

(4) ω_Δ *is a reflexive as a left or right* Δ*-module.*
(5) ω_Δ *is invertible in codimension one. Moreover, as bimodules*

$$\omega_\Delta = \left(\omega_Z \otimes_Z \prod_{\mathfrak{p} \in \mathfrak{P}_1(Z)} \left(\Delta \cap \mathrm{rad}(\Delta_\mathfrak{p})\right) \cdot \mathfrak{p}^{(-1)}\right)^{**}.$$

(6) *There is a canonical isomorphism* $Z(\omega_\Delta) = \omega_Z$.

Proof. (1) The proof of [Ye2, Proposition 5.7] shows that $\mathrm{RHom}_R(\Delta, D_R)$ is a rigid dualizing complex for Δ and so the result follows by the uniqueness of D_Δ.

(2) Take cohomology of (1).

(3) By [YZ2, Theorem 3.8] $D_{\Delta_\mathcal{C}} \cong \Delta_\mathcal{C} \overset{L}{\otimes}_\Delta D_\Delta \overset{L}{\otimes} \Delta_\mathcal{C}$ as Δ-bimodules. Now take cohomology, using the fact that, as mentioned above, each $\mathrm{H}^q(D_\Delta)$ is Z-central.

(4) By part (2) and [Si, Lemma 1.5] it suffices to prove the result for ω_R. This case is well-known, but here is an easy proof. By part (3) we may assume that R is an affine k-algebra. By Noether normalization R is a finitely generated module over some polynomial subring R_0 and it is a tame R_0-order since it is normal. It is standard [YZ2, Example 3.13] that $\omega_{R_0} \cong R_0$ as bimodules and so [Si, Lemma 1.5] and Lemma 2.1 imply that $\omega_R \cong \mathrm{Hom}_{R_0}(R, \omega_{R_0})$ is a reflexive R-module.

(5) The first assertion follows, for example, from [CR, Corollary 37.9] combined with part (2). In codimension one the displayed equation also follows easily from part (2) and the general case then follows from parts (4) and (3).

(6) This follows from part (5). □

Remark 2.5. We emphasize that our definition of ω_R does coincide with the usual commutative notion $\bigwedge^d(\Omega_{R/k})^{**}$ when R is essentially affine with $\operatorname{GKdim} R = d$. To see this, set $\omega'_R = \bigwedge^d(\Omega_{R/k})^{**}$. Then [Ye2, Lemma 5.4] shows that $\omega_S = \omega'_S$ holds for any regular, essentially finite domain S. As R is normal, it is regular in codimension one and so Lemma 2.4(3) implies that $(\omega_R)_{\mathfrak{p}} = (\omega'_R)_{\mathfrak{p}}$ for all height one prime ideals $\mathfrak{p}$. By Lemma 2.4(4), ω_R and ω'_R are reflexive, and hence $\omega_R = \omega'_R$.

Proposition 2.6. *Assume that Δ is a prime affine k-algebra. Then Δ is homologically homogeneous of dimension d if and only if $\operatorname{gl\,dim}\Delta < \infty$ and $D_\Delta = \Omega[d]$ for some invertible Δ-bimodule Ω. If this holds then $\Omega = \omega_\Delta$.*

Remark 2.7. In the notation of [VB1, Section 8], the proposition states that Δ is homologically homogeneous of dimension d if and only if $\operatorname{gl\,dim}\Delta < \infty$ and Δ is AS-Gorenstein. See [SZ, Theorems 1.3 and 1.4] for a closely related result.

Proof. Assume first that Δ is homologically homogeneous of dimension d. Since the statement of the proposition is independent of the choice of R we may, by Noether normalization, assume that R is a polynomial ring. By Theorem 2.2(1,3), Δ is CM and hence free as an R-module. But now $D_R = \omega_R[d] \cong R[d]$ and so $D_\Delta = \operatorname{RHom}_R(\Delta, D_R)$ lives purely in dimension $-d$, whence $D_\Delta = \omega_\Delta[d]$. Lemma 2.4(2) implies that ω_Δ is free and hence CM as an R-module and so [BH, Corollary 3.1] implies that ω_Δ is a projective Δ-module on either side. On the other hand, as Δ is a free R-module it is a tame R-order and so Lemma 2.4(5) implies that ω_Δ is invertible in codimension one. Together with [Si, Proposition 3.1], these observations imply that ω_Δ is invertible, finishing the proof in this direction.

Conversely, assume that $\operatorname{gl\,dim}\Delta < \infty$ and $D_\Delta = \Omega[d]$ for some invertible bimodule Ω. We will show that every simple Δ-module S has projective dimension d. By [YZ1, Corollary 6.9] D_Δ is Auslander and GKdim-Macaulay in the sense of [YZ1, Definitions 2.1 and 2.24]. Since S is finite dimensional the Macaulay property means that $\operatorname{Ext}^d_\Delta(S, \Omega) \neq 0$. If $\operatorname{gl\,dim}\Delta = e > d$ then, by [Ra, Theorem 8], there exists a simple Δ-module S with $\operatorname{Ext}^e_\Delta(S, \Delta) \neq 0$. Since Ω is invertible, this implies that $E = \operatorname{Ext}^e_\Delta(S', \Omega) \neq 0$ for $S' = \Omega \otimes_\Delta S$. By the Auslander property, this means that $\operatorname{Cdim}_{D_\Delta}(E) \leq -e$ which, by the GKdim-Macaulay property, implies that $\operatorname{GKdim} S < 0$. This is absurd. □

The following formulæ will be useful.

Corollary 2.8. *Assume that R is essentially affine with $\operatorname{GKdim} R = d$ and let Δ be a tame R-order. Then*

$$\omega_\Delta^{(-1)} = \operatorname{Ext}^d_{\Delta^e}(\Delta, \Delta^e)^{**}. \tag{2.9}$$

If Δ is homologically homogeneous then

$$\omega_\Delta^{(-1)} = \operatorname{RHom}_{\Delta^e}(\Delta, \Delta^e)[d]. \tag{2.10}$$

Proof. If Δ is homologically homogeneous then it has dimension d by Theorem 2.2. Thus [VB1, Proposition 8.4] and Remark 2.7 combine to prove (2.10).

Now suppose that Δ is a tame R-order and set $\Gamma = \Delta_{\mathfrak{p}}$, for some $\mathfrak{p} \in \mathfrak{P}_1(Z)$. Then Γ is an hereditary order and, by [MR, Theorem 13.10.1], $Z(\Gamma)$ is a local PID. By Lemma 2.4(4,5), ω_Δ is invertible and hence, just as in the proof of Theorem 2.6, $D_\Delta = \omega_\Delta[d]$. Thus, [VB1, Proposition 8.4] can again be applied to show that $\omega_\Gamma^{(-1)} = \mathrm{Ext}^d_{\Gamma^e}(\Gamma, \Gamma^e)^{**}$. In other words, (2.9) holds in codimension one. Since both sides of that equation are reflexive, it holds everywhere. □

3. Reduction to the Calabi-Yau case

Let Δ be a homologically homogeneous ring. In Section 4 we will use the structure of ω_Δ to show that Z has rational singularities, but this is awkward to prove when ω_Δ is not cyclic. In this section we show how to use a trick from [NV, Theorem 3.1] to (locally) replace Δ by an order for which ω_Δ is generated by a single central element. This is a noncommutative generalization of a well known technique in algebraic geometry where one constructs a Gorenstein cover of a $\mathbb{Q}$-Gorenstein singularity.

Given a tame R-order Γ in A and $I \in \mathrm{Div}(\Gamma)$, the *Rees ring* $\Gamma[I]$ of Γ is defined to be the subring $\sum_{n=-\infty}^{\infty} I^{(n)} x^n$ of the Laurent polynomial ring $A[x, x^{-1}]$.

Proposition 3.1. *Assume that Δ is homologically homogeneous. For some $n \geq 1$, suppose that $\omega_\Delta^{\otimes n} \cong \Delta$ as bimodules and choose n minimal with this property. Write*

$$\Lambda = \Delta \oplus \omega_\Delta \oplus \omega_\Delta^{\otimes 2} \oplus \cdots \oplus \omega_\Delta^{\otimes n-1}$$

where the multiplication is defined using the isomorphism $\omega_\Delta^{\otimes n} \cong \Delta$. Then:

1. *Λ is a prime homologically homogeneous ring;*
2. *$\omega_\Lambda \cong \Lambda$, as Λ-bimodules.*

Proof. (1) By Theorem 2.2(3,4), Z is an affine normal domain, and Δ is a tame Z-order in its simple artinian ring of fractions A. By [YZ1, Corollary 3.6], ω_Δ is Z-central and so Lemma 2.4(4,5) implies that ω_Δ is isomorphic to a divisorial fractional ideal I. Therefore, $I^{(n)} = \Lambda a$ for some $a \in L = Z(A)$ and so $\Lambda \cong \Delta[I]/(1 - ax^n)$. The field of fractions of Λ is therefore

$$B = A \otimes_L L[x]/(1 - ax^n)$$

By [Si, Theorem 2.3], $\mathrm{Div}(\Delta)$ is a free abelian group. Therefore, if $a = b^m$ for some $m > 1$ and $b \in L$ then we would have $m \mid n$ and $I^{(n/m)} = \Lambda b$ contradicting the minimality of n. If follows that $L[x]/(1 - ax^n)$ is a field and thus B is a central simple algebra. Consequently Λ is prime.

The ring Λ is strongly graded and hence $\mathrm{gl\,dim}\,\Lambda = \mathrm{gl\,dim}\,\Delta$ follows from [MR, Corollary 7.6.18] together with the fact that the categories of Δ-modules and graded Λ-modules are equivalent. Thus $\mathrm{gl\,dim}\,\Lambda = \mathrm{gl\,dim}\,\Delta = \mathrm{GKdim}\,\Delta = \mathrm{GKdim}\,\Lambda$ by Theorem 2.2(2) and [MR, Proposition 8.2.9(ii)]. By Theorem 2.2(1), Δ is CM as a Z-module and hence so is each $\omega_\Delta^{\otimes j}$ and Λ. Thus Λ is homologically homogeneous by Lemma 2.3.

(2) Using the formula $\omega_\Lambda = \mathrm{Hom}_R(\Lambda, \omega_R)$ we compute that

$$\omega_\Lambda = \omega_\Delta \oplus \omega_\Delta^{\otimes 2} \oplus \cdots \oplus \omega_\Delta^{\oplus n-1} \oplus \Delta$$

as $\mathbb{Z}/n\mathbb{Z}$-graded Λ-bimodules. Forgetting the grading gives the result. □

Remarks 3.2. (1) Assume that Z an essentially affine k-algebra. Following [Br] or [Gi], Δ is called *Calabi-Yau of dimension d* if $D_\Delta \cong \Delta[d]$ in $D(\Delta^e)$. (Some authors also require Calabi-Yau algebras to have finite global dimension; see, for example, [IR, Theorem 3.2(iii)].) For a survey on the Calabi-Yau property in an algebraic context see [Gi].

By Proposition 2.6, an affine Calabi-Yau algebra of finite global dimension is automatically homologically homogeneous. Conversely, Proposition 3.1 can be regarded as a reduction to the Calabi-Yau case.

(2) Proposition 3.1 can also be regarded as a reduction to the case of orders unramified in codimension one. In order to explain this, recall that a tame order Δ is *unramified in codimension one* if $\mathfrak{p}\Delta_\mathfrak{p} = \operatorname{rad}\Delta_\mathfrak{p}$ for all $p \in \mathfrak{P}_1(Z)$. Given a tame Caladi-Yau order Δ, then Lemma 2.4(6) implies that $Z \cong Z(\omega_\Delta) = \omega_Z$ and so Lemma 2.4(5) implies that $\mathfrak{p}\Delta_\mathfrak{p} = \operatorname{rad}\Delta_\mathfrak{p}$ for all $p \in \mathfrak{P}_1(Z)$.

Even when Δ is homologically homogeneous, there is no reason for ω_Δ to have finite order and so Proposition 3.1 cannot be applied directly. However, ω_Δ has finite order locally, which will be sufficient for our applications. Before stating the result, we prove some elementary facts.

Lemma 3.3. *If S is a ring with Jacobson radical* $\operatorname{rad}(S)$ *and P is an invertible S-bimodule then* $\operatorname{rad}(S)P = P\operatorname{rad}(S)$.

Proof. We claim that the image of composition

$$\chi : P^{-1} \otimes_S \operatorname{rad}(S) \otimes_S P \ \to \ P^{-1} \otimes_S S \otimes_S P \ \cong \ S \tag{3.4}$$

lies in $\operatorname{rad}(S)$. This proves the inclusion $\operatorname{rad}(S)P \subseteq P\operatorname{rad}(S)$. To prove the opposite inclusion interchange P and P^{-1}.

In order to prove the claim we will prove that the image of χ annihilates all simple S-modules. Let M be a simple S-module. We must show that the map

$$P^{-1} \otimes_S \operatorname{rad}(S) \otimes_S P \otimes_S M \to M \tag{3.5}$$

is zero. Tensoring (3.5) on the left by P we obtain the map

$$\operatorname{rad}(S) \otimes_S P \otimes_S M \to P \otimes_S M \tag{3.6}$$

Since $P \otimes_S -$ is an autoequivalence of $\operatorname{Mod}(S)$, $P \otimes_S M$ is a simple module and hence (3.6) is indeed the zero map. □

Lemma 3.7. *Assume that R is local and that Γ is a tame R-order in A, with $R = Z(\Gamma)$. If P is an R-central invertible Γ-bimodule, then there exists an integer $n > 0$ such that $P^{\otimes n} \cong \Gamma$ as Γ-bimodules.*

Proof. Since P is invertible, tensor powers, symbolic powers and ordinary powers all coincide, so we will drop the tensor product sign.

We first prove that $P^n \cong \Gamma$ as left Γ-modules. By Lemma 3.3, $P/\operatorname{rad}(\Gamma)P$ is an invertible bimodule over $\Gamma/\operatorname{rad}(\Gamma)$. Since $\Gamma/\operatorname{rad}(\Gamma)$ is semi-simple, it is easy to see that there exists $n > 0$ such that

$$P^n/\operatorname{rad}(\Gamma)P^n = (P/\operatorname{rad}(\Gamma)P)^n \cong \Gamma/\operatorname{rad}(\Gamma)$$

as left $\Gamma/\operatorname{rad}(\Gamma)$-modules. By Nakayama's Lemma it follows that $P^n \cong \Gamma$, again as left Γ-modules.

Let K denote the fraction field of R. Since P is R-central, $K \otimes_R P$ is an invertible A-bimodule. After choosing an isomorphism $K \otimes_R P \cong A$ we may assume that P

is a divisorial fractional R-ideal. By [LVV, Proposition II.4.20], some power $P^{(e)}$ of P lies in the image of $\mathrm{Div}(Z(\Gamma))$ in $\mathrm{Div}(\Gamma)$; that is, $P^{(e)} \cong (\Gamma I)^{**}$ for some reflexive ideal I of R. By the last paragraph, we may also assume that $P^e \cong \Gamma$ as left Γ modules.

Now let $u = \mathrm{rk}_R \Gamma$. Then $\big(\bigwedge_R^u(\Gamma I)\big)^{**} \cong \bigwedge_R^u P^e \cong \big(\bigwedge_R^u(\Gamma)\big)^{**}$ as R-modules. On the other hand,

$$\big(\bigwedge^u(\Gamma I)\big)^{**} = \big((\bigwedge^u \Gamma) I^u\big)^{**} = \big(\bigwedge^u \Gamma\big)^{**} \big(I^u\big)^{**}$$

and so $\big(\bigwedge^u \Gamma\big)^{**}\big(I^u\big)^{**} = \big(\bigwedge^u(\Gamma)\big)^{**}$. Cancelling $\big(\bigwedge_R^u \Gamma\big)^{**}$ gives $\big(I^u\big)^{**} \cong R$ as R-modules. Since $P^e \cong (\Gamma I)^{**}$ as Γ-bimodules we obtain $P^{eu} \cong \Gamma$ as Γ-bimodules. □

Corollary 3.8. *Suppose that Δ is homologically homogeneous k-algebra. Then for every maximal ideal $\mathfrak{m}$ of Z there exist $f \in Z \smallsetminus \mathfrak{m}$ and $n > 0$ with the property that $\omega_{\Delta_f}^{\otimes n} \cong \Delta_f$ as Δ_f-bimodules.*

Proof. By Proposition 2.6, ω_Δ is invertible and, as has been observed before, it is Z-central. By Lemma 2.4(3) and Theorem 2.2(4), we can therefore apply Lemma 3.7 to $P = \omega_{\Delta_\mathfrak{m}}$ and conclude that $\omega_{\Delta_\mathfrak{m}}^{\otimes n} \cong \Delta_\mathfrak{m}$ as $\Delta_\mathfrak{m}$-bimodules. As usual this isomorphism may be "spread out" on a neighbourhood of $\mathfrak{m}$ in $\mathrm{Spec}\, Z$. □

4. The centre of homologically homogeneous rings

In this section we prove Theorem 1.1 from the introduction. We start with two preparatory lemmas, the first of which gives a useful algebraic criterion for a ring to have rational singularities.

Lemma 4.1. *Let Z be an affine normal CM k-domain with field of fractions K. Then Z has rational singularities if and only if, for all regular affine k-algebras S satisfying $Z \subseteq S \subset K$, we have $\omega_Z \subseteq \omega_S$ inside ω_K.*

Proof. Let $X = \mathrm{Spec}\, Z$. By Remark 2.5, ω_X in the sense of [KKMS, KM] is equal to ω_Z in the sense of this paper and so, by Lemma 2.4(4), ω_X is reflexive. According to [KKMS, p. 50] or [KM, Theorem 5.10], X has rational singularities if and only if for one (or for all) resolution(s) of singularities $f : Y \to X$ we have $f_*\omega_Y = \omega_X$ inside ω_K. Since ω_X and ω_Y are reflexive this is equivalent to $\omega_X \subseteq f_*\omega_Y$ and the latter condition is equivalent to $(f^*\omega_X)^{**} \subseteq \omega_Y$. This can be checked locally on Y.

So assume that $\omega_Z \subseteq \omega_S$ for all affine regular k-algebras S satisfying $Z \subseteq S \subset K$. Pick Y by the last paragraph and an open affine subset $U \subset Y$. Then $\omega_Z \subseteq \omega_S$ for $S = \mathcal{O}(U)$ and hence $(S \otimes_Z \omega_Z)^{**} \subseteq \omega_S$. Globalizing gives $(f^*\omega_X)^{**} \subseteq \omega_Y$ and so Z has rational singularities.

Conversely assume that Z has rational singularities and let $Z \subseteq S$ be as in the statement of the lemma. Put $U = \mathrm{Spec}\, S$. We may compactify the map $g : U \to X$ to a projective map $\bar{g} : Y' \to X$. A priori Y' will not be smooth but we can resolve it further without touching U (see [KM, Theorem 0.2]) to arrive at a resolution of singularities $f : Y \to X$. The fact that $(f^*\omega_X)^{**} \subseteq \omega_Y$ yields $(S \otimes_Z \omega_Z)^{**} \subseteq \omega_S$ after restricting to U. Thus $\omega_Z \subseteq \omega_S$. □

Lemma 4.2. *Let Λ_1 and Λ_2 be affine k-algebras of finite global dimension that satisfy a polynomial identity. Then $\Lambda_1 \otimes_k \Lambda_2$ has finite global dimension.*

Proof. By the Nullstellensatz [MR, Theorem 13.10.3], every primitive factor ring of Λ_i is isomorphic to a full matrix ring over k. Hence every primitive factor ring Γ of $\Lambda = \Lambda_1 \otimes_k \Lambda_2$ decomposes as $\Gamma = \Gamma_1 \otimes_k \Gamma_2$ for primitive factor rings Γ_i of Λ_i. Thus any simple Λ-module M can be written as $M = M_1 \otimes_k M_2$, where each M_i is a simple Λ_i-module. Now use [CE, Proposition IX.2.6]. □

Theorem 4.3. *If Δ is a homologically homogeneous k-algebra, then $Z = Z(\Delta)$ has rational singularities.*

Proof. It is enough to prove the result locally, so by Corollary 3.8 we can replace Δ by some Δ_f and assume that $\omega_\Delta^{\otimes n} \cong \Delta$ as Δ-bimodules. By Proposition 3.1, the algebra $\Lambda = \Delta \oplus \omega_\Delta \oplus \omega_\Delta^{\otimes 2} \oplus \cdots \oplus \omega_\Delta^{\otimes n-1}$ satisfies $\omega_\Lambda \cong \Lambda$ as Λ-bimodules. Then Λ and hence $Z(\Lambda)$ are $\mathbb{Z}/n\mathbb{Z}$-graded. Moreover, as ω_Δ is Z-central, clearly Z commutes with each $\omega_\Delta^{\otimes j}$ and so $Z \subseteq Z(\Lambda)_0$. Since the other inclusion is trivial, $Z = Z(\Lambda)_0$ and Z is a module-theoretic summand of $Z(\Lambda)$. Since a direct summand of a ring with rational singularities has rational singularities [Bo] we may therefore replace Δ by Λ and assume that that $\omega_\Delta \cong \Delta$ as bimodules. By Proposition 3.1(1) Δ remains homologically homogeneous.

We will use Lemma 4.1, so fix a ring $Z \subseteq S \subset K$ as in the lemma and let Γ be a maximal, and therefore tame S-order containing $S\Delta$ inside the simple artinian ring of fractions A of Δ. Our discussion in Section 2 on dualizing complexes also applies to Γ, so $\omega_\Gamma = \mathrm{Hom}_S(\Gamma, \omega_S)$ and $\omega_S = Z(\omega_\Gamma)$ in the notation developed there. We will show that $\omega_\Delta \subseteq \omega_\Gamma$ inside ω_A. Since $S \subset K$, this will yield $Z(\omega_\Delta) \subseteq Z(\omega_\Gamma)$ as subgroups of $Z(\omega_A)$ and so Lemma 2.4(6) will imply that $\omega_Z \subseteq \omega_S$, as required.

In order to prove that $\omega_\Delta \subseteq \omega_\Gamma$ we may as well prove that $\omega_\Delta\Gamma \subseteq \omega_\Gamma$. The bimodule isomorphism $\omega_\Delta \cong \Delta$ means that $\omega_\Delta = c\Delta$ for some central element $c \in \omega_A$. From this we deduce that $\omega_\Delta\Gamma = \Gamma\omega_\Delta$ is an invertible Γ-bimodule with inverse $\omega_\Delta^{(-1)}\Gamma = c^{-1}\Gamma$. By Lemma 2.4(4), ω_Γ is reflexive and so it suffices to prove that $\omega_\Gamma^{(-1)} \subseteq \omega_\Delta^{(-1)}\Gamma$ inside $\omega_A^{(-1)}$.

We claim that

$$\Gamma \overset{L}{\otimes}_\Delta \mathrm{RHom}_{\Delta^e}(M, \Delta^e) \overset{L}{\otimes}_\Delta \Gamma = \mathrm{RHom}_{\Gamma^e}(\Gamma \overset{L}{\otimes}_\Delta M \overset{L}{\otimes}_\Delta \Gamma, \Gamma^e) \tag{4.4}$$

for any object M in $D^b(\Delta^e)$ with finitely generated cohomology. To prove this recall that, by Lemma 4.2, $\mathrm{gl\,dim}\,\Delta^e < \infty$. Thus we can replace M by a finite projective resolution of Δ^e-modules and it then suffices prove the claim for $M = \Delta^e$. This case is obvious.

Applying (4.4) with $M = \Delta$ and using the formula $\omega_\Delta^{(-1)} = \mathrm{RHom}_{\Delta^e}(\Delta, \Delta^e)[d]$ from (2.10) we obtain

$$\Gamma \overset{L}{\otimes}_\Delta \omega_\Delta^{(-1)} \overset{L}{\otimes}_\Delta \Gamma = \mathrm{RHom}_{\Gamma^e}(\Gamma \overset{L}{\otimes}_\Delta \Gamma, \Gamma^e)[d].$$

Using the fact that the derived tensor product maps to the ordinary tensor this induces a composed map

$$\mathrm{RHom}_{\Gamma^e}(\Gamma, \Gamma^e)[d] \;\to\; \mathrm{RHom}_{\Gamma^e}(\Gamma \overset{L}{\otimes}_\Delta \Gamma, \Gamma^e)[d] \;=\; \Gamma \overset{L}{\otimes}_\Delta \omega_\Delta^{(-1)} \overset{L}{\otimes}_\Delta \Gamma \;\to\; \omega_\Delta^{(-1)}\Gamma.$$

Taking cohomology in degree zero and then biduals gives a map

$$\mathrm{Ext}^d_{\Gamma^e}(\Gamma, \Gamma^e)^{**} \to (\omega_\Delta^{(-1)}\Gamma)^{**} = \omega_\Delta^{(-1)}\Gamma.$$

Using (2.9) this induces a map

$$\omega_\Gamma^{(-1)} \to \omega_\Delta^{(-1)}\Gamma. \tag{4.5}$$

Now we could have done these computations after tensoring with the field of fractions K of Z. Since $K = KZ = KS$ and $K\Delta = K\Gamma = A$, all morphisms would then have been (canonically) the identity. From this we deduce that (4.5) is an inclusion which takes place inside $\omega_A^{(-1)}$. This means we are done. □

Remarks 4.6. (1) Suppose that Δ is an affine Calabi-Yau k-algebra of finite global dimension. Then Theorem 4.3 and Remark 3.2(1) combine to prove that Z has rational singularities.

(2) Homologically homogeneous rings were defined in [BH] for orders in semisimple rather than simple artinian rings. However, by [BH, Theorem 5.3], these more general algebras are direct sums of prime homologically homogeneous rings and so the more general case also follows from this theorem. Similarly, one can weaken the hypothesis that Δ be finitely generated as a module over its centre to the assumption that it be an affine algebra satisfying a polynomial identity since, by [SZ, Theorem 5.6(iv)], this already forces Δ to be a finitely generated Z-module.

5. Examples

Here we give two examples to illustrate the earlier results. The first shows that [VB2, Lemma 4.2] cannot be improved while the second one shows that Theorem 1.1 can fail in finite characteristic.

In addition to our standing hypotheses, suppose that R is an affine Gorenstein k-algebra and that $\Delta = \mathrm{End}_R(M)$ for some finitely generated reflexive R-module M. Then it follows from [VB2, Lemma 4.2] that Δ is homologically homogeneous if and only if $\mathrm{gl\,dim}\,\Delta < \infty$ and Δ is a CM R-module. This is useful for the theory of noncommutative crepant resolutions, so it would be useful if we could weaken the hypotheses in this result. In our first example, we show that the Gorenstein condition is necessary.

Here is the example. Let T be a one-dimensional torus acting on the generators of the polynomial ring $S = k[x_1, x_2, x_3, x_4, x_5]$ with weights $1, 1, 1, -1, -1$ and let $R = S^T$. We may also view R as the coordinate ring of the variety of 2×3-matrices of rank ≤ 1.

The T-weights give a grading $S = \bigoplus_{\ell=-\infty}^{\infty} S_\ell$ with $S_0 = R$. According to the proof of [VB2, Lemma 8.8] the S_i are isomorphic to reflexive ideals of R with $S_{a+b} = (S_a S_b)^{**}$ for all $a, b \in \mathbb{Z}$. Furthermore it is easy to see that S_i is not a projective R-module when $i \neq 0$.

It follows from [VB2, Lemma 8.1] that S_{-2}, S_{-1}, R and S_1 are CM R-modules while R is certainly normal. It follows from [VB2, Lemma 8.2 and Theorem 8.6] that

$$\Delta = \mathrm{End}_R(R \oplus S_1) = \begin{pmatrix} R & S_1 \\ S_{-1} & R \end{pmatrix}.$$

has finite global dimension and hence is a tame order over its centre R. By [Kn, Korollar 2], the dualizing module ω_R is isomorphic to S_{-1} (where -1 represents minus the sum of the weights of the generators of S) from which we deduce that

$$\omega_\Delta = \mathrm{Hom}_R(\Delta, \omega_R) \cong \begin{pmatrix} S_{-1} & R \\ S_{-2} & S_{-1} \end{pmatrix}.$$

Both Δ and ω_Δ are graded for the standard grading on R. For this choice of grading, Δ is graded semi-local and ω_Δ is (as left or right module) not a direct sum of indecomposable graded Δ-projectives. Consequently, ω_Δ is not projective.

By Proposition 2.6, Δ is therefore not homologically homogeneous.

Remarks 5.1. (1) By the proof of [DV, Proposition A.2] it follows that ω_Δ defines an element of the *derived* Picard group of Δ.

(2) The methods of [BLV] allow one to treat this example in the context of determinantal varieties. It follows from the results given there that one of the simple graded Δ-modules has projective dimension 4 and the other has projective dimension 5.

The example leads naturally to the following question.

Question 5.2. Assume that $Z = Z(\Delta)$ is an affine normal k-domain and that Δ is a finitely generated CM Z-module with finite global dimension. Then, does Z have rational singularities?

We now turn to an example in finite characteristic of a homologically homogeneous ring whose centre is CM but which does not have rational singularities in any reasonable sense.

Assume that F is a field of characteristic 2 and let $C = F[u, v, x, y]/(p, q)$ where

$$p = x + u^2 + x^2u \qquad \text{and} \qquad q = y + v^2 + y^2v.$$

As the Jacobian matrix of p, q with respect to x, y is invertible, $C/F[x, y]$ is étale and hence C is regular. Consider the action of $G = \mathbb{Z}/(2) = \{1, \sigma\}$ on C by $\sigma(u) = u + x^2$, $\sigma(v) = v + y^2$, $\sigma(x) = x$, and $\sigma(y) = y$. Clearly $B = C^G$ is an affine normal domain of Krull dimension two and hence it is CM.

Resolutions of singularities are known exist for surfaces in all characteristics and there is a corresponding satisfactory theory of rational singularities. We will show that B does not have rational singularities. Let $\mathfrak{m} = (u, v) \subset C$ and notice that $\mathfrak{m} = (u, v, x, y)$ is maximal; thus $\widehat{C}_\mathfrak{m} = F[[u, v]]$. It suffices to prove that $\widehat{B}_\mathfrak{n}$, for $\mathfrak{n} = B \cap \mathfrak{m}$, does not have rational singularities. Since $uu^\sigma = u^2 + ux^2 = x \in \widehat{C}^G_\mathfrak{m} = \widehat{B}_\mathfrak{n}$ and $vv^\sigma = y \in \widehat{B}_\mathfrak{n}$, our notation conforms with that of [Ar, Theorem]. Now the fact that $u^2 + x^2u + x = 0 = v^2 + y^2v + y$ means that $\widehat{B}_\mathfrak{n}$ does not have rational singularities by the observation from [Ar, p. 64].

Finally, let $\Lambda = C[x; \sigma]$ be the twisted polynomial ring; thus $xc = c^\sigma x$ for all $c \in C$. By the Nullstellensatz, every simple Λ-module is finite dimensional and so, by [MR, Theorem 7.9.16], Λ is homologically homogeneous of dimension 3. As $\sigma^2 = 1$, the element x^2 is central. It follows routinely that $Z(\Lambda) = B[x^2]$. Thus, $Z(\Lambda)$ also does not have rational singularities.

The basic reason why such counterexamples exist in bad characteristic is that a fixed ring S^G need not be a summand of the ring S. The example [BHM, Example 7.3] of a homologically homogeneous ring with a non-CM centre occurs for a similar reason. So, it is natural to ask:

Question 5.3. Suppose that Λ is a homologically homogeneous ring whose centre $Z(\Lambda)$ is an affine F-algebra for field F of characteristic $p > 0$. If $Z(\Lambda)$ is a $Z(\Lambda)$-module summand of Λ, then does $Z(\Lambda)$ have rational singularities?

It was conjectured in [VB3] and proved in [VB2, Theorem 6.6.3] that a 3-dimensional k-variety with terminal singularities has a noncommutative crepant

resolution if and only if it has a commutative one (see also [IR, Corollary 8.8]). We end by noting that this is not true in higher dimensions. One way to produce counterexamples is with the fixed ring $R = \mathbb{C}[V]^G$ of a finite group $G \subset SL(V)$, where $V = \mathbb{C}^n$. In this case, the twisted group ring $\mathbb{C}[V] * G \cong \mathrm{End}_R(\mathbb{C}[V])$ is a noncommutative crepant resolution of R [VB2, Example 1.1], but it is well-known that such a ring R need not have a commutative crepant resolution (see, for example, [Kl1, Theorem 1.7]).

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